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Author(s)	Kato, Kazuya; Nakayama, Chikara; Usui, Sampei
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CLASSIFYING SPACES OF DEGENERATING MIXED HODGE STRUCTURES, III: SPACES OF NILPOTENT ORBITS

Kazuya Kato[1], Chikara Nakayama[2], Sampei Usui[3]

Abstract. We construct toroidal partial compactifications of the moduli spaces of mixed Hodge structures with polarized graded quotients. They are moduli spaces of log mixed Hodge structures with polarized graded quotients. We construct them as the spaces of nilpotent orbits.

Contents

Introduction

This is Part III of our series of papers to study degeneration of mixed Hodge structures, and an expanded, full detailed version of our previous three announcements [KNU10a], [KNU10b], and [KNU10c]. In this part, we construct toroidal partial compactifications of the moduli spaces of mixed Hodge structures with polarized graded quotients, study their properties, and prove that they are moduli spaces of log mixed Hodge structures with polarized graded quotients. We also apply them to investigate Néron models including degenerations of intermediate Jacobians.

We explain the above more precisely. For a complex analytic space S with an fs log structure, a log mixed Hodge structure (LMH, for short) on S is, roughly speaking, an analytic family parametrized by S of "mixed Hodge structures which may have logarithmic degeneration".

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The main subject of this paper is to construct a moduli space of LMH. (Actually, to construct it, we fix certain data such as the Hodge type of graded quotients.) A morphism from S to this moduli space is in one-to-one correspondence with an isomorphism class of LMH on S. This space gives a toroidal partial compactification of the moduli space of mixed Hodge structure. This moduli space is not necessarily a complex analytic space, but it belongs to a category $\mathcal{B}(\log)$ which contains the category of fs log analytic spaces as a full subcategory. (An object of $\mathcal{B}(\log)$ is a space which is locally isomorphic to a strong subspace of an fs log analytic space. See 1.1.) In the category $\mathcal{B}(\log)$, the space is a fine moduli space. See Theorems 2.6.6 and 5.3.3.

This paper is the mixed Hodge-theoretic version of the main part of [KU09] (a summary of which is [KU99]), where the degeneration of pure Hodge structures was studied. That is, we generalize results in [KU09] to the mixed Hodge-theoretic contexts. In particular, we prove that the above moduli space has nice properties, especially, that it is a log manifold. See Theorems 2.5.1–2.5.6 and 5.2.8. Every proof goes in a parallel way to the pure case [KU09].

In addition, we further enhance the story by replacing "fan" in [KU09] with "weak fan", which is introduced in [KNU10c]. See 2.2.3 and 5.1.6 for the definition of a weak fan. Roughly speaking, a weak fan is a set of cones which admits an overlapping under a certain condition, i.e., the intersection of two members may not be a face of them under a certain condition. With this relaxed concept "weak fan", the theory can be applied more vastly. An example of such applications is the study of degenerations of intermediate Jacobians. To construct Néron models of intermediate Jacobians by our method (see §1.4 for an outline), the concept of fan is not enough, and that of weak fan is indispensable (see §7.2 for detail examples). In this paper, we illustrate a few of such applications, but this important subject of degenerations of intermediate Jacobians should be investigated more in a forthcoming paper. Thus, this paper is more general than [KU09] even in the pure case, though the proofs are still parallel.

We also consider a relative moduli theory fixing a base S and graded pieces in the following sense. That is, we assume that we are given a collection $(H_{(w)})_{w\in\mathbf{Z}}$, where $H_{(w)}$ is a pure, polarized log Hodge structure of weight w on an object S of $\mathcal{B}(\log)$. We construct the moduli space of LMH with the given graded quotients $(H_{(w)})_{w\in\mathbf{Z}}$ (5.3.3).

The relationship between the absolute moduli explained previously and these relative moduli is roughly as follows. If S_0 is the product of the moduli of pure log Hodge structures on each graded pieces constructed in [KU09], the absolute moduli space is the relative moduli space over S_0 for the collection $(H_{(w)})_{w\in\mathbf{Z}}$ of universal families of pure log Hodge structures $H_{(w)}$ of weight w on S_0. In this sense, the relative theory plus the pure theory implies the absolute theory. Conversely, if the given collection $(H_{(w)})_{w\in\mathbf{Z}}$ on S corresponds to a period map $S \to S_0$, the relative moduli space is the pullback of the absolute moduli space by this period map. Actually, the relative theory has more advantages than the absolute one in the study of Néron models because sometimes the period map exists only after blowing up the base S. This is the reason why we develop the theory of Néron models based on the relative theory (as is explained in 1.4.4), not on the absolute theory. See 5.5.3 for more details of this last point.

We organize the paper as follows. §1 is preliminary. §2 is the absolute theory with proofs in §3–§4. §5 is the relative theory. In §6–§7, we gather applications, examples, and complements.

More closely, each section is as follows.

In §1, we review the notion of log mixed Hodge structures, explaining the concept of admissibility of the local monodromy, which appears in the mixed Hodge case. In §1.4, using these terminology, we outline the construction of relative moduli in some important cases including Néron models of intermediate Jacobians to motivate the paper.

In §2, we state the absolute theory, that is, we consider the moduli space of log mixed Hodge structures with polarized graded quotients for the weight filtration. Let D be a classifying space of mixed Hodge structures with polarized graded quotients ([U84]). It is a complex analytic manifold, and is a mixed Hodge-theoretic generalization of a classifying space of polarized Hodge structures defined by Griffiths ([G68a]). The absolute moduli we consider are toroidal partial compactifications $\Gamma\backslash D_\Sigma$ of $\Gamma\backslash D$ for discrete subgroups Γ of $\mathrm{Aut}(D)$. We list several nice properties (2.5.1–2.5.6) of this space and state a theorem (2.6.6) that it is a moduli space of LMH with polarized graded quotients. Here D_Σ is a space of nilpotent orbits in the directions in Σ. This space sits in the following fundamental diagram.

$$\begin{array}{ccccccc}
 & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS},\mathrm{val}} \\
 & & & & \downarrow & & \downarrow \\
D_{\Sigma,\mathrm{val}} & \leftarrow & D^\sharp_{\Sigma,\mathrm{val}} & \stackrel{\psi}{\to} & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \\
\downarrow & & \downarrow & & & & \\
D_\Sigma & \leftarrow & D^\sharp_\Sigma & & & &
\end{array}$$

We constructed the Borel-Serre space D_{BS} in Part I of this series of papers ([KNU09]), and the space of SL(2)-orbits $D_{\mathrm{SL}(2)}$ in Part II ([KNU11]). The left four spaces are variants of a space of nilpotent orbits and constructed in this paper. The most important map ψ associates an SL(2)-orbit to a nilpotent orbit by the mixed Hodge-theoretic version ([KNU08]) of the SL(2)-orbit theorem by Cattani-Kaplan-Schmid ([CKS86]).

For the degeneration of polarized (pure) Hodge structures, this diagram was constructed in [KU09].

The results of this section 2 was announced in [KNU10b], but only in the case of fan, not weak fan. Hence, this section is essentially the same as [KNU10b] rewritten by using weak fan.

The proofs of the main results in §2 are given in §3 and §4, by using the theory of SL(2)-orbits in Part II. As is said above, the proofs are parallel to those in the pure case.

In §5, we prove the relative theory in the sense explained in the above. As was said, we assumed that all $H_{(w)}$ are polarizable, but we expect that the construction works without assuming $H_{(w)}$ are polarizable. In some cases, we can show that this is true (see 7.1.4).

The relation of results of §2 (absolute theory) and those of §5 (relative theory) is discussed in 5.5.

In §6, we give the theory of Néron models which appear for example when an intermediate Jacobian degenerates.

In §7, we give examples and explain the theories of log abelian varieties and log complex tori in [KKN08] from the point of view of this paper. We also explain the necessity of weak fan and discuss completeness of weak fans and extensions of period maps.

We greatly appreciate valuable comments from the referee.

Notations.

Λ	fixed data including Hodge numbers (2.1.1, 5.1.1)
D	classifying space of mixed Hodge structures with polarized graded quotients (2.1.2)
$G_{\mathbf{Z}}$	group of automorphisms of the lattice (2.1.3)
$\mathfrak{g}_{\mathbf{Q}}$	Lie algebra (2.1.3)
Σ	weak fan (2.2.3, 5.1.6)
D_{Σ}	space of nilpotent orbits (2.2.5)
$D_{S,\Sigma}$	space of nilpotent orbits (5.1.10)
$G'_{\mathbf{Z}}$	$G_{\mathbf{Z}}(\mathrm{gr}^W)$ (5.1.1)
Γ', σ'	fixed data in the relative case (5.1.1)
Γ	subgroup of $G_{\mathbf{Z}}$ or $G' \times_{G'_{\mathbf{Z}}} G_{\mathbf{Z}}$ (2.2.6, 5.1.7)
J_{Σ}	$\Gamma\backslash D_{S,\Sigma}$ (5.3.2)
J_1	Néron model (6.1.1)
J_0	connected Néron model (6.1.1)
$\Sigma(\Upsilon)$	weak fan associated to a subgroup Υ of $G_{\mathbf{Q},u}$ (6.2.1)

§1. Log mixed Hodge structures

§1.1. The category $\mathcal{B}(\log)$ and relative log manifolds

We review the category $\mathcal{B}(\log)$ introduced in [KU09], which plays a central role in this paper. We introduce a notion "relative log manifold" (1.1.6), which is a relative version of the notion "log manifold" introduced in [KU09] and which becomes important in §5.

1.1.1. Strong topology.

Let Z be a local ringed space over $\mathbf{C}$, and S a subset of Z. The *strong topology* of S in Z is defined as follows. A subset U of S is open for this topology if and only if for any analytic space A (analytic space means a complex analytic space) and any morphism $f : A \to Z$ of local ringed spaces over $\mathbf{C}$ such that $f(A) \subset S$, $f^{-1}(U)$ is open in A. This topology is stronger than or equal to the topology as a subspace of Z.

Remark. In [KU09], we only consider the strong topology of a subset of an analytic space. Here we work with any local ringed space over $\mathbf{C}$. Thus, we can also consider the strong topology of a subset of, say, a log manifold. This change increases the flexibility.

1.1.2. Let Z be a local ringed space (resp. an fs log local ringed space, i.e., a local ringed space endowed with an fs log structure ([KU09] 2.1.5)) over $\mathbf{C}$, and S a subset of Z. The *strong subspace of Z defined by S* is the subset S with the strong topology in Z (1.1.1) and with the inverse image of $\mathcal{O}_Z$ (resp. the inverse images of $\mathcal{O}_Z$ and M_Z, where M_Z denotes the log structure of Z). A morphism $\iota : S \to Z$ of local ringed spaces (resp. fs log local ringed spaces) over $\mathbf{C}$ is a *strong immersion* if S is isomorphic to a strong subspace of Z via ι. If this is the case, it is easy to see by definition that any morphism $A \to Z$ from an analytic space (resp. an fs log analytic space) uniquely factors through S if and only if it factors through S set-theoretically.

Lemma 1.1.3. *A composite of two strong immersions is a strong immersion.*

Proof. It is enough to show the following: Let Z be a local ringed space over $\mathbf{C}$ and $T \subset S \subset Z$ be two subsets of Z. Endow S with the strong topology in Z. Then, the strong topology of T in Z coincides with the strong topology of T in S.

First, let U be an open subset of T with respect to the strong topology of T in Z. Let $A \to S$ be a morphism from an analytic space, whose image is contained in T. Then, the image of the composite $A \to S \to Z$ is also contained in T and the inverse image of U in A is open. Hence, U is open with respect to the strong topology of T in S. Conversely, let U be an open subset of T with respect to the strong topology of T in S. Let $A \to Z$ be a morphism from an analytic space. If its image is contained in T, then it is contained in S, and $A \to Z$ factors through $A \to S$ (1.1.2). Since the image of the last morphism is contained in T, the inverse image of U in A is open. Therefore, U is open with respect to the strong topology of T in Z. □

1.1.4. The category $\mathcal{B}(\log)$.

As in [KU09], in the theory of moduli spaces of log Hodge structures, we have to enlarge the category of analytic spaces, for the moduli spaces are often not analytic spaces.

Let $\mathcal{A}$ be the category of analytic spaces and let $\mathcal{A}(\log)$ be the category of fs log analytic spaces (i.e., analytic spaces endowed with an fs log structure). We enlarge $\mathcal{A}$ and $\mathcal{A}(\log)$ to $\mathcal{B}$ and $\mathcal{B}(\log)$, respectively, as follows. In this paper, we will work mainly in the category $\mathcal{B}(\log)$, and, as we did in [KU09], we will find the moduli spaces in $\mathcal{B}(\log)$.

$\mathcal{B}$ (resp. $\mathcal{B}(\log)$) is the category of all local ringed spaces S over $\mathbf{C}$ (resp. local ringed spaces over $\mathbf{C}$ endowed with a log structure) having the following property:

S is locally isomorphic to a strong subspace of an analytic space (resp. fs log analytic space).

Note that these definitions are restatements of those in [KU09] 3.2.4. The category $\mathcal{B}(\log)$ has fiber products ([KU09] 3.5.1).

1.1.5. Log manifold ([KU09] 3.5.7).

In [KU09], we saw that moduli spaces of polarized log Hodge structures were logarithmic manifolds (abbreviated as log manifolds in this paper).

An object S of $\mathcal{B}(\log)$ is a *log manifold* if, locally on S, there are an fs log analytic space Z which is log smooth over $\mathbf{C}$ ([KU09] 2.1.11), elements $\omega_1, \dots, \omega_n$ of $\Gamma(Z, \omega^1_Z)$, and an open immersion from S to the strong subspace of Z defined by the subset $\{z \in Z \mid \omega_1, \dots, \omega_n \text{ are zero in } \omega^1_z\}$. Here ω^1_Z denotes the sheaf of log differential 1-forms on Z ([KU09] 2.1.7), and ω^1_z denotes the space of log differential 1-forms on the point $z = \operatorname{Spec}(\mathbf{C})$ which is endowed with the inverse image of the log structure of Z.

Example. $S = ((\mathbf{C} \times \mathbf{C}) \smallsetminus (\{0\} \times \mathbf{C})) \cup \{(0,0)\}$ has a natural structure of a log manifold. See [KU09] 0.4.17.

1.1.6. *Relative log manifold* over an object S of $\mathcal{B}(\log)$.

It is an object T of $\mathcal{B}(\log)$ over S such that, locally on S and on T, there exist a log smooth morphism $Y \to X$ of fs log analytic spaces over $\mathbf{C}$, a morphism $S \to X$, an open subspace U of $S \times_X Y$, elements $\omega_1, \dots, \omega_n \in \Gamma(U, \omega^1_U)$, and an open immersion from T to the strong subspace of U defined by the subset $\{u \in U \mid \omega_1, \dots, \omega_n \text{ are zero in } \omega^1_u\}$.

§1.2. Relative monodromy filtrations

1.2.1. Let V be an abelian group, let $W = (W_w)_{w \in \mathbf{Z}}$ be a finite increasing filtration on V, and let $N \in \operatorname{End}(V, W)$. Assume that N is nilpotent.

Then a finite increasing filtration $M = (M_w)_{w \in \mathbf{Z}}$ on V is called the *relative monodromy filtration of N with respect to W* if it satisfies the following conditions (1) and (2).

(1) $NM_w \subset M_{w-2}$ for any $w \in \mathbf{Z}$.

(2) For any $w \in \mathbf{Z}$ and $m \geq 0$, we have an isomorphism

$$N^m : \operatorname{gr}^M_{w+m} \operatorname{gr}^W_w \xrightarrow{\sim} \operatorname{gr}^M_{w-m} \operatorname{gr}^W_w .$$

The relative monodromy filtration of N with respect to W need not exist. If it exists, it is unique ([D80] 1.6.13). If it exists, we denote it by $M(N, W)$ as in [SZ85] (2.5). It exists in the case where W is pure (i.e. $W_w = V$ and $W_{w-1} = 0$ for some w) ([D80] 1.6.1).

Relative monodromy filtration is also called relative weight filtration, or relative monodromy weight filtration.

If V is a vector space over a field K and the W_w are K-vector subspaces of V, and if N is a K-linear map, then the relative monodromy filtration, if it exists, is formed by K-subspaces of V.

Lemma. *If K is of characteristic 0, for a finite increasing filtration M on V consisting of K-vector subspaces, M is the relative monodromy filtration $M(N, W)$ if and only if the above (1) is satisfied and, for any $w \in \mathbf{Z}$, there is an action of the Lie algebra $\mathfrak{sl}(2, K)$ on gr^W_w satisfying the following conditions (3) and (4). Let $Y := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{sl}(2, K)$.*

(3) *For $k \in \mathbf{Z}$, $M_k \operatorname{gr}_w^W$ is the sum of $\{v \in \operatorname{gr}_w^W \mid Yv = \ell v\}$ for all integers $\ell \le k - w$.*

(4) *$N : \operatorname{gr}_w^W \to \operatorname{gr}_w^W$ coincides with the action of* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, K)$.

Proof. This reduces to [KNU08] 2.2. □

We have the following compatibility of relative monodromy filtrations with various operators (direct sum, tensor product, etc.). Much of this is in Deligne's appendix to [SZ85]. For $j = 1, 2$, let V_j be a vector space over a field K endowed with a finite increasing filtration $W_\bullet V_j$ consisting of K-vector subspaces, and let $N_j \in \operatorname{End}_K(V_j, W_\bullet V_j)$ be nilpotent.

1.2.1.1. Assume that the relative monodromy filtration $M_\bullet V_j = M(N_j, W_\bullet V_j)$ exists. Then, the filtration $M_\bullet V_1 \oplus M_\bullet V_2$ on $V_1 \oplus V_2$ is the relative monodromy filtration of $N_1 \oplus N_2 \in \operatorname{End}(V_1 \oplus V_2, W_\bullet V_1 \oplus W_\bullet V_2)$. Assume furthermore that K is of characteristic 0. Then, the filtration $M_\bullet V_1 \otimes M_\bullet V_2$ on $V_1 \otimes V_2$ is the relative monodromy filtration of $N_1 \otimes 1 + 1 \otimes N_2 \in \operatorname{End}(V_1 \otimes V_2, W_\bullet V_1 \otimes W_\bullet V_2)$. (Here the w-th filter of the filtration $M_\bullet V_1 \otimes M_\bullet V_2$ is $\sum_{j+k=w} M_j V_1 \otimes M_k V_2$, and the filtration $W_\bullet V_1 \otimes W_\bullet V_2$ is defined similarly.) The filtration M on $H := \operatorname{Hom}_K(V_1, V_2)$ defined by $M_w = \{f \in \operatorname{Hom}_K(V_1, V_2) \mid f(M_k V_1) \subset M_{k+w} V_2 \text{ for all } k\}$ is the relative monodromy filtration of $N \in \operatorname{End}(H, W)$, where W is the filtration on H defined by $W_w H = \{f \in \operatorname{Hom}_K(V_1, V_2) \mid f(W_k V_1) \subset W_{k+w} V_2 \text{ for all } k\}$ and $N : H \to H$ is the map $f \mapsto N_2 f - f N_1$.

1.2.1.2. Assume that the relative monodromy filtration M of $N_1 \oplus N_2 \in \operatorname{End}(V_1 \oplus V_2, W_\bullet V_1 \oplus W_\bullet V_2)$ exists. Then the relative monodromy filtration $M_\bullet V_j = M(N_j, W_\bullet V_j)$ exists for $j = 1, 2$, and $M = M_\bullet V_1 \oplus M_\bullet V_2$.

By 1.2.1.1 and 1.2.1.2, we have the compatibility with symmetric powers and exterior powers because these are direct summands of tensor products.

These 1.2.1.1 and 1.2.1.2 are certainly well-known. They are proved as follows.

Proof of 1.2.1.1. We give the proof for the tensor product. The proof for Hom is similar and the proof for the direct sum is easy. For $j = 1, 2$ and $w \in \mathbf{Z}$, let $\rho_{j,w}$ be the action of $\mathfrak{sl}(2, K)$ on $\operatorname{gr}_w^W V_j$ satisfying the conditions (3) and (4) in Lemma in 1.2.1 (with M and W in (3) and (4) replaced by $M_\bullet V_j$ and $W_\bullet V_j$, respectively, and with N in (4) replaced by N_j). For $w \in \mathbf{Z}$, let ρ_w be the representation of $\mathfrak{sl}(2, K)$ on $\bigoplus_{k+\ell=w} \operatorname{gr}_k^W V_1 \otimes \operatorname{gr}_\ell^W V_2$ defined to be $\bigoplus_{k+\ell=w} \rho_{1,k} \otimes \rho_{2,\ell}$. Then ρ_w satisfies the conditions (3) and (4) in Lemma in 1.2.1 (with M and W in (3) and (4) replaced by $M_\bullet V_1 \otimes M_\bullet V_2$ and $W_\bullet V_1 \otimes W_\bullet V_2$, respectively, and with N in (4) replaced by $N_1 \otimes 1 + 1 \otimes N_2$). Hence $M_\bullet V_1 \otimes M_\bullet V_2$ is the relative monodromy filtration of $N_1 \otimes 1 + 1 \otimes N_2$ with respect to $W_\bullet V_1 \otimes W_\bullet V_2$. □

Proof of 1.2.1.2. By the uniqueness of the relative monodromy filtration, M is invariant under the maps $V_1 \oplus V_2 \xrightarrow{\sim} V_1 \oplus V_2$, $(x, y) \mapsto (ax, by)$ for all $a, b \in K^\times$. If K is not $\mathbf{F}_2$, this shows that M is the direct sum of a filtration M' on V_1 and a filtration M'' on V_2. Also, in the case $K = \mathbf{F}_2$, too, the last property can be proved by using a finite extension of K to get a, b such that $a \neq b$. It is easy to see that M' (resp. M'') is the relative monodromy filtration of N_1 (resp. N_2) with respect to $W_\bullet V_1$ (resp. $W_\bullet V_2$). □

Remark. If the characteristic of K is $p > 0$, 1.2.1.1 need not hold for tensor products and Hom. For example, let V_1 be of dimension p and with basis $e_1, \dots, e_p$ such that $N_1(e_1) = 0$ and $N_1(e_j) = e_{j-1}$ for $2 \le j \le p$, let V_2 be of dimension 2 with basis e'_1, e'_2 such that $N_2(e'_1) = 0$ and $N_2(e'_2) = e'_1$, and let $W_0 V_j = V_j$ and $W_{-1} V_j = 0$ for $j = 1, 2$. Then for $M = M_{\bullet} V_1 \otimes M_{\bullet} V_2$, gr^M_p (resp. gr^M_{-p}) is of dimension 1 and generated by the class of $e_p \otimes e'_2$ (resp. $e_1 \otimes e'_1$). But $(N_1 \otimes 1 + 1 \otimes N_2)^p$ induces the zero map $\mathrm{gr}^M_p \to \mathrm{gr}^M_{-p}$ because $(N_1 \otimes 1 + 1 \otimes N_2)^p = N_1^p \otimes 1 + 1 \otimes N_2^p$ and $N_1^p(e_p) = 0$, $N_2^p(e'_2) = 0$.

In §6, in the proof of Theorem 6.2.1, we will use the following fact concerning the relative monodromy filtration.

1.2.1.3. Let (V, W) and N be at the beginning of 1.2.1, and assume that the relative monodromy filtration $M = M(N, W)$ exists. Then we have

$$\mathrm{Ker}(N) \cap W_w \subset M_w \quad \text{for all } \ w \in \mathbf{Z}.$$

Proof of 1.2.1.3. We prove $\mathrm{Ker}(N) \cap W_w \subset M_w + W_k$ $(w, k \in \mathbf{Z})$ by downward induction on k starting from the trivial case $k = w$. Assume that we have already shown $\mathrm{Ker}(N) \cap W_w \subset M_w + W_k$ with some $k \le w$. Let $x \in \mathrm{Ker}(N) \cap W_w$, and write $x = y + z$ with $y \in M_w$ and $z \in W_k$. Since $N : \mathrm{gr}^M_j(\mathrm{gr}^W_k) \to \mathrm{gr}^M_{j-2}(\mathrm{gr}^W_k)$ are injective for all $j > k$, we have that the map $N : \mathrm{gr}^W_k / M_j \, \mathrm{gr}^W_k \to \mathrm{gr}^W_k / M_{j-2} \, \mathrm{gr}^W_k$ is injective if $j \ge k$. Since $N(z) = -N(y) \in M_{w-2}$ and $w \ge k$, this shows that the class of z in gr^W_k belongs to $M_w \, \mathrm{gr}^W_k$. Hence we can write $z = y' + z'$ with $y' \in M_w$ and $z' \in W_{k-1}$. Hence $x = (y + y') + z' \in M_w + W_{k-1}$. $\square$

1.2.2. Let σ be a cone in some $\mathbf{R}$-vector space (this means that $0 \in \sigma$, σ is stable under the addition, and stable under the multiplication by elements of $\mathbf{R}_{\ge 0}$), and let $\sigma_{\mathbf{R}}$ be the $\mathbf{R}$-linear span of σ. Assume that the cone σ is finitely generated. Let V be another $\mathbf{R}$-vector space endowed with a finite increasing filtration W (by $\mathbf{R}$-linear subspaces). Assume that we are given an $\mathbf{R}$-linear map

$$h : \sigma_{\mathbf{R}} \to \mathrm{End}_{\mathbf{R}}(V, W)$$

whose image consists of mutually commuting nilpotent operators.

We say that the action of σ on V via h is *admissible (with respect to W)* if there exists a family $(M(\tau, W))_\tau$ of finite increasing filtrations $M(\tau, W)$ on V given for each face τ of σ satisfying the following conditions (1)–(4).

(1) $M(\sigma \cap (-\sigma), W) = W$.

(2) For any face τ of σ, any $N \in \sigma$ and any $w \in \mathbf{Z}$, we have $h(N) M(\tau, W)_w \subset M(\tau, W)_w$.

(3) For any face τ of σ, any $N \in \tau$ and any $w \in \mathbf{Z}$, we have

$$h(N) M(\tau, W)_w \subset M(\tau, W)_{w-2}.$$

(4) For any faces τ, τ' of σ and for any $N \in \sigma$ such that τ' is the smallest face of σ containing τ and N, $M(\tau', W)$ is the relative monodromy filtration of $h(N)$ with respect to $M(\tau, W)$.

(We remark that in [KKN08] 2.1.4, the above condition (2) was missed.)

We call the above filtration $M(\tau, W)$ the *relative monodromy filtration of τ with respect to W*. The condition (4) (by considering the case where τ and $\sigma \cap (-\sigma)$ are taken as τ' and τ in (4), respectively) implies that $M(\tau, W) = M(N, W)$ for any N in the interior of τ.

By 1.2.1.1 and 1.2.1.2, we have the following compatibility of this admissibility with various operators. For $j = 1, 2$, let V_j be a vector space over $\mathbf{R}$ endowed with a finite increasing filtration $W_\bullet V_j$, and assume that we are given an $\mathbf{R}$-linear map $\sigma_{\mathbf{R}} \to \mathrm{End}_{\mathbf{R}}(V_j, W_\bullet V_j)$ whose image consists of mutually commuting nilpotent operators.

1.2.2.1. Assume that for $j = 1, 2$, the action of σ on V_j is admissible with respect to $W_\bullet V_j$. Then, the diagonal action of σ on $V_1 \oplus V_2$ is admissible with respect to the filtration $W_\bullet V_1 \oplus W_\bullet V_2$, and $M(\tau, W_\bullet V_1 \oplus W_\bullet V_2) = M(\tau, W_\bullet V_1) \oplus M(\tau, W_\bullet V_2)$ for any face τ of σ. The action of σ on $V_1 \otimes V_2$ ($N \in \sigma$ acts by sending $x \otimes y$ to $Nx \otimes y + x \otimes Ny$) is admissible with respect to the filtration $W_\bullet V_1 \otimes W_\bullet V_2$, and $M(\tau, W_\bullet V_1 \otimes W_\bullet V_2) = M(\tau, W_\bullet V_1) \otimes M(\tau, W_\bullet V_2)$. The action of σ on $H := \mathrm{Hom}_{\mathbf{R}}(V_1, V_2)$ ($N \in \sigma$ acts by $f \mapsto Nf - fN$) is admissible with respect to the filtration $W_\bullet H$ defined by $W_w H = \{f \in \mathrm{Hom}_{\mathbf{R}}(V_1, V_2) \mid f(W_k V_1) \subset W_{k+w} V_2 \text{ for all } k\}$, and $M(\tau, W_\bullet H)$ on H is given by $M(\tau, W_\bullet H)_w = \{f \in \mathrm{Hom}_{\mathbf{R}}(V_1, V_2) \mid f(M(\tau, W)_k V_1) \subset M(\tau, W)_{k+w} V_2 \text{ for all } k\}$.

1.2.2.2. Assume that the diagonal action of σ on $V_1 \oplus V_2$ is admissible with respect to $W_\bullet V_1 \oplus W_\bullet V_2$. Then the action of σ on V_j is admissible with respect to $W_\bullet V_j$, and $M(\tau, W_\bullet V_1 \oplus W_\bullet V_2) = M(\tau, W_\bullet V_1) \oplus M(\tau, W_\bullet V_2)$.

Lemma 1.2.3. *Let the notation be as in* 1.2.2*, let σ' be a finitely generated cone, and assume that we are given an $\mathbf{R}$-linear map $\sigma'_{\mathbf{R}} \to \sigma_{\mathbf{R}}$ which sends σ' in σ.*

(i) *Assume that the action of σ on V is admissible with respect to W. Then the action of σ' on V is also admissible with respect to W. For a face τ' of σ', $M(\tau', W) = M(\tau, W)$, where τ denotes the smallest face of σ which contains the image of τ'.*

(ii) *Assume that the action of σ' on V is admissible with respect to W and that the map $\sigma' \to \sigma$ is surjective. Then the action of σ on V is admissible with respect to W. For a face τ of σ, $M(\tau, W) = M(\tau', W)$, where τ' denotes the inverse image of τ in σ'.*

Proof. (i) This is essentially included in [KKN08] 2.1.5. The proof is easy.

(ii) Denote the given map $\sigma' \to \sigma$ by f. Let α and β be faces of σ, let $N \in \beta$, and assume that β is the smallest face of σ which contains α and N. It is sufficient to prove that $M(f^{-1}(\beta), W)$ is the relative monodromy filtration of $h(N)$ with respect to $M(f^{-1}(\alpha), W)$. Let x be an element of the interior of $f^{-1}(\beta)$. Since β is the smallest face of σ which contains α and N, there are $y \in \beta$, $z \in \alpha$, $c \in \mathbf{R}_{>0}$ such that $f(x) + y = z + cN$. Take $y' \in f^{-1}(\beta)$ and $z' \in f^{-1}(\alpha)$ such that $f(y') = y$ and $f(z') = z$, and let $u = x + y'$. Since u is in the interior of $f^{-1}(\beta)$, $M(f^{-1}(\beta), W)$ is the relative monodromy filtration of $h(f(u))$ with respect to $M(f^{-1}(\alpha), W)$. Since

$f(u) = f(z') + cN$, this shows that $M(f^{-1}(\beta), W)$ is the relative monodromy filtration of $h(N)$ with respect to $M(f^{-1}(\alpha), W)$. $\square$

1.2.4. Let S be an object of $\mathcal{B}(\log)$, and let the topological space $S^{\log}$ and the continuous proper map $\tau : S^{\log} \to S$ be as in [KN99], [KU09] 2.2. Let $L = (L, W)$ be a locally constant sheaf L of finite dimensional $\mathbf{R}$-vector spaces over $S^{\log}$ endowed with a finite increasing filtration W consisting of locally constant $\mathbf{R}$-subsheaves. We say that the local monodromy of L is *admissible* if the following two conditions (1) and (2) are satisfied for any $s \in S$ and any $t \in s^{\log} = \tau^{-1}(s) \subset S^{\log}$.

(1) For any $\gamma \in \pi_1(s^{\log})$, the action of γ on the stalk L_t is unipotent.

(2) Let $C(s) := \mathrm{Hom}((M_S/\mathcal{O}_S^\times)_s, \mathbf{R}^{\mathrm{add}}_{\geq 0})$, where $\mathbf{R}^{\mathrm{add}}_{\geq 0}$ denotes $\mathbf{R}_{\geq 0}$ regarded as an additive monoid. Then the action of the cone $C(s)$ on L_t via $C(s)_{\mathbf{R}} = \mathbf{R} \otimes \pi_1(s^{\log}) \to \mathrm{End}_{\mathbf{R}}(L_t, W)$, defined by $\gamma \mapsto \log(\gamma)$ $(\gamma \in \pi_1(s^{\log}))$, is admissible with respect to W.

Lemma 1.2.5. *Admissibility of local monodromy is preserved by pulling back by a morphism $S' \to S$ in $\mathcal{B}(\log)$.*

Proof. This follows from 1.2.3 (i). $\square$

1.2.6. Let the situation be as in 1.2.2. There are weaker versions of the notion of admissibility of a cone with respect to W.

For example, one version is

(1) The relative monodromy filtration $M(N, W)$ exists for any $N \in \sigma$.

Another version is

(2) The relative monodromy filtration $M(N, W)$ exists for any $N \in \sigma$, and it depends only on the smallest face of σ which contains N.

In the situation with polarized graded quotients explained in 1.3.3 below, it is shown that these weaker admissibilities coincide with the admissibility considered in this subsection §1.2. See 1.3.2–1.3.4.

In the papers [KNU10a], [KNU10b], and [KNU10c], we adopted the admissibility in (2) above. But the difference with the admissibility in this subsection is not essential for this paper. See 2.2.5, Remark 2.

§1.3. Log mixed Hodge structures

1.3.1. Log mixed Hodge structure (LMH, for short) on S ([KKN08] 2.3, [KU09] 2.6).

Let S be an object in $\mathcal{B}(\log)$. Recall that we have a sheaf of commutative rings $\mathcal{O}_S^{\log}$ on $S^{\log}$ which is an algebra over the inverse image $\tau^{-1}(\mathcal{O}_S)$ of $\mathcal{O}_S$ ([KN99], [KU09] 2.2).

A *pre-log mixed Hodge structure (pre-LMH) on S* is a triple $(H_{\mathbf{Z}}, W, H_{\mathcal{O}})$, where

$H_{\mathbf{Z}}$ is a locally constant sheaf of finitely generated free $\mathbf{Z}$-modules on $S^{\log}$,

W is a finite increasing filtration on $H_{\mathbf{R}}$ by locally constant rational $\mathbf{R}$-subsheaves, and

$H_{\mathcal{O}}$ is an $\mathcal{O}_S$-module on S which is locally free of finite rank, endowed with a decreasing filtration $(F^p H_{\mathcal{O}})_{p \in \mathbf{Z}}$ and endowed with an isomorphism of $\mathcal{O}_S^{\log}$-modules

$\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}} \simeq \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} H_{\mathcal{O}}$ on $S^{\log}$, satisfying the condition that $F^p H_{\mathcal{O}}$ and $H_{\mathcal{O}}/F^p H_{\mathcal{O}}$ are locally free of finite rank for any $p \in \mathbf{Z}$.

Denote $F^p := \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} F^p H_{\mathcal{O}}$ ($p \in \mathbf{Z}$) the filtration on $\mathcal{O}_S^{\log}$-module $\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} H_{\mathcal{O}} \simeq \mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$. The filtrations $F^p H_{\mathcal{O}}$ and F^p determine each other (in fact, $F^p H_{\mathcal{O}} = \tau_* F^p$) so that we can define a pre-LMH as a triple $(H_{\mathbf{Z}}, W, F)$, where F is a decreasing filtration on $\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$, satisfying the corresponding conditions as in [KU09] 2.6.1. In the rest of this paper, we freely use both formulations.

Let $s \in S$ and let $t \in \tau^{-1}(s) \subset S^{\log}$. By a *specialization* at t, we mean a ring homomorphism $\mathcal{O}_{S,t}^{\log} \to \mathbf{C}$ such that the composition $\mathcal{O}_{S,s} \to \mathcal{O}_{S,t}^{\log} \to \mathbf{C}$ is the evaluation at s. Since the evaluation at s is the unique $\mathbf{C}$-algebra homomorphism $\mathcal{O}_{S,s} \to \mathbf{C}$, any $\mathbf{C}$-algebra homomorphism $\mathcal{O}_{S,t}^{\log} \to \mathbf{C}$ is a specialization at t.

Let s be an fs log point (i.e., an fs log analytic space whose underlying analytic space is a one-point set endowed with the ring $\mathbf{C}$).

A pre-LMH $(H_{\mathbf{Z}}, W, H_{\mathcal{O}})$ on s is a *log mixed Hodge structure (LMH) on* s, if it satisfies the following conditions (1)–(3).

(1) The local monodromy of $(H_{\mathbf{R}}, W)$ is admissible in the sense of 1.2.4.

(2) Let $\nabla = d \otimes 1_{H_{\mathbf{Z}}} : \mathcal{O}_s^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}} \to \omega_s^{1,\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$. Then

$$\nabla F^p \subset \omega_s^{1,\log} \otimes_{\mathcal{O}_s^{\log}} F^{p-1} \quad \text{for all } p.$$

(3) Let $t \in s^{\log}$. For a specialization $a : \mathcal{O}_{s,t}^{\log} \to \mathbf{C}$ at t, let $F(a) := \mathbf{C} \otimes_{\mathcal{O}_{s,t}^{\log}} F_t$ be the filtration on $\mathbf{C} \otimes_{\mathbf{Z}} H_{\mathbf{Z},t}$. If a is sufficiently twisted in the sense explained below, then, for any face τ of the cone $C(s)$ in 1.2.4, $(H_{\mathbf{Z},t}, M(\tau, W), F(a))$ is a mixed Hodge structure in the usual sense.

Here, fixing a finite family $(q_j)_{1 \le j \le n}$ of elements of $M_s \smallsetminus \mathbf{C}^\times$ such that $(q_j \bmod \mathbf{C}^\times)_j$ generates $M_s/\mathbf{C}^\times$ (note that $\mathbf{C}^\times = \mathcal{O}_s^\times \subset M_s$), we say a is sufficiently twisted if $\exp(a(\log(q_j)))$ are sufficiently near 0. This notion "sufficiently twisted" is in fact independent of the choice of $(q_j)_j$.

Let S be an object in $\mathcal{B}(\log)$ again.

A pre-LMH on S is a *log mixed Hodge structure (LMH)* on S if its pullback to each fs log point $s \in S$ is an LMH on s ([KKN08] 2.3, [KU09] 2.6).

Remark. Direct sums, tensor products, symmetric powers, exterior powers, and inner-Hom of pre-LMH are defined in the evident ways. By using 1.2.2.1 and 1.2.2.2, we see that these operations preserve LMH.

For $w \in \mathbf{Z}$, we call a pre-LMH (resp. LMH) such that $W_w = H_{\mathbf{R}}$ and $W_{w-1} = 0$ simply a pre-log Hodge structure (pre-LH) (resp. log Hodge structure (LH)) of weight w. In this case, we often omit the W.

1.3.2. Polarized log Hodge structure (PLH) on S ([KU99] §5, [KKN08] 2.5, [KU09] 2.4).

Let w be an integer.

A *pre-polarized log Hodge structure (pre-PLH) of weight* w is a triple $(H_{\mathbf{Z}}, H_{\mathcal{O}}, \langle\ ,\ \rangle)$, where

$(H_{\mathbf{Z}}, H_{\mathcal{O}})$ is a pre-LH of weight w, and

$\langle\ ,\ \rangle : H_{\mathbf{R}} \times H_{\mathbf{R}} \to \mathbf{R}$ is a rational non-degenerate $(-1)^w$-symmetric bilinear form, satisfying $\langle F^p, F^q\rangle = 0$ when $p + q > w$.

A pre-PLH $(H_{\mathbf{Z}}, H_{\mathcal{O}}, \langle\ ,\ \rangle)$ of weight w is a *polarized log Hodge structure (PLH) of weight* w if for any $s \in S$, the pullback of $(H_{\mathbf{Z}}, H_{\mathcal{O}}, \langle\ ,\ \rangle)$ to the fs log point s has the property 1.3.1 (2) and the following property: In the notation t, a and $F(a)$ in 1.3.1 (3), $(H_{\mathbf{Z},t}, \langle\ ,\ \rangle, F(a))$ is a polarized Hodge structure of weight w in the usual sense for any sufficiently twisted specialization a (1.3.1).

A PLH of weight w is an LH of weight w. This follows from results of [CK82] and [Scm73].

Remark. Equivalently, a pre-PLH $(H_{\mathbf{Z}}, H_{\mathcal{O}}, \langle\ ,\ \rangle)$ of weight w is a PLH if for any $s \in S$, its pullback to s has the following properties (1) and (2). This is by [CKS] (4.66). Let $C(s)$ be the cone in 1.2.4. Fix a set of generators $\gamma_1, \dots, \gamma_n$ of $C(s)$. Let $t \in s^{\log}$. Let a be a specialization. By [KU09] 2.3.3, (1) in 1.2.4 is satisfied and we can define $N_j := \log(\gamma_j) \in \mathrm{End}(H_{\mathbf{R},t})$ for all j.

(1) Let $N = y_1N_1 + \cdots + y_nN_n$ for $y_1, \dots, y_n > 0$. Then the filtration $W(N)$ is independent of $y_1, \dots, y_n$.

(2) $(H_{\mathbf{Z},t}, \langle\ ,\ \rangle, W(N)[-w], F(a))$ is polarized by N.

1.3.3. By results of Cattani-Kaplan, Schmid, and Kashiwara, under the existence of polarizations on graded quotients, the definition of LMH becomes simpler as follows.

Let s be an fs log point and let $(H_{\mathbf{Z}}, W, H_{\mathcal{O}})$ be a pre-LMH on s. Assume that $(H_{\mathbf{Z}}, W, H_{\mathcal{O}})$ satisfies the condition 1.3.1 (2). Assume moreover that, for each $w \in \mathbf{Z}$, a rational non-degenerate $(-1)^w$-symmetric bilinear form $\langle\ ,\ \rangle_w : H(\mathrm{gr}^W_w)_{\mathbf{R}} \times H(\mathrm{gr}^W_w)_{\mathbf{R}} \to \mathbf{R}$ is given and that the triple $(H(\mathrm{gr}^W_w)_{\mathbf{Z}}, H(\mathrm{gr}^W_w)_{\mathcal{O}}, \langle\ ,\ \rangle_w)$ is a PLH. Let $t \in s^{\log}$. Let $\gamma_1, \dots, \gamma_n$ and $N_1, \dots, N_n$ be as in the remark in 1.3.2.

Proposition 1.3.4. *Let the assumption be as in* 1.3.3*. Then, the following are equivalent.*

(1) $(H_{\mathbf{Z}}, W, H_{\mathcal{O}})$ *is an LMH.*

(2) *The local monodromy of* $(H_{\mathbf{R}}, W)$ *is admissible.*

(3) *The relative monodromy filtrations* $M(N_j, W)$ *exist for all* j*.*

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are evident.

We prove (2) $\Rightarrow$ (1). Assume (2). Since each gr is a PLH, it is also an LH. Hence 1.3.1 (3) is satisfied for each gr. By [K86] 5.2.1, 1.3.1 (3) is satisfied, that is, (1) holds.

We prove (3) $\Rightarrow$ (2). Assume (3). Let F be the specialization (1.3.1) of the Hodge filtration with respect to some $a : \mathcal{O}^{\log}_{s,t} \to \mathbf{C}$ at t. Then, $(H_{\mathbf{C},t}; W_{\mathbf{C},t}; F, \bar{F}; N_1, \dots, N_n)$ is a pre-IMHM in the sense of [K86] 4.2. By 4.4.1 of [K86], it is an IMHM. In particular, for any subset $J \subset \{1, \dots, n\}$, $M(\sum_{j\in J} N_j, W)$ exists. Let $M(\tau, W) := M(\sum_{j\in J} N_j, W)$, where τ is the face of $C(s)$ generated by γ_j for j running in J. Then, this family satisfies the condition 1.2.2 (4) by 5.2.5 of [K86]. Hence, the local monodromy of $(H_{\mathbf{R}}, W)$ is admissible. □

Remark. The above proposition says that the concept of LMH with polarized graded quotients over a log point is essentially the same as the concept of IMHM in [K86]. See 2.2.2 Remark below.

1.3.5. Let S be an object of $\mathcal{B}(\log)$. An *LMH with polarized graded quotients on* S is a quadruple $H = (H_{\mathbf{Z}}, W, H_{\mathcal{O}}, (\langle\ ,\ \rangle_w)_w)$ such that $(H_{\mathbf{Z}}, W, H_{\mathcal{O}})$ is an LMH on S and $(H(\mathrm{gr}^W_w)_{\mathbf{Z}}, H(\mathrm{gr}^W_w)_{\mathcal{O}}, \langle\ ,\ \rangle_w)$ is a PLH of weight w for any w.

The following lemma will be used later in 4.5.7.

Lemma 1.3.6. *Let* S *be an object of* $\mathcal{B}(\log)$*. Let* H *be an LMH over* S*. Let* γ *be an automorphism of* H*. Then we have the following.*
(1) *If* $\mathrm{gr}(\gamma)$ *is the identity, then* γ *is also the identity.*
(2) *If* H *has polarized graded quotients, then the order of* γ *is finite.*

Proof. (1) In general, a morphism between pure log Hodge structures of different weights is zero. To see this, it is enough to show that the morphism of lattices is zero. This is reduced to the non-log case (that is, the fact that a morphism of pure Hodge structures of different weights is zero) by the condition (3) in 1.3.1. Hence, $\gamma - 1$ is zero, that is, γ is the identity.
(2) Under the assumption, $\mathrm{gr}(\gamma)$ is of finite order. Hence (2) is reduced to (1).

Example 1.3.7. Let $S = \Delta^{n+t}$, where Δ is the unit disk. Let $S^* = (\Delta^*)^n \times \Delta^t$, where $\Delta^* = \Delta - \{0\}$. Let X be an analytic space and $X \to S$ be a projective morphism which is smooth over S^*. Let $X^* = X \times_S S^*$. Let E be a divisor on X such that $E \cap X^*$ is relatively normal crossing over S^*. Let $m \in \mathbf{Z}$. Let $h : X^* \smallsetminus E \to S^*$. Assume that the local monodromy of $H_{\mathbf{R}} := R^m h_*(\mathbf{R})$ along E is unipotent. Then, the natural variation of mixed Hodge structure (VMHS) with polarized graded quotients on S^* underlain by $H_{\mathbf{Z}} := R^m h_*(\mathbf{Z})$ canonically extends to an LMH on S with polarized graded quotients, where S is endowed with the log structure defined by the divisor $S \smallsetminus S^*$. This was proved by Steenbrink-Zucker [SZ85], Kashiwara [K86], Saito [Sa90], and Fujino [F04]. See the explanation of the last part of [KNU08] 12.10.

Remark. In the above, we assumed that the local monodromy is unipotent. Thus, the theory on Néron models in this paper can be applied to a general case of quasi-unipotent local monodromy only after base change by a finite ramified covering. If one wants to treat such case directly without base change, it would suffice to formulate LMH on the ket site ([IKN05]) (which is LMH in the above sense ket locally) as in [KMN02].

§1.4. Moduli of log mixed Hodge structures (some examples)

We illustrate how the moduli spaces (classifying spaces) of log mixed Hodge structures look like. We review that an intermediate Jacobian is an example of a moduli space of mixed Hodge structures (see 1.4.1 Example 2), and we illustrate how we construct a degenerating intermediate Jacobian as a moduli space of log mixed Hodge structures (see 1.4.2 Example 2). We hope this subsection serves as a more precise version of Introduction of this paper.

1.4.1. Let H' be a mixed Hodge structure. By $\mathrm{Ext}^1(\mathbf{Z}, H')$, we denote the set of all isomorphism classes of mixed Hodge structures H endowed with an exact sequence $0 \to H' \to H \to \mathbf{Z} \to 0$. Here $\mathbf{Z}$ is regarded as a pure Hodge structure of weight 0 as usual, and the exactness means that the sequence $0 \to H'_{\mathbf{Z}} \to H_{\mathbf{Z}} \to \mathbf{Z} \to 0$ of $\mathbf{Z}$-structures, the sequences of W_w ($w \in \mathbf{Z}$), and the sequences of F^p ($p \in \mathbf{Z}$) are all exact. As is well-known, if H' is of weight $w \le -1$, we have a bijection

$$\mathrm{Ext}^1(\mathbf{Z}, H') \simeq H'_{\mathbf{Z}} \backslash H'_{\mathbf{C}} / F^0 H'_{\mathbf{C}}.$$

Here the class of H corresponds to the class of the element $e' - e \in H'_{\mathbf{C}}$ in $H'_{\mathbf{Z}} \backslash H'_{\mathbf{C}} / F^0 H'_{\mathbf{C}}$, where e is a lifting of $1 \in \mathbf{Z}$ to $H_{\mathbf{Z}}$ and e' is a lifting of $1 \in \mathbf{Z}$ to $F^0 H_{\mathbf{C}}$.

Example 1. Consider the case H' is the pure Hodge structure $\mathbf{Z}(1)$ of weight -2. In this case, $\mathrm{Ext}^1(\mathbf{Z}, \mathbf{Z}(1)) \simeq \mathbf{C}^\times$. In fact, for $H' = \mathbf{Z}(1)$, $H'_{\mathbf{Z}} \backslash H'_{\mathbf{C}} / F^0 H'_{\mathbf{C}} = \mathbf{Z}(1) \backslash \mathbf{C} \xrightarrow[\exp]{\sim} \mathbf{C}^\times$.

Example 2. Let X be a projective smooth algebraic variety over $\mathbf{C}$, let $r \ge 1$, and let H' be the Hodge structure $H^{2r-1}(X)(r)$ of weight -1. Then $\mathrm{Ext}^1(\mathbf{Z}, H') = H'_{\mathbf{Z}} \backslash H'_{\mathbf{C}} / F^0 H'_{\mathbf{C}}$ is the r-th intermediate Jacobian of X. (In this case, $H'_{\mathbf{Z}} = H^{2r-1}(X, \mathbf{Z})(r)/(\text{torsion})$, $H'_{\mathbf{C}} = H^{2r-1}(X, \mathbf{C})$, and $F^0 H'_{\mathbf{C}} = F^r H^{2r-1}(X, \mathbf{C})$ with F^r the Hodge filtration.)

1.4.2. Let S be an object of $\mathcal{B}(\log)$ and let H' be a log mixed Hodge structure on S. By $\mathcal{E}xt^1(\mathbf{Z}, H')$, we denote the sheafification of the functor $S' \mapsto \mathrm{Ext}^1_{\mathrm{LMH}/S'}(\mathbf{Z}, H'|_{S'})$ on the category $\mathcal{B}(\log)/S$ of objects of $\mathcal{B}(\log)$ over S. Here $H'|_{S'}$ denotes the pullback of H' to S' and $\mathrm{Ext}^1_{\mathrm{LMH}/S'}(\mathbf{Z}, H'|_{S'})$ is the set of all isomorphism classes of log mixed Hodge structures H on S' endowed with an exact sequence $0 \to H'|_{S'} \to H \to \mathbf{Z} \to 0$, where the exactness means that the sequence $0 \to (H'|_{S'})_{\mathbf{Z}} \to H_{\mathbf{Z}} \to \mathbf{Z} \to 0$ of $\mathbf{Z}$-structures, the sequences of W_w ($w \in \mathbf{Z}$), and the sequences of F^p ($p \in \mathbf{Z}$) are all exact. (The functor $S' \mapsto \mathrm{Ext}^1_{\mathrm{LMH}/S'}(\mathbf{Z}, H'|_{S'})$ is already a sheaf if H' is of weight ≤ -1.)

In general, $\mathcal{E}xt^1(\mathbf{Z}, H')$ is not necessarily representable. By using fans or more generally weak fans, we can construct objects of $\mathcal{B}(\log)$ over S (called models) which represent subsheaves of $\mathcal{E}xt^1(\mathbf{Z}, H')$. (By using weak fans not only fans, we can construct more models.)

Assume that H' is a polarizable log Hodge structure of weight ≤ -1. Then in §6, we construct two special models, called the Néron model and the connected Néron model, respectively.

Example 1. Consider the case $S = \mathrm{Spec}(\mathbf{C})$ with the trivial log structure and $H' = \mathbf{Z}(1)$. As a sheaf on $\mathcal{B}(\log)$, we have $\mathcal{E}xt^1(\mathbf{Z}, \mathbf{Z}(1)) \cong \mathbf{G}_{m,\log}$, where for an object S' of $\mathcal{B}(\log)$, $\mathbf{G}_{m,\log}(S') := \Gamma(S', M^{\mathrm{gp}}_{S'})$. This will be explained in 7.1.1. This sheaf is not representable. In this case, the Néron model = the connected Néron model = $\mathbf{C}^\times$ which represents the subfunctor $\mathbf{G}_m$ of $\mathbf{G}_{m,\log}$, where $\mathbf{G}_m(S') = \Gamma(S', \mathcal{O}^\times_{S'})$.

As in 7.1.1, we have a bigger model $\mathbf{P}^1(\mathbf{C})$ with the log structure associated to the divisor $\{0, \infty\}$. This model represents the subsheaf $S' \mapsto \Gamma(S', M_{S'} \cup M_{S'}^{-1})$ of $\mathbf{G}_{m,\log}$ on $\mathcal{B}(\log)$, which is bigger than $\mathbf{G}_m$.

Example 2. Let X and S be as in 1.3.7 with $E \cap X^* = \emptyset$. The variation of polarized Hodge structure $\{H^{2r-1}(X_s)(r)\}_{s \in S^*}$ on S^* extends to a polarized log Hodge structure H' on S of weight -1. The restriction of $\mathcal{E}xt^1(\mathbf{Z}, H')$ to $\mathcal{B}(\log)/S^*$ is represented by the family $\{J^r_{X_s}\}_{s \in S^*}$ of intermediate Jacobians over S^*. By using weak fans, we have various models of $\mathcal{E}xt^1(\mathbf{Z}, H')$ on the whole S, especially the Néron model and the connected Néron model.

In the case $r = 1$, the restriction of $\mathcal{E}xt^1(\mathbf{Z}, H')$ to $\mathcal{B}(\log)/S^*$ is represented by the (relative) Picard variety of X^* over S^* which is a family of abelian varieties over S^*, and if $\dim(S) = 1$, on the whole S, our Néron model (resp. connected Néron model) in §6 coincides with the Néron model (resp. connected Néron model) in the classical theory of degeneration of abelian varieties.

We explain that for Example 2 in 1.4.2, to construct a model of $\mathcal{E}xt^1(\mathbf{Z}, H')$, this paper provides two methods, one is that of §2 (explained in 1.4.3 below) and the other is that of §5 (explained in 1.4.4 below).

1.4.3. In the paper [KNU10a], for Example 2 in 1.4.2, we constructed a model of $\mathcal{E}xt^1(\mathbf{Z}, H')$ as the fiber product of $S \to \Gamma' \backslash D'_{\Sigma'} \leftarrow \Gamma \backslash D_{\Sigma}$, where $\Gamma' \backslash D'_{\Sigma'}$ is a moduli space of polarized log Hodge structures of weight -1 constructed by using a fan Σ', $S \to \Gamma' \backslash D'_{\Sigma'}$ is the period map defined by H', $\Gamma \backslash D_{\Sigma}$ is a moduli space of LMH with polarized graded quotients whose gr^W_0 is $\mathbf{Z}$ and whose gr^W_w is 0 for $w \neq 0, -1$ constructed by using a fan Σ, and the map $\Gamma \backslash D_{\Sigma} \to \Gamma' \backslash D'_{\Sigma'}$ is to take gr^W_{-1}. The space $\Gamma' \backslash D'_{\Sigma'}$ is given in [KU09] and the space $\Gamma \backslash D_{\Sigma}$ is given in [KNU10b] and in §2 of this paper. (In §2 of this paper, we use weak fans Σ to obtain more models.)

1.4.4. In this paper, for Example 2 in 1.4.2, we construct a model of $\mathcal{E}xt^1(\mathbf{Z}, H')$ also by the following method of §5. In §5, for an object S of $\mathcal{B}(\log)$ and for a family of polarized log Hodge structures $H_{(w)}$ of weight w on S given for each $w \in \mathbf{Z}$ (we assume $H_{(w)} = 0$ for almost all w), by using a weak fan, we construct a moduli space over S of LMH whose gr^W_w is $H_{(w)}$ for each w. For Example 2 in 1.4.2, we take $H_{(0)} = \mathbf{Z}$, $H_{(-1)} = H'$, and $H_{(w)} = 0$ for $w \neq 0, -1$. Then we obtain a model of $\mathcal{E}xt^1(\mathbf{Z}, H')$. The method of §5 provides more models than that of §2. In particular, the Néron model and the connected Néron model constructed in §6 by the method of §5 are not necessarily constructed by the method of §2.

§2. Moduli spaces of log mixed Hodge structures with polarized graded quotients

§2.1. Classifying space D

2.1.1. In this §2, we fix $\Lambda = (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$, where

H_0 is a finitely generated free $\mathbf{Z}$-module,

W is a finite increasing rational filtration on $H_{0,\mathbf{R}} = \mathbf{R} \otimes H_0$,

$\langle\ ,\ \rangle_w$ for each $w \in \mathbf{Z}$ is a rational non-degenerate $\mathbf{R}$-bilinear form $\mathrm{gr}^W_w \times \mathrm{gr}^W_w \to \mathbf{R}$ which is symmetric if w is even and is anti-symmetric if w is odd,

$h^{p,q}$ are non-negative integers given for each $(p,q) \in \mathbf{Z}^2$ satisfying the following conditions (1)–(3).

(1) $\sum_{p,q} h^{p,q} = \mathrm{rank}_{\mathbf{Z}}(H_0)$,

(2) $\sum_{p+q=w} h^{p,q} = \dim_{\mathbf{R}}(\mathrm{gr}^W_w)$ for any $w \in \mathbf{Z}$,

(3) $h^{p,q} = h^{q,p}$ for any (p,q).

2.1.2. Let D be the classifying space of gradedly polarized mixed Hodge structures in [U84] associated with the data fixed in 2.1.1. As a set, D consists of all increasing filtrations F on $H_{0,\mathbf{C}} = \mathbf{C} \otimes H_0$ such that $(H_0, W, (\langle\ ,\ \rangle_w)_w, F)$ is a gradedly polarized mixed Hodge structures with $\dim_{\mathbf{C}} F^p(\mathrm{gr}^W_{p+q})/F^{p+1}(\mathrm{gr}^W_{p+q}) = h^{p,q}$ for all p,q. Let $\check{D}$ be its "compact dual". Then D and $\check{D}$ are complex analytic manifolds, and D is open in $\check{D}$. Cf. [KNU09] 1.5.

2.1.3. We recall the notation used in this series of papers.

As in [KNU09] 1.6, for $A = \mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}$, let G_A be the group of the A-automorphisms of $(H_{0,A}, W)$ whose gr^W_w are compatible with $\langle\ ,\ \rangle_w$ for all w. Then $G_{\mathbf{C}}$ (resp. $G_{\mathbf{R}}$) acts on $\check{D}$ (resp. D).

As in [KNU09] 1.7, for $A = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, let $\mathfrak{g}_A$ be the associated Lie algebra of G_A, which is identified with the set of all A-endomorphisms N of $(H_{0,A}, W)$ whose gr^W_w satisfies $\langle \mathrm{gr}^W_w(N)(x), y\rangle_w + \langle x, \mathrm{gr}^W_w(N)(y)\rangle_w = 0$ for all w, x, y.

As in [KNU09] 1.6 and 1.7, let $G_{A,u} = \{\gamma \in G_A \mid \mathrm{gr}^W(g) = 1\}$, $\mathfrak{g}_{A,u} = \{N \in \mathfrak{g}_A \mid \mathrm{gr}^W(N) = 0\}$. Then $G_A/G_{A,u}$ is isomorphic to $G_A(\mathrm{gr}^W) := \prod_w G_A(\mathrm{gr}^W_w)$ and $\mathfrak{g}_A/\mathfrak{g}_{A,u}$ is isomorphic to $\mathfrak{g}_A(\mathrm{gr}^W) := \prod_w \mathfrak{g}_A(\mathrm{gr}^W_w)$, where $G_A(\mathrm{gr}^W_w)$ (resp. $\mathfrak{g}_A(\mathrm{gr}^W_w)$) is "the G_A (resp. $\mathfrak{g}_A$) for gr^W_w".

§2.2. The sets D_Σ and $D^\sharp_\Sigma$

Here we define the sets D_Σ and $D^\sharp_\Sigma$ in our mixed Hodge case. See [KU09] for the pure case. Everything is parallel to and generalize the pure case, except that we use weak fans instead of fans.

2.2.1. Nilpotent cone. A subset σ of $\mathfrak{g}_{\mathbf{R}}$ is called a *nilpotent cone* if the following conditions (1) and (2) are satisfied.

(1) All elements of σ are nilpotent and $NN' = N'N$ for any $N, N' \in \sigma$, as linear maps $H_{0,\mathbf{R}} \to H_{0,\mathbf{R}}$.

(2) σ is finitely generated. That is, there is a finite family $(N_j)_j$ of elements of σ such that $\sigma = \sum_j \mathbf{R}_{\geq 0} N_j$.

A nilpotent cone is said to be *rational* if we can take $N_j \in \mathfrak{g}_{\mathbf{Q}}$ in the above condition (2).

For a nilpotent cone σ, let $\sigma_{\mathbf{R}}$ be the $\mathbf{R}$-linear span of σ in $\mathfrak{g}_{\mathbf{R}}$, and let $\sigma_{\mathbf{C}}$ be the $\mathbf{C}$-linear span of σ in $\mathfrak{g}_{\mathbf{C}}$.

We say that σ is *admissible* if the action of σ on $H_{0,\mathbf{R}}$ is admissible with respect to W in the sense of 1.2.2.

2.2.2. Let $N_1, \dots, N_n \in \mathfrak{g}_{\mathbf{R}}$ be mutually commuting nilpotent elements and let $F \in \check{D}$. We say $(N_1, \dots, N_n, F)$ *generates a nilpotent orbit* if the following (1)–(3) are satisfied.

(1) The action of the cone $\sum_{j=1}^{n} \mathbf{R}_{\geq 0} N_j$ on $H_{0,\mathbf{R}}$ is admissible (1.2.2) with respect to W.

(2) $N_j F^p \subset F^{p-1}$ for $1 \leq j \leq n$ and for any $p \in \mathbf{Z}$.

(3) If $y_j \in \mathbf{R}_{>0}$ and y_j are sufficiently large, we have $\exp(\sum_{j=1}^{n} iy_j N_j)F \in D$.

Note that these conditions depend only on the cone $\sigma := \sum_{j=1}^{n} \mathbf{R}_{\geq 0} N_j$ and F. We say also that the pair (σ, F) generates a nilpotent orbit if these conditions are satisfied.

Let σ be a nilpotent cone. A subset Z of $\check{D}$ is called a *σ-nilpotent orbit* (resp. *σ-nilpotent i-orbit*) if the following conditions (4) and (5) are satisfied for some $F \in Z$.

(4) $Z = \exp(\sigma_{\mathbf{C}})F$ (resp. $Z = \exp(i\sigma_{\mathbf{R}})F$).

(5) (σ, F) generates a nilpotent orbit.

Such a pair (σ, Z) is called a *nilpotent orbit* (resp. *nilpotent i-orbit*).

Note that if (4) and (5) are satisfied for one $F \in Z$, they are satisfied for all $F \in Z$.

Remark. The notion of nilpotent orbit is essentially the same as the notion of LMH with polarized graded quotients on an fs log point. This is the mixed Hodge-theoretic version of [KU09] §2.5. The proof of the mixed Hodge case is similar to the pure case written in [KU09] §2.5, especially ibid. Proposition 2.5.5, where there are full explanations of the relationship among the exterior differential of the sheaf of rings of logarithms, Gauss-Manin connection, and monodromy logarithms, and also the relationship between the difference of two specializations and coordinates of the universal covering.

Consequently, in view of Remark in 1.3.4, the notion of nilpotent orbit is also essentially the same as the notion of infinitesimal mixed Hodge module (IMHM) by Kashiwara ([K86]). A precise relation is as follows. (The statement in [KNU10b] 2.1.3 was not accurate.)

Let H_0 and W be as in 2.1.1. Let F be a decreasing filtration on $H_{0,\mathbf{C}}$. Let $\bar{F}$ be its complex conjugation. Let $N_1, \dots, N_n$ be a mutually commuting set of nilpotent endomorphisms of $H_{0,\mathbf{Q}}$. Then, $(H_{0,\mathbf{C}}; W_{\mathbf{C}}; F, \bar{F}; N_1, \dots, N_n)$ is an IMHM if and only if there exists a Λ (2.1.1) extending (H_0, W) such that $N_1, \dots, N_n \in \mathfrak{g}_{\mathbf{R}}$ and $F \in \check{D}$, and that $(N_1, \dots, N_n, F)$ generates a nilpotent orbit. This was seen in the course of the proof of 1.3.4.

2.2.3. Weak fan in $\mathfrak{g}_{\mathbf{Q}}$ ([KNU10c]).

A *weak fan Σ in $\mathfrak{g}_{\mathbf{Q}}$* is a non-empty set of sharp rational nilpotent cones in $\mathfrak{g}_{\mathbf{R}}$ satisfying the following conditions (1) and (2). Here a nilpotent cone σ is said to be *sharp* if $\sigma \cap (-\sigma) = \{0\}$.

(1) If $\sigma \in \Sigma$, all faces of σ belong to Σ.

(2) Let $\sigma, \sigma' \in \Sigma$, and assume that σ and σ' have a common interior point. Assume that there is an $F \in \check{D}$ such that (σ, F) and (σ', F) generate nilpotent orbits. Then $\sigma = \sigma'$.

A *fan* Σ *in* $\mathfrak{g}_{\mathbf{Q}}$ is defined similarly by replacing the condition (2) in the above with the condition: If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face of σ.

For simplicity, we call a weak fan (resp. a fan) in $\mathfrak{g}_{\mathbf{Q}}$ a weak fan (resp. a fan).

An example of a fan is

$$\Xi = \{\mathbf{R}_{\geq 0} N \mid N \in \mathfrak{g}_{\mathbf{Q}},\ N \text{ is nilpotent}\}.$$

An example of a weak fan which is not a fan appears in 7.2.1.

The next lemma, which was announced in [KNU10c] 1.7, implies that a fan in $\mathfrak{g}_{\mathbf{Q}}$ is a weak fan.

Lemma 2.2.4. *In the definition* 2.2.3*, under the condition* (1)*, the condition* (2) *is equivalent to* $(2)'$ *below and also is equivalent to* $(2)''$ *below.*

$(2)'$ *If* $\sigma, \sigma' \in \Sigma$ *and if there is an* $F \in \check{D}$ *such that* (σ, F)*,* (σ', F)*, and* $(\sigma \cap \sigma', F)$ *generate nilpotent orbits, then* $\sigma \cap \sigma'$ *is a face of* σ*.*

$(2)''$ *Let* τ *be a nilpotent cone. Let* (τ, F) $(F \in \check{D})$ *generate a nilpotent orbit. If the set* $A := \{\sigma \in \Sigma \,|\, \tau \subset \sigma, (\sigma, F) \text{ generates a nilpotent orbit}\}$ *is not empty,* A *has a smallest element, and it is a face of any element of* A*.*

Proof. $(2)'' \Rightarrow (2)'$. Let σ, σ' and F be as in $(2)'$. Let $\tau = \sigma \cap \sigma'$. Then, σ and σ' belong to A. Let σ_0 be the smallest element of A. Then, σ_0 is a face of σ and also a face of σ', and hence $\sigma_0 \subset \sigma \cap \sigma' = \tau$. On the other hand, since $\sigma_0 \in A$, we have $\tau \subset \sigma_0$ by definition of A. Thus, $\sigma_0 = \tau$. Hence, $\sigma \cap \sigma' = \tau = \sigma_0$ is a face of σ.

$(2)' \Rightarrow (2)$. Let σ, σ' and F be as in (2). Since $\sigma \cap \sigma'$ contains an interior point of σ, $(\sigma \cap \sigma', F)$ generates a nilpotent orbit. Hence, $(2)'$ implies that $\sigma \cap \sigma'$ is a face of σ, but again, since $\sigma \cap \sigma'$ contains an interior point of σ, we have $\sigma \cap \sigma' = \sigma$. By symmetry, $\sigma \cap \sigma' = \sigma'$. Therefore, $\sigma = \sigma'$.

$(2) \Rightarrow (2)''$. Let (τ, F) be as in $(2)''$. For an element $\sigma \in A$, we write as σ_0 the face of σ spanned by τ. Then, (σ_0, F) generates a nilpotent orbit because both (τ, F) and (σ, F) generate nilpotent orbits. Thus, $\sigma_0 \in A$. It is enough to show $\sigma_0 = \sigma'_0$ for any $\sigma, \sigma' \in A$, because, then, this common σ_0 is the smallest element of A as soon as A is not empty, and it is indeed a face of any $\sigma \in A$. But, since an interior point of τ is a common interior point of σ_0 and σ'_0, we have certainly $\sigma_0 = \sigma'_0$ by (2). □

2.2.5. Let Σ be a weak fan in $\mathfrak{g}_{\mathbf{Q}}$. Let D_Σ (resp. $D_\Sigma^\sharp$) be the set of all nilpotent orbits (resp. nilpotent i-orbits) (σ, Z) with $\sigma \in \Sigma$.

We have embeddings

$$D \subset D_\Sigma, \quad F \mapsto (\{0\}, \{F\}),$$

$$D \subset D_\Sigma^\sharp, \quad F \mapsto (\{0\}, \{F\}).$$

We have a canonical surjection

$$D^\sharp_\Sigma \to D_\Sigma, \quad (\sigma, Z) \mapsto (\sigma, \exp(\sigma_{\mathbf{C}})Z),$$

which is compatible with the above embeddings of D.

Remark 1. If $'\Sigma \subset \Sigma$ denotes the subset of Σ consisting of all admissible $\sigma \in \Sigma$, and if $''\Sigma \subset {}'\Sigma$ denotes the subset of Σ consisting of all $\sigma \in \Sigma$ such that a σ-nilpotent orbit exists, then $'\Sigma$ and $''\Sigma$ are weak fans, and we have $D_\Sigma = D_{'\Sigma} = D_{''\Sigma}$, $D^\sharp_\Sigma = D^\sharp_{'\Sigma} = D^\sharp_{''\Sigma}$.

Remark 2. In our previous papers [KNU10b] and [KNU10c], we adopted the following formulations (1)–(3) which are slightly different from those in the present paper. In those papers:

(1) Nilpotent cone was assumed to be sharp.

(2) In the definition of nilpotent orbit, we adopted the admissibility condition 1.2.6 (2) on the cone instead of the condition 2.2.2 (1).

(3) In the definition of weak fan (resp. fan), in [KNU10c] (resp. [KNU10b] and [KNU10c]), we put the condition that any cone in a weak fan (resp. fan) should satisfy the admissibility condition 1.2.6 (2).

These changes of the formulations are not essential in this paper. In fact, for sharp cones, the definition of nilpotent orbit in [KNU10b] and [KNU10c], explained in the above (2), is equivalent to that in this paper. Cf. 1.2.6, 1.3.4. Further, by Remark 1 above, these changes do not make any difference for the space of nilpotent orbits D_Σ and for the moduli spaces $\Gamma\backslash D_\Sigma$ of LMH which appear in §2.4 below. Thus, we do not impose any admissible condition on cones of (weak) fan in the present paper (cf. the above (3)), for we prefer a simpler definition here.

Note that, in the pure case, the definitions of nilpotent cone, nilpotent orbit, and fan in this paper coincide with those in [KU09].

2.2.6. Compatibility of a weak fan and a subgroup of $G_{\mathbf{Z}}$.

Let Σ be a weak fan in $\mathfrak{g}_{\mathbf{Q}}$ and let Γ be a subgroup of $G_{\mathbf{Z}}$.

We say that Σ and Γ are *compatible* if the following condition (1) is satisfied.

(1) If $\gamma \in \Gamma$ and $\sigma \in \Sigma$, then $\mathrm{Ad}(\gamma)\sigma \in \Sigma$.

If Σ and Γ are compatible, Γ acts on D_Σ and also on $D^\sharp_\Sigma$ by $(\sigma, Z) \mapsto (\mathrm{Ad}(\gamma)\sigma, \gamma Z)$ $(\gamma \in \Gamma)$.

We say that Σ and Γ are *strongly compatible* if they are compatible and furthermore the following condition (2) is satisfied.

(2) If $\sigma \in \Sigma$, any element of σ is a sum of elements of the form aN, where $a \in \mathbf{R}_{\geq 0}$ and $N \in \sigma$ satisfying $\exp(N) \in \Gamma$.

2.2.7. For a sharp rational nilpotent cone σ, let face(σ) be the set of all faces of σ. It is a fan in $\mathfrak{g}_{\mathbf{Q}}$. Let

$$D_\sigma := D_{\mathrm{face}(\sigma)}, \quad D^\sharp_\sigma := D^\sharp_{\mathrm{face}(\sigma)}.$$

Let

$$\Gamma(\sigma) := \Gamma \cap \exp(\sigma) \subset \Gamma(\sigma)^{\mathrm{gp}} = \Gamma \cap \exp(\sigma_{\mathbf{R}}) = \{ab^{-1} \mid a, b \in \Gamma(\sigma)\} \subset G_{\mathbf{Z}}.$$

Then $\Gamma(\sigma)$ is a sharp torsion-free fs (i.e., finitely generated, integral, and saturated) monoid and $\Gamma(\sigma)^{\mathrm{gp}}$ is a finitely generated free abelian group. Here a monoid is said to be *sharp* if there is no invertible elements other than the unit element (cf. 2.2.3).

Let Σ and Γ be as in 2.2.6. Assume that they are strongly compatible. Then, for any $\sigma \in \Sigma$, the fan face(σ) and the group $\Gamma(\sigma)^{\mathrm{gp}}$ are strongly compatible.

We will use the following fact in the proof of Theorem 6.2.1.

Proposition 2.2.8. *Let $\sigma \subset \mathfrak{g}_{\mathbf{R}}$ be an admissible nilpotent cone. Then the adjoint action of σ on $\mathfrak{g}_{\mathbf{R}}$ is admissible.*

Proof. By 1.2.2.1, the adjoint action of σ on $A := \mathrm{Hom}_{\mathbf{R}}(H_{0,\mathbf{R}}, H_{0,\mathbf{R}})$ is admissible with respect to the filtration of A induced by W. For a face τ of σ, let $M(\tau)$ be the relative monodromy filtration of τ on A for this admissible action. The adjoint action of σ on $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_w)$ for $w < 0$ is admissible, whose relative monodromy filtrations are induced by $M(\tau)$, because $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_w) = A(\mathrm{gr}^W_w)$. By 1.2.2.2, the adjoint action of σ on $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_0)$ is admissible, whose relative monodromy filtrations are induced by $M(\tau)$, because $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_0)$ is a direct summand of $A(\mathrm{gr}^W_0)$ for the adjoint action of σ. It follows that the adjoint action of σ on $\mathfrak{g}_{\mathbf{R}}$ is admissible, whose relative monodromy filtrations are induced by $M(\tau)$. □

§2.3. The sets E_σ and $E^\sharp_\sigma$

We introduce the sets E_σ and $E^\sharp_\sigma$, which will be used to give various structures to D_σ and $D^\sharp_\sigma$.

2.3.1. Toric varieties.

Assume that we are given Σ and Γ as in 2.2.6. We assume that they are strongly compatible.

Fix $\sigma \in \Sigma$ in the following. Let $P(\sigma) = \mathrm{Hom}(\Gamma(\sigma), \mathbf{N})$. Define

$$\mathrm{toric}_\sigma := \mathrm{Hom}(P(\sigma), \mathbf{C}^{\mathrm{mult}}) \supset |\mathrm{toric}|_\sigma := \mathrm{Hom}(P(\sigma), \mathbf{R}^{\mathrm{mult}}_{\geq 0}).$$

Here $\mathbf{C}^{\mathrm{mult}}$ (resp. $\mathbf{R}^{\mathrm{mult}}_{\geq 0}$) denotes the set $\mathbf{C}$ (resp. $\mathbf{R}_{\geq 0}$) regarded as a multiplicative monoid. (Cf. [KU09] 3.3.2 for toric_σ in the pure case.) Let

$$\mathrm{torus}_\sigma := \mathbf{C}^\times \otimes_{\mathbf{Z}} \Gamma(\sigma)^{\mathrm{gp}} \supset |\mathrm{torus}|_\sigma := \mathbf{R}_{>0} \otimes_{\mathbf{Z}} \Gamma(\sigma)^{\mathrm{gp}}.$$

We regard torus_σ as an open set of toric_σ and $|\mathrm{torus}|_\sigma$ as an open set of $|\mathrm{toric}|_\sigma$ via the embeddings

$$\mathrm{torus}_\sigma = \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{C}^\times) \subset \mathrm{toric}_\sigma, \quad |\mathrm{torus}|_\sigma = \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{R}_{>0}) \subset |\mathrm{toric}|_\sigma .$$

We have a natural action of torus_σ on toric_σ, and a natural action of $|\text{torus}|_\sigma$ on $|\text{toric}|_\sigma$. We have an exact sequence

$$0 \to \Gamma(\sigma)^{\text{gp}} \xrightarrow{\log} \sigma_{\mathbf{C}} \xrightarrow{\mathbf{e}} \text{torus}_\sigma \to 0,$$

where

$$\mathbf{e}(z \otimes \log(\gamma)) = e^{2\pi i z} \otimes \gamma \ \text{ for } \ z \in \mathbf{C}, \gamma \in \Gamma(\sigma).$$

For a face τ of σ, the surjective homomorphism $P(\sigma) \to P(\tau)$ induces an embedding $\text{toric}_\tau \to \text{toric}_\sigma$. Let $0_\tau \in \text{toric}_\tau \subset \text{toric}_\sigma$ be the point corresponding to the homomorphism $P(\tau) \to \mathbf{C}^{\text{mult}}$ which sends $1 \in P(\tau)$ to 1 and all the other elements of $P(\tau)$ to 0. (This point $0_\tau \in \text{toric}_\sigma$ is denoted by 1_τ in [KU09], but we change the notation.)

Any element q of toric_σ (resp. $|\text{toric}|_\sigma$) is written in the form $q = \mathbf{e}(a) \cdot 0_\tau$ (resp. $q = \mathbf{e}(ib) \cdot 0_\tau$) for $a \in \sigma_{\mathbf{C}}$ (resp. $b \in \sigma_{\mathbf{R}}$) and for a face τ of σ. The face τ is determined by q, and a modulo $\tau_{\mathbf{C}} + \log(\Gamma(\sigma)^{\text{gp}})$ (resp. b modulo $\tau_{\mathbf{R}}$) is determined by q.

2.3.2. Let

$$\check{E}_\sigma := \text{toric}_\sigma \times \check{D}.$$

(See [KU09] 3.3.2 for the pure case.)

We endow $\check{E}_\sigma$ with the inverse image ([KU09] 2.1.3) of the log structure of toric_σ ([KU09] 2.1.6 (ii)).

We define a canonical pre-LMH on $\check{E}_\sigma$. See [KU09] 3.3.3 for the pure case. Let $(H_{\sigma,\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w)$ be the locally constant sheaf of $\mathbf{Z}$-modules on $\text{toric}_\sigma^{\log}$ endowed with a weight filtration and polarizations on the graded quotients which is characterized by the property that the stalk at the unit point $1 \in \text{torus}_\sigma \subset \text{toric}_\sigma^{\log}$ is $(H_0, W, (\langle\ ,\ \rangle_w)_w)$ and that the action of $\pi_1(\text{toric}_\sigma^{\log}) = \Gamma(\sigma)^{\text{gp}}$ is given by $\Gamma(\sigma)^{\text{gp}} \subset G_{\mathbf{Z}}$. We consider its pullback to $\check{E}_\sigma$ still denoted by the same symbol $(H_{\sigma,\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w)$. Similarly as in the pure case, [KU09] 2.3.7 gives a canonical isomorphism $\mathcal{O}^{\log}_{\check{E}_\sigma} \otimes H_{\sigma,\mathbf{Z}} = \mathcal{O}^{\log}_{\check{E}_\sigma} \otimes H_0$. The $\mathcal{O}_{\check{D}} \otimes H_0$ has the universal Hodge filtration, which induces by pulling back a filtration on $\mathcal{O}^{\log}_{\check{E}_\sigma} \otimes H_0$, and by τ_* a filtration on $H_{\sigma,\mathcal{O}} := \tau_*(\mathcal{O}^{\log}_{\check{E}_\sigma} \otimes H_0)$. Then, the triple $H_\sigma := (H_{\sigma,\mathbf{Z}}, W, H_{\sigma,\mathcal{O}})$ is a pre-LMH, which we call a canonical pre-LMH on $\check{E}_\sigma$. Note that each graded quotient of this pre-LMH together with $\langle\ ,\ \rangle_w$ is regarded as a pre-PLH of weight w.

2.3.3. We define the subsets $\tilde{E}_\sigma$, E_σ, and $E^\sharp_\sigma$ of $\check{E}_\sigma$. Let H_σ be the canonical pre-LMH on $\check{E}_\sigma$ defined in 2.3.2. Then, the subset $\tilde{E}_\sigma$ (resp. E_σ) is defined as the set of points $x \in \check{E}_\sigma$ such that the pullback of H_σ on x satisfies the Griffiths transversality (1.3.1 (2)) (resp. is an LMH with polarized graded quotients (1.3.5)). Thus, E_σ is contained in $\tilde{E}_\sigma$.

Consider the subset $|\text{toric}|_\sigma$ of toric_σ, let $\check{E}^\sharp_\sigma := |\text{toric}|_\sigma \times \check{D}$, and let $E^\sharp_\sigma := E_\sigma \cap \check{E}^\sharp_\sigma$.

E_σ and $E^\sharp_\sigma$ are characterized as follows (cf. [KNU10b] 2.2.3).

For $q \in \text{toric}_\sigma$ (resp. $q \in |\text{toric}|_\sigma$), write $q = \mathbf{e}(a) \cdot 0_\tau$ (resp. $q = \mathbf{e}(ib) \cdot 0_\tau$), where $a \in \sigma_{\mathbf{C}}$ ($b \in \sigma_{\mathbf{R}}$), τ is the face of σ and 0_τ is as in 2.3.1. Then, a point (q, F) of $\check{E}_\sigma = \text{toric}_\sigma \times \check{D}$ (resp. $\check{E}^\sharp_\sigma = |\text{toric}|_\sigma \times \check{D}$) belongs to E_σ (resp. $E^\sharp_\sigma$) if and only if $(\tau, \exp(a)F)$ (resp. $(\tau, \exp(ib)F)$) generates a nilpotent orbit.

The proofs of these facts are similar to the pure case (cf. [KU09] 3.3.7), and based on a key observation of the equivalence between a log mixed Hodge structure with polarized graded quotients on an fs log point and a nilpotent orbit, mentioned in Remark in 2.2.2.

2.3.4. In the above notation, we have canonical projections $\varphi : E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ and $\varphi^\sharp : E^\sharp_\sigma \to D^\sharp_\sigma$ by

$$\varphi(q, F) = (\tau,\, \exp(\tau_{\mathbf{C}}) \exp(a)F) \bmod \Gamma(\sigma)^{\mathrm{gp}},$$
$$\varphi^\sharp(q, F) = (\tau,\, \exp(i\tau_{\mathbf{R}}) \exp(ib)F).$$

§2.4. Topology, complex structures, and log structures

Let Σ and Γ be as in 2.2.6. We assume that they are strongly compatible.

We define a structure of log local ringed space over $\mathbf{C}$ on $\Gamma\backslash D_\Sigma$ and a topology on $D^\sharp_\Sigma$.

2.4.1. As in 2.4.1 of [KNU10b], for each $\sigma \in \Sigma$, we endow the subsets E_σ and $\tilde{E}_\sigma$ of $\check{E}_\sigma$ in 2.3.3 with the following structures of log local ringed spaces over $\mathbf{C}$. The topology is the strong topology in $\check{E}_\sigma$. The sheaf $\mathcal{O}$ of rings and the log structure M are the inverse images of $\mathcal{O}$ and M of $\check{E}_\sigma$, respectively.

We endow a topology on $E^\sharp_\sigma$ as a subspace of E_σ.

2.4.2. We endow $\Gamma\backslash D_\Sigma$ with the strongest topology for which the maps $\pi_\sigma : E_\sigma \xrightarrow{\varphi} \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to \Gamma\backslash D_\Sigma$ are continuous for all $\sigma \in \Sigma$. Here φ is as in 2.3.4. We endow $\Gamma\backslash D_\Sigma$ with the following sheaf of rings $\mathcal{O}_{\Gamma\backslash D_\Sigma}$ over $\mathbf{C}$ and the following log structure $M_{\Gamma\backslash D_\Sigma}$. For any open set U of $\Gamma\backslash D_\Sigma$ and for any $\sigma \in \Sigma$, let $U_\sigma := \pi_\sigma^{-1}(U)$ and define $\mathcal{O}_{\Gamma\backslash D_\Sigma}(U)$ (resp. $M_{\Gamma\backslash D_\Sigma}(U)$) $:= \{\text{map } f : U \to \mathbf{C} \mid f \circ \pi_\sigma \in \mathcal{O}_{E_\sigma}(U_\sigma)$ (resp. $\in M_{E_\sigma}(U_\sigma)$) $(\forall \sigma \in \Sigma)\}$. Here we regard $M_{E_\sigma}(U_\sigma)$ as a subset of $\mathcal{O}_{E_\sigma}(U_\sigma)$ via the structural map $M_{E_\sigma}(U_\sigma) \to \mathcal{O}_{E_\sigma}(U_\sigma)$, which is injective. (Note that, then, any $f \in \mathcal{O}_{\Gamma\backslash D_\Sigma}$ is a continuous map.)

2.4.3. We introduce on $D^\sharp_\Sigma$ the strongest topology for which the maps $E^\sharp_\sigma \xrightarrow{\varphi^\sharp} D^\sharp_\sigma \to D^\sharp_\Sigma$ $(\sigma \in \Sigma)$ are continuous. Here $\varphi^\sharp$ is as in 2.3.4. Note that the surjection $D^\sharp_\Sigma \to \Gamma\backslash D_\Sigma$ (cf. 2.2.5) becomes continuous.

§2.5. Properties of $\Gamma\backslash D_\Sigma$

In this §2.5, let Σ be a weak fan in $\mathfrak{g}_{\mathbf{Q}}$ and let Γ be a subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Σ (2.2.6).

The following Theorem 2.5.1–Theorem 2.5.6 are the mixed Hodge-theoretic versions of 4.1.1 Theorem A (i)–(vi) of [KU09].

The proofs of these theorems will be given in §4 after a preparation in §3.

Theorem 2.5.1. *For $\sigma \in \Sigma$, E_σ is open in $\tilde{E}_\sigma$ in the strong topology of $\tilde{E}_\sigma$ in $\check{E}_\sigma$. Both $\tilde{E}_\sigma$ and E_σ are log manifolds.*

We say that Γ is *neat* if for any $\gamma \in \Gamma$, the subgroup of $\mathbf{C}^\times$ generated by all eigenvalues of the action of γ on $H_{0,\mathbf{C}}$ is torsion-free. It is known that there is a neat subgroup of $G_{\mathbf{Z}}$ of finite index. (See [Bor69].)

Theorem 2.5.2. *If Γ is neat, then $\Gamma\backslash D_\Sigma$ is a log manifold.*

Theorem 2.5.3. *Let $\sigma \in \Sigma$ and define the action of $\sigma_{\mathbf{C}}$ on E_σ over $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ by*

$$a \cdot (q, F) := (\mathbf{e}(a) \cdot q, \exp(-a)F) \quad (a \in \sigma_{\mathbf{C}}, (q, F) \in E_\sigma),$$

where $\mathbf{e}(a) \in \mathrm{torus}_\sigma$ is as in 2.3.1 and $\mathbf{e}(a) \cdot q$ is defined by the natural action of torus_σ on toric_σ. Then, $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ is a $\sigma_{\mathbf{C}}$-torsor in the category of log manifolds. That is, locally on the base $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$, E_σ is isomorphic as a log manifold to the product of $\sigma_{\mathbf{C}}$ and the base endowed with the evident action of $\sigma_{\mathbf{C}}$.

Theorem 2.5.4. *If Γ is neat, then, for any $\sigma \in \Sigma$, the map*

$$\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to \Gamma\backslash D_\Sigma$$

is locally an isomorphism of log manifolds.

Theorem 2.5.5. *The topological space $\Gamma\backslash D_\Sigma$ is Hausdorff.*

Theorem 2.5.6. *If Γ is neat, then there is a homeomorphism of topological spaces*

$$(\Gamma\backslash D_\Sigma)^{\log} \simeq \Gamma\backslash D_\Sigma^\sharp,$$

which is compatible with $\tau : (\Gamma\backslash D_\Sigma)^{\log} \to \Gamma\backslash D_\Sigma$ and the projection $\Gamma\backslash D_\Sigma^\sharp \to \Gamma\backslash D_\Sigma$ induced by $D_\Sigma^\sharp \to D_\Sigma$ in 2.2.5.

§2.6. Moduli

We define the moduli functor of LMH with polarized graded quotients.

2.6.1. Fix $\Phi = (\Lambda, \Sigma, \Gamma)$, where Λ is as in 2.1.1 and Σ and Γ are as in 2.5.

Let S be an object of $\mathcal{B}(\log)$. Recall that an LMH with polarized graded quotients on S is a quadruple $H = (H_{\mathbf{Z}}, W, H_{\mathcal{O}}, (\langle\ ,\ \rangle_w)_w)$ such that $(H_{\mathbf{Z}}, W, H_{\mathcal{O}})$ is an LMH on S and $(H(\mathrm{gr}^W_w)_{\mathbf{Z}}, H(\mathrm{gr}^W_w)_{\mathcal{O}}, \langle\ ,\ \rangle_w)$ is a PLH of weight w for any w.

2.6.2. Let S be an object of $\mathcal{B}(\log)$. By an *LMH with polarized graded quotients of type Φ on S*, we mean an LMH with polarized graded quotients $H = (H_{\mathbf{Z}}, W, H_{\mathcal{O}}, (\langle\ ,\ \rangle_w)_w)$ endowed with a global section μ of the sheaf $\Gamma\backslash\mathcal{I}som((H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w), (H_0, W, (\langle\ ,\ \rangle_w)_w))$ on $S^{\log}$ (called a Γ-*level structure*) which satisfies the following conditions (1) and (2).

(1) $\mathrm{rank}_{\mathbf{Z}}(H_{\mathbf{Z}}) = \sum_{p,q} h^{p,q}$, $\quad \mathrm{rank}_{\mathcal{O}_S}(F^pH(\mathrm{gr}^W_w)_{\mathcal{O}}/F^{p+1}H(\mathrm{gr}^W_w)_{\mathcal{O}}) = h^{p,w-p}$ for all w, p.

(2) For any $s \in S$ and $t \in S^{\log}$ lying over s, if $\tilde{\mu}_t : (H_{\mathbf{Z},t}, W, (\langle\ ,\ \rangle_w)_w) \overset{\sim}{\to} (H_0, W, (\langle\ ,\ \rangle_w)_w)$ is a representative of the germ of μ at t, then there exists $\sigma \in \Sigma$ such that the image of the composite map

$$\mathrm{Hom}((M_S/\mathcal{O}_S^\times)_s, \mathbf{N}) \hookrightarrow \pi_1(s^{\log}) \to \mathrm{Aut}(H_{\mathbf{Z},t}, W, (\langle\ ,\ \rangle_w)_w) \xrightarrow{\text{by } \tilde{\mu}_t} \mathrm{Aut}(H_0, W, (\langle\ ,\ \rangle_w)_w)$$

is contained in $\exp(\sigma)$ and such that the $\exp(\sigma_{\mathbf{C}})$-orbit Z including $\tilde{\mu}_t(\mathbf{C}\otimes_{\mathcal{O}^{\log}_{S,t}} F_t)$, which is independent of the choice of a specialization $\mathcal{O}^{\log}_{S,t} \to \mathbf{C}$ at t (1.3.1), is a σ-nilpotent orbit.

We call an LMH with polarized graded quotients of type Φ on S also an *LMH with polarized graded quotients, with global monodromy in* Γ*, and with local monodromy in* Σ.

Remark. There is a mistake in the condition (1) in [KNU10b] 3.2.2. The correct condition is the above (1). In fact, the above (1) implies condition (1) there, but the converse is not necessarily true.

2.6.3. Moduli functor LMH_Φ.

Let $\mathrm{LMH}_\Phi : \mathcal{B}(\log) \to (\mathrm{set})$ be the contravariant functor defined as follows: For an object S of $\mathcal{B}(\log)$, $\mathrm{LMH}_\Phi(S)$ is the set of isomorphism classes of LMH with polarized graded quotients of type Φ on S.

2.6.4. Period maps.

We will have some period maps associated to an LMH with polarized graded quotients of type Φ. In the following, assume that Γ is neat.

(1) $\mathrm{LMH}_\Phi \to \ \mathrm{Map}(-, \Gamma\backslash D_\Sigma)$.

Set-theoretically, this period map is described as follows. Let $S \in \mathcal{B}(\log)$. Let $H \in \mathrm{LMH}_\Phi(S)$. Then, the image of $s \in S$ by the map corresponding to H is $((\sigma, Z) \bmod \Gamma) \in \Gamma\backslash D_\Sigma$. Here σ is the smallest cone of Σ satisfying 2.6.2 (2), which exists by 2.2.4 (2)$''$, and Z is the associated $\exp(\sigma_{\mathbf{C}})$-orbit which appeared in 2.6.2 (2).

(2) $\mathrm{LMH}_\Phi \to \ \mathrm{Map}(-^{\log}, \Gamma\backslash D^\sharp_\Sigma)$.

This is the composite of (1) and $(S \xrightarrow{f} \Gamma\backslash D_\Sigma) \mapsto (S^{\log} \xrightarrow{f^{\log}} (\Gamma\backslash D_\Sigma)^{\log} \simeq \Gamma\backslash D^\sharp_\Sigma)$ (cf. 2.5.6). Set-theoretically, this is described as follows. Let $S \in \mathcal{B}(\log)$. Let $H \in \mathrm{LMH}_\Phi(S)$. Then, the image of $t \in S^{\log}$ by the map corresponding to H is $((\sigma, Z) \bmod \Gamma) \in \Gamma\backslash D^\sharp_\Sigma$. Here σ is the same as in (1), and Z is the $\exp(i\sigma_{\mathbf{R}})$-orbit including $\tilde{\mu}_t(\mathbf{C} \otimes_{\mathcal{O}^{\log}_{S,t}} F_t)$ which appeared in 2.6.2 (2).

(3) A variant of (2). Let $\mathcal{B}(\log)^{\log}$ be the category of pairs (S, U), where S is an object of $\mathcal{B}(\log)$ and U is an open set of $S^{\log}$. Let $\mathrm{LMH}^{\log}_\Phi$ be the functor on $\mathcal{B}(\log)^{\log}$ defined as follows. $\mathrm{LMH}^{\log}_\Phi((S, U))$ is the set of isomorphism classes of an $H \in \mathrm{LMH}_\Phi(S)$ plus a representative $\tilde{\mu} : (H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w) \simeq (H_0, W, (\langle\ ,\ \rangle_w)_w)$ of μ given on U. Then the map $\mathrm{LMH}^{\log}_\Phi((S, U)) \to \ \mathrm{Map}(U, D^\sharp_\Sigma)$ is given, whose set-theoretical description is similar to (2).

2.6.5. The idea of the definition of the period map

$$\mathrm{LMH}_\Phi \to \ \mathrm{Map}(-, \Gamma\backslash D_\Sigma)$$

of functors on $\mathcal{B}(\log)$ in 2.6.4 (1) is the same as in [KU09]: Let $S \in \mathcal{B}(\log)$. Let $H \in \mathrm{LMH}_\Phi(S)$. Then, locally on S, the corresponding $S \to \Gamma\backslash D_\Sigma$ comes from a morphism $S \to E_\sigma$ for some $\sigma \in \Sigma$. We define it in the course of the proof of the following theorem.

Theorem 2.6.6. *Let* $\Phi = (\Lambda, \Sigma, \Gamma)$ *be as in* 2.6.1*. Assume that* Γ *is neat. Then* LMH_Φ *is represented by* $\Gamma\backslash D_\Sigma$ *in* $\mathcal{B}(\log)$*.*

The proof will be given in §3–§4.

§3. Nilpotent orbits and associated SL(2)-orbits

In the next §4, we will prove main results in §2. This §3 is a preparation for §4. We review some necessary definitions and results on the space $D_{\mathrm{SL}(2)}$ of SL(2)-orbits in [KNU11], introduce the most important map (called the CKS map) $D^\sharp_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ in the fundamental diagram in Introduction, and prove that it is continuous, which will be crucial in §4.

§3.1. $D_{\mathrm{SL}(2)}$

We review from [KNU11] the definition of SL(2)-orbit, the space $D_{\mathrm{SL}(2)}$ and the SL(2)-orbit associated to an $(n+1)$-tuple $(N_1, \dots, N_n, F)$ which generates a nilpotent orbit.

3.1.1. SL(2)-orbit ([KNU11] 2.3).

Assume first that we are in the pure case (originally considered by [Scm73], [CKS86]), that is, $W_w = H_{0,\mathbf{R}}$ and $W_{w-1} = 0$ for some w. Then an SL(2)*-orbit in* n *variables* is a pair (ρ, φ), where $\rho : \mathrm{SL}(2, \mathbf{C})^n \to G_\mathbf{C}$ is a homomorphism of algebraic groups defined over $\mathbf{R}$ and $\varphi : \mathbf{P}^1(\mathbf{C})^n \to \check{D}$ is a holomorphic map, satisfying the following three conditions ([KU02], [KU09]).

(1) $\varphi(gz) = \rho(g)\varphi(z)$ for any $g \in \mathrm{SL}(2, \mathbf{C})^n$, $z \in \mathbf{P}^1(\mathbf{C})^n$.

(2) $\varphi(\mathfrak{h}^n) \subset D$.

(3) $\rho_*(\mathrm{fil}^p_z(\mathfrak{sl}(2, \mathbf{C})^{\oplus n})) \subset \mathrm{fil}^p_{\varphi(z)}(\mathfrak{g}_\mathbf{C})$ for any $z \in \mathbf{P}^1(\mathbf{C})^n$ and any $p \in \mathbf{Z}$.

Here in (2), $\mathfrak{h} \subset \mathbf{P}^1(\mathbf{C})$ is the upper-half plane. In (3), ρ_* denotes the Lie algebra homomorphism induced by ρ, and $\mathrm{fil}_z(\mathfrak{sl}(2, \mathbf{C})^{\oplus n})$, $\mathrm{fil}_{\varphi(z)}(\mathfrak{g}_\mathbf{C})$ are the filtrations induced by the Hodge filtrations at z, $\varphi(z)$, respectively.

Now we consider the general mixed Hodge situation. A *non-degenerate* SL(2)*-orbit of rank* n is a pair $((\rho_w, \varphi_w)_{w\in\mathbf{Z}}, \mathbf{r})$, where (ρ_w, φ_w) is an SL(2)-orbit in n variables for gr^W_w (that is, an SL(2)-orbit for the pure case) for each $w \in \mathbf{Z}$ and $\mathbf{r}$ is an element of D satisfying the following conditions (4)–(6).

(4) $\mathbf{r}(\mathrm{gr}^W_w) = \varphi_w(\mathbf{i})$ for any $w \in \mathbf{Z}$, where $\mathbf{i} = (i, \dots, i) \in \mathbf{P}^1(\mathbf{C})^n$.

(5) If $2 \le j \le n$, there exists $w \in \mathbf{Z}$ such that the j-th component of ρ_w is a non-trivial homomorphism.

(6) If $\mathbf{r} \in D_{\mathrm{spl}}$ and $n \ge 1$, there exists $w \in \mathbf{Z}$ such that the 1-st component of ρ_w is a non-trivial homomorphism. Here, D_{spl} denotes the subset of D consisting of $\mathbf{R}$-split mixed Hodge structures.

Let $((\rho_w, \varphi_w)_w, \mathbf{r})$ be a non-degenerate SL(2)-orbit of rank n.

Then, the associated homomorphism of algebraic groups over $\mathbf{R}$

$$\tau : \mathbf{G}^n_{m,\mathbf{R}} \to \mathrm{Aut}_\mathbf{R}(H_{0,\mathbf{R}}, W)$$

and the associated set (or family) of weight filtrations are defined as follows.

The canonical splitting (over $\mathbf{R}$) of the weight filtration of a mixed Hodge structure was originally defined in [CKS86]. It was reviewed in [KNU08] §1. See also [KNU09] §4 and [KNU11] §1.2. By associating the canonical splitting $\mathrm{spl}_W(F)$ of W to a mixed Hodge structure $F \in D$, we have a continuous map

$$\mathrm{spl}_W : D \to \mathrm{spl}(W),$$

where $\mathrm{spl}(W)$ denotes the set of all splittings of W.

Let $s_{\mathbf{r}} := \mathrm{spl}_W(\mathbf{r}) : \mathrm{gr}^W \xrightarrow{\sim} H_{0,\mathbf{R}}$ be the canonical splitting of W associated to $\mathbf{r}$. Then

$$\tau(t_1, \dots, t_n) = s_{\mathbf{r}} \circ \Big(\bigoplus_{w \in \mathbf{Z}} \big(\prod_{j=1}^n t_j\big)^w \rho_w(g_1, \dots, g_n) \text{ on } \mathrm{gr}^W_w \Big) \circ s_{\mathbf{r}}^{-1}$$

$$\text{with} \quad g_j = \begin{pmatrix} 1/\prod_{k=j}^n t_k & 0 \\ 0 & \prod_{k=j}^n t_k \end{pmatrix}.$$

For $1 \le j \le n$, let $\tau_j : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ be the j-th component of τ.

For $1 \le j \le n$, let $W^{(j)}$ be the finite increasing filtration on $H_{0,\mathbf{R}}$ defined by

$$W^{(j)}_w = \bigoplus_{k \le w} \{v \in H_{0,\mathbf{R}} \mid \tau_j(a)v = a^k v \ (\forall\, a \in \mathbf{R}^\times)\} \quad (w \in \mathbf{Z}).$$

The associated set (resp. family; this term is used for an indexed set) of weight filtrations is defined to be the set $\{W^{(1)}, \cdots, W^{(n)}\}$ (resp. the family $(W^{(j)})_{1 \le j \le n}$).

3.1.2. $D_{\mathrm{SL}(2)}$ ([KNU11] 2.5).

Two non-degenerate SL(2)-orbits $p = ((\rho_w, \varphi_w)_w, \mathbf{r})$ and $p' = ((\rho'_w, \varphi'_w)_w, \mathbf{r}')$ of rank n are said to be *equivalent* if there is a $t \in \mathbf{R}^n_{>0}$ such that

$$\rho'_w = \mathrm{Int}(\mathrm{gr}^W_w(\tau(t))) \circ \rho_w, \quad \varphi'_w = \mathrm{gr}^W_w(\tau(t)) \circ \varphi_w \quad (\forall\, w \in \mathbf{Z}), \quad \mathbf{r}' = \tau(t)\mathbf{r},$$

where $\mathrm{Int}(g)$ is the automorphism $x \mapsto gxg^{-1}$. This is actually an equivalence relation.

The associated homomorphism of algebraic groups $\tau : \mathbf{G}^n_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$, and the associated set (resp. family) of weight filtrations depend only on the equivalence class.

Let $D_{\mathrm{SL}(2)}$ be the set of all equivalence classes p of non-degenerate SL(2)-orbits of various ranks satisfying the following condition (1).

(1) If W' is a member of the set of weight filtrations associated to p, then the $\mathbf{R}$-spaces $W'_k \mathrm{gr}^W_w$ are rational (i.e. defined over $\mathbf{Q}$) for any k and w.

In [KNU11], we defined two topologies of $D_{\mathrm{SL}(2)}$, which we call the stronger topology and the weaker topology, respectively, and denoted by $D^I_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ the set $D_{\mathrm{SL}(2)}$ endowed with the stronger topology and the weaker topology, respectively. In this paper, we mainly use $D^I_{\mathrm{SL}(2)}$.

These topologies coincide in the pure case. In the mixed case, the identity map $D^I_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$ is continuous. Both topologies are Hausdorff ([KNU11] 3.5.7 (i)). In [KNU11], we defined some kind of real analytic structures on $D^I_{\mathrm{SL}(2)}$ and on $D^{II}_{\mathrm{SL}(2)}$, but we do not use them in the present paper.

A finite set Ψ of finite increasing filtrations on $H_{0,\mathbf{R}}$ is called an admissible set of weight filtrations if Ψ is the associated set of weight filtrations of some $p \in D_{\mathrm{SL}(2)}$. For an admissible set Ψ of weight filtrations, let $D^I_{\mathrm{SL}(2)}(\Psi)$ be the subset of $D_{\mathrm{SL}(2)}$ consisting of all points p of $D_{\mathrm{SL}(2)}$ such that the set of weight filtrations associated to p is a subset of Ψ. Then $\{D^I_{\mathrm{SL}(2)}(\Psi)\}_\Psi$ is an open covering of $D^I_{\mathrm{SL}(2)}$.

3.1.3. Associated SL(2)-orbits ([KNU11] 2.4.2).

Let $(N_1, \dots, N_n, F)$ generate a nilpotent orbit (2.2.2). Then, we associate to it a non-degenerate SL(2)-orbit $((\rho'_w, \varphi'_w)_w, \mathbf{r}_1)$ as follows. (In the following review of [KNU11] 2.4.2, the construction is by limits. For an alternative construction by finite algebraic steps, see ibid. 2.4.6.)

First, by [CKS86], for each $w \in \mathbf{Z}$, we have the SL(2)-orbit (ρ_w, φ_w) in n variables for gr^W_w associated to $(\mathrm{gr}^W_w(N_1), \dots, \mathrm{gr}^W_w(N_n), F(\mathrm{gr}^W_w))$, which generates a nilpotent orbit for gr^W_w. Let $k = \min(\{j \mid 1 \le j \le n,\ N_j \ne 0\} \cup \{n+1\})$. Let

$$J' = \{j \mid 1 \le j \le n, \text{ the } j\text{-th component of } \rho_w \text{ is non-trivial for some } w \in \mathbf{Z}\}.$$

Let $J = J' = \emptyset$ if $k = n+1$, and let $J = J' \cup \{k\}$ if otherwise. Let $J = \{a(1), \dots, a(r)\}$ with $a(1) < \dots < a(r)$. Then (ρ'_w, φ'_w) is an SL(2)-orbit on gr^W_w characterized by

$$\rho'_w(g_{a(1)}, \dots, g_{a(r)}) := \rho_w(g_1, \dots, g_n), \quad \varphi'_w(z_{a(1)}, \dots, z_{a(r)}) := \varphi_w(z_1, \dots, z_n).$$

Next, if $y_j \in \mathbf{R}_{>0}$ and $y_j/y_{j+1} \to \infty$ ($1 \le j \le n$, y_{n+1} means 1), the canonical splitting $\mathrm{spl}_W(\exp(\sum_{j=1}^n iy_jN_j)F)$ of W (3.1.1) associated to $\exp(\sum_{j=1}^n iy_jN_j)F$ converges in $\mathrm{spl}(W)$ by the main theorem 0.5 of [KNU08]. Let $s \in \mathrm{spl}(W)$ be the limit.

Let $\tau : \mathbf{G}^n_{m,\mathbf{R}} \to \mathrm{Aut}_\mathbf{R}(H_{0,\mathbf{R}}, W)$ be the homomorphism of algebraic groups defined by

$$\tau(t_1, \dots, t_n) = s \circ \Big(\bigoplus_{w\in\mathbf{Z}} \big(\prod_{j=1}^n t_j\big)^w \rho_w(g_1, \dots, g_n) \text{ on } \mathrm{gr}^W_w \Big) \circ s^{-1},$$

where g_j is as in 3.1.1. Then, as $y_j > 0$, $y_1 = \dots = y_k$, $y_j/y_{j+1} \to \infty$ ($k \le j \le n$, y_{n+1} means 1),

$$\tau\left(\sqrt{\frac{y_2}{y_1}}, \dots, \sqrt{\frac{y_{n+1}}{y_n}}\right)^{-1} \exp(\textstyle\sum_{j=1}^n iy_jN_j)F$$

converges in D. Let $\mathbf{r}_1 \in D$ be the limit (cf. [KNU11] 2.4.2 (ii)).

§3.2. Valuative spaces

3.2.1. D_{val} and $D^\sharp_{\mathrm{val}}$.

Let D_{val} be the set of all triples (A, V, Z), where

A is a $\mathbf{Q}$-subspace of $\mathfrak{g}_\mathbf{Q}$ consisting of mutually commutative nilpotent elements. Let $A_\mathbf{R} = \mathbf{R} \otimes_\mathbf{Q} A$ and $A_\mathbf{C} = \mathbf{C} \otimes_\mathbf{Q} A$.

V is an (additive) submonoid of $A^* = \mathrm{Hom}_{\mathbf{Q}}(A, \mathbf{Q})$ satisfying the following conditions: $V \cap (-V) = \{0\}$, $V \cup (-V) = A^*$.

Z is a subset of $\check{D}$ such that $Z = \exp(A_{\mathbf{C}})F$ for any $F \in Z$ and that there exists a finitely generated rational subcone τ of $A_{\mathbf{R}}$ satisfying the following condition (1).

(1) Z is a τ-nilpotent orbit, and V contains any element h of A^* such that $h : A \to \mathbf{Q}$ sends $A \cap \tau$ to $\mathbf{Q}_{\geq 0}$.

Let $D^{\sharp}_{\mathrm{val}}$ be the set of all triples (A, V, Z), where A and V are as above, and Z is a subset of $\check{D}$ satisfying the condition gotten from the condition above by replacing "$\exp(A_{\mathbf{C}})F$" by "$\exp(iA_{\mathbf{R}})F$" and "nilpotent orbit" by "nilpotent i-orbit".

We have a canonical map

$$D^{\sharp}_{\mathrm{val}} \to D_{\mathrm{val}}, \quad (A, V, Z) \mapsto (A, V, \exp(A_{\mathbf{C}})Z),$$

which is surjective.

See [KU09] 5.3.3 for D_{val} and $D^{\sharp}_{\mathrm{val}}$ in the pure case. See [KU09] 3.6.17 for some typical examples of valuative submonoids.

3.2.2. $D_{\Sigma,\mathrm{val}}$ and $D^{\sharp}_{\Sigma,\mathrm{val}}$.

Let Σ be a weak fan (2.2.3) in $\mathfrak{g}_{\mathbf{Q}}$.

Let $D_{\Sigma,\mathrm{val}}$ (resp. $D^{\sharp}_{\Sigma,\mathrm{val}}$) be the subset of D_{val} (resp. $D^{\sharp}_{\mathrm{val}}$) consisting of all $(A, V, Z) \in D_{\mathrm{val}}$ (resp. $D^{\sharp}_{\mathrm{val}}$) such that there exists $\sigma \in \Sigma$ satisfying the following conditions (1) and (2).

(1) $\exp(\sigma_{\mathbf{C}})Z$ is a σ-nilpotent orbit (resp. $\exp(i\sigma_{\mathbf{R}})Z$ is a σ-nilpotent i-orbit).

(2) V contains any element h of A^* such that $h : A \to \mathbf{Q}$ sends $A \cap \sigma$ to $\mathbf{Q}_{\geq 0}$.

Note that (2) implies $A \subset \sigma_{\mathbf{R}}$.

The canonical map $D^{\sharp}_{\mathrm{val}} \to D_{\mathrm{val}}$ (3.2.1) induces a canonical map $D^{\sharp}_{\Sigma,\mathrm{val}} \to D_{\Sigma,\mathrm{val}}$ which is surjective.

For a sharp rational nilpotent cone σ, let $D_{\sigma,\mathrm{val}} = D_{\mathrm{face}(\sigma),\mathrm{val}}$ and $D^{\sharp}_{\sigma,\mathrm{val}} = D^{\sharp}_{\mathrm{face}(\sigma),\mathrm{val}}$ (cf. 2.2.7).

Our next subject is to define canonical maps $D_{\Sigma,\mathrm{val}} \to D_{\Sigma}$ and $D^{\sharp}_{\Sigma,\mathrm{val}} \to D^{\sharp}_{\Sigma}$ for a weak fan Σ.

Lemma 3.2.3. *Let Σ be a weak fan and let $(A, V, Z) \in D_{\Sigma,\mathrm{val}}$. Let S be the set of all $\sigma \in \Sigma$ satisfying the conditions* (1) *and* (2) *in* 3.2.2. *Then S has a smallest element.*

Proof. Let T be the set of all rational nilpotent cones σ which satisfy (1) and (2) in 3.2.2. Hence $S = T \cap \Sigma$. Let T_1 be the set of all finitely generated subcones of $A_{\mathbf{R}}$ satisfying the condition 3.2.1 (1).

Then we have

Claim. If $\sigma_1 \in T_1$ and $\sigma_2 \in T$, then $\sigma_1 \cap \sigma_2 \in T_1$.

We prove Claim. By $(\sigma_1 \cap \sigma_2)^{\vee} = \sigma_1^{\vee} + \sigma_2^{\vee}$ ([O88] A.1 (2)), 3.2.2 (2) is satisfied for $\sigma := \sigma_1 \cap \sigma_2$. Here $(\ \ )^{\vee}$ denotes the dual of a cone. Then, $\dim \sigma = \dim \sigma_1$. On the other hand, Z generates a σ_1-nilpotent orbit by 3.2.1 (1) for σ_1. Thus Z also generates a σ-nilpotent orbit. Hence $\sigma \in T_1$.

We now prove 3.2.3. Let σ_1 and σ_2 be minimal elements of S. We prove $\sigma_1 = \sigma_2$. Since $(A, V, Z) \in D_{\mathrm{val}}$, the set T_1 is not empty. Let τ be an element of T_1, and let $\tau' = \tau \cap \sigma_1 \cap \sigma_2$. Then $\tau' \in T_1$ by Claim. Since Σ is a weak fan, by 2.2.4 (2)$''$, the set $\{\alpha \in S \mid \tau' \subset \alpha\}$ has a smallest element σ. Since σ_1 and σ_2 belong to this set, $\sigma \subset \sigma_1$ and $\sigma \subset \sigma_2$. Since σ_1 and σ_2 are minimal in S, we have $\sigma_1 = \sigma = \sigma_2$. □

3.2.4. Let Σ be a weak fan. We define the maps $D_{\Sigma,\mathrm{val}} \to D_\Sigma$ and $D^\sharp_{\Sigma,\mathrm{val}} \to D^\sharp_\Sigma$ as

$$(A, V, Z) \mapsto (\sigma, \exp(\sigma_{\mathbf{C}})Z), \quad (A, V, Z) \mapsto (\sigma, \exp(i\sigma_{\mathbf{R}})Z),$$

respectively, where σ is the smallest element of the set S in 3.2.3.

3.2.5. The sets $E_{\sigma,\mathrm{val}}$ and $E^\sharp_{\sigma,\mathrm{val}}$.

Let E_σ and $E^\sharp_\sigma$ be as in 2.3.3, and let $\mathrm{toric}_{\sigma,\mathrm{val}}$ over toric_σ and $|\mathrm{toric}|_{\sigma,\mathrm{val}}$ be defined as in [KU09] 5.3.6. Define

$$E_{\sigma,\mathrm{val}} = \mathrm{toric}_{\sigma,\mathrm{val}} \times_{\mathrm{toric}_\sigma} E_\sigma, \quad E^\sharp_{\sigma,\mathrm{val}} = |\mathrm{toric}|_{\sigma,\mathrm{val}} \times_{|\mathrm{toric}|_\sigma} E^\sharp_\sigma.$$

We have the canonical projections $E_{\sigma,\mathrm{val}} \to E_\sigma$ and $E^\sharp_{\sigma,\mathrm{val}} \to E^\sharp_\sigma$.

3.2.6. Analogously as in [KU09] 5.3.7, we have the projections

$$\begin{aligned} E_{\sigma,\mathrm{val}} &\to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_{\sigma,\mathrm{val}}; & (A, V', z, F) &\mapsto (A, V', \exp(A_{\mathbf{C}})\exp(z)F), \\ E^\sharp_{\sigma,\mathrm{val}} &\to D^\sharp_{\sigma,\mathrm{val}}; & (A, V', y, F) &\mapsto (A, V', \exp(iA_{\mathbf{R}})\exp(iy)F), \end{aligned}$$

where $F \in \check{D}$, and $(A, V', z) \in \mathrm{toric}_{\sigma,\mathrm{val}}$ (resp. $(A, V', y) \in |\mathrm{toric}|_{\sigma,\mathrm{val}}$) (cf. [KU09] 5.3.6, 5.3.7).

3.2.7. By the topology of $E^\sharp_\sigma$ given in 2.4.1, we define on $E^\sharp_{\sigma,\mathrm{val}}$ and on $D^\sharp_{\Sigma,\mathrm{val}}$ topologies, analogously as in [KU09] 5.3.6–5.3.8. In the same way, by using the structure of E_σ given in 2.4.1 and the structure of $\mathrm{toric}_{\sigma,\mathrm{val}}$ given in [KU09] 3.6.23, we define on $E_{\sigma,\mathrm{val}}$ and on $\Gamma\backslash D_{\Sigma,\mathrm{val}}$ structures of log local ringed spaces over $\mathbf{C}$.

§3.3. The map $D^\sharp_{\mathrm{val}} \to D_{\mathrm{SL}(2)}$

We have the following two theorems 3.3.1 and 3.3.2. These are the mixed Hodge-theoretic analogues of [KU09] 5.4.3 and 5.4.4, respectively, whose proofs are given in 6.4 of [KU09]. Since the proofs of 3.3.1 and 3.3.2 are parallel to this pure case, we only give some details of key steps in 3.3.3–3.3.7 below.

Theorem 3.3.1. *Let $p = (A, V, Z) \in D^{\sharp}_{\mathrm{val}}$.*

(i) *There exists a family $(N_j)_{1 \le j \le n}$ of elements of $A_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} A$ satisfying the following two conditions*:

(1) *If $F \in Z$, $(N_1, \dots, N_n, F)$ generates a nilpotent orbit.*

(2) *Via $(N_j)_{1 \le j \le n} : A^* = \mathrm{Hom}_{\mathbf{Q}}(A, \mathbf{Q}) \to \mathbf{R}^n$, V coincides with the set of all elements of A^* whose images in $\mathbf{R}^n$ are ≥ 0 with respect to the lexicographic order.*

(ii) *Take $(N_j)_{1 \le j \le n}$ as in* (i), *let $F \in Z$ and define $\psi(p) \in D_{\mathrm{SL}(2)}$ to be the class of the* SL(2)*-orbit associated to $(N_1, \dots, N_n, F)$. Then $\psi(p)$ is independent of the choices of $(N_j)_{1 \le j \le n}$ and F.*

By Theorem 3.3.1, we obtain a map

(3) $$\psi : D^{\sharp}_{\mathrm{val}} \to D_{\mathrm{SL}(2)}.$$

We call this map the *CKS map*, like in the pure case in [KU09]. This map is the most important ingredient in our series of works. (CKS stands for Cattani-Kaplan-Schmid from whose work [CKS86] the map ψ in the pure case was defined in [KU09].)

Theorem 3.3.2. *Let Σ be a weak fan in $\mathfrak{g}_{\mathbf{Q}}$. Then, $\psi : D^{\sharp}_{\Sigma,\mathrm{val}} \to D^{I}_{\mathrm{SL}(2)}$ is continuous.*

Thus $\psi : D^{\sharp}_{\Sigma,\mathrm{val}} \to D^{I}_{\mathrm{SL}(2)}$ is the unique continuous extension of the identity map of D.

We plan to discuss the C^{∞}-property of the CKS map over the boundary (cf. [KNU11] 4.4.8 and 4.4.9) in a future paper.

As in [KU09] 6.4, a key step for these theorems is Proposition 3.3.4 below, which is an analog of [KU09] 6.4.2. We only prove this proposition.

3.3.3. Let $(N_s)_{s \in S}$ be a finite family of mutually commuting nilpotent elements of $\mathfrak{g}_{\mathbf{R}}$, let $F \in \check{D}$, and assume that $((N_s)_{s \in S}, F)$ generates a nilpotent orbit. Let $a_s \in \mathbf{R}_{>0}$ for $s \in S$. Assume that S is the disjoint union of non-empty subsets S_j ($1 \le j \le n$). Denote $S_{\le j} := \bigsqcup_{k \le j} S_k$ and $S_{\ge j} := \bigsqcup_{k \ge j} S_k$. For $1 \le j \le n$, let $\check{D}_j$ be the subset of $\check{D}$ consisting of all $F' \in \check{D}$ such that $((N_s)_{s \in S_{\le j}}, F')$ generates a nilpotent orbit. Let L be a directed ordered set (not necessarily an ordinary sequence $\{1, 2, 3, \dots\}$), let $F_\lambda \in \check{D}$ ($\lambda \in L$), $y_{\lambda,s} \in \mathbf{R}_{>0}$ ($\lambda \in L, s \in S$), and assume that the following five conditions are satisfied.

(1) F_λ converges to F.

(2) $y_{\lambda,s} \to \infty$ for any $s \in S$.

(3) If $1 \le j < n$, $s \in S_{\le j}$ and $t \in S_{\ge j+1}$, then $\frac{y_{\lambda,s}}{y_{\lambda,t}} \to \infty$.

(4) If $1 \le j \le n$ and $s, t \in S_j$, then $\frac{y_{\lambda,s}}{y_{\lambda,t}} \to \frac{a_s}{a_t}$.

(5) For $1 \le j \le n$ and $e \ge 0$, there exist $F^*_\lambda \in \check{D}$ ($\lambda \in L$) and $y^*_{\lambda,t} \in \mathbf{R}_{>0}$ ($\lambda \in L, t \in S_{\ge j+1}$) such that

$$\exp\bigl(\textstyle\sum_{t \in S_{\ge j+1}} i y^*_{\lambda,t} N_t\bigr) F^*_\lambda \in \check{D}_j \quad (\lambda : \text{sufficiently large}),$$

$$y^e_{\lambda,s} d(F_\lambda, F^*_\lambda) \to 0 \quad (\forall s \in S_j),$$

$$y^e_{\lambda,s} |y_{\lambda,t} - y^*_{\lambda,t}| \to 0 \quad (\forall s \in S_j, \forall t \in S_{\ge j+1}).$$

Here d is a metric on a neighborhood of F in $\check{D}$ which is compatible with the analytic topology of $\check{D}$.

For each $1 \leq j \leq n$, take $c_j \in S_j$ and denote $N_j := \sum_{s \in S_j} \frac{a_s}{a_{c_j}} N_s$.

Then, $(N_1, \cdots, N_n, F)$ generates a nilpotent orbit.

Proposition 3.3.4. *Let the assumption be as in* 3.3.3. *Let* $\mathbf{r}_1 \in D$ *be the point and* τ *be the homomorphism which are associated to* $(N_1, \ldots, N_n, F)$ *as in* 3.1.3.

Then we have the following convergences (1) *and* (2).

(1) $\tau\left(\sqrt{\frac{y_{\lambda,c_1}}{y_{\lambda,c_2}}}, \ldots, \sqrt{\frac{y_{\lambda,c_n}}{y_{\lambda,c_{n+1}}}}\right) \exp(\sum_{s \in S} i y_{\lambda,s} N_s) F_\lambda \to \mathbf{r}_1$ *in* D*, where* $y_{\lambda,c_{n+1}} = 1$.

(2) $\mathrm{spl}_W\big(\exp(\sum_{s \in S} i y_{\lambda,s} N_s) F_\lambda\big) \to \mathrm{spl}_W(\mathbf{r}_1)$ *in* $\mathrm{spl}(W)$.

Here, for $F' \in D$, $\mathrm{spl}_W(F') \in \mathrm{spl}(W)$ *denotes the canonical splitting of* W *associated to* F' *as in* 3.1.1.

Remark. By [KNU11] 3.2.12 (i) which characterizes the topology of $D^I_{\mathrm{SL}(2)}$, this proposition implies the convergence

$$\exp(\textstyle\sum_{s \in S} i y_{\lambda,s} N_s) F_\lambda \to \mathrm{class}(\psi(N_1, \cdots, N_n, F)) \quad \text{in } D^I_{\mathrm{SL}(2)}.$$

A new feature in the mixed Hodge situation is the condition (2) in 3.3.4.

3.3.5. In the rest of this subsection, we prove this proposition. First we prove a lemma which includes (1) of 3.3.4.

Let the notation be as in 3.3.3. Before stating the lemma in 3.3.6, we recall the definition of $\hat{F}_{(j)}$ and τ_j associated to $(N_1, \ldots, N_n, F)$ (cf. [KNU08] 10.1.1, [KNU11] 2.4.6).

Let $W^{(0)} = W$ and, for $1 \leq j \leq n$, let $W^{(j)} = M(N_1 + \cdots + N_j, W)$.

For $0 \leq j \leq n$, we define an $\mathbf{R}$-split mixed Hodge structure $(W^{(j)}, \hat{F}_{(j)})$ and the associated splitting $s^{(j)}$ of $W^{(j)}$ inductively starting from $j = n$ and ending at $j = 0$. First, $(W^{(n)}, F)$ is a mixed Hodge structure as is proved by Deligne (see [K86] 5.2.1). Let $(W^{(n)}, \hat{F}_{(n)})$ be the $\mathbf{R}$-split mixed Hodge structure associated to the mixed Hodge structure $(W^{(n)}, F)$. Then $(W^{(n-1)}, \exp(iN_n)\hat{F}_{(n)})$ is a mixed Hodge structure. (As explained in [KNU08] 10.1, this is shown by reducing to the case of [CKS86] where W is pure.) Let $(W^{(n-1)}, \hat{F}_{(n-1)})$ be the $\mathbf{R}$-split mixed Hodge structure associated to $(W^{(n-1)}, \exp(iN_n)\hat{F}_{(n)})$. Then $(W^{(n-2)}, \exp(iN_{n-1})\hat{F}_{(n-1)})$ is a mixed Hodge structure. This process continues. In this way we define $\hat{F}_{(j)}$ downward inductively as the $\mathbf{R}$-split mixed Hodge structure associated to the mixed Hodge structure $(W^{(j)}, \exp(iN_{j+1})\hat{F}_{(j+1)})$, and define $s^{(j)} : \mathrm{gr}^{W^{(j)}} \simeq H_{0,\mathbf{R}}$ to be the splitting of $W^{(j)}$ associated to $\hat{F}_{(j)}$. The splitting s in 3.1.3 is nothing but $s^{(0)}$ ([KNU08] 10.1.2). The j-th component

$$\tau_j : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W) \quad (0 \leq j \leq n)$$

of the homomorphism τ associated to $(N_1, \ldots, N_n, F)$ (3.1.3) is characterized as follows. For $a \in \mathbf{R}^\times$ and $w \in \mathbf{Z}$, $\tau_j(a)$ acts on $s^{(j)}(\mathrm{gr}^{W^{(j)}}_w)$ as the multiplication by a^w.

Lemma 3.3.6. *Let the notation be as in* 3.3.3 *and* 3.3.5*. Let* $1 \le j \le n$*. Then*

$$\Big(\prod_{j\le k\le n} \tau_k\Big(\sqrt{\tfrac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}}\Big)\Big) \exp(\textstyle\sum_{s\in S_{\ge j}} iy_{\lambda,s}N_s)F_\lambda \to \exp(iN_j)\hat{F}_{(j)} \quad \text{in } \check{D},$$

where $y_{\lambda,c_{n+1}} := 1$*. (Recall* $N_j := \sum_{s\in S_j} \frac{a_s}{a_{c_j}} N_s$*.)*

Proof. This is a mixed Hodge version of [KU09] Lemma 6.4.2, and proved by the same method in the proof of that lemma. We repeat the argument for readers' convenience.

We prove 3.3.6 by downward induction on j.

First we show that we may assume

$$\text{(1)} \qquad \exp(\textstyle\sum_{s\in S_{\ge j+1}} iy_{\lambda,s}N_s)F_\lambda \in \check{D}_j \quad (\forall\lambda).$$

Let $u_\lambda = \Big(\prod_{j\le k\le n} \tau_k\Big(\sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}}\Big)\Big) \exp\big(\sum_{s\in S_{\ge j}} iy_{\lambda,s}N_s\big)$. Since the N_s are nilpotent and they mutually commute, $\exp(\sum_{s\in S_{\ge j}} iy_{\lambda,s}N_s)$ is a polynomial in the $y_{\lambda,s}$ with coefficients in $\mathfrak{g}_{\mathbf{C}}$. From this, we see that, if $e \ge 0$ is a sufficiently large integer, then the following holds: If $x_\lambda \in \mathfrak{g}_{\mathbf{C}}$ converges to 0 satisfying $y^e_{\lambda,c_j}x_\lambda \to 0$, then $\mathrm{Ad}(u_\lambda)(x_\lambda) \to 0$. Take such e and take F^*_λ $(\lambda \in L)$ and $y^*_{\lambda,t}$ $(\lambda \in L, t \in S_{\ge j+1})$ as in 3.3.3 (5). Then (1) is satisfied if we replace $y_{\lambda,s}$ by $y^*_{\lambda,s}$ and F_λ by F^*_λ. Let $y^*_{\lambda,s} = y_{\lambda,s}$ $(\lambda \in L, s \in S_j)$ and define u^*_λ in the same way as u_λ replacing $y_{\lambda,s}$ by $y^*_{\lambda,s}$. Then $F_\lambda = \exp(x_\lambda)F^*_\lambda$ for some $x_\lambda \in \mathfrak{g}_{\mathbf{C}}$ such that $y^e_{\lambda,c_j}x_\lambda \to 0$. We have $u_\lambda F_\lambda = u_\lambda \exp(x_\lambda)F^*_\lambda = \exp(\mathrm{Ad}(u_\lambda)(x_\lambda))(u_\lambda(u^*_\lambda)^{-1})u^*_\lambda F^*_\lambda$, and $\mathrm{Ad}(u_\lambda)(x_\lambda) \to 0$, $u_\lambda(u^*_\lambda)^{-1} \to 1$. Hence we can assume (1).

We assume (1). By [KNU08] 10.3, we have

$$\begin{aligned}&\Big(\prod_{k\ge j} \tau_k\Big(\sqrt{\tfrac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}}\Big)\Big) \exp(\textstyle\sum_{s\in S_{\ge j}} iy_{\lambda,s}N_s)F_\lambda \\ &= \exp\big(\textstyle\sum_{s\in S_j} i\frac{y_{\lambda,s}}{y_{\lambda,c_j}}N_s\big)\Big(\prod_{k\ge j} \tau_k\Big(\sqrt{\tfrac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}}\Big)\Big) \exp(\textstyle\sum_{t\in S_{\ge j+1}} iy_{\lambda,t}N_t)F_\lambda.\end{aligned}$$

If $j < n$, the hypothesis of induction is

$$\text{(2)} \qquad \Big(\prod_{k\ge j+1} \tau_k\Big(\sqrt{\tfrac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}}\Big)\Big) \exp(\textstyle\sum_{t\in S_{\ge j+1}} iy_{\lambda,t}N_t)F_\lambda \to \exp(iN_{j+1})\hat{F}_{(j+1)}.$$

On the other hand, in the case $j = n$, consider the following convergence:

$$\text{(3)} \qquad F_\lambda \to F.$$

By (1), in the case $j < n$ (resp. $j = n$), (2) (resp. (3)) is a convergence of mixed Hodge structures for the weight filtration $W^{(j)}$. By [KU09] 6.1.11 (ii), it follows

$$\text{(4)} \qquad \Big(\prod_{k\ge j} \tau_k\Big(\sqrt{\tfrac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}}\Big)\Big) \exp(\textstyle\sum_{t\in S_{\ge j+1}} iy_{\lambda,t}N_t)F_\lambda \to \hat{F}_{(j)}.$$

On the other hand, by 3.3.3 (4), we have

(5) $$\sum_{s\in S_j} \frac{y_{\lambda,s}}{y_{\lambda,c_j}} N_s \to N_j.$$

Applying the exponential of i-times of (5) to (4), we obtain Lemma 3.3.6. □

3.3.7. We prove Proposition 3.3.4.

(1) of 3.3.4 is the case $j = k$ (k is as in 3.1.3) of 3.3.6 by $\mathbf{r}_1 = \exp(iN_k)\hat{F}_{(k)}$ ([KNU11] 2.4.8, Claim).

For 3.3.4 (2), we prove the following assertion (A_j) by downward induction on j. (Note that (A_0) is what we want to prove.)

(A_j) : Proposition 3.3.4 (2) is true in the case where $\exp\big(\sum_{t\in S_{\geq j+1}} iy_{\lambda,t}N_t\big)F_\lambda \in \check{D}_j$ for all λ. Here $\check{D}_j$ for $1 \leq j \leq n$ are defined in 3.3.3, and $\check{D}_0 := D$.

Proof. Let $0 \leq j \leq n$.

Let $p_\lambda = \exp(\sum_{s\in S} iy_{\lambda,s}N_s)F_\lambda$, $t_{\lambda,j} = \prod_{j<k\leq n} \tau_k\Big(\sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}}\Big)$, and $p_{\lambda,j} = t_{\lambda,j}p_\lambda$. Then, by [KNU08] 10.3,

$$p_{\lambda,j} = t_{\lambda,j}p_\lambda = \exp\big(\sum_{s\in S_{\leq j}} i\frac{y_{\lambda,s}}{y_{\lambda,c_{j+1}}}N_s\big)U_{\lambda,j},$$

$$\text{where} \quad U_{\lambda,j} := t_{\lambda,j}\exp\big(\sum_{t\in S_{\geq j+1}} iy_{\lambda,t}N_t\big)F_\lambda.$$

Assume $\exp\big(\sum_{t\in S_{\geq j+1}} iy_{\lambda,t}N_t\big)F_\lambda \in \check{D}_j$. Then $(N'_1, \ldots, N'_j, U_{\lambda,j})$ generates a nilpotent orbit, where $N'_k = \sum_{s\in S_k} \frac{y_{\lambda,s}}{y_{\lambda,c_k}}N_s$ $(1 \leq k \leq j)$. Let s_λ be the associated limit splitting. By [KNU08] 0.5 (2) and 10.8 (1), there is a convergent power series u_λ whose constant term is 1 and whose coefficients depend on $U_{\lambda,j}$ and $y_{\lambda,s}/y_{\lambda,c_k}$ $(1 \leq k \leq j,\ s \in S_k)$ real analytically such that $\mathrm{spl}_W(p_{\lambda,j}) = u_\lambda\Big(\frac{y_{\lambda,c_2}}{y_{\lambda,c_1}}, \ldots, \frac{y_{\lambda,c_{j+1}}}{y_{\lambda,c_j}}\Big)s_\lambda$. Since s_λ also depends real analytically on $U_{\lambda,j}$, we have

(1) $\mathrm{spl}_W(p_{\lambda,j})$ converges to $\mathrm{spl}_W(\mathbf{r}_1)$.

This already showed (A_n).

Next, assume $j < n$ and assume that (A_{j+1}) is true. We prove (A_j) is true.

Choose a sufficiently big $e > 0$ depending on $\tau_{j+1}, \ldots, \tau_n$.

Take F^*_λ and $y^*_{\lambda,t}$ $(t \in S_{\geq j+2})$ by the assumption 3.3.3 (5). Let $y^*_{\lambda,t} = y_{\lambda,t}$ $(t \in S_{\leq j+1})$. Define p^*_λ, $t^*_{\lambda,j}$ and $p^*_{\lambda,j}$ similarly as p_λ, $t_{\lambda,j}$ and $p_{\lambda,j}$, respectively.

Then we have

(2) $\mathrm{spl}_W(p^*_{\lambda,j})$ converges to $\mathrm{spl}_W(\mathbf{r}_1)$, and $y^e_{\lambda,c_{j+1}}d(\mathrm{spl}_W(p_{\lambda,j}), \mathrm{spl}_W(p^*_{\lambda,j})) \to 0$.

By downward induction hypothesis on j, we have

(3) $\mathrm{spl}_W(p^*_\lambda) = t^{*-1}_{\lambda,j}\,\mathrm{spl}_W(p^*_{\lambda,j})\,\mathrm{gr}^W(t^*_{\lambda,j})$ converges to $\mathrm{spl}_W(\mathbf{r}_1)$.

By (1)–(3), we have $\mathrm{spl}_W(p_\lambda) = t^{-1}_{\lambda,j}\,\mathrm{spl}_W(p_{\lambda,j})\,\mathrm{gr}^W(t_{\lambda,j})$ also converges to $\mathrm{spl}_W(\mathbf{r}_1)$. □

§4. Proofs of the main results

In this section, we prove the results in §2. The proofs are the mixed Hodge-theoretic and weak fan versions of [KU09] §7, and arguments are parallel to those in [KU09]. We replace the results and lemmas used there with the corresponding ones here (e.g. the Hausdorffness of $D_{\mathrm{SL}(2)}$ in the pure case [KU02] 3.14 (ii) is replaced by the corresponding result in the mixed Hodge situation [KNU11]), and make obvious modifications (e.g. fans are replaced by weak fans, the associated filtration $W(-)$ is replaced by $M(-,W)$, $\tilde{\rho}$ is replaced by τ, ...). For the readers' conveniences, we include several important arguments rewritten from those of [KU09] §7, even in the case where they are more or less repetitions.

§4.1. Proof of Theorem 2.5.1

We prove Theorem 2.5.1.

4.1.1. The fact that $\tilde{E}_\sigma$ is a log manifold is proved by the similar arguments in the proof of the pure case [KU09] 3.5.10. In fact, the proof there essentially uses only the horizontality which remains true in the mixed Hodge case.

4.1.2. We prove that E_σ is a log manifold.

To see this, it is sufficient to show that E_σ is open in $\tilde{E}_\sigma$ for the strong topology of $\tilde{E}_\sigma$ in $\check{E}_\sigma$.

For a face τ of σ, let $U(\tau)$ be the open set of toric_σ defined to be $\mathrm{Spec}(\mathbf{C}[P])_{\mathrm{an}}$ where $P \subset \mathrm{Hom}(\Gamma(\sigma)^{\mathrm{gp}}, \mathbf{Z})$ is the inverse image of $\Gamma(\tau)^\vee \subset \mathrm{Hom}(\Gamma(\tau)^{\mathrm{gp}}, \mathbf{Z})$ under the canonical map $\mathrm{Hom}(\Gamma(\sigma)^{\mathrm{gp}}, \mathbf{Z}) \to \mathrm{Hom}(\Gamma(\tau)^{\mathrm{gp}}, \mathbf{Z})$. Let $V(\tau)$ be the open set of $\tilde{E}_\sigma$ defined to be $\tilde{E}_\sigma \cap (U(\tau) \times \check{D})$. Let V be the union of $V(\tau)$, where τ ranges over all admissible faces of σ. Then V is open in $\tilde{E}_\sigma$, and $E_\sigma \subset V$. Concerning the canonical projection $V \to \prod_w \tilde{E}_{\sigma(\mathrm{gr}^W_w)}$, E_σ is the inverse image of $\prod_w E_{\sigma(\mathrm{gr}^W_w)}$ in V. By [KU09] 4.1.1 Theorem A (i), each $E_{\sigma(\mathrm{gr}^W_w)}$ is open in $\tilde{E}_{\sigma(\mathrm{gr}^W_w)}$ in the strong topology of $\tilde{E}_{\sigma(\mathrm{gr}^W_w)}$ in $\check{E}_{\sigma(\mathrm{gr}^W_w)}$. Hence E_σ is open in $\tilde{E}_\sigma$ in the strong topology of $\tilde{E}_\sigma$ in $\check{E}_\sigma$. Hence E_σ is a log manifold.

§4.2. Study of $D^\sharp_\sigma$

Let Σ be a weak fan, Γ a subgroup of $G_{\mathbf{Z}}$, and assume that they are strongly compatible. Let $\sigma \in \Sigma$.

We prove some results on $D^\sharp_\sigma$.

4.2.1. The actions of $\sigma_{\mathbf{C}}$

$$\sigma_{\mathbf{C}} \times E_\sigma \to E_\sigma, \quad (a,(q,F)) \mapsto (\mathbf{e}(a)q, \exp(-a)F),$$

where $a \in \sigma_{\mathbf{C}}$, $q \in \mathrm{toric}_\sigma$, $F \in \check{D}$ and $(q,F) \in E_\sigma$ (for $\mathbf{e}(a)$, see 2.3.1), and

$$\sigma_{\mathbf{C}} \times E_{\sigma,\mathrm{val}} \to E_{\sigma,\mathrm{val}}, \quad (a,(q,F)) \mapsto (\mathbf{e}(a)q, \exp(-a)F),$$

where $a \in \sigma_{\mathbf{C}}$, $q \in \mathrm{toric}_{\sigma,\mathrm{val}}$, $F \in \check{D}$ and $(q, F) \in E_{\sigma,\mathrm{val}}$, are proved to be continuous by the same argument in the pure case [KU09] 7.2.1. (The key point is how the strong topology behaves under products (cf. [KU09] 3.1.8).) They induce the actions of $i\sigma_{\mathbf{R}}$ on $E_\sigma^\sharp$ and $E_{\sigma,\mathrm{val}}^\sharp$.

The quotient spaces for these actions are identified as

$$\sigma_{\mathbf{C}}\backslash E_\sigma = \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma, \quad \sigma_{\mathbf{C}}\backslash E_{\sigma,\mathrm{val}} = \Gamma(\sigma)^{\mathrm{gp}}\backslash D_{\sigma,\mathrm{val}},$$
$$i\sigma_{\mathbf{R}}\backslash E_\sigma^\sharp = D_\sigma^\sharp, \quad i\sigma_{\mathbf{R}}\backslash E_{\sigma,\mathrm{val}}^\sharp = D_{\sigma,\mathrm{val}}^\sharp.$$

We can verify the first and the third equivalences by using the results in 2.3.3. Recalling 3.2.5–3.2.7, we can also verify the second and the fourth.

Proposition 4.2.2. (i) *The action of $\sigma_{\mathbf{C}}$ on E_σ (resp. $E_{\sigma,\mathrm{val}}$) is free.*

(ii) *The action of $i\sigma_{\mathbf{R}}$ on $E_\sigma^\sharp$ (resp. $E_{\sigma,\mathrm{val}}^\sharp$) is free.*

Here and in the following, free action always means set-theoretically free action. That is, an action of a group H on a set S is free if and only if $hs \neq s$ for any $h \in H - \{1\}$ and $s \in S$.

Before proving 4.2.2, we state a lemma, whose variant in the pure case is [KU09] 6.1.7 (2).

Lemma 4.2.2.1. *Let $(H_{\mathbf{R}}, W, F)$ be an $\mathbf{R}$-mixed Hodge structure, let*

$$\delta(F) \in L_{\mathbf{R}}^{-1,-1}(F(\mathrm{gr}^W)) \subset \mathrm{End}_{\mathbf{R}}(H(\mathrm{gr}^W)_{\mathbf{R}})$$

be as in [CKS86] (2.20), [KU09] §4, [KNU11] 1.2.2, *and let $s' \in \mathrm{spl}(W)$ be the splitting of W characterized by $F = s'(\exp(i\delta(F))F(\mathrm{gr}^W))$ (ibid.).*

Let $N \in \mathrm{End}_{\mathbf{R}}(H_{\mathbf{R}})$ and assume that $(s')^{-1}Ns' \in L_{\mathbf{R}}^{-1,-1}(F(\mathrm{gr}^W))$ and that $(s')^{-1}Ns'$ commutes with $\delta(F)$. Let $z \in \mathbf{C}$. Then, we have $\delta(\exp(zN)F) = \delta(F) + \mathrm{Im}(z)(s')^{-1}Ns'$.

Proof. Write $z = x + iy$ with x and y real. Let $\delta = \delta(F)$. Apply $\exp(zN)$ to $F = s'(\exp(i\delta)F(\mathrm{gr}^W))$. Then, we have $\exp(zN)F = (\exp(xN)s')\exp(iy(s')^{-1}Ns')\exp(i\delta)F(\mathrm{gr}^W)$. By the commutativity assumption, $\exp(iy(s')^{-1}Ns')\exp(i\delta) = \exp(i(y(s')^{-1}Ns' + \delta))$, and the conclusion follows. □

Proof of 4.2.2. We show here that the action of $\sigma_{\mathbf{C}}$ on E_σ is free. The rest is proved in the similar way.

Assume $a \cdot (q, F) = (q, F)$ ($a \in \sigma_{\mathbf{C}}$, $(q, F) \in E_\sigma$), that is, $\mathbf{e}(a) \cdot q = q$ and $\exp(-a)F = F$.

By the pure case considered in [KU09] 7.2.9, the image of a in $\mathfrak{g}_{\mathbf{C}}(\mathrm{gr}^W)$ is zero. The rest of the proof of 4.2.2 is parallel to the argument in [KU09] 7.2.9 as follows.

By $\mathbf{e}(a) \cdot q = q$, we have $a = b + c$ ($b \in \sigma(q)_{\mathbf{C}} \cap \mathfrak{g}_{\mathbf{C},u}$, $c \in \log(\Gamma(\sigma)^{\mathrm{gp}} \cap G_{\mathbf{Z},u})$), where $\sigma(q)$ is the unique face of σ such that $q \in \mathbf{e}(\sigma_{\mathbf{C}}) \cdot 0_{\sigma(q)}$. On the other hand, taking $\delta(\ \ ) = \delta(M(\sigma(q), W),\ \ )$ of $\exp(-a)F = F$, we have $\delta(F) = \delta(\exp(-a)F) = -\mathrm{Im}(b) + \delta(F)$ by 4.2.2.1. It follows $\mathrm{Im}(b) = 0$ and hence $\exp(\mathrm{Re}(b) + c)F = F$. Take an element y of the interior of $\sigma(q)$ such that $F' := \exp(iy)F \in D$. Then $\exp(\mathrm{Re}(b) + c)F' = F'$.

Since $\mathrm{Re}(b) + c \in \mathfrak{g}_{\mathbf{R},u}$, we have $\exp(\mathrm{Re}(b)+c)\,\mathrm{spl}_W(F') = \mathrm{spl}_W(F')$. This proves $\mathrm{Re}(b) + c = 0$ and hence $a = 0$. $\square$

For a sharp rational nilpotent cone σ, we denote by $E^\sharp_{\sigma,\mathrm{val},\mathrm{triv}}$ the part of $E^\sharp_{\sigma,\mathrm{val}}$ where the log structure of $E_{\sigma,\mathrm{val}}$ is trivial.

The definition of an excellent basis [KU09] 6.3.8 for a point $(A, V, Z) \in D^\sharp_{\mathrm{val}}$ obviously extends to the mixed Hodge case, and its existence is seen similarly to [KU09] 6.3.9.

Proposition 4.2.3. *Let σ and σ' be sharp rational nilpotent cones. Let $\alpha \in E^\sharp_{\sigma,\mathrm{val}}$ and $\alpha' \in E^\sharp_{\sigma',\mathrm{val}}$. Assume that $(\mathbf{e}(iy_\lambda), F_\lambda) \in E^\sharp_{\sigma,\mathrm{val},\mathrm{triv}}$ ($y_\lambda \in \sigma_{\mathbf{R}}$, $F_\lambda \in \check{D}$) (resp. $(\mathbf{e}(iy'_\lambda), F'_\lambda) \in E^\sharp_{\sigma',\mathrm{val},\mathrm{triv}}$ ($y'_\lambda \in \sigma'_{\mathbf{R}}$, $F'_\lambda \in \check{D}$)) converges to α (resp. α') in the strong topology, and that*

$$\exp(iy_\lambda)F_\lambda = \exp(iy'_\lambda)F'_\lambda \ \text{ in } D.$$

Then we have:

(i) *The images of α and α' in $D^\sharp_{\mathrm{val}}$ coincide.*

(ii) *$y_\lambda - y'_\lambda$ converges in $\mathfrak{g}_{\mathbf{R}}$.*

Proof. First of all, we remark that the proof below is just a revision of that of [KU09] 7.2.10.

Since the composite map $E^\sharp_{\sigma,\mathrm{val}}$ (resp. $E^\sharp_{\sigma',\mathrm{val}}$) $\to D^\sharp_{\mathrm{val}} \xrightarrow{\psi} D^I_{\mathrm{SL}(2)}$ is continuous by 3.2.7 and Theorem 3.3.2, the image of α (resp. α') under this composite map is the limit of $\exp(iy_\lambda)F_\lambda$ (resp. $\exp(iy'_\lambda)F'_\lambda$) in $D^I_{\mathrm{SL}(2)}$. Since $\exp(iy_\lambda)F_\lambda = \exp(iy'_\lambda)F'_\lambda$ and since $D^I_{\mathrm{SL}(2)}$ is Hausdorff ([KNU11] 3.5.17 (i)), these images coincide, say $p \in D^I_{\mathrm{SL}(2)}$. Let m be the rank of p and let $\Psi = (W^{(k)})_{1\le k\le m}$ be the associated family of weight filtrations (3.1.1, 3.1.2). Let (A, V, Z) (resp. (A', V', Z')) be the image of α (resp. α') in $D^\sharp_{\mathrm{val}}$. Take an excellent basis $(N_s)_{s\in S}$ (resp. $(N'_s)_{s\in S'}$) for (A, V, Z) (resp. (A', V', Z')), and let $(a_s)_{s\in S}$ and $(S_j)_{1\le j\le n}$ (resp. $(a'_s)_{s\in S'}$ and $(S'_j)_{1\le j\le n'}$) be as in (the mixed Hodge version of) [KU09] 6.3.3. For each l with $1 \le l \le m$, let $f(l)$ (resp. $f'(l)$) be the smallest integer such that $1 \le f(l) \le n$ (resp. $1 \le f'(l) \le n'$), and that $W^{(l)} = M(\sum_{s\in S_{\le f(l)}} \mathbf{R}_{\ge 0}N_s, W)$ (resp. $= M(\sum_{s\in S'_{\le f'(l)}} \mathbf{R}_{\ge 0}N'_s, W)$). Let $g(l)$ (resp. $g'(l)$) be any element of $S_{f(l)}$ (resp. $S'_{f'(l)}$).

Take an $\mathbf{R}$-subspace B of $\sigma_{\mathbf{R}}$ (resp. B' of $\sigma'_{\mathbf{R}}$) such that $\sigma_{\mathbf{R}} = A_{\mathbf{R}} \oplus B$ (resp. $\sigma'_{\mathbf{R}} = A'_{\mathbf{R}} \oplus B'$). Write

$$y_\lambda = \textstyle\sum_{s\in S} y_{\lambda,s}N_s + b_\lambda, \quad y'_\lambda = \sum_{s\in S'} y'_{\lambda,s}N'_s + b'_\lambda$$

with $y_{\lambda,s}$, $y'_{\lambda,s} \in \mathbf{R}$, $b_\lambda \in B$, $b'_\lambda \in B'$. We may assume $y_{\lambda,s}$, $y'_{\lambda,s} > 0$. Then, by the mixed Hodge version of [KU09] 6.4.11, proved similarly, $y_{\lambda,s}$ ($s \in S$), $y'_{\lambda,s}$ ($s \in S'$), $\frac{y_{\lambda,s}}{y_{\lambda,t}}$ ($s \in S_{\le j}$, $t \in S_{\ge j+1}$) and $\frac{y'_{\lambda,s}}{y'_{\lambda,t}}$ ($s \in S'_{\le j}$, $t \in S'_{\ge j+1}$) tend to ∞, $\frac{y_{\lambda,s}}{y_{\lambda,t}}$ ($s, t \in S_j$) tends to $\frac{a_s}{a_t}$, $\frac{y'_{\lambda,s}}{y'_{\lambda,t}}$ ($s, t \in S'_j$) tends to $\frac{a'_s}{a'_t}$, and b_λ and b'_λ converge.

Claim A. $\frac{y'_{\lambda,g'(l)}}{y_{\lambda,g(l)}}$ *converges to an element of* $\mathbf{R}_{>0}$ *for* $1 \le l \le m$.

We prove this claim. The assumption in 3.3.3 is satisfied if we take $(N_s)_{s\in S}$, a_s, S_j, $y_{\lambda,s}$ as above the same ones in 3.3.3, the above $\exp(b_\lambda)F_\lambda$ as F_λ in 3.3.3, the limit of the above $\exp(b_\lambda)F_\lambda$ as F in 3.3.3. Let $\tau : (\mathbf{R}^\times)^m \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ be the homomorphism associated to p (3.1.1, 3.1.2). Then, by 3.3.4 (1), we have

$$\tau\Big(\sqrt{\frac{y_{\lambda,g(1)}}{y_{\lambda,g(2)}}}, \dots, \sqrt{\frac{y_{\lambda,g(m)}}{y_{\lambda,g(m+1)}}}\Big)\exp(iy_\lambda)F_\lambda \to \mathbf{r}_1 \quad \text{in } D.$$

See 3.1.3 for $\mathbf{r}_1$. Similarly, for some $t = (t_1, \dots, t_m) \in \mathbf{R}^m_{>0}$, we have

$$\tau\Bigg(\sqrt{\frac{y'_{\lambda,g'(1)}}{y'_{\lambda,g'(2)}}}, \dots, \sqrt{\frac{y'_{\lambda,g'(m)}}{y'_{\lambda,g'(m+1)}}}\Bigg)\exp(iy'_\lambda)F'_\lambda \to \tau(t)\cdot\mathbf{r}_1 \quad \text{in } D$$

(3.1.2). Here $y_{\lambda,g(m+1)} = y'_{\lambda,g'(m+1)} = 1$. Recall that $\Psi = \{W^{(1)}, \dots, W^{(m)}\}$ is the set of weight filtrations associated to p. In the case $W \notin \Psi$ (resp. $W \in \Psi$), take a distance to Ψ-boundary

$$\beta : D \to \mathbf{R}^m_{>0} \quad (\text{resp. } D_{\mathrm{nspl}} := D \smallsetminus D_{\mathrm{spl}} \to \mathbf{R}^m_{>0})$$

in [KNU11] 3.2.5. This means that β is a real analytic map satisfying $\beta(\alpha(t)x) = t\beta(x)$ ($x \in D$ (resp. D_{nspl}), $t \in \mathbf{R}^m_{>0}$) for any splitting α of Ψ (cf. [KNU11] 3.2.13 for a simple example of the distance to the boundary). In particular, $\beta(\tau(t)x) = t\beta(x)$ ($x \in D$ (resp. D_{nspl}), $t \in \mathbf{R}^m_{>0}$). Note that, in the case $W \in \Psi$, $\mathbf{r}_1 \in D_{\mathrm{nspl}}$. Applying β to the above convergences, taking their ratios, and using $\exp(iy_\lambda)F_\lambda = \exp(iy'_\lambda)F'_\lambda$, we have

$$\frac{y'_{\lambda,g'(l)}}{y_{\lambda,g(l)}} \to t_l^2 \quad (1 \le l \le m).$$

Claim A is proved.

Next, we prove the following Claims B_l and C_l ($1 \le l \le m+1$) by induction on l.

Claim B_l. $y^{-1}_{\lambda,g(l)}\big(\sum_{s\in S_{\le f(l)-1}} y_{\lambda,s}N_s - \sum_{s\in S'_{\le f'(l)-1}} y'_{\lambda,s}N'_s\big)$ *converges.*

Claim C_l. $\sum_{s\in S_{\le f(l)-1}} \mathbf{Q}N_s = \sum_{s\in S'_{\le f'(l)-1}} \mathbf{Q}N'_s$.

Here $y_{\lambda,g(m+1)} := 1$, $S_{\le f(m+1)-1} := S$, $S'_{\le f'(m+1)-1} := S'$.

Note that Proposition 4.2.3 follows from Claims B_{m+1} and C_{m+1}. In fact, $A = A'$ follows from Claim C_{m+1}, $V = V'$ follows from Claim B_{m+1}, and $Z = Z'$ follows from the facts that the limit of $\exp(b_\lambda)F_\lambda$ (resp. $\exp(b'_\lambda)F'_\lambda$) is an element of Z (resp. Z') and that these limits coincide by Claim B_{m+1}, Claim C_{m+1} and the assumption $\exp(iy_\lambda)F_\lambda = \exp(iy'_\lambda)F'_\lambda$.

We prove these claims. First, Claims B_1 and C_1 are trivial by definition. Assume $l > 1$. By the hypothesis Claim C_{l-1} of induction, N_s ($s \in S_{\le f(l-1)-1}$) and elements of

σ' are commutative. Hence, by the formula $\exp(x_1+x_2)=\exp(x_1)\exp(x_2)$ if $x_1x_2=x_2x_1$ and by the assumption $\exp(iy_\lambda)F_\lambda=\exp(iy'_\lambda)F'_\lambda$, we have

$$\exp\big(iy_\lambda-\textstyle\sum_{s\in S_{\le f(l-1)-1}} iy_{\lambda,s}N_s\big)F_\lambda=\exp\big(iy'_\lambda-\textstyle\sum_{s\in S_{\le f(l-1)-1}} iy_{\lambda,s}N_s\big)F'_\lambda.$$

Applying $\prod_{k=l}^m \tau_k\Big(\sqrt{\frac{y_{\lambda,g(k)}}{y_{\lambda,g(k+1)}}}\Big)$ to this and using [KNU08] 10.3, we obtain

$$(1)\qquad \begin{aligned}&\exp\big(\textstyle\sum_{f(l-1)\le j<f(l)}\sum_{s\in S_j} i\frac{y_{\lambda,s}}{y_{\lambda,g(l)}}N_s\big)\\ &\qquad\cdot\textstyle\prod_{k\ge l}\tau_k\Big(\sqrt{\frac{y_{\lambda,g(k)}}{y_{\lambda,g(k+1)}}}\Big)\exp\big(\sum_{s\in S_{\ge f(l)}} iy_{\lambda,s}N_s+ib_\lambda\big)F_\lambda\\ =\ &\exp\big(\textstyle\sum_{s\in S'_{\le f'(l)-1}} i\frac{y'_{\lambda,s}}{y_{\lambda,g(l)}}N'_s-\sum_{s\in S_{\le f(l-1)-1}} i\frac{y_{\lambda,s}}{y_{\lambda,g(l)}}N_s\big)\\ &\qquad\cdot\textstyle\prod_{k\ge l}\tau_k\Big(\sqrt{\frac{y_{\lambda,g(k)}}{y_{\lambda,g(k+1)}}}\Big)\exp\big(\sum_{s\in S'_{\ge f'(l)}} iy'_{\lambda,s}N'_s+ib'_\lambda\big)F'_\lambda.\end{aligned}$$

By Lemma 3.3.6,

$$(2)\qquad \begin{aligned}&\textstyle\prod_{k\ge l}\tau_k\Big(\sqrt{\frac{y_{\lambda,g(k)}}{y_{\lambda,g(k+1)}}}\Big)\exp\big(\sum_{s\in S_{\ge f(l)}} iy_{\lambda,s}N_s+ib_\lambda\big)F_\lambda\ \text{ and}\\ &\textstyle\prod_{k\ge l}\tau_k\Big(\sqrt{\frac{y_{\lambda,g(k)}}{y_{\lambda,g(k+1)}}}\Big)\exp\big(\sum_{s\in S'_{\ge f'(l)}} iy'_{\lambda,s}N'_s+ib'_\lambda\big)F'_\lambda\ \text{ converge in } \check{D}.\end{aligned}$$

Let d (resp. d') be a metric on a neighborhood in $\check{D}$ of the limit of F_λ (resp. F'_λ) which is compatible with the analytic structure. Let $e\ge 1$ be an integer. Then, since $(\mathbf{e}(iy_\lambda),F_\lambda)$ (resp. $(\mathbf{e}(iy'_\lambda),F'_\lambda)$) converges to α (resp. α') in the strong topology, there exist, by [KU09] 3.1.6, $y^*_\lambda=\sum_{s\in S}y^*_{\lambda,s}N_s+b^*_\lambda\in\sigma_{\mathbf{R}}$ ($b^*_\lambda\in B_{\mathbf{R}}$), $y'^*_\lambda=\sum_{s\in S'}y'^*_{\lambda,s}N'_s+b'^*_\lambda\in\sigma'_{\mathbf{R}}$ ($b'^*_\lambda\in B'_{\mathbf{R}}$) and $F^*_\lambda,F'^*_\lambda\in\check{D}$ having the following three properties:

(3) $y^e_{\lambda,g(l-1)}(y_\lambda-y^*_\lambda)$, $y^e_{\lambda,g(l-1)}(y'_\lambda-y'^*_\lambda)$, $y^e_{\lambda,g(l-1)}d(F_\lambda,F^*_\lambda)$, and $y^e_{\lambda,g(l-1)}d'(F'_\lambda,F'^*_\lambda)$ converge to 0.

(4) $y_{\lambda,s}=y^*_{\lambda,s}$ ($s\in S_{\le f(l-1)}$) and $y'_{\lambda,s}=y'^*_{\lambda,s}$ ($s\in S'_{\le f'(l-1)}$).

(5) $\big((N_s)_{s\in S_{\le f(l)-1}},\exp\big(\sum_{s\in S_{\ge f(l)}} iy^*_{\lambda,s}N_s+b^*_\lambda\big)F^*_\lambda\big)$ and $\big((N'_s)_{s\in S'_{\le f'(l)-1}},\exp\big(\sum_{s\in S'_{\ge f'(l)}} iy'^*_{\lambda,s}N'_s+b'^*_\lambda\big)F'^*_\lambda\big)$ generate nilpotent orbits.

Take e sufficiently large. Then, by the hypothesis Claim B$_{l-1}$ of induction and by (1) and (2), the difference between $\delta(W^{(l-1)},\ )$ of

$$\begin{aligned}&\exp\Big(\textstyle\sum_{f(l-1)\le j<f(l)}\sum_{s\in S_j} i\frac{y_{\lambda,s}}{y_{\lambda,g(l)}}N_s\Big)\\ &\qquad\cdot\textstyle\prod_{k\ge l}\tau_k\Big(\sqrt{\frac{y_{\lambda,g(k)}}{y_{\lambda,g(k+1)}}}\Big)\exp\big(\sum_{s\in S_{\ge f(l)}} iy^*_{\lambda,s}N_s+ib^*_\lambda\big)F^*_\lambda\end{aligned}$$

and $\delta(W^{(l-1)},\ )$ of

$$\begin{aligned}&\exp\Big(\textstyle\sum_{s\in S'_{\le f'(l)-1}} i\frac{y'_{\lambda,s}}{y_{\lambda,g(l)}}N'_s-\sum_{s\in S_{\le f(l-1)-1}} i\frac{y_{\lambda,s}}{y_{\lambda,g(l)}}N_s\Big)\\ &\qquad\cdot\textstyle\prod_{k\ge l}\tau_k\Big(\sqrt{\frac{y_{\lambda,g(k)}}{y_{\lambda,g(k+1)}}}\Big)\exp\big(\sum_{s\in S'_{\ge f'(l)}} iy'^*_{\lambda,s}N'_s+ib'^*_\lambda\big)F'^*_\lambda\end{aligned}$$

converges to 0. By 4.2.2.1, the former is equal to

$$\sum_{f(l-1)\le j<f(l)} \sum_{s\in S_j} i\frac{y_{\lambda,s}}{y_{\lambda,g(l)}} N_s + \text{(a term which converges)},$$

and the latter is equal to

$$\sum_{s\in S'_{\le f'(l)-1}} i\frac{y'_{\lambda,s}}{y_{\lambda,g(l)}} N'_s - \sum_{s\in S_{\le f(l-1)-1}} i\frac{y_{\lambda,s}}{y_{\lambda,g(l)}} N_s + \text{(a term which converges)}.$$

This proves Claim B_l. Claim C_l follows from Claim B_l easily. □

We recall the notion of proper action. In this series of papers, a continuous map of topological spaces is said to be proper if it is proper in the sense of [Bou66] and is separated (cf. [KU09] 0.7.5).

Definition 4.2.4 ([Bou66] Ch.3 §4 no.1 Definition 1)**.** Let H be a Hausdorff topological group acting continuously on a topological space X. H is said to act properly on X if the map

$$H \times X \to X \times X, \quad (h, x) \mapsto (x, hx),$$

is proper.

As is explained above, the meaning of properness of a continuous map here is slightly different from that in [Bou66]. However, for a continuous action of a Hausdorff topological group H on a topological space X, the map $H \times X \to X \times X, (h, x) \mapsto (x, hx)$, is always separated, and hence properness of this map in both senses coincide.

The following 4.2.4.1–4.2.4.7 hold. Recall that when we say an action is free, it means free set-theoretically (cf. 4.2.2).

For proofs of 4.2.4.1, 4.2.4.2, 4.2.4.3, see [Bou66] Ch.3 §4 no.2 Proposition 3, ibid. Ch.3 §4 no.4 Corollary, ibid. Ch.3 §4 no.2 Proposition 5, respectively. 4.2.4.4 is [KU09] Lemma 7.2.7. See [KU09] for the proof of it. We prove 4.2.4.5–4.2.4.7 later.

4.2.4.1. If a Hausdorff topological group H acts properly on a topological space X, then the quotient space $H\backslash X$ is Hausdorff.

4.2.4.2. If a discrete group H acts properly and freely on a Hausdorff space X, then the projection $X \to H\backslash X$ is a local homeomorphism.

4.2.4.3. Let H be a Hausdorff topological group acting continuously on topological spaces X and X'. Let $\psi : X \to X'$ be an equivariant continuous map.

(i) If ψ is proper and surjective and if H acts properly on X, then H acts properly on X'.

(ii) If H acts properly on X' and if X is Hausdorff, then H acts properly on X.

4.2.4.4. Assume that a Hausdorff topological group H acts on a topological space X continuously and freely. Let X' be a dense subset of X. Then, the following two conditions (1) and (2) are equivalent.

(1) The action of H on X is proper.

(2) Let $x, y \in X$, L be a directed set, $(x_\lambda)_{\lambda\in L}$ be a family of elements of X' and $(h_\lambda)_{\lambda\in L}$ be a family of elements of H, such that $(x_\lambda)_\lambda$ (resp. $(h_\lambda x_\lambda)_\lambda$) converges to x (resp. y). Then $(h_\lambda)_\lambda$ converges to an element h of H and $y = hx$.

If X is Hausdorff, these equivalent conditions are also equivalent to the following condition (3).

(3) Let L be a directed set, $(x_\lambda)_{\lambda\in L}$ be a family of elements of X' and $(h_\lambda)_{\lambda\in L}$ be a family of elements of H, such that $(x_\lambda)_\lambda$ and $(h_\lambda x_\lambda)_\lambda$ converge in X. Then $(h_\lambda)_\lambda$ converges in H.

4.2.4.5. Let H be a Hausdorff topological group acting continuously on Hausdorff topological spaces X_1 and X_2. Let H_1 be a closed normal subgroup of H, and assume that the action of H on X_2 factors through $H_2 := H/H_1$. Assume that for $j = 1, 2$, the action of H_j on X_j is proper and free. Assume further that there are a neighborhood U of 1 in H_2 and a continuous map $s : U \to H$ such that the composition $U \xrightarrow{s} H \to H_2$ is the inclusion map. Then the diagonal action of H on $X_1 \times X_2$ is proper and free.

4.2.4.6. Let H be a topological group acting continuously on a Hausdorff topological space X. Assume that $Y := H\backslash X$ is Hausdorff and assume that X is an H-torsor over Y in the category of topological spaces. Then the action of H on X is proper.

The following 4.2.4.7 is not related to a group action, but we put it here for 4.2.4.1 and 4.2.4.3 (i) imply it in the case $H = \{1\}$.

4.2.4.7. Let $f : X \to Y$ be a continuous map between topological spaces and assume that f is proper and surjective. Assume X is Hausdorff. Then Y is Hausdorff.

4.2.4.5 is proved as follows. The freeness is clear. We prove the properness. Let L be a directed set, let $(x_\lambda, y_\lambda)_{\lambda\in L}$ be a family of elements of $X_1 \times X_2$, let $(h_\lambda)_{\lambda\in L}$ be a family of elements of H. Assume that x_λ, y_λ, $h_\lambda x_\lambda$, $h_\lambda y_\lambda$ converge. By 4.2.4.4, it is sufficient to prove that h_λ converges. Let $\bar{h}_\lambda$ be the image of h_λ in H_2. Since the action of H_2 on X_2 is proper, $\bar{h}_\lambda$ converges by 4.2.4.4. Let $h \bmod H_1$ ($h \in H$) be the limit. By replacing h_λ by $h^{-1}h_\lambda$, we may assume $\bar{h}_\lambda \to 1$. Replacing h_λ by $s(\bar{h}_\lambda)^{-1}h_\lambda$, we may assume $h_\lambda \in H_1$. Since the action of H_1 on X_1 is proper, h_λ converges by 4.2.4.4.

4.2.4.6 is proved as follows. Let L be a directed set, let $(x_\lambda)_{\lambda\in L}$ be a family of elements of X, and let $(h_\lambda)_{\lambda\in L}$ be a family of elements of H. Assume that x_λ converges to $x \in X$ and $h_\lambda x_\lambda$ converges to $y \in X$. By 4.2.4.4, it is sufficient to prove that h_λ converges. Since $Y = H\backslash X$ is Hausdorff, the images of x and y in Y coincide. Let $z \in Y$ be their image. Replacing Y by a sufficiently small neighborhood of z, we may assume that $X = H \times Y$ with the evident action of H. Then the convergence of $(h_\lambda)_\lambda$ is clear.

Proposition 4.2.5. *The action of $i\sigma_{\mathbf{R}}$ on $E_\sigma^\sharp$ (resp. $E_{\sigma,\mathrm{val}}^\sharp$) is proper.*

Proof. The proof is exactly the same as in the pure case [KU09] 7.2.11, that is, it reduces to 4.2.2 (ii) and 4.2.3 (ii) by 4.2.4.3 (i) and 4.2.4.4. □

By 4.2.5 and 4.2.4.1, we have the following result.

Corollary 4.2.6. *The spaces* $D^\sharp_\sigma$ *and* $D^\sharp_{\sigma,\mathrm{val}}$ *are Hausdorff.*

This corollary will be generalized in §4.3 by replacing σ with Σ (4.3.3).

The case (b) of the following lemma will be used in §4.4 later.

Lemma 4.2.7. *Let* $\mathcal{C}$ *be either one of the following two categories*:

(a) *the category of topological spaces.*

(b) *the category of log manifolds* (1.1.5).

In the case (a) (*resp.* (b)), *let* H *be a topological group* (*resp. a complex analytic group*), X *an object of* $\mathcal{C}$ *and assume that we have an action* $H \times X \to X$ *in* $\mathcal{C}$. (*In the case* (b), *we regard* H *as having the trivial log structure.*) *Assume this action is proper topologically* (4.2.4) *and is free set-theoretically. Assume moreover the following condition* (1) *is satisfied.*

(1) *For any point* $x \in X$, *there exist an object* S *of* $\mathcal{C}$, *a morphism* $\iota : S \to X$ *of* $\mathcal{C}$ *whose image contains* x *and an open neighborhood* U *of* 1 *in* H *such that* $U \times S \to X$, $(h, s) \mapsto h\iota(s)$, *induces an isomorphism in* $\mathcal{C}$ *from* $U \times S$ *onto an open set of* X.

Then:

(i) *In the case* (b), *the quotient topological space* $H\backslash X$ *has a unique structure of an object of* $\mathcal{C}$ *such that, for an open set* V *of* $H\backslash X$, $\mathcal{O}_{H\backslash X}(V)$ (*resp.* $M_{H\backslash X}(V)$) *is the set of all functions on* V *whose pullbacks to the inverse image* V' *of* V *in* X *belong to* $\mathcal{O}_X(V')$ (*resp.* $M_X(V')$).

(ii) $X \to H\backslash X$ *is an* H*-torsor in the category* $\mathcal{C}$.

Proof. This is [KU09] Lemma 7.3.3. See [KU09] for the proof.

Proposition 4.2.8. $E^\sharp_\sigma \to i\sigma_{\mathbf{R}}\backslash E^\sharp_\sigma = D^\sharp_\sigma$ *and* $E^\sharp_{\sigma,\mathrm{val}} \to i\sigma_{\mathbf{R}}\backslash E^\sharp_{\sigma,\mathrm{val}} = D^\sharp_{\sigma,\mathrm{val}}$ *are* $i\sigma_{\mathbf{R}}$*-torsors in the category of topological spaces.*

Proof. This is proved by using the arguments in the pure case [KU09] 7.3.5. We explain the case of $E^\sharp_\sigma \to D^\sharp_\sigma$. The other case is similar. We apply Lemma 4.2.7 by taking $H = i\sigma_{\mathbf{R}}$, $X = E^\sharp_\sigma$, and $\mathcal{C}$ to be the category of topological spaces as in 4.2.7 (a). By Proposition 4.2.5, Proposition 4.2.2 and Lemma 4.2.7, it is sufficient to prove that the condition 4.2.7 (1) is satisfied. Let $x = (q, F) \in E^\sharp_\sigma$. Let $S_1 \subset E_\sigma$ be the log manifold containing x, constructed in the same way as the log manifold denoted by S in the pure case [KU09] 7.3.5. (In the argument to construct this log manifold, we replace $W(\sigma(q))[-w]$ there by $M(\sigma(q), W)$ here. Cf. the proof of 4.4.3.) Let $S_2 = E^\sharp_\sigma \cap S_1$, let U be a sufficiently small neighborhood of 0 in $\sigma_{\mathbf{R}}$, and let $S = \{(q', \exp(a)F') \mid (q', F') \in S_2, a \in U\}$. Then S has the desired property. □

§4.3. Study of $D^\sharp_\Sigma$

Let Σ be a weak fan. We prove some results on $D^\sharp_\Sigma$.

Theorem 4.3.1. *For $\sigma \in \Sigma$, the inclusion maps $D^\sharp_\sigma \to D^\sharp_\Sigma$ and $D^\sharp_{\sigma,\mathrm{val}} \to D^\sharp_{\Sigma,\mathrm{val}}$ are open maps.*

Proof. We prove that $D^\sharp_\sigma \to D^\sharp_\Sigma$ is an open map. The val version is proved similarly.

For $\sigma, \tau \in \Sigma$, $D^\sharp_\sigma \cap D^\sharp_\tau$ is the union of $D^\sharp_\alpha$, where α ranges over all elements of Σ having the following property: α is a face of σ and a face of τ at the same time. Hence, by the definition of the topology of $D^\sharp_\Sigma$, it is sufficient to prove the following (1).

(1) If $\sigma \in \Sigma$ and τ is a face of σ, the inclusion $D^\sharp_\tau \to D^\sharp_\sigma$ is a continuous open map.

We prove (1). Let the open set $U(\tau)$ of toric_σ be as in 4.1.2, and let $|U|(\tau)$ be the open set $U(\tau) \cap |\mathrm{toric}|_\sigma$ of $|\mathrm{toric}|_\sigma$. Then $|\mathrm{toric}|_\tau \subset |U|(\tau)$ as subsets of $|\mathrm{toric}|_\sigma$. Let $|\tilde{U}|(\tau) \subset E^\sharp_\sigma$ be the inverse image of $|U|(\tau)$ under $E^\sharp_\sigma \to |\mathrm{toric}|_\sigma$. We have commutative diagrams of topological spaces

$$\begin{array}{ccccc} E^\sharp_\tau & \xrightarrow{\subset} & |\tilde{U}|(\tau) & \xrightarrow{\subset} & E^\sharp_\sigma \\ \downarrow & & \downarrow & & \downarrow \\ |\mathrm{toric}|_\tau & \xrightarrow{\subset} & |U|(\tau) & \xrightarrow{\subset} & |\mathrm{toric}|_\sigma, \end{array} \qquad \begin{array}{ccc} E^\sharp_\tau & \xrightarrow{\subset} & E^\sharp_\sigma \\ \downarrow & & \downarrow \\ D^\sharp_\tau & \xrightarrow{\subset} & D^\sharp_\sigma. \end{array}$$

Furthermore, the inverse image of $D^\sharp_\tau$ under $E^\sharp_\sigma \to D^\sharp_\sigma$ coincides with $|\tilde{U}|(\tau)$. The second diagram shows that $D^\sharp_\tau \to D^\sharp_\sigma$ is continuous. Let B be an $\mathbf{R}$-vector subspace of $\sigma_{\mathbf{R}}$ such that $\tau_{\mathbf{R}} \oplus B = \sigma_{\mathbf{R}}$. Then we have a homeomorphism

$$|\mathrm{toric}|_\tau \times B \xrightarrow{\sim} |U|(\tau), \quad (a, b) \mapsto \mathbf{e}(ib)a.$$

From this, we see that the projection $|\tilde{U}|(\tau) \to D^\sharp_\tau$ factors through $|\tilde{U}|(\tau) \to E^\sharp_\tau$ which sends $(\mathbf{e}(ib)a, F)$ to $(a, \exp(-ib)F)$. This shows that a subset U of $D^\sharp_\tau$ is open in $D^\sharp_\sigma$, that is, its inverse image in $E^\sharp_\sigma$ is open, if and only if it is open in $D^\sharp_\tau$, that is, its inverse image in $E^\sharp_\tau$ is open. This completes the proof of (1). □

Proposition 4.3.2. *The map $D^\sharp_{\Sigma,\mathrm{val}} \to D^\sharp_\Sigma$ is proper.*

Proof. By Theorem 4.3.1, we are reduced to the case $\Sigma = \{\text{face of } \sigma\}$. In this case, by Proposition 4.2.8, we are reduced to the fact that $E^\sharp_{\sigma,\mathrm{val}} \to E^\sharp_\sigma$ is proper, and hence to the fact that $|\mathrm{toric}|_{\sigma,\mathrm{val}} \to |\mathrm{toric}|_\sigma$ is proper (cf. 3.2.5). □

Proposition 4.3.3. *$D^\sharp_\Sigma$ and $D^\sharp_{\Sigma,\mathrm{val}}$ are Hausdorff spaces.*

Proof. The statement for $D^\sharp_\Sigma$ follows from that for $D^\sharp_{\Sigma,\mathrm{val}}$, since $D^\sharp_{\Sigma,\mathrm{val}} \to D^\sharp_\Sigma$ is proper by Proposition 4.3.2 and is surjective.

We prove that $D^\sharp_{\Sigma,\mathrm{val}}$ is Hausdorff. By Theorem 4.3.1, it is sufficient to prove the following (1).

(1) Let σ and σ' be sharp rational nilpotent cones and let $\beta \in D^\sharp_{\sigma,\mathrm{val}}$ and $\beta' \in D^\sharp_{\sigma',\mathrm{val}}$. Assume $x_\lambda \in D$ converges to β in $D^\sharp_{\sigma,\mathrm{val}}$ and to β' in $D^\sharp_{\sigma',\mathrm{val}}$. Then $\beta = \beta'$ in $D^\sharp_{\mathrm{val}}$.

We prove this. By Proposition 4.2.8, there exist an open neighborhood U of β in $D^{\sharp}_{\sigma,\mathrm{val}}$ (resp. U' of β' in $D^{\sharp}_{\sigma',\mathrm{val}}$) and a continuous section $s_\sigma : U \to E^{\sharp}_{\sigma,\mathrm{val}}$ (resp. $s_{\sigma'} : U' \to E^{\sharp}_{\sigma',\mathrm{val}}$) of the projection $E^{\sharp}_{\sigma,\mathrm{val}} \to D^{\sharp}_{\sigma,\mathrm{val}}$ (resp. $E^{\sharp}_{\sigma',\mathrm{val}} \to D^{\sharp}_{\sigma',\mathrm{val}}$). Write

$$s_\sigma(x_\lambda) = (\mathbf{e}(iy_\lambda), F_\lambda), \quad s_{\sigma'}(x_\lambda) = (\mathbf{e}(iy'_\lambda), F'_\lambda).$$

Write $\alpha = s_\sigma(\beta)$, $\alpha' = s_{\sigma'}(\beta')$. Then the assumption of Proposition 4.2.3 is satisfied. Hence we have $\beta = \beta'$ by Proposition 4.2.3 (i). □

Lemma 4.3.4. *Let V be a vector space over a field K and let $N, h : V \to V$ be K-linear maps such that $Nh = hN$. Let I be a finite increasing filtration on V with $I_k = 0$ for $k \ll 0$ and $I_k = V$ for $k \gg 0$. Assume that $NI_k \subset I_k$ and $hI_k \subset I_{k-1}$ for all k, and that the monodromy filtration M of N relative to I exists. Then $\mathrm{gr}^M(h) = 0$.*

Proof. We prove the following statement (A_j) for $j \geq 1$, by induction on j.

(A_j) $hI_k \,\mathrm{gr}^M_l \subset I_{k-j} \,\mathrm{gr}^M_l$ for any $k, l \in \mathbf{Z}$.

First, (A_1) holds by the assumption $hI_k \subset I_{k-1}$. Let $j \geq 1$, and assume that (A_j) holds. We prove that (A_{j+1}) holds. It is sufficient to prove that the map $\mathrm{gr}^I_k \mathrm{gr}^M_l \to \mathrm{gr}^I_{k-j} \mathrm{gr}^M_l$ induced by h is the zero map for any $k, l \in \mathbf{Z}$. Note $\mathrm{gr}^I_k \mathrm{gr}^M_l = (I_k \cap M_l)/(I_{k-1} \cap M_l + I_k \cap M_{l-1}) = \mathrm{gr}^M_l \mathrm{gr}^I_k$. We prove that the map $\mathrm{gr}^M_l \mathrm{gr}^I_k \to \mathrm{gr}^M_l \mathrm{gr}^I_{k-j}$ induced by h is the zero map. If $l \geq k$, we denote the kernel of $N^{(l-k)+1} : \mathrm{gr}^M_l \mathrm{gr}^I_k \to \mathrm{gr}^M_{2k-l-2} \mathrm{gr}^I_k$ by $P(\mathrm{gr}^M_l \mathrm{gr}^I_k)$. By [D80] 1.6, $\mathrm{gr}^M_l \mathrm{gr}^I_k$ is the sum of the images of $N^s : P(\mathrm{gr}^M_{l'} \mathrm{gr}^I_k) \to \mathrm{gr}^M_l \mathrm{gr}^I_k$, where (s, l') ranges over all pairs of integers such that $l' \geq k$, $s \geq 0$, and $l = l' - 2s$. Hence it is sufficient to prove that the map $P(\mathrm{gr}^M_l \mathrm{gr}^I_k) \to \mathrm{gr}^M_l \mathrm{gr}^I_{k-j}$ induced by h is the zero map for any $k, l \in \mathbf{Z}$ such that $l \geq k$. Let $x \in \mathrm{gr}^M_l \mathrm{gr}^I_{k-j}$ be an element of the image of $P(\mathrm{gr}^M_l \mathrm{gr}^I_k)$ under this map. By the definition of $P(\mathrm{gr}^M_l \mathrm{gr}^I_k)$ and by $Nh = hN$, the map $N^{l-k+1} : \mathrm{gr}^M_l \mathrm{gr}^I_{k-j} \to \mathrm{gr}^M_{l-2(l-k+1)} \mathrm{gr}^I_{k-j}$ annihilates x. On the other hand, since M is the monodromy filtration of N relative to I, the map $N^{l-(k-j)} : \mathrm{gr}^M_l \mathrm{gr}^I_{k-j} \to \mathrm{gr}^M_{l-2(l-(k-j))} \mathrm{gr}^I_{k-j}$ is an isomorphism. Since $l - (k-j) \geq l - k + 1$, we have $x = 0$. □

Let Γ be a subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Σ.

Theorem 4.3.5. *Assume that Γ is neat.*

(i) *Let $p \in D^{\sharp}_{\Sigma}$, $\gamma \in \Gamma$, and assume $\gamma p = p$. Then $\gamma = 1$.*

(ii) *Let $p = (\sigma, Z) \in D_\Sigma$, $\gamma \in \Gamma$, and assume $\gamma p = p$. Then $\gamma \in \Gamma(\sigma)^{\mathrm{gp}}$.*

Proof. Let $(\sigma, Z) \in D_\Sigma$ and $\gamma \in \Gamma$. Assume $\gamma(\sigma, Z) = (\sigma, Z)$, that is, $\mathrm{Ad}(\gamma)(\sigma) = \sigma$, $\gamma Z = Z$. We prove first

(1) $\gamma N = N\gamma$ for any $N \in \sigma_{\mathbf{C}}$.

Since $\gamma\Gamma(\sigma)\gamma^{-1} = \Gamma(\sigma)$, we have an automorphism $\mathrm{Int}(\gamma) : y \mapsto \gamma y \gamma^{-1}$ $(y \in \Gamma(\sigma))$ of the sharp fs monoid $\Gamma(\sigma)$. Since the automorphism group of a sharp fs monoid is finite (see for example, [KU09] 7.4.4; an alternative proof: see the generators of the 1-faces), this automorphism of $\Gamma(\sigma)$ is of finite order. Since $\sigma_{\mathbf{C}}$ is generated over $\mathbf{C}$ by $\log(\Gamma(\sigma))$, the $\mathbf{C}$-linear map $\mathrm{Ad}(\gamma) : \sigma_{\mathbf{C}} \to \sigma_{\mathbf{C}}$, $y \mapsto \gamma y \gamma^{-1}$, is of finite order. On the other hand, any eigenvalue a of this $\mathbf{C}$-linear map is equal to bc^{-1} for some eigenvalues b, c of the $\mathbf{C}$-linear map $\gamma : H_{0,\mathbf{C}} \to H_{0,\mathbf{C}}$, and hence the neat property of Γ shows $a = 1$. Thus we have $\gamma y \gamma^{-1} = y$ for any $y \in \sigma_{\mathbf{C}}$.

Let $M := M(\sigma, W)$. By (1), we have $\gamma M = M$.

Take $F \in Z$. Since $\gamma Z = Z$, there exists $N \in \sigma_{\mathbf{C}}$ such that

(2) $$\gamma F = \exp(N)F.$$

Recall that the pure Hodge-theoretic version of 4.3.5 is proved in [KU09] 7.4.5, and its proof shows $\log(\mathrm{gr}^W(\gamma)) = \mathrm{gr}^W(N)$. Hence $\mathrm{gr}^W(\gamma)$ is in the image of $\Gamma(\sigma)^{\mathrm{gp}}$, and we may assume $\mathrm{gr}^W(\gamma) = 1$.

Since $\exp(N)$ acts on $\mathrm{gr}^M := (\mathrm{gr}^M)_{\mathbf{C}}$ trivially, $\mathrm{gr}^M(\gamma)F(\mathrm{gr}^M) = F(\mathrm{gr}^M)$ follows.

Claim 1. $\mathrm{gr}^M(\gamma) = 1$.

This follows from (1) and Lemma 4.3.4 applied to $h = \gamma - 1$.

Claim 2. $\log(\gamma)M_k \subset M_{k-2} \quad (\forall k \in \mathbf{Z})$.

In fact, by (2), we have

(3) $$(\log(\gamma) - N)F^p \subset F^p \quad (\forall p \in \mathbf{Z}).$$

Since $N(M_k) \subset M_{k-2}$, (3) shows that the map $\mathrm{gr}_k^M \to \mathrm{gr}_{k-1}^M$ induced by $\log(\gamma)$, which we denote as $\mathrm{gr}_{-1}^M(\log(\gamma)) : \mathrm{gr}_k^M \to \mathrm{gr}_{k-1}^M$, satisfies

(4) $$\mathrm{gr}_{-1}^M(\log(\gamma))F^p(\mathrm{gr}_k^M) \subset F^p(\mathrm{gr}_{k-1}^M) \quad (\forall k, \forall p).$$

By taking the complex conjugation of (4), we have

(5) $$\mathrm{gr}_{-1}^M(\log(\gamma))\overline{F}^p(\mathrm{gr}_k^M) \subset \overline{F}^p(\mathrm{gr}_{k-1}^M) \quad (\forall k, \forall p).$$

Since

$$\mathrm{gr}_k^M = \textstyle\bigoplus_{p+q=k} F^p(\mathrm{gr}_k^M) \cap \overline{F}^q(\mathrm{gr}_k^M) \quad \text{and}$$
$$F^p(\mathrm{gr}_{k-1}^M) \cap \overline{F}^q(\mathrm{gr}_{k-1}^M) = 0 \quad \text{for } p + q = k > k - 1,$$

(4) and (5) show that the map $\mathrm{gr}_{-1}^M(\log(\gamma)) : \mathrm{gr}_k^M \to \mathrm{gr}_{k-1}^M$ is the zero map. This proves Claim 2.

By (3), $\log(\gamma)F^p \subset F^{p-1}$, $\log(\gamma)\overline{F}^p \subset \overline{F}^{p-1}$ $(\forall p \in \mathbf{Z})$. These and Claim 2 show, by [KU09] Lemma 6.1.8 (iv),

$$\log(\gamma), N \in L^{-1,-1}(M, F).$$

(See [KU09] 6.1.2 for the definition of $L^{-1,-1}(M, F)$.) Hence, by [KU09] Lemma 6.1.8 (iii), $\gamma F = \exp(N)F$ proves $\log(\gamma) = N$. This proves $\gamma \in \Gamma(\sigma)^{\mathrm{gp}}$.

If $\gamma(\sigma, Z') = (\sigma, Z')$ for some $(\sigma, Z') \in D_\Sigma^\sharp$, then, for $Z := \exp(\sigma_{\mathbf{R}})Z'$, we have $(\sigma, Z) \in D_\Sigma$ and $\gamma(\sigma, Z) = (\sigma, Z)$. In the above argument, we take $N \in i\sigma_{\mathbf{R}}$, and have $\log(\gamma) = N$. Since $\log(\gamma)$ is real and N is purely imaginary, this shows that $\log(\gamma) = 0$. Hence $\gamma = 1$. $\square$

Theorem 4.3.6. (i) *The actions of* Γ *on* $D^\sharp_\Sigma$ *and* $D^\sharp_{\Sigma,\mathrm{val}}$ *are proper. In particular, the quotient spaces* $\Gamma\backslash D^\sharp_\Sigma$ *and* $\Gamma\backslash D^\sharp_{\Sigma,\mathrm{val}}$ *are Hausdorff.*

(ii) *Assume* Γ *is neat. Then the canonical maps* $D^\sharp_\Sigma \to \Gamma\backslash D^\sharp_\Sigma$ *and* $D^\sharp_{\Sigma,\mathrm{val}} \to \Gamma\backslash D^\sharp_{\Sigma,\mathrm{val}}$ *are local homeomorphisms.*

Proof. We first prove (i). By [KNU11] 3.5.17, the action of Γ on $D^I_{\mathrm{SL}(2)}$ is proper. By 4.2.4.3 (ii), the statement for $D^\sharp_{\Sigma,\mathrm{val}}$ follows from this, from the continuity of $\psi : D^\sharp_{\Sigma,\mathrm{val}} \to D^I_{\mathrm{SL}(2)}$, and from Proposition 4.3.3. By 4.2.4.3 (i), the statement for $D^\sharp_\Sigma$ follows from the above result, since $D^\sharp_{\Sigma,\mathrm{val}} \to D^\sharp_\Sigma$ is proper by Proposition 4.3.2 and is surjective.

Next, by 4.2.4.2, (ii) follows from (i) and Theorem 4.3.5. □

§4.4. Study of $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$

Let Σ be a weak fan, let $\sigma \in \Sigma$, and let Γ be a subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Σ.

In this subsection, we prove Theorem 2.5.3 and the local cases (the cases where $\Sigma = \mathrm{face}(\sigma)$) of Theorems 2.5.2, 2.5.5, and 2.5.6.

Lemma 4.4.1. *Let* $\sigma \in \Sigma$ *and assume that a* σ*-nilpotent orbit exists. Let* $\mathcal{I}$ *be the set of all admissible sets* Ψ *of weight filtrations on* $H_{0,\mathbf{R}}$ (3.1.2) *such that* $M(\sigma, W) \in \Psi$*. Then, the image of the CKS map* $D^\sharp_{\sigma,\mathrm{val}} \to D^I_{\mathrm{SL}(2)}$ *is contained in the open set* $\bigcup_{\Psi\in\mathcal{I}} D^I_{\mathrm{SL}(2)}(\Psi)$ (3.1.2) *of* $D^I_{\mathrm{SL}(2)}$.

Proof. Let $\alpha = (A, V, Z) \in D^\sharp_{\sigma,\mathrm{val}}$. Then there exist $N_1, \dots, N_n \in A \cap \sigma$ which generate A over $\mathbf{Q}$ such that, for the injective homomorphism $(N_j)_{1\le j\le n} : \mathrm{Hom}_{\mathbf{Q}}(A, \mathbf{Q}) \to \mathbf{R}^n$, V coincides with the inverse image of the subset of $\mathbf{R}^n$ consisting of all elements which are ≥ 0 for the lexicographic order. Take elements $N_{n+1}, \dots, N_m$ of $\mathfrak{g}_{\mathbf{Q}} \cap \sigma$ such that $N_1, \dots, N_m$ generate the cone σ. By assumption, there is an $F \in \check{D}$ which generates a σ-nilpotent orbit. Let $\Psi := \{M(N_1 + \cdots + N_j, W) \mid 1 \le j \le m\} \supset \Psi' := \{M(N_1 + \cdots + N_j, W) \mid 1 \le j \le n\}$. Then Ψ coincides with the set of weight filtrations associated to the class of the SL(2)-orbit associated to $((N_j)_{1\le j\le m}, F)$, and Ψ' coincides with the set of weight filtrations associated to the image p of α in $D_{\mathrm{SL}(2)}$. We have $p \in D^I_{\mathrm{SL}(2)}(\Psi)$ and $M(\sigma, W) = M(N_1 + \cdots + N_m, W) \in \Psi$. □

The following lemma will be improved as Proposition 4.4.6 later.

Lemma 4.4.2. *Let* $\sigma \in \Sigma$*, let* τ *be a face of* σ *such that a* τ*-nilpotent orbit exists, let* $U(\tau)$ *be the open set of* toric_σ *defined in* 4.1.2*, and let* $\tilde{U}(\tau)$ *(resp.* $\tilde{U}(\tau)_{\mathrm{val}}$*) be the inverse image of* $U(\tau)$ *under* $E_\sigma \to \mathrm{toric}_\sigma$ *(resp.* $E_{\sigma,\mathrm{val}} \to \mathrm{toric}_\sigma$*). Then the action* (4.2.1) *of* $\sigma_{\mathbf{C}}$ *on* $\tilde{U}(\tau)$ *(resp.* $\tilde{U}(\tau)_{\mathrm{val}}$*) is proper.*

Proof. Since $\tilde{U}(\tau)_{\mathrm{val}} \to \tilde{U}(\tau)$ is proper and surjective, it is sufficient to consider $\tilde{U}(\tau)_{\mathrm{val}}$ (4.2.4.3 (i)).

Let W' be the filtration $M(\tau, W)(\mathrm{gr}^W)$ on gr^W induced by $M(\tau, W)$. We consider five continuous maps

$$f_1 : \tilde{U}(\tau)_{\mathrm{val}} \to E^{\sharp}_{\sigma,\mathrm{val}}, \quad f_2 : \tilde{U}(\tau)_{\mathrm{val}} \to D^{\sharp}_{\tau,\mathrm{val}},$$

$$f_3 : \tilde{U}(\tau)_{\mathrm{val}} \to \mathrm{spl}(W), \quad f_4 : \tilde{U}(\tau)_{\mathrm{val}} \to \mathrm{spl}(W'), \text{and}$$

$$f_5 : \tilde{U}(\tau)_{\mathrm{val}} \to \sigma_{\mathbf{R}}/(\tau_{\mathbf{R}} + \log(\Gamma(\sigma)^{\mathrm{gp}})).$$

Here f_1 is induced by $|-| : E_{\sigma,\mathrm{val}} \to E^{\sharp}_{\sigma,\mathrm{val}}$, $(q, F) \mapsto (|q|, F)$, where for $q : \Gamma(\sigma)^{\vee} \to \mathbf{C}^{\mathrm{mult}}$, $|q|$ denotes the composition of q and $\mathbf{C} \to \mathbf{R}_{\geq 0}$, $a \mapsto |a|$. The map f_2 is induced from f_1 via the canonical map $E^{\sharp}_{\sigma,\mathrm{val}} \to D^{\sharp}_{\sigma,\mathrm{val}}$. The map f_3 is induced from f_2, the CKS map $D^{\sharp}_{\tau,\mathrm{val}} \to D^I_{\mathrm{SL}(2)}$, and the canonical map $\mathrm{spl}_W : D^I_{\mathrm{SL}(2)} \to \mathrm{spl}(W)$ ([KNU11] §3.2). The map f_4 is induced from the map f_2, the CKS map $D^{\sharp}_{\tau,\mathrm{val}} \to \bigcup_{\Psi\in\mathcal{I}} D^I_{\mathrm{SL}(2)}(\Psi)$, where $\mathcal{I}$ denotes the set of all admissible sets of weight filtrations Ψ such that $M(\tau, W) \in \Psi$ (here we apply 4.4.1 replacing σ there by τ), and the Borel-Serre splitting $\mathrm{spl}^{\mathrm{BS}}_{W'} : \bigcup_{\Psi\in\mathcal{I}} D^I_{\mathrm{SL}(2)}(\Psi) \to \mathrm{spl}(W')$ ([KNU11] §3.2). The map f_5 is the composition

$$\tilde{U}(\tau)_{\mathrm{val}} \to U(\tau) \to \mathbf{C}^{\times} \otimes (\Gamma(\sigma)^{\mathrm{gp}}/\Gamma(\tau)^{\mathrm{gp}}) \to \sigma_{\mathbf{R}}/(\tau_{\mathbf{R}} + \log(\Gamma(\sigma)^{\mathrm{gp}})),$$

where the second arrow is defined by the fact

$$\mathrm{Ker}(\mathrm{Hom}(\Gamma(\sigma)^{\mathrm{gp}}, \mathbf{Z}) \to \mathrm{Hom}(\Gamma(\tau)^{\mathrm{gp}}, \mathbf{Z})) \subset P$$

(for P, see 4.1.2), and the last arrow is the homomorphism $e^{2\pi i z} \otimes \gamma \mapsto \mathrm{Re}(z) \log(\gamma)$ ($z \in \mathbf{C}$, $\gamma \in \Gamma(\sigma)^{\mathrm{gp}}$).

These maps f_j have the following compatibilities with the action of $\sigma_{\mathbf{C}}$ on $\tilde{U}(\tau)$:

For any $a \in \sigma_{\mathbf{C}}$ and any $x \in \tilde{U}(\tau)_{\mathrm{val}}$, $f_j(a \cdot x) = a \cdot f_j(x)$,

where $\sigma_{\mathbf{C}}$ acts on the target space of f_j as follows. For $x = (q, F) \in E^{\sharp}_{\sigma,\mathrm{val}}$, $a \cdot x = (\mathbf{e}(i\,\mathrm{Im}(a)) \cdot q, \exp(-a)F)$. Note that, in the case $a \in i\sigma_{\mathbf{R}}$, this action coincides with the original action of a on $E^{\sharp}_{\sigma,\mathrm{val}}$. For $x = (A, V, Z) \in D^{\sharp}_{\tau,\mathrm{val}}$, $a \cdot x = (A, V, \exp(-\mathrm{Re}(a))Z)$. For $s \in \mathrm{spl}(W)$, $a \cdot s = \exp(-\mathrm{Re}(a)) \circ s \circ \mathrm{gr}^W(\exp(\mathrm{Re}(a)))$. For $s \in \mathrm{spl}(W')$, $a \cdot s = \mathrm{gr}^W(\exp(-\mathrm{Re}(a))) \circ s \circ \mathrm{gr}^{W'}(\mathrm{gr}^W(\exp(\mathrm{Re}(a))))$. For $x \in \sigma_{\mathbf{R}}/(\tau_{\mathbf{R}} + \log(\Gamma(\sigma)^{\mathrm{gp}}))$, $a \cdot x = x + \mathrm{Re}(a)$.

Let $H = \sigma_{\mathbf{C}}$, and define closed subgroups $H(j)$ $(0 \leq j \leq 5)$ of H such that $0 = H(0) \subset H(1) \subset \cdots \subset H(5) = H$ as follows. $H(1) = i\sigma_{\mathbf{R}}$, $H(2) = H(1) + \log(\Gamma(\sigma)^{\mathrm{gp}})$, $H(3) = H(2) + \tau_{\mathbf{R},u}$, where $\tau_{\mathbf{R},u} = \tau_{\mathbf{R}} \cap \mathfrak{g}_{\mathbf{R},u}$, $H(4) = H(3) + \tau_{\mathbf{R}}$. Define the spaces X_j $(1 \leq j \leq 5)$ with actions of H as follows: $X_1 = E^{\sharp}_{\sigma,\mathrm{val}}$, $X_2 = D^{\sharp}_{\tau,\mathrm{val}}$, X_3 is the quotient space of $\mathrm{spl}(W)$ under the action of $\Gamma(\sigma)^{\mathrm{gp}}$ given by $s \mapsto \gamma \circ s \circ \mathrm{gr}^W(\gamma)^{-1}$ ($s \in \mathrm{spl}(W)$, $\gamma \in \Gamma(\sigma)^{\mathrm{gp}}$), X_4 is the quotient space of $\mathrm{spl}(W')$ under the action of $\Gamma(\sigma)^{\mathrm{gp}}$ given by $s \mapsto \mathrm{gr}^W(\gamma) \circ s \circ \mathrm{gr}^{W'}(\mathrm{gr}^W(\gamma))^{-1}$ ($s \in \mathrm{spl}(W')$, $\gamma \in \Gamma(\sigma)^{\mathrm{gp}}$), and $X_5 = \sigma_{\mathbf{R}}/(\tau_{\mathbf{R}} + \log(\Gamma(\sigma)^{\mathrm{gp}}))$. Then, for $1 \leq j \leq 5$, the action of $H(j)$ on X_j factors through $H_j := H(j)/H(j-1)$, and the action of H_j on X_j is proper and free. In fact, for $j=1$, this is by Proposition 4.2.5 and Proposition 4.2.2 (ii). For $j=2$, this is by Theorem 4.3.6 (i) and Theorem 4.3.5 (i). For $j=3,4,5$, this is easily seen. Hence by 4.2.4.5, the action of H on $X_1 \times X_2 \times X_3 \times X_4 \times X_5$ is proper. By 4.2.4.3 (ii), this proves that the action of $\sigma_{\mathbf{C}}$ on $\tilde{U}(\tau)_{\text{val}}$ is proper. □

Proposition 4.4.3. *The log local ringed space $\Gamma(\sigma)^{\text{gp}}\backslash D_\sigma$ over $\mathbf{C}$ is a log manifold.*

We prove this proposition together with Theorem 2.5.3 (that is, E_σ is a $\sigma_{\mathbf{C}}$-torsor over $\Gamma(\sigma)^{\text{gp}}\backslash D_\sigma$).

Proof. Recall that, for a face τ of σ, toric_τ is embedded in toric_σ as in 2.3.1, that $U(\tau)$ is the torus_σ-orbit of toric_τ in toric_σ (4.1.2), and that $\tilde{U}(\tau)$ is the pullback of $U(\tau)$ in E_σ (4.4.2). Since $E_\sigma = \bigcup_\tau \tilde{U}(\tau)$, where τ ranges over all faces of σ such that a τ-nilpotent orbit exists, it is sufficient to prove that, for such τ, $\Gamma(\sigma)^{\text{gp}}\backslash D_\tau = \sigma_{\mathbf{C}}\backslash\tilde{U}(\tau)$ is a log manifold and $\tilde{U}(\tau)$ is a $\sigma_{\mathbf{C}}$-torsor over $\Gamma(\sigma)^{\text{gp}}\backslash D_\tau$.

For such τ, we apply Lemma 4.2.7 by taking $H=\sigma_{\mathbf{C}}$, $X=\tilde{U}(\tau)$ and $\mathcal{C}$ to be the category of log manifolds as in 4.2.7 (b). Since we have already seen that the action of H on X is proper by Lemma 4.4.2 and free by Proposition 4.2.2 (i), it is sufficient to prove that the condition 4.2.7 (1) is satisfied. For this part of the proof, the argument in the pure case [KU09] 7.3.5 works just by replacing $W(\sigma(q))[-w]$ there by $M(\sigma(q),W)$ here. But, for readers' convenience, we sketch here the construction of a local section $\iota\colon S\to X$ in the condition 4.2.7 (1), since this is an essential part for understanding the structure of $\Gamma(\sigma)^{\text{gp}}\backslash D_\sigma$ and hence of $\Gamma\backslash D_\Sigma$.

Let $x=(q,F)\in\tilde{U}(\tau)$. Let $T_{\check{D}}(F)$ be the tangent space of $\check{D}$ at F. Then the morphism $G_{\mathbf{C}}\to\check{D}$, $g\mapsto gF$, induces a surjective homomorphism $\mathfrak{g}_{\mathbf{C}}\to T_{\check{D}}(F)$ whose kernel consists of all elements N of $\mathfrak{g}_{\mathbf{C}}$ such that $NF^p\subset F^p$ for all $p\in\mathbf{Z}$. We have:

(1) The homomorphism $\sigma(q)_{\mathbf{C}}\to T_{\check{D}}(F)$ is injective.

In fact, if $N\in\sigma(q)_{\mathbf{C}}$ is in the kernel, then $N\in L^{-1,-1}(M(\sigma(q),W),F)$ and $\exp(N)F=F$, and hence $N=0$ by [KU09] Lemma 6.1.8 (iii). (1) is proved.

Now take a $\mathbf{C}$-subspace B_1 of $\mathfrak{g}_{\mathbf{C}}$ such that $\sigma(q)_{\mathbf{C}}\oplus B_1\to T_{\check{D}}(F)$ is an isomorphism. Then, we can find an open neighborhood U_0 (resp. U_1, resp. U) of 0 in $\text{toric}_{\sigma(q)}$ (resp. B_1, resp. $\sigma_{\mathbf{C}}$) and an element b of $\sigma_{\mathbf{C}}$ such that $q=\mathbf{e}(b)0_{\sigma(q)}$ and that

$$U\times U_0\times U_1\to\check{E}_\sigma,\ \ (h,a_0,a_1)\mapsto(\mathbf{e}(h+b)a_0,\exp(-h)\exp(a_1)F)=h\cdot(\mathbf{e}(b)a_0,\exp(a_1)F),$$

is a continuous injective open map which sends $(0,0,0)$ to x. Let A be the intersection of the image of the above map with $\{h=0\}$, and let $S=A\cap\tilde{U}(\tau)$. Then it can be seen that S has the desired property.

For more details, see [KU09] 7.3.5. □

Since $\Gamma(\sigma)^{\text{gp}}\backslash D_\sigma$ belongs to $\mathcal{B}(\log)$ by 4.4.3, the topological space $(\Gamma(\sigma)^{\text{gp}}\backslash D_\sigma)^{\log}$ is defined.

Proposition 4.4.4. *We have a canonical homeomorphism*

$$\Gamma(\sigma)^{\text{gp}}\backslash D_\sigma^\sharp\simeq(\Gamma(\sigma)^{\text{gp}}\backslash D_\sigma)^{\log}$$

over $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$.

Proof. We define a canonical homeomorphism $(\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma)^{\log} \simeq \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma^\sharp$ over $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$. We have

$$(\mathrm{toric}_\sigma)^{\log} \simeq \mathrm{Hom}(\Gamma(\sigma)^\vee, \mathbf{S}^1 \times \mathbf{R}_{\geq 0}^{\mathrm{mult}}) \simeq (\mathbf{S}^1 \otimes_{\mathbf{Z}} \Gamma(\sigma)^{\mathrm{gp}}) \times |\mathrm{toric}|_\sigma .$$

Hence the homeomorphisms

$$(E_\sigma)^{\log} \simeq (\mathrm{toric}_\sigma)^{\log} \times_{\mathrm{toric}_\sigma} E_\sigma, \quad E_\sigma^\sharp \simeq |\mathrm{toric}|_\sigma \times_{\mathrm{toric}_\sigma} E_\sigma$$

induce a homeomorphism

$$(1) \qquad (E_\sigma)^{\log} \simeq (\mathbf{S}^1 \otimes_{\mathbf{Z}} \Gamma(\sigma)^{\mathrm{gp}}) \times E_\sigma^\sharp.$$

This homeomorphism is compatible with the actions of $\sigma_{\mathbf{C}}$. Here $z = x+iy \in \sigma_{\mathbf{C}}$ ($x, y \in \sigma_{\mathbf{R}}$) acts on the left-hand side of (1) as the map $(E_\sigma)^{\log} \to (E_\sigma)^{\log}$ induced by the action of z on E_σ (4.2.1) and on the right-hand side as $(u, q, F) \mapsto (\mathbf{e}(x)u, \mathbf{e}(iy)q, \exp(-z)F)$ ($u \in \mathbf{S}^1 \otimes_{\mathbf{Z}} \Gamma(\sigma)^{\mathrm{gp}}$, $q \in |\mathrm{toric}|_\sigma$, $F \in \check{D}$). The homeomorphism between the quotient spaces of these actions is

$$(\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma)^{\log} \simeq \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma^\sharp. \quad \square$$

Corollary 4.4.5. *The space* $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ *is Hausdorff.*

The space $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma^\sharp$ is Hausdorff by 4.3.6 (i) and the map $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma^\sharp \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ is proper and surjective by 4.4.4. Hence $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ is Hausdorff by 4.2.4.7.

Proposition 4.4.6. *The action of* $\sigma_{\mathbf{C}}$ *on* E_σ (*resp.* $E_{\sigma,\mathrm{val}}$) *is proper.*

The result for E_σ follows from Theorem 2.5.3 and Corollary 4.4.5 by 4.2.4.6. The result for $E_{\sigma,\mathrm{val}}$ follows from it by 4.2.4.3 (ii).

§4.5. Study of $\Gamma\backslash D_\Sigma$

In this subsection, we prove Theorems 2.5.2, 2.5.4, 2.5.5, 2.5.6, and 2.6.6.

Lemma 4.5.1. *Let* X *be a topological space with a continuous action of a discrete group* Γ *and let* Y *be a set with an action of* Γ. *Let* $f : X \to Y$ *be a* Γ*-equivariant surjective map. Let* Γ' *be a subgroup of* Γ. *We introduce the quotient topologies of* X *on* $\Gamma'\backslash Y$ *and on* $\Gamma\backslash Y$. *Let* V *be an open set of* $\Gamma'\backslash Y$ *and let* U *be the inverse image of* V *in* $\Gamma'\backslash X$. *We assume moreover the three conditions* (1)–(3) *below. Then,* $V \to \Gamma\backslash Y$ *is a local homeomorphism.*

(1) $X \to \Gamma\backslash X$ *is a local homeomorphism and* $\Gamma\backslash X$ *is Hausdorff.*

(2) $U \to V$ *is proper.*

(3) *If* $x \in X$ *and* $\gamma \in \Gamma$, *and if the images of* γx *and* x *in* $\Gamma'\backslash Y$ *are contained in* V *and they coincide, then* $\gamma \in \Gamma'$.

Proof. This is [KU09] Lemma 7.4.7. See [KU09] for a proof. □

4.5.2. *Proof of Theorem 2.5.4.* We prove that $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to \Gamma\backslash D_\Sigma$ is locally an isomorphism.

We use Lemma 4.5.1 for $X = D^\sharp_\Sigma$, $Y = D_\Sigma$, $\Gamma = \Gamma$, $\Gamma' = \Gamma(\sigma)^{\mathrm{gp}}$, $V = \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ and $U = \Gamma(\sigma)^{\mathrm{gp}}\backslash D^\sharp_\sigma$. Theorem 4.3.6 for $D^\sharp_\Sigma$ shows that the condition (1) in Lemma 4.5.1 is satisfied. Theorem 2.5.6 for $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$, proved in §4.4, shows that the condition (2) in 4.5.1 is satisfied. Theorem 4.3.5 shows that the condition (3) in 4.5.1 is satisfied. □

4.5.3. *Proof of Theorem 2.5.2.* We prove that $\Gamma\backslash D_\Sigma$ is a log manifold.

This follows from Theorem 2.5.2 for $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ (4.4.3) by Theorem 2.5.4. □

4.5.4. *Proof of Theorem 2.5.6.* We prove that $(\Gamma\backslash D_\Sigma)^{\log} \simeq \Gamma\backslash D^\sharp_\Sigma$ over $\Gamma\backslash D_\Sigma$.

By Theorem 2.5.4, $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to \Gamma\backslash D_\Sigma$ ($\sigma \in \Sigma$) are local isomorphisms of log manifolds. Hence the canonical homeomorphisms $(\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma)^{\log} \simeq \Gamma(\sigma)^{\mathrm{gp}}\backslash D^\sharp_\sigma$ over $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$, which were given in 4.4.4, glue up uniquely to a homeomorphism $(\Gamma\backslash D_\Sigma)^{\log} \simeq \Gamma\backslash D^\sharp_\Sigma$ over $\Gamma\backslash D_\Sigma$. □

4.5.5. *Proof of Theorem 2.5.5.* We prove that $\Gamma\backslash D_\Sigma$ is a Hausdorff space.

Replacing Γ by a subgroup of finite index, we may assume that Γ is neat. Then, by Theorem 2.5.6 proved in 4.5.4, the map $\Gamma\backslash D^\sharp_\Sigma \to \Gamma\backslash D_\Sigma$ is proper and surjective. Since $\Gamma\backslash D^\sharp_\Sigma$ is Hausdorff by Theorem 4.3.6 (i), it follows by 4.2.4.7 that $\Gamma\backslash D_\Sigma$ is Hausdorff. □

A local description of the log manifold $\Gamma\backslash D_\Sigma$ is given by the following.

Theorem 4.5.6. *Let Σ be a weak fan and let Γ be a neat subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Σ. Let $(\sigma, Z) \in D_\Sigma$ and let p be its image in $\Gamma\backslash D_\Sigma$.*

(i) *Let $F_{(0)} \in Z$. Then there exists a locally closed analytic submanifold Y of $\check{D}$ satisfying the following conditions* (1) *and* (2)*.*

(1) $F_{(0)} \in Y$.

(2) *The canonical map $T_Y(F_{(0)}) \oplus \sigma_{\mathbf{C}} \to T_{\check{D}}(F_{(0)})$ is an isomorphism. Here $T_Y(F_{(0)})$ denotes the tangent space of Y at $F_{(0)}$ and $T_{\check{D}}(F_{(0)})$ denotes the tangent space of $\check{D}$ at $F_{(0)}$.*

(ii) *Let $F_{(0)}$ and Y be as in* (i)*. Let $X := \mathrm{toric}_\sigma \times Y$, and let S be the log manifold defined by*

$$S := \{(q, F) \in X \mid N(F^p) \subset F^{p-1}\ (\forall N \in \sigma(q), \forall p \in \mathbf{Z})\},$$

where $\sigma(q)$ denotes the face of σ corresponding to q. Then, there exist an open neighborhood U_1 of $(0, F_{(0)})$ in S in the strong topology, an open neighborhood U_2 of p in $\Gamma\backslash D_\Sigma$, and an isomorphism $U_1 \xrightarrow{\sim} U_2$ of log manifolds sending $(0, F_{(0)})$ to p.

Proof. This is the mixed Hodge-theoretic version of [KU09] 7.4.13 and proved similarly by using the proof of 2.5.3 in 4.4.3 and 2.5.4. □

4.5.7. We prove Theorem 2.6.6. This is the mixed Hodge-theoretic version of Theorem B in [KU09] 4.2, which was stated and proved in the category $\overline{\mathcal{A}}_2(\log)$, but we work here in the category $\mathcal{B}(\log)$ (cf. Remark below).

Since everything is prepared in §2.5, especially Theorem 2.5.3, the proof of 2.6.6 is exactly parallel to the pure case [KU09] 8.2. We sketch an outline of the proof.

Let $\sigma \in \Sigma$. Let $\Phi_\sigma = (\Lambda, \{\text{face of } \sigma\}, \Gamma(\sigma)^{\mathrm{gp}})$. Put $C := \mathrm{LMH}_\Phi$, $C_\sigma := \mathrm{LMH}_{\Phi_\sigma}$. Define contravariant functors $\check{B}_\sigma, B_\sigma : \mathcal{B}(\log) \to (\mathrm{set})$ as follows. For $S \in \mathcal{B}(\log)$, define $\check{B}_\sigma(S)$ (resp. $B_\sigma(S)$) to be the set of all pairs (θ, F) with a morphism $\theta : S \to \mathrm{toric}_\sigma$ and $F \in \check{D}$ such that F paired with the inverse image of the canonical log local system (cf. 2.3.2) on toric_σ by $\theta^{\log} : S^{\log} \to \mathrm{toric}_\sigma^{\log}$ is a pre-LMH (resp. LMH with polarized graded quotients) on S of type Λ (resp. type Φ_σ).

As functors on $\mathcal{B}(\log)$,

Step 1. $B_\sigma \simeq \mathrm{Mor}(\quad, E_\sigma)$ (strong open in $\check{E}_\sigma$),

Step 2. $C_\sigma \simeq \mathrm{Mor}(\quad, \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma)$ (quotient),

Step 3. $C \simeq \mathrm{Mor}(\quad, \Gamma \backslash D_\Sigma)$ (gluing; this is Theorem 2.6.6).

To prove Step 1, we first show $\check{B}_\sigma \simeq \mathrm{Mor}(\quad, \check{E}_\sigma)$, then show that their respective subfunctors B_σ and $\mathrm{Mor}(\quad, E_\sigma)$ coincide.

An outline of the proof of Step 2 is as follows. First, note that C_σ is a sheaf-functor on $\mathcal{B}(\log)$ by 1.3.6. Consider the quotient $\mathrm{Mor}(\quad, \sigma_{\mathbf{C}}) \backslash B_\sigma$. By the exact sequence $1 \longrightarrow \Gamma(\sigma)^{\mathrm{gp}} \longrightarrow \sigma_{\mathbf{C}} \longrightarrow \mathrm{torus}_\sigma \longrightarrow 1$, we have

$$\mathrm{Mor}(\quad, \sigma_{\mathbf{C}}) \backslash B_\sigma \simeq \mathrm{Mor}(\quad, \mathrm{torus}_\sigma) \backslash (\Gamma(\sigma)^{\mathrm{gp}} \backslash B_\sigma).$$

By the exact sequence $1 \to \mathcal{O}_S^\times \to M_S \to M_S/\mathcal{O}_S^\times \to 1$, we have

$$\begin{array}{ccc} \mathrm{Mor}(S, \mathrm{torus}_\sigma) & \xrightarrow{\sim} & \mathrm{Hom}(\Gamma(\sigma)^\vee, \mathcal{O}_S^\times) \\ \cap & & \cap \\ \mathrm{Mor}(S, \mathrm{toric}_\sigma) & \xrightarrow{\sim} & \mathrm{Hom}(\Gamma(\sigma)^\vee, M_S) \\ \downarrow & & \downarrow \\ \overline{C}_\sigma(S) & \xrightarrow{\sim} & \mathrm{Hom}(\Gamma(\sigma)^\vee, M_S/\mathcal{O}_S^\times) \end{array}$$

for any $S \in \mathcal{B}(\log)$. From this, we have a cartesian diagram of $\mathrm{Mor}(\quad, \mathrm{torus}_\sigma)$-torsors

$$\begin{array}{ccc} \Gamma(\sigma)^{\mathrm{gp}} \backslash B_\sigma & \longrightarrow & \mathrm{Mor}(\quad, \mathrm{toric}_\sigma) \\ \downarrow & & \downarrow \\ C_\sigma & \longrightarrow & \overline{C}_\sigma, \end{array}$$

and hence $\mathrm{Mor}(\quad, \sigma_{\mathbf{C}}) \backslash B_\sigma \simeq C_\sigma$ (the left-hand side of Step 2). On the other hand, by the result of Step 1 and Theorem 2.5.3, we have

$$\mathrm{Mor}(\quad, \sigma_{\mathbf{C}}) \backslash B_\sigma \simeq \mathrm{Mor}(\quad, \sigma_{\mathbf{C}}) \backslash \mathrm{Mor}(\quad, E_\sigma) \simeq \mathrm{Mor}(\quad, \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma)$$

(the right-hand side of Step 2).

An outline of the proof of Step 3 is as follows. By the result of Step 2 and Theorem 2.5.4, we have

$$(\bigsqcup_{\sigma\in\Sigma} C_\sigma)/\sim \;\xrightarrow{\sim}\; \big(\bigsqcup_{\sigma\in\Sigma} \mathrm{Mor}(\quad, \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma)\big)/\sim \;\xrightarrow{\sim}\; \mathrm{Mor}(\quad, \Gamma\backslash D_\Sigma),$$

where, in the middle, "$/\sim$" is the quotient in the category of sheaf-functors by the equivalence relation $\sim$ generated by the following relation: For a triple (σ,τ,γ) with $\sigma,\tau\in\Sigma$ and $\gamma\in\Gamma$ such that $\mathrm{Ad}(\gamma)(\tau)\subset\sigma$, and for $S\in\mathcal{B}(\log)$, $a\in\mathrm{Mor}\big(S,\Gamma(\tau)^{\mathrm{gp}}\backslash D_\tau\big)$ is related by $\sim$ to $\gamma\circ a\in\mathrm{Mor}\big(S,\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma\big)$, where the γ denotes the morphism $\Gamma(\tau)^{\mathrm{gp}}\backslash D_\tau\to\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ in $\mathcal{B}(\log)$ induced by γ. Similarly defined "$/\sim$" in the left. The bijectivity

$$(\bigsqcup_{\sigma\in\Sigma} C_\sigma)/\sim \;\xrightarrow{\sim}\; C \tag{1}$$

is shown in the following way. C is a sheaf-functor on $\mathcal{B}(\log)$ by 1.3.6. First,

$$\big(\bigsqcup_{\sigma\in\Sigma} \overline{C}_\sigma\big)/\sim \;\xrightarrow{\sim}\; \overline{C} \tag{2}$$

is proved by reducing to the case: $\big(\bigsqcup_{\sigma\in\Sigma}\overline{C}_\sigma(p)\big)/\sim\;\xrightarrow{\sim}\;\overline{C}(p)$ for any fs log point p. The surjectivity of (1) follows from the surjectivity of (2) and the surjectivity of $C_\sigma\to\bar{C}_\sigma\times_{\bar{C}}C$. The injectivity of (1) is proved by examining plainly the equivalence relation $\sim$ in (1) and by using the injectivity of (2). □

Remark. In order to discuss log Hodge theory, the category of analytic spaces was enlarged in several ways in [KU09] 3.2. The categories $\overline{\mathcal{A}}_2(\log)$ and $\mathcal{B}(\log)$ are two of them. The latter is a full subcategory of the former. Theorem B in [KU09] was stated and proved in $\overline{\mathcal{A}}_2(\log)$, but Theorem 0.4.27 in [KU09] and Theorem 2.6.6 in the present paper are done in $\mathcal{B}(\log)$. For the reason why we do so and for the relationship among all these categories, see [KU09] 3.2.6.

§5. Moduli spaces of log mixed Hodge structures with given graded quotients

Let S be an object of $\mathcal{B}(\log)$, and assume that we are given a family $Q=(H_{(w)})_{w\in\mathbf{Z}}$, where $H_{(w)}$ is a pure LMH on S of weight w for each w and $H_{(w)}=0$ for almost all w. Shortly speaking, the subject of this section is to construct spaces over S which classify LMH H with $H(\mathrm{gr}^W_w)=H_{(w)}$ for any w.

We assume that all $H_{(w)}$ are polarizable, though we expect that the method of this section works (with some suitable modifications) without assuming the polarizability of $H_{(w)}$. It actually works successfully in some non-polarizable case; cf. 7.1.4.

More precisely, let

$$\mathrm{LMH}_Q$$

be the contravariant functor on the category $\mathcal{B}(\log)/S$ defined as follows. For an object S' of $\mathcal{B}(\log)$ over S, $\mathrm{LMH}_Q(S')$ is the set of all isomorphism classes of an LMH H on S' endowed with an isomorphism $H(\mathrm{gr}^W_w)\simeq H_{(w)}|_{S'}$ for each $w\in\mathbf{Z}$. Here $H_{(w)}|_{S'}$ denotes the pullback of $H_{(w)}$ on S'. The functor LMH_Q is usually not representable. (For example, $\mathcal{E}xt^1_{\mathrm{LMH}}(\mathbf{Z},\mathbf{Z}(1)) = (U \mapsto M^{\mathrm{gp}}(U))$ (cf. 7.1.1) is not representable, though its subfunctors $(U \mapsto M(U))$ and $(U \mapsto M(U) \cup M(U)^{-1})$ are represented by the affine line with the log by the origin and the projective line with the log by the origin and the infinity, respectively.) Our subject is to construct relative log manifolds over S which represent big subfunctors of LMH_Q.

We can often construct such a relative log manifold over S by the method of §2: As is explained in [KNU10a], we can construct such a space as the fiber product of $S \to \prod_{w\in\mathbf{Z}} \Gamma_w\backslash D(\mathrm{gr}^W_w)_{\sigma_w} \leftarrow \Gamma\backslash D_\Sigma$, where $S \to \Gamma_w\backslash D(\mathrm{gr}^W_w)_{\sigma_w}$ is the period map of $H_{(w)}$ (if it exists) and D_Σ is a moduli space in §2 (see §5.5 for details). However, this method has a disadvantage that the period map $S \to \Gamma_w\backslash D(\mathrm{gr}^W_w)_{\sigma_w}$ can be defined in general only after blowing up S as is explained in [KU09] §4.3. (Cf. 5.5.3.)

The main purpose of this §5 is to construct more relative log manifolds over S which represent big subfunctors of LMH_Q, by improving the method of §2. The difference between §2 and §5 lies in the situation that cones considered in §2 were inside $\mathfrak{g}_\mathbf{R}$, whereas the cones considered in §5 exist outside $\mathfrak{g}_\mathbf{R}$. In particular, by the method of this §5, we will obtain in §6 the Néron model and the connected Néron model, an open subspace of the Néron model, both of which represent big subfunctors of LMH_Q (see Theorem 6.1.1). In this method, we do not need blow up S.

The proofs for this section is in 5.4, where we reduce the results in this section to those in §2.

§5.1. Relative formulation of cones, and set $D_{S,\Sigma}$ of nilpotent orbits

Here we give a formulation of cones which is suitable for the study of the above functor LMH_Q of log mixed Hodge structures relative to the prescribed Q.

5.1.1. Assume that we are given an object S of $\mathcal{B}(\log)$ and a polarized log Hodge structure $H_{(w)}$ on S of weight w for each $w \in \mathbf{Z}$. Assume that $H_{(w)} = 0$ for almost all w. Write $Q = (H_{(w)})_{w\in\mathbf{Z}}$.

Then, locally on S, we can find the following (1)–(5).

(1) $\Lambda = (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$ as in 2.1.1.

(2) A sharp fs monoid P.

(3) A homomorphism

$$\Gamma' := \mathrm{Hom}(P^{\mathrm{gp}}, \mathbf{Z}) \to G'_\mathbf{Z} := G_\mathbf{Z}(\mathrm{gr}^W) = \textstyle\prod_w G_\mathbf{Z}(\mathrm{gr}^W_w)$$

whose image consists of unipotent automorphisms.

((4) and (5) are stated after a preparation.) Let

$$\sigma' := \mathrm{Hom}(P, \mathbf{R}^{\mathrm{add}}_{\geq 0}), \quad \mathrm{toric}_{\sigma'} := \mathrm{Spec}(\mathbf{C}[P])_{\mathrm{an}}.$$

(The symbol σ will be used later for cones in $\sigma' \times_{\mathfrak{g}'_\mathbf{R}} \mathfrak{g}_\mathbf{R}$.) Let

$$\sigma'_\mathbf{R} \to \mathfrak{g}'_\mathbf{R} := \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W)$$

be the $\mathbf{R}$-linear map induced by the logarithm of the homomorphism in (3).

We define the set $E'_{\sigma'}$. The definition is parallel to the definition of E_σ in 2.3.3. Let $E'_{\sigma'}$ be the set of all points $s \in \mathrm{toric}_{\sigma'} \times \check{D}(\mathrm{gr}^W)$, where $\check{D}(\mathrm{gr}^W) = \prod_w \check{D}(\mathrm{gr}^W_w)$, such that for each w, the pullback of the canonical pre-PLH of weight w on $\mathrm{toric}_{\sigma'} \times \check{D}(\mathrm{gr}^W_w)$ to the fs log point s is a PLH. Here the canonical pre-PLH of weight w on $\mathrm{toric}_{\sigma'} \times \check{D}(\mathrm{gr}^W_w)$ is defined in the same way as in 2.3.2.

In other words, $E'_{\sigma'}$ is the set of all pairs $(q', F') \in \mathrm{toric}_{\sigma'} \times \check{D}(\mathrm{gr}^W)$ satisfying the following condition:

If we write $q' = \mathbf{e}(a)\cdot 0_{\tau'}$ with τ' a face of σ' and $a \in \sigma'_{\mathbf{C}}$ (2.3.1), then $(\tau'_{\mathfrak{g}'}, \exp(a_{\mathfrak{g}'})F')$ generates a nilpotent orbit, where $(-)_{\mathfrak{g}'}$ denotes the image under the $\mathbf{C}$-linear map $\sigma'_{\mathbf{C}} \to \mathfrak{g}'_{\mathbf{C}} := \mathfrak{g}_{\mathbf{C}}(\mathrm{gr}^W)$ induced by the above $\mathbf{R}$-linear map $\sigma'_{\mathbf{R}} \to \mathfrak{g}'_{\mathbf{R}}$.

We endow $E'_{\sigma'}$ with the structure of an object of $\mathcal{B}(\log)$ by using the embedding $E'_{\sigma'} \subset \mathrm{toric}_{\sigma'} \times \check{D}(\mathrm{gr}^W)$.

(4) A strict morphism $S \to E'_{\sigma'}$ in $\mathcal{B}(\log)$.

Here a morphism $X \to Y$ in $\mathcal{B}(\log)$ is said to be *strict* if via this morphism, the log structure of X coincides with the inverse image of the log structure of Y.

(5) An isomorphism of PLH for each $w \in \mathbf{Z}$ between $H_{(w)}$ and the pullback of the canonical PLH of weight w on $E'_{\sigma'}$ under the morphism in (4).

5.1.2. The local existence of (1)–(5) in 5.1.1 follows from a general theory of PLH in [KU09].

In fact, take $s_0 \in S$, and take a point $t_0 \in S^{\log}$ lying over s_0. Let $H_0 := \bigoplus_w H_{(w),\mathbf{Z},t_0}$ with the evident weight filtration W on $H_{0,\mathbf{R}}$ and with the bilinear form $\langle\ ,\ \rangle_w$ given by the polarization of $H_{(w)}$. For each w, let $(h^{p,q})_{p,q}$, $p+q=w$, be the Hodge type of $H_{(w)}$ at s_0.

Let $P := (M_S/\mathcal{O}_S^\times)_{s_0}$. Then $\Gamma' := \mathrm{Hom}(P^{\mathrm{gp}}, \mathbf{Z})$ is identified with the fundamental group $\pi_1(s_0^{\log})$, and the actions of $\pi_1(s_0^{\log})$ on $H_{(w),\mathbf{Z},t_0}$ for $w \in \mathbf{Z}$ give the homomorphism $\Gamma' \to G'_{\mathbf{Z}}$.

On an open neighborhood of s_0 in S, there is a homomorphism $P \to M_S$ which induces the identity map $P \to (M_S/\mathcal{O}_S^\times)_{s_0}$ and which is a chart ([KU09] 2.1.5) of the fs log structure M_S. We replace S by this neighborhood of s_0. Then this homomorphism induces a strict morphism $S \to \mathrm{toric}_{\sigma'}$. By [KU09] 8.2.1–8.2.3, we obtain a strict morphism $S \to E'_{\sigma'}$ over $\mathrm{toric}_{\sigma'}$ and the isomorphism (5) in 5.1.1.

5.1.3. Assume that we are given S, $Q = (H_{(w)})_{w\in\mathbf{Z}}$, and (1)–(5) of 5.1.1 and fix them.

We will fix a splitting $H_0 = \bigoplus_w H_0(\mathrm{gr}^W_w)$ of the filtration W on H_0. We will denote $H_0(\mathrm{gr}^W_w)$ also by $H_{0,(w)}$.

Let μ' be the Γ'-level structure on $H_{\mathbf{Z}}(\mathrm{gr}^W) := \bigoplus_w H_{(w),\mathbf{Z}}$, i.e., the class of local isomorphisms $H_{\mathbf{Z}}(\mathrm{gr}^W) \simeq H_0$ on $S^{\log}$ which is global mod Γ', defined by the isomorphism (5) in 5.1.1.

We will denote $E'_{\sigma'}$ often by S_0.

In §5, the notions of nilpotent cone, nilpotent orbit, fan, and weak fan are different from those in §2. Recall $\sigma' = \mathrm{Hom}(P, \mathbf{R}^{\mathrm{add}}_{\geq 0})$ together with the linear map $\sigma'_{\mathbf{R}} \to$

$\mathfrak{g}'_{\mathbf{R}} := \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)$. In §5, nilpotent cones, fans, and weak fans appear in the fiber product $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$, and nilpotent orbits appear in the product $\sigma'_{\mathbf{C}} \times \check{D}$, as explained below.

5.1.4. Nilpotent cone.

In §5, a *nilpotent cone* means a finitely generated cone σ in the fiber product

$$\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$$

satisfying the following (1) and (2).

(1) For any $N \in \sigma$, the $\mathfrak{g}_{\mathbf{R}}$-component $N_{\mathfrak{g}} \in \mathfrak{g}_{\mathbf{R}}$ of N is nilpotent.

(2) For any $N_1, N_2 \in \sigma$, $N_{1,\mathfrak{g}} N_{2,\mathfrak{g}} = N_{2,\mathfrak{g}} N_{1,\mathfrak{g}}$.

That is, σ is a nilpotent cone if the image $\sigma_{\mathfrak{g}}$ of σ in $\mathfrak{g}_{\mathbf{R}}$ is a nilpotent cone in the sense of 2.2.1.

We say a nilpotent cone σ is *admissible* if the following condition (3) is satisfied.

(3) The action of σ on $H_{0,\mathbf{R}}$ via $\sigma \to \mathfrak{g}_{\mathbf{R}}$, $N \mapsto N_{\mathfrak{g}}$, is admissible with respect to W in the sense of 1.2.2.

By 1.2.3, this is equivalent to that $\sigma_{\mathfrak{g}}$ is admissible in the sense of 2.2.1.

5.1.5. Nilpotent orbit.

Let σ be a nilpotent cone. A *σ-nilpotent orbit* is a subset Z of $\sigma'_{\mathbf{C}} \times \check{D}$ satisfying the following conditions (1) and (2).

(1) If $(a, F) \in Z$ ($a \in \sigma'_{\mathbf{C}}, F \in \check{D}$), then $Z = \{(a + b', \exp(b_{\mathfrak{g}})F) \mid b \in \sigma_{\mathbf{C}}\}$ (here b' denotes the image of b in $\sigma'_{\mathbf{C}}$).

(2) If $(a, F) \in Z$, then $(\sigma_{\mathfrak{g}}, F)$ generates a nilpotent orbit (2.2.2).

Note that for $F \in \check{D}$ and $a \in \sigma'_{\mathbf{C}}$, $\{(a+b', \exp(b_{\mathfrak{g}})F) \mid b \in \sigma_{\mathbf{C}}\}$ is a σ-nilpotent orbit if and only if the above condition (2) is satisfied, which depends on F but not on a. If this condition (2) is satisfied, we say F *generates a σ-nilpotent orbit* (or (σ, F), or further, $(N_1, \dots, N_n, F)$ *generates a nilpotent orbit* when σ is generated by $N_1, \dots, N_n$).

5.1.6. Fan and weak fan.

In §5, a *fan* is a non-empty set of sharp rational nilpotent cones in $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$ which is closed under the operations of taking a face and taking the intersection.

A *weak fan* is a non-empty set Σ of sharp rational nilpotent cones in $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$ which is closed under the operation of taking a face and satisfies the following condition.

(1) Let $\sigma, \tau \in \Sigma$, and assume that σ and τ have a common interior point. Assume that there is an $F \in \check{D}$ such that (σ, F) and (τ, F) generate nilpotent orbits in the sense of 5.1.5. Then $\sigma = \tau$.

A fan is a weak fan. (This is proved in the same way as 2.2.4.)

An example of a fan is

$$\Xi = \{\mathbf{R}_{\geq 0} N \mid N \in \sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}, N \text{ is rational, } N_{\mathfrak{g}} \text{ is nilpotent}\}.$$

5.1.7. Recall

$$\Gamma' = \mathrm{Hom}(P^{\mathrm{gp}}, \mathbf{Z}).$$

We denote the given homomorphism $\Gamma' \to G'_{\mathbf{Z}}$ (5.1.1 (3)) by $a \mapsto a_{G'}$. We regard $G'_{\mathbf{Z}}$ as a subgroup of $G_{\mathbf{Z}}$ by the fixed splitting of W on H_0.

As is easily seen, there is a one-to-one correspondence between

(1) A subgroup Γ_u of $G_{\mathbf{Z},u}$ such that $\gamma_{G'}\Gamma_u\gamma_{G'}^{-1} = \Gamma_u$ for all $\gamma \in \Gamma'$. Here $G_{\mathbf{Z},u}$ is the unipotent radical of $G_{\mathbf{Z}}$.

(2) A subgroup Γ of $\Gamma' \times_{G'_{\mathbf{Z}}} G_{\mathbf{Z}}$ containing $\{(a, a_{G'}) \mid a \in \Gamma'\} \cong \Gamma'$.

The correspondence is:
From (1) to (2). $\Gamma := \{(a, a_{G'}b) \mid a \in \Gamma', b \in \Gamma_u\}$.
From (2) to (1). $\Gamma_u := \mathrm{Ker}(\Gamma \to \Gamma')$.
In what follows, we will denote the projection $\Gamma' \times_{G'_{\mathbf{Z}}} G_{\mathbf{Z}} \to G_{\mathbf{Z}}$ by $a \mapsto a_G$.

5.1.8. Let Σ be a weak fan, and let Γ be a subgroup of $\Gamma' \times_{G'_{\mathbf{Z}}} G_{\mathbf{Z}}$ satisfying (2) in 5.1.7.

We say that Σ and Γ are *compatible* if $\mathrm{Ad}(\gamma)(\sigma) \in \Sigma$ for any $\gamma \in \Gamma$ and $\sigma \in \Sigma$. Here $\mathrm{Ad}(\gamma)$ is $(x, y) \mapsto (x, \mathrm{Ad}(\gamma_G)y)$ ($x \in \sigma'$, $y \in \mathfrak{g}_{\mathbf{R}}$).

We say that Σ and Γ are *strongly compatible* if they are compatible and, for any $\sigma \in \Sigma$, any element of σ is a sum of elements of the form $a\log(\gamma)$ with $a \in \mathbf{R}_{\geq 0}$ and $\gamma \in \Gamma(\sigma)$. Here log is the map $\Gamma \to \Gamma' \times \mathfrak{g}_{\mathbf{R}}$, where $\Gamma \to \Gamma'$ is the projection and $\Gamma \to \mathfrak{g}_{\mathbf{R}}$ is $\gamma \mapsto \log(\gamma_G)$, and $\Gamma(\sigma)$ is defined by

$$\Gamma(\sigma) := \{\gamma \in \Gamma \mid \log(\gamma) \in \sigma\}.$$

Remark. It is easy to see that if Γ_u is of finite index in $G_{\mathbf{Z},u}$ and Σ is compatible with Γ, then Σ is strongly compatible with Γ.

5.1.9. Γ-level structure on H.

Let Γ be as in 5.1.8. Let Γ_G be the image of Γ in $G_{\mathbf{Z}}$.

Let $S' \in \mathcal{B}(\log)/S$. Let H be an LMH on S'. Assume that an isomorphism between $H(\mathrm{gr}^W_w)$ and $H_{(w)}|_{S'}$ is given for each w.

Then, there is a one-to-one correspondence between the following (1) and (2).

(1) The class (mod Γ_G) of a collection of local isomorphisms $(H_{\mathbf{Z}}, W) \simeq (H_0, W)$ which is compatible with the given Γ'-level structure μ' (5.1.3) on $H_{\mathbf{Z}}(\mathrm{gr}^W)$ and which is global mod Γ_G.

(2) The class (mod Γ_G) of a collection of local splittings $H_{\mathbf{Z}}(\mathrm{gr}^W) \simeq H_{\mathbf{Z}}$ of the filtration W on the local system $H_{\mathbf{Z}}$ on $S^{\log}$ which is global mod Γ_G.

Such a class is called a Γ-*level structure* on $H_{\mathbf{Z}}$.

5.1.10. Assume that Σ and Γ as in 5.1.8 are given. Assume that Σ is strongly compatible with Γ. Define

$$D_{S,\Sigma} = \{(s, \sigma, Z) \mid s \in S, \sigma \in \Sigma, Z \subset \sigma'_{\mathbf{C}} \times \check{D} \text{ satisfying (1) and (2) below}\}.$$

(1) Z is a σ-nilpotent orbit (5.1.5).

(2) For any $(a, F) \in Z$, the image of s under $S \to E'_{\sigma'}$ coincides with $(\mathbf{e}(a) \cdot 0_\sigma(\mathrm{gr}^W), \exp(-a_{\mathfrak{g}'})F(\mathrm{gr}^W))$. Here $0_\sigma(\mathrm{gr}^W)$ denotes the image of $0_\sigma \in \mathrm{toric}_\sigma$ in $\mathrm{toric}_{\sigma'}$. (That is, if α denotes the smallest face of σ' which contains the image of $\sigma \to \sigma'$, then $0_\sigma(\mathrm{gr}^W) = 0_\alpha$.)

Note that under the condition (1), the element $(\mathbf{e}(a) \cdot 0_\sigma(\mathrm{gr}^W), \exp(-a_{\mathfrak{g}'})F(\mathrm{gr}^W))$ of $E'_{\sigma'}$ in the condition (2) is independent of the choice of $(a, F) \in Z$.

If $'\Sigma \subset \Sigma$ denotes the subset of Σ consisting of all admissible $\sigma \in \Sigma$, and if $''\Sigma \subset {'\Sigma}$ denotes the subset of Σ consisting of all $\sigma \in \Sigma$ such that a σ-nilpotent orbit exists, then $'\Sigma$ and $''\Sigma$ are weak fans, and we have $D_{S,\Sigma} = D_{S,'\Sigma} = D_{S,''\Sigma}$.

§5.2. Toroidal partial compactifications $\Gamma\backslash D_{S,\Sigma}$

Notation is as in §5.1.

Assume that Σ is a weak fan and is strongly compatible with Γ. In this §5.2, we endow the set $\Gamma\backslash D_{S,\Sigma}$ with a structure of an object of $\mathcal{B}(\log)$. Here $\gamma \in \Gamma$ acts on $D_{S,\Sigma}$ as $(s, \sigma, Z) \mapsto (s, \mathrm{Ad}(\gamma)(\sigma), \gamma Z)$, where γZ is defined by the action of γ on $\sigma'_{\mathbf{C}} \times \check{D}$ given by $(a, F) \mapsto (a - \gamma', \gamma_G F)$ with γ' being the image of γ in Γ' (5.1.7).

5.2.1. Let $\sigma \in \Sigma$. Let

$$\mathrm{toric}_\sigma = \mathrm{Spec}(\mathbf{C}[\Gamma(\sigma)^\vee])_{\mathrm{an}},$$

where $\Gamma(\sigma)^\vee = \mathrm{Hom}(\Gamma(\sigma), \mathbf{N})$.

Let S be an object of $\mathcal{B}(\log)$ and let

$E_{S,\sigma} = \{(s, z, q, F) \mid s \in S, z \in \sigma'_{\mathbf{C}}, q \in \mathrm{toric}_\sigma, F \in \check{D}$ satisfying the following (1) and (2)$\}$.

(1) If we write $q = \mathbf{e}(b) \cdot 0_\tau$ with τ a face of σ and $b \in \sigma_{\mathbf{C}}$ (2.3.1), then $\exp(b_{\mathfrak{g}})F$ generates a τ-nilpotent orbit (5.1.5).

(2) The image of s under $S \to E'_{\sigma'}$ coincides with $(\mathbf{e}(z)q(\mathrm{gr}^W), \exp(-z_{\mathfrak{g}'})F(\mathrm{gr}^W))$, where $q(\mathrm{gr}^W)$ is the image of q in $\mathrm{toric}_{\sigma'}$.

5.2.2. Note that, for $\sigma \in \Sigma$, we have a canonical map

$$E_{S,\sigma} \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_{S,\sigma}, \quad (s, z, q, F) \mapsto \mathrm{class}(s, \tau, Z).$$

Here $D_{S,\sigma}$ denotes $D_{S,\mathrm{face}(\sigma)}$, and (τ, Z) is as follows. Write $q = \mathbf{e}(b) \cdot 0_\tau$ (2.3.1). This determines τ. Further, Z is the unique τ-nilpotent orbit containing $(z + b', \exp(b_{\mathfrak{g}})F)$.

Proposition 5.2.3. *Set-theoretically, $E_{S,\sigma}$ is a $\sigma_{\mathbf{C}}$-torsor over $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_{S,\sigma}$ with respect to the following action of $\sigma_{\mathbf{C}}$ on $E_{S,\sigma}$. For $a \in \sigma_{\mathbf{C}}$, a acts on $E_{S,\sigma}$ as $(s, z, q, F) \mapsto (s, z - a',\ \mathbf{e}(a)q,\ \exp(-a_{\mathfrak{g}})F)$, where a' denotes the image of a in $\sigma'_{\mathbf{C}}$.*

This is proved easily.

5.2.4. Let $S_0 = E'_{\sigma'}$ as in 5.1.3. We consider the case that the morphism $S \to S_0$ is an isomorphism (we will express this situation as the case $S = S_0$). We endow $E_{S_0,\sigma}$ with the structure of an object of $\mathcal{B}(\log)$ by using the embedding

$$E_{S_0,\sigma} \subset \sigma'_{\mathbf{C}} \times \mathrm{toric}_\sigma \times \check{D}, \quad (s, z, q, F) \mapsto (z, q, F).$$

That is, the topology of $E_{S_0,\sigma}$ is the strong topology in $\sigma'_{\mathbf{C}} \times \mathrm{toric}_\sigma \times \check{D}$, and $\mathcal{O}$ and the log structure M of $E_{S_0,\sigma}$ are the inverse images of $\mathcal{O}$ and M of $\sigma'_{\mathbf{C}} \times \mathrm{toric}_\sigma \times \check{D}$, respectively.

5.2.5. We endow $\Gamma\backslash D_{S_0,\Sigma}$ with the structure of a log local ringed space over $\mathbf{C}$ as follows.

We define the topology of $\Gamma\backslash D_{S_0,\Sigma}$ as follows. A subset U of $\Gamma\backslash D_{S_0,\Sigma}$ is open if and only if, for any $\sigma \in \Sigma$, the inverse image U_σ of U in $E_{S_0,\sigma}$ is open.

We define the sheaf of rings $\mathcal{O}$ on $\Gamma\backslash D_{S_0,\Sigma}$ as a subsheaf of the sheaf of $\mathbf{C}$-valued functions, as follows. For an open set U of $\Gamma\backslash D_{S_0,\Sigma}$ and for a map $f : U \to \mathbf{C}$, f belongs to $\mathcal{O}$ if and only if, for any $\sigma \in \Sigma$, the pullback of f on U_σ belongs to the $\mathcal{O}$ of $E_{S_0,\sigma}$.

We define the log structure of $\Gamma\backslash D_{S_0,\Sigma}$ as a subsheaf of $\mathcal{O}$ of $\Gamma\backslash D_{S_0,\Sigma}$, as follows. For an open set U of $\Gamma\backslash D_{S_0,\Sigma}$ and for an $f \in \mathcal{O}(U)$, f belongs to the log structure if and only if, for any $\sigma \in \Sigma$, the pullback of f on U_σ belongs to the log structure of $E_{S_0,\sigma}$. Here we regard the log structure of $E_{S_0,\sigma}$ as a subsheaf of $\mathcal{O}$ of $E_{S_0,\sigma}$.

Proposition 5.2.6. (i) *$E_{S_0,\sigma}$ and $\Gamma\backslash D_{S_0,\Sigma}$ are objects of $\mathcal{B}(\log)$.*

(ii) *$E_{S_0,\sigma}$ is a $\sigma_{\mathbf{C}}$-torsor over $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_{S_0,\sigma}$ in $\mathcal{B}(\log)$.*

This will be proved in §5.4.

5.2.7. Note that, as a set, $\Gamma\backslash D_{S,\Sigma}$ is the fiber product of $S \to S_0 \leftarrow \Gamma\backslash D_{S_0,\Sigma}$.

We endow $\Gamma\backslash D_{S,\Sigma}$ with the structure as an object of $\mathcal{B}(\log)$ by regarding it as the fiber product of $S \to S_0 \leftarrow \Gamma\backslash D_{S_0,\Sigma}$ in $\mathcal{B}(\log)$. The following theorem will be proved in §5.4.

Theorem 5.2.8. (i) *$\Gamma\backslash D_{S,\Sigma}$ is a relative log manifold over S.*

(ii) *For $\sigma \in \Sigma$, the morphism $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_{S,\sigma} \to \Gamma\backslash D_{S,\Sigma}$ is locally an isomorphism in $\mathcal{B}(\log)$.*

(iii) *If S is Hausdorff, $\Gamma\backslash D_{S,\Sigma}$ is Hausdorff.*

5.2.9. We define $D^\sharp_{S,\Sigma}$ in the evident way, replacing nilpotent orbit by nilpotent i-orbit which is defined in the evident way. Then $\Gamma\backslash D^\sharp_{S,\Sigma}$ is identified with $(\Gamma\backslash D_{S,\Sigma})^{\log}$ via the natural map.

§5.3. Moduli

5.3.1. Let Σ be a weak fan in $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$, where $\sigma' = \mathrm{Hom}(P, \mathbf{R}^{\mathrm{add}}_{\geq 0})$.

Assume that we are given a subgroup Γ_u of $G_{\mathbf{Z},u}$ which satisfies the condition in 5.1.7 (1), and let Γ be the corresponding group as in 5.1.7. Assume that Σ and Γ are strongly compatible.

5.3.2. Let $\mathrm{LMH}^{(\Sigma)}_{Q,\Gamma}$ be the following functor on the category $\mathcal{B}(\log)/S$. For any object S' of $\mathcal{B}(\log)/S$, $\mathrm{LMH}^{(\Sigma)}_{Q,\Gamma}(S')$ is the set of all isomorphism classes of the following H.

H is an LMH on S' endowed with a Γ-level structure μ (5.1.9), whose gr^W is identified with $(H_{(w)})_w$ endowed with the given Γ'-level structure (5.1.3), satisfying the following condition.

For any $s \in S'$, any $t \in (S')^{\log}$ lying over s, any representative $\mu_t : H_{\mathbf{Z},t} \simeq H_0$ of μ, and for any specialization a at t (1.3.1), there exists $\sigma \in \Sigma$ such that the image of $\mathrm{Hom}((M_{S'}/\mathcal{O}_{S'}^\times)_s, \mathbf{N}) \to \sigma' \times \mathfrak{g}_{\mathbf{R}}$ is contained in σ and $\mu_t(F(a))$ generates a σ-nilpotent orbit (5.1.5). Here F denotes the Hodge filtration of H.

We have the canonical morphism $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)} \to \mathrm{LMH}_Q$.

In the case $\Gamma_u = G_{\mathbf{Z},u}$, $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)}$ is the following functor.

For any object S' of $\mathcal{B}(\log)/S$, $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)}(S')$ is the set of all isomorphism classes of the following H.

H is an LMH on S', whose gr^W is identified with $(H_{(w)})_w$, satisfying the following condition.

For any $s \in S'$, any $t \in (S')^{\log}$ lying over s, any isomorphism $\mu_t : H_{\mathbf{Z},t} \simeq H_0$ perserving W whose gr^W belongs to the given Γ'-level structure (5.1.3), and for any specialization a at t (1.3.1), there exists $\sigma \in \Sigma$ such that the image of $\mathrm{Hom}((M_{S'}/\mathcal{O}_{S'}^\times)_s, \mathbf{N}) \to \sigma' \times \mathfrak{g}_{\mathbf{R}}$ is contained in σ and $\mu_t(F(a))$ generates a σ-nilpotent orbit (5.1.5). Here F denotes the Hodge filtration of H.

From this, we see that in the case $\Gamma_u = G_{\mathbf{Z},u}$, $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)} \to \mathrm{LMH}_Q$ is injective and hence $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)}$ is regarded as a subfunctor of LMH_Q. We will denote this subfunctor by $\mathrm{LMH}_Q^{(\Sigma)}$. (This injectivity is explained also from the fact that, in the case $\Gamma_u = G_{\mathbf{Z},u}$, H with the given $\mathrm{gr}^W = (H_{(w)})_w$ is automatically endowed with a unique Γ-level structure.)

In the case $\Gamma_u = G_{\mathbf{Z},u}$, we denote $\Gamma \backslash D_{S,\Sigma}$ by

$$J_{S,\Sigma}, \quad \text{or simply by } \; J_\Sigma.$$

Theorem 5.3.3. *As an object of* $\mathcal{B}(\log)$ *over* S, $\Gamma \backslash D_{S,\Sigma}$ *represents* $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)}$.

This will be proved in §5.4.

§5.4. Proofs

The results in §5.2–5.3 are reduced to the ones in §2. We explain this reduction here.

Let the notation be as in §5.1–5.3.

5.4.1. We define a modification $\Lambda^e = (H_0^e, W^e, (\langle\ ,\ \rangle_w^e)_w, (h^{e,p,q})_{p,q})$ of $\Lambda = (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q}$ in 5.1.1 (1). Then, the results in §5.2–5.3 are reduced to the results in §2 for Λ^e.

Take an integer r such that $W_{2r} = 0$. Let $H_{0,(w)}^e$ be Γ' if $w = 2r - 2$, be $\mathbf{Z}$ if $w = 2r$, and be $H_{0,(w)}$ if $w \neq 2r - 2, 2r$. Let $H_0^e = \bigoplus_{w \in \mathbf{Z}} H_{0,(w)}^e = \Gamma' \oplus \mathbf{Z} \oplus H_0$. Let the finite increasing filtration W^e on $H_{0,\mathbf{R}}^e$ be the evident one.

Define the bilinear form $\langle\ ,\ \rangle^e_w : H^e_{0,(w),\mathbf{R}} \times H^e_{0,(w),\mathbf{R}} \to \mathbf{R}$ as follows. If $w \neq 2r-2, 2r$, it is just the bilinear form $\langle\ ,\ \rangle_w$ on $H_{0,(w)}$. If $w = 2r-2$, it is any fixed positive definite symmetric bilinear form $\mathbf{R} \otimes \Gamma' \times \mathbf{R} \otimes \Gamma' \to \mathbf{R}$ which is rational over $\mathbf{Q}$. If $w = 2r$, it is the form $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$, $(x, y) \mapsto xy$.

Let $h^{e,p,q}$ be $\operatorname{rank}(\Gamma')$ if $p = q = r-1$, be 1 if $p = q = r$, and be $h^{p,q}$ otherwise.

We denote by D^e the classifying space in §2 for $\Lambda^e = (H^e_0, W^e, (\langle\ ,\ \rangle^e_w)_w, (h^{e,p,q})_{p,q})$.

For a nilpotent cone σ in §5.1, we regard σ as a nilpotent cone in §2 for Λ^e in the following way. For $N \in \sigma$, we regard it as a linear map $H^e_{0,\mathbf{R}} \to H^e_{0,\mathbf{R}}$ whose restriction to $H_{0,\mathbf{R}}$ is $H_{0,\mathbf{R}} \overset{N_{\mathfrak{g}}}{\to} H_{0,\mathbf{R}} \subset H^e_{0,\mathbf{R}}$, whose restriction to $\mathbf{R} \otimes \Gamma' = H^e_{0,(2r-2),\mathbf{R}}$ is 0, and which sends $1 \in \mathbf{R} = H^e_{0,(2r)}$ to the image of N in $\mathbf{R} \otimes \Gamma' = H^e_{0,(2r-2),\mathbf{R}} \subset H^e_{0,\mathbf{R}}$ under $\sigma \to \sigma' \subset \sigma'_{\mathbf{R}} \simeq \mathbf{R} \otimes \Gamma'$ (5.1.1). (Note that, by 1.2.2.1, if the original nilpotent cone in the sense of §5.1 is admissible, then it is also admissible when regarded as a cone in the sense of §2.)

By this, a weak fan Σ in §5.1 is regarded as a weak fan in §2 for Λ^e.

We denote the objects E_σ, $G_{\mathbf{Z}}$ etc. of §2 for Λ^e, by E^e_σ, $G^e_{\mathbf{Z}}$, etc.

Γ in §5.1 is regarded as a subgroup of $G^e_{\mathbf{Z}}$ in the following way. We regard $(a, b) \in \Gamma$ ($a \in \Gamma'$, $b \in G_{\mathbf{Z}}$) as an element of $G^e_{\mathbf{Z}}$ whose restriction to H_0 is b, whose restriction to $\Gamma' = H^e_{0,(2r-2)}$ is the identity map, and which sends $1 \in \mathbf{Z} = H^e_{0,(2r)}$ to the sum of $1 \in H^e_{0,(2r)}$ and the element a of $\Gamma' = H^e_{0,(2r-2)}$. Σ and Γ are compatible (resp. strongly compatible) in the sense of §5.1 if and only if they are compatible (resp. strongly compatible) in the sense of §2 for Λ^e. The notation $\Gamma(\sigma)$ in §5.1 and that in §2 for Λ^e are compatible.

We define a morphism $\sigma'_{\mathbf{C}} \times \check{D} \to \check{D}^e$ as follows. An element $a \in \sigma'_{\mathbf{C}}$ corresponds to a decreasing filtration $f(a)$ on $V_{\mathbf{C}} := \mathbf{C} \otimes \Gamma' \oplus \mathbf{C} = H^e_{0,(2r-2),\mathbf{C}} \oplus H^e_{0,(2r),\mathbf{C}}$ given by

$$f(a)^{r+1} = 0 \subset f(a)^r = \mathbf{C} \cdot (a, 1) \subset f(a)^{r-1} = V_{\mathbf{C}},$$

where a in $(a, 1)$ is identified with an element of $\mathbf{C} \otimes \Gamma'$ via the canonical isomorphism $\sigma'_{\mathbf{C}} \simeq \mathbf{C} \otimes \Gamma'$. The map $\sigma'_{\mathbf{C}} \times \check{D} \to \check{D}^e$ sends an element (a, F) of $\sigma'_{\mathbf{C}} \times \check{D}$ to the element $f(a) \oplus F$ of $\check{D}^e$ defined as the direct sum of $f(a)$ and F on $V_{\mathbf{C}} \oplus H_{0,\mathbf{C}} = H^e_{0,\mathbf{C}}$.

Note that, via the above morphism, $\sigma'_{\mathbf{C}} \times \check{D}$ is a closed analytic submanifold of $\check{D}^e$.

Proposition 5.4.2. *Let $S_0 = E'_{\sigma'}$ as in* 5.1.1–5.1.3. *Then the following diagram is cartesian in* $\mathcal{B}(\log)$.

$$\begin{array}{ccc} E_{S_0,\sigma} & \longrightarrow & E^e_\sigma \\ \downarrow & & \downarrow \\ \sigma'_{\mathbf{C}} \times \check{D} & \longrightarrow & \check{D}^e. \end{array}$$

Here the lower horizontal arrow sends (z, F) ($z \in \sigma'_{\mathbf{C}}, F \in \check{D}$) to $f(z) \oplus F$ as in 5.4.1, *the left vertical arrow sends (s, z, q, F) to (z, F) ($s \in S_0, z \in \sigma'_{\mathbf{C}}$, $q \in \operatorname{toric}_\sigma$, $F \in \check{D}$), the right vertical arrow sends (q, F) to F ($q \in \operatorname{toric}_\sigma$, $F \in \check{D}^e$), and the upper horizontal arrow sends (s, z, q, F) to $(q, f(z) \oplus F)$.*

This is seen easily.

5.4.3. We give preliminaries to Propositions 5.4.4 and 5.4.5.

Let X, Y be topological spaces, let $r : X \to Y$ be a continuous map, and assume that, locally on Y, there is a continuous map $s : Y \to X$ such that $r \circ s$ is the identity map of Y. Let B be a subspace of Y and let $A = r^{-1}(B)$.

5.4.3.1. Endow A with the topology as a subspace of X. Then as is easily seen, the following two topologies on B coincide. The image of the topology of A under $r : A \to B$, and the inverse image of the topology of Y.

5.4.3.2. Assume furthermore that X (resp. Y) is endowed with a subsheaf $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) of rings over $\mathbf{C}$ of the sheaf of $\mathbf{C}$-valued continuous functions on X (resp. Y). Assume that the stalks of $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) are local rings. Assume that r respects these sheaves and assume that, locally on Y, we can take s as above which respects these sheaves.

Let $\mathcal{O}_A$ be the subsheaf of the sheaf of $\mathbf{C}$-valued continuous functions on A defined as follows. For a $\mathbf{C}$-valued continuous function f on an open set U of A, f belongs to $\mathcal{O}_A$ if and only if, for any $x \in U$, there are an open neighborhood U' of x in U, an open neighborhood V of x in X such that $U' \subset V$, and an element g of $\mathcal{O}_X(V)$ such that the restrictions of f and g to U' coincide.

Then it is easy to see the coincidence of the following two subsheaves (1) and (2) of the sheaf of $\mathbf{C}$-valued continuous functions on B: Let f be a $\mathbf{C}$-valued continuous function on an open set U of B. f belongs to the sheaf (1) if and only if $f \circ r$ belongs to $\mathcal{O}_A(r^{-1}(U))$. f belongs to the sheaf (2) if and only if, for any $y \in U$, there are an open neighborhood U' of y in U, an open neighborhood V of y in Y such that $U' \subset V$, and an element g of $\mathcal{O}_Y(V)$ such that the restrictions of f and g to U' coincide.

Let $\mathcal{O}_B$ be this sheaf on B.

5.4.3.3. Assume further that we have elements $f_1, \dots, f_n$ of $\mathcal{O}_X(X)$ such that $A = \{x \in X \mid f_1(x) = \dots = f_n(x) = 0\}$ and such that the kernel of $\mathcal{O}_X|_A \to \mathcal{O}_A$ is the ideal generated by $f_1, \dots, f_n$. Here $\mathcal{O}_X|_A$ is the sheaf-theoretic inverse image of $\mathcal{O}_X$ by $A \hookrightarrow X$. Then locally on Y, for s as above, $B = \{y \in Y \mid f_1 \circ s(y) = \dots = f_n \circ s(y) = 0\}$ and the kernel of $\mathcal{O}_Y|_B \to \mathcal{O}_B$ is the ideal generated by $f_1 \circ s, \dots, f_n \circ s$.

5.4.3.4. Assume further that Y is endowed with a log structure M_Y. Let M_X (resp. M_A) be the inverse image of M_Y on X (resp. A) and assume that the structural map $M_A \to \mathcal{O}_A$ of log structure is injective.

Then the following two log structures (1) and (2) on B coincide. (1) For an open set U of B, $M_B(U)$ is the subset of $\mathcal{O}_B(U)$ consisting of all elements whose pullback to $r^{-1}(U)$ belongs to the image of $M_A \to \mathcal{O}_A$. (2) The inverse image of M_Y on B.

5.4.3.5. Assume furthermore that X and Y belong to $\mathcal{B}(\log)$. Let H be a complex Lie group. Assume that an action of H on X in $\mathcal{B}(\log)$ is given, and assume that X is an H-torsor over Y in $\mathcal{B}(\log)$. Note that A belongs to $\mathcal{B}(\log)$. By the above, B also belongs to $\mathcal{B}(\log)$. Assume that H acts also on A in $\mathcal{B}(\log)$ and assume that this action is compatible with the action of H on X. Then A is an H-torsor over B in $\mathcal{B}(\log)$.

Proof of 5.4.3.5. Considering locally on B, we may assume that we have $s : Y \to X$ as above. Hence we have $H \times Y \to X$, $(h, y) \mapsto h \cdot s(y)$. Let $t : X \to H$ be the composition $X \simeq H \times Y \to H$. Then we have a morphism $A \to H \times B$, $a \mapsto (t(a), r(a))$ which is the inverse map of the morphism $H \times B \to A$, $(h, b) \mapsto h \cdot s(b)$. These morphisms are inverse to each other in $\mathcal{B}(\log)$. This proves that A is an H-torsor over B in $\mathcal{B}(\log)$. □

It follows that the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

in $\mathcal{B}(\log)$ is cartesian.

Proposition 5.4.4. (i) $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_{S_0,\sigma}$ *is a relative log manifold over* S_0*.*

(ii) $E_{S_0,\sigma}$ *is a relative log manifold over* S_0*.*

(iii) $E_{S_0,\sigma} \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_{S_0,\sigma}$ *is a* $\sigma_{\mathbf{C}}$*-torsor in the category* $\mathcal{B}(\log)$*.*

Proof. We apply 5.4.3 by taking

$$X = E_\sigma^e, \quad Y = \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma^e, \quad A = E_{S_0,\sigma}, \quad B = \Gamma(\sigma)^{\mathrm{gp}}\backslash D_{S_0,\sigma}.$$

By 5.4.3, $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_{S_0,\sigma} \in \mathcal{B}(\log)$ and we have (iii). By (iii), (i) and (ii) are equivalent. Hence it is sufficient to prove (ii). This is reduced to the fact that the morphism

$$\sigma'_{\mathbf{C}} \times \mathrm{toric}_\sigma \to \mathrm{toric}_{\sigma'}, \quad (z, q) \mapsto \mathbf{e}(z) \cdot q(\mathrm{gr}^W)$$

is log smooth. We prove this. Let $\mathcal{S} = \Gamma(\sigma)^\vee$, $\mathcal{S}' = \Gamma'(\sigma')^\vee$. Then, the last morphism is isomorphic to $\mathrm{Spec}(\mathbf{C}[\mathcal{S}'^{\mathrm{gp}} \times \mathcal{S}])_{\mathrm{an}} \to \mathrm{Spec}(\mathbf{C}[\mathcal{S}'])_{\mathrm{an}}$ induced by an injective homomorphism $h : \mathcal{S}' \to \mathcal{S}'^{\mathrm{gp}} \times \mathcal{S}$. Hence the morphism is log smooth. □

In the following, Σ is a weak fan in the sense of §5.1. Let $\sigma \in \Sigma$.

Proposition 5.4.5. (i) $\Gamma\backslash D_{S_0,\Sigma}$ *is a relative log manifold over* S_0*.*

(ii) *The morphism* $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_{S_0,\sigma} \to \Gamma\backslash D_{S_0,\Sigma}$ *in* $\mathcal{B}(\log)$ *is locally an isomorphism.*

(iii) $\Gamma\backslash D_{S_0,\Sigma}$ *is Hausdorff.*

Proof. By the next lemma, we can apply 5.4.3 by taking

$$X = \textstyle\coprod_{\sigma\in\Sigma} \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma^e, \quad Y = \Gamma\backslash D_\Sigma^e, \quad A = \textstyle\coprod_{\sigma\in\Sigma} \Gamma(\sigma)^{\mathrm{gp}}\backslash D_{S_0,\sigma}, \quad B = \Gamma\backslash D_{S_0,\Sigma}.$$

By 5.4.3.1–5.4.3.4, $\Gamma\backslash D_{S_0,\Sigma}$ belongs to $\mathcal{B}(\log)$ and we have (ii) by 2.5.4. (ii) and 5.4.4 (i) imply (i). (iii) follows from the fact that $\Gamma\backslash D_\Sigma^e$ is Hausdorff (2.5.5). □

Lemma 5.4.6. *For $\sigma \in \Sigma$, the following diagrams of sets are cartesian.*

$$\begin{array}{ccc} D_{S,\sigma} & \longrightarrow & D_{S,\Sigma} \\ \downarrow & & \downarrow \\ D^e_\sigma & \longrightarrow & D^e_\Sigma, \end{array} \qquad \begin{array}{ccc} \Gamma(\sigma)^{\mathrm{gp}}\backslash D_{S,\sigma} & \longrightarrow & \Gamma\backslash D_{S,\Sigma} \\ \downarrow & & \downarrow \\ \Gamma(\sigma)^{\mathrm{gp}}\backslash D^e_\sigma & \longrightarrow & \Gamma\backslash D^e_\Sigma. \end{array}$$

The proof of this lemma is easy.

We prove 5.2.8. Recall that we endowed $\Gamma\backslash D_{S,\Sigma}$ with the structure of an object of $\mathcal{B}(\log)$ by regarding it as the fiber product of $S \to S_0 \leftarrow \Gamma\backslash D_{S_0,\Sigma}$, where $S \to S_0$ is a strict morphism in $\mathcal{B}(\log)$ as in 5.1.1 (4). Hence Theorem 5.2.8 follows from Proposition 5.4.5.

5.4.7. We prove Theorem 5.3.3. By 2.6.6 for Λ^e, $\Gamma\backslash D^e_\Sigma$ represents the moduli functor LMH_{Φ^e} for $\Phi^e := (\Lambda^e, \Sigma, \Gamma)$. From 5.4.2, we have that the subspace $\Gamma\backslash D_{S_0,\Sigma}$ of $\Gamma\backslash D^e_\Sigma$ represents the subfunctor of LMH_{Φ^e} consisting of the classes of LMH which are the direct sum of an LMH H of weight $> 2r$ and an LMH h of weight $\leq 2r$. The projection $E_{S_0,\sigma} \to S_0 \subset \mathrm{toric}_{\sigma'} \times \check{D}(\mathrm{gr}^W)$ represents $h \oplus H \mapsto (h, H(\mathrm{gr}^W))$. Hence over S_0, $\Gamma\backslash D_{S_0,\Sigma}$ is the moduli space of this H with given $H(\mathrm{gr}^W)$, and hence represents $\mathrm{LMH}^{(\Sigma)}_{Q,\Gamma}$. □

§5.5. Relation with §2

5.5.1. We explain the relation between the method in §2 and the method in §5, starting from the former. Let the situation be as in §2.

Let Σ be a weak fan, and let Γ be a neat subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Σ.

Let σ_w be a sharp rational nilpotent cone in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_w)$ for each w. Assume that, for any $\sigma \in \Sigma$ and for any w, the image of σ in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_w)$ is contained in σ_w.

Let Γ_w be a neat subgroup of $G_{\mathbf{Z}}(\mathrm{gr}^W_w)$ for each w. Assume that for each w, the image of Γ in $G_{\mathbf{Z}}(\mathrm{gr}^W_w)$ is contained in Γ_w.

Assume that the fan face(σ_w) and Γ_w are strongly compatible.

Let S be an object of $\mathcal{B}(\log)$. Assume that we are given a morphism $S \to \Gamma_w\backslash D(\mathrm{gr}^W_w)_{\sigma_w}$ for each w. As in the papers [KNU10a] and [KNU10c], we consider the fiber product of

$$S \to \prod_w \Gamma_w\backslash D(\mathrm{gr}^W_w)_{\sigma_w} \leftarrow \Gamma\backslash D_\Sigma.$$

Assume that $S \to \prod_w \Gamma_w\backslash D(\mathrm{gr}^W_w)_{\sigma_w}$ is strict. Assume also that the homomorphism $\Gamma \to \prod_w \Gamma_w$ is surjective. Then, from this fiber product, the space in this section is constructed as follows.

Let $H_{(w)}$ be the pullback on S of the universal polarized log Hodge structure of weight w on $\Gamma_w\backslash D(\mathrm{gr}^W_w)_{\sigma_w}$, and let $Q = (H_{(w)})_w$.

Locally on S, the morphism $S \to \Gamma_w\backslash D(\mathrm{gr}^W_w)_{\sigma_w}$ factors as $S \to E(\mathrm{gr}^W_w)_{\sigma_w} \to \Gamma_w\backslash D(\mathrm{gr}^W_w)_{\sigma_w}$. We have the situation of 5.1.1 by taking $H_{0,(w)} = H_0(\mathrm{gr}^W_w)$ and P the dual fs monoid of $\prod_w \Gamma_w(\sigma_w)$. We have $\sigma' = \prod_w \sigma_w$, $\Gamma' = \prod_w \Gamma_w(\sigma_w)^{\mathrm{gp}}$, and Σ can be regarded as a weak fan in the sense of 5.1.6 by identifying $\sigma \in \Sigma$ with $\{(N_{\sigma'}, N) \mid N \in \sigma\}$, where $N_{\sigma'}$ is the image of N in σ'.

Let $\tilde{\Gamma} \subset \Gamma$ be the inverse image of $\Gamma' \subset \prod_w \Gamma_w$.

Proposition 5.5.2. *We have a canonical isomorphism*

$$\tilde{\Gamma}\backslash D_{S,\Sigma} \simeq S \times_{\prod_w \Gamma_w\backslash D(\mathrm{gr}^W_w)_{\sigma_w}} \Gamma\backslash D_\Sigma$$

in the category $\mathcal{B}(\log)/S$.

Proof. We have a canonical map $D_{S,\Sigma} \to D_\Sigma$ which sends (s, σ, Z) to $(\sigma, \exp(\sigma_{\mathbf{C}})F)$, where F is any element of $\check{D}$ such that $(a, F) \in Z$ for some $a \in \sigma'_{\mathbf{C}}$.

The induced map $\tilde{\Gamma}\backslash D_{S,\Sigma} \to Y := S \times_{\prod_w \Gamma_w\backslash D(\mathrm{gr}^W_w)_{\sigma_w}} \Gamma\backslash D_\Sigma$ is bijective. In fact, it is easy to see the surjectivity, and the injectivity is reduced to the following Claim 1.

Claim 1. Assume $(s, \sigma, Z_j) \in D_{S,\Sigma}$ and $(a_j, F) \in Z_j$ for $j = 1, 2$, with $s \in S$, $\sigma \in \Sigma$, $Z_j \subset \sigma'_{\mathbf{C}} \times \check{D}$, $a_j \in \sigma'_{\mathbf{C}} = \prod_w \sigma_{w,\mathbf{C}}$, $F \in \check{D}$. Then $a_1 = a_2$ (hence $Z_1 = Z_2$).

We prove Claim 1. By the definition of $D_{S,\Sigma}$, for $j = 1, 2$, the image of s in $S_0 = E'_{\sigma'}$ is $(\mathbf{e}(a_j)\cdot 0_\sigma(\mathrm{gr}^W), \exp(-a_j)F(\mathrm{gr}^W))$. Hence $\exp(-a_1)F(\mathrm{gr}^W) = \exp(-a_2)F(\mathrm{gr}^W)$. Since $(\sigma', F(\mathrm{gr}^W))$ generates a nilpotent orbit, this implies $a_1 = a_2$ by [KU09] Proposition 7.2.9 (i).

We will prove

Claim 2. In the case $S = S_0 = \prod_w E(\mathrm{gr}^W_w)_{\sigma_w}$, the above bijection $\tilde{\Gamma}\backslash D_{S,\Sigma} \to Y$ is in fact an isomorphism $\tilde{\Gamma}\backslash D_{S_0,\Sigma} \xrightarrow{\sim} Y$ in $\mathcal{B}(\log)$.

By the base change by $S \to S_0$, the isomorphism in Claim 2 induces an isomorphism $\tilde{\Gamma}\backslash D_{S,\Sigma} \xrightarrow{\sim} Y$ in $\mathcal{B}(\log)/S$ in general.

We prove Claim 2. By Theorem 2.5.4 and Theorem 5.2.8 (ii), it is sufficient to prove that for each $\sigma \in \Sigma$, the morphism of $\mathcal{B}(\log)$ from $\tilde{\Gamma}(\sigma)^{\mathrm{gp}}\backslash D_{S_0,\sigma}$ to the fiber product of

$$S_0 \to \prod_w \Gamma_w(\sigma_w)^{\mathrm{gp}}\backslash D(\mathrm{gr}^W_w)_{\sigma_w} \leftarrow \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$$

is locally an isomorphism. By Theorem 2.5.3, $S_0 = E'_{\sigma'} \to \prod_w \Gamma_w(\sigma_w)^{\mathrm{gp}}\backslash D(\mathrm{gr}^W_w)_{\sigma_w}$ is a $\sigma'_{\mathbf{C}}$-torsor and E_σ in §2 is a $\sigma_{\mathbf{C}}$-torsor over $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$. Furthermore, $E_{S_0,\sigma}$ is a $\sigma_{\mathbf{C}}$-torsor over $\tilde{\Gamma}(\sigma)^{\mathrm{gp}}\backslash D_{S_0,\sigma}$ by Proposition 5.2.6 (ii). Hence Claim 2 is reduced to the evident fact that the canonical morphism $E_{S_0,\sigma} \to \sigma'_{\mathbf{C}} \times E_\sigma$ is an isomorphism. □

5.5.3. Thus, to construct the moduli space of LMH with given graded quotients of the weight filtration, there are two methods. One is the method in [KNU10a] and [KNU10c] to take the fiber product as in the above 5.5.1, and the other is the method of this §5. As advantages of the method of §5, we have:

(1) As is shown in §6, the Néron model and the connected Néron model can be constructed by the method of §5 even in the case of higher dimensional base S. In the method of §2, to obtain such models, the period map $S \to \Gamma'\backslash D_{\Sigma'}$ of the gr^W is necessary to define the fiber product. But, since such a period map of the gr^W can exist in general only after some blowing-up (see §4.3 of [KU09]), we may have to blow up the base S in the method of §2.

(2) The case with no polarization on gr^W can be sometimes treated. (See 7.1.4 for example.)

§6. Néron models

In this section, we construct the Néron model and the connected Néron model in degeneration of Hodge structures. In the case of Hodge structures corresponding to abelian varieties of semi-stable reductions, these coincide with the classical Néron model and the connected Néron model in the theory of degeneration of abelian varieties. We expect that the constructions of such models are useful in the study of degeneration of intermediate Jacobians. In the case of degeneration of intermediate Jacobians of Griffiths [G68a] and [G68b], our connected Néron model coincides with the model of Zucker [Z76] (which need not be Hausdorff) modified by putting slits (then we have a Hausdorff space), and under certain assumptions, our Néron model coincides with the models constructed by Clemens [C83] and by Green-Griffiths-Kerr ([GGK10]), respectively. See 6.3.5 for further relationships with other works.

In §6.4, we give the proofs of the results in [KNU10c] which were omitted there.

§6.1. Results on Néron models and connected Néron models

In §6.1–§6.3, as in §5, let S be an object of $\mathcal{B}(\log)$, and assume that for each $w \in \mathbf{Z}$, we are given a polarized log Hodge structure $H_{(w)}$ of weight w on S. Assume $H_{(w)} = 0$ for almost all w. Let S° be S regarded as an object of $\mathcal{B}$ by forgetting the log structure.

In this section, we prove the following Theorem 6.1.1: the existence of the connected Néron model (resp. Néron model) which represents the functor of isomorphism classes of LMH whose $\mathbf{Z}$-structure $H_{\mathbf{Z}}$ splits (resp. whose $\mathbf{Q}$-structure $H_{\mathbf{Q}} := H_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q}$ splits).

Theorem 6.1.1. *There is a relative log manifold J_1 (resp. J_0) over S which is strict over S and which represents the following functor on $\mathcal{B}/S^\circ$:*

$\mathcal{B}/S^\circ \ni S' \mapsto$ *the set of all isomorphism classes of LMH H on S' with $H(\mathrm{gr}^W_w) = H_{(w)}$ $(w \in \mathbf{Z})$ satisfying the following condition: Locally on S', there is a splitting $H_{\mathbf{Q}}(\mathrm{gr}^W) \simeq H_{\mathbf{Q}}$ (resp. $H_{\mathbf{Z}}(\mathrm{gr}^W) \simeq H_{\mathbf{Z}}$) of the weight filtration W on the local system $H_{\mathbf{Q}}$ (resp. $H_{\mathbf{Z}}$) on $(S')^{\log}$.*

In the description of the functor in 6.1.1, $S' \in \mathcal{B}/S^\circ$ is endowed with the inverse image of the log structure of S.

The proof of Theorem 6.1.1 is given in §6.2.

6.1.2. We call the relative log manifold J_1 the *Néron model*, and J_0 the *connected Néron model*.

In the case of degeneration of abelian varieties, these terminologies agree with the classical ones (see 6.1.7).

6.1.3. Let $w \in \mathbf{Z}$, $w < 0$. Assume $H_{(k)} = 0$ for $k \neq w, 0$, and that $H_{(0)} = \mathbf{Z}$. Let $H' = H_{(w)}$. Consider the exact sequence on $S^{\log}$

$$0 \to H'_{\mathbf{Z}} \to H'_{\mathcal{O}^{\log}}/F^0 \to H'_{\mathbf{Z}} \backslash H'_{\mathcal{O}^{\log}}/F^0 \to 0.$$

Descending this by $\tau : S^{\log} \to S$, we have an exact sequence of sheaves of abelian groups on $\mathcal{B}(\log)/S$

$$0 \to \tau_* H'_{\mathbf{Z}} \backslash (H'_{\mathcal{O}}/F^0 H'_{\mathcal{O}})^{sG} \to \mathcal{E}xt^1(\mathbf{Z}, H') \to R^1\tau_* H'_{\mathbf{Z}}.$$

Here $(\ )^{sG}$ is the small Griffiths part, i.e., the part restricted by the Griffiths transversality after pulling-back to each point $s \in S$ (cf. [KU09] 2.4.9), and $\mathcal{E}xt^1(\mathbf{Z}, H') := \mathcal{E}xt^1_{\mathrm{LMH}/S}(\mathbf{Z}, H')$ is the Ext-sheaf of the category of LMH which is identified with the subgroup of $\tau_*(H'_{\mathbf{Z}}\backslash H'_{\mathcal{O}^{\log}}/F^0)^{sG}$ restricted by the condition of the admissibility of local monodromy.

Proposition 6.1.4. *In the situation of* 6.1.3:

(i) *On $\mathcal{B}/S^\circ$, the Néron model represents*

$$\mathrm{Ker}(\mathcal{E}xt^1(\mathbf{Z}, H') \to R^1\tau_*H'_{\mathbf{Z}} \to R^1\tau_*H'_{\mathbf{Q}}).$$

(ii) *On $\mathcal{B}/S^\circ$, the connected Néron model represents*

$$\mathrm{Ker}(\mathcal{E}xt^1(\mathbf{Z}, H') \to R^1\tau_*H'_{\mathbf{Z}}) = \tau_*H'_{\mathbf{Z}}\backslash(H'_{\mathcal{O}}/F^0H'_{\mathcal{O}})^{sG}.$$

The proof is given in §6.3.

6.1.5. *Remark 1.* In the situation of 6.1.3, Zucker ([Z76]) constructed a model, which is $\tau_*H'_{\mathbf{Z}}\backslash H'_{\mathcal{O}}/F^0H'_{\mathcal{O}}$ in the notation above. Hence, 6.1.4 (ii) shows that in the situation of 6.1.3, the connected Néron model is the Zucker model modified by putting slits induced by the small Griffiths transversality. Cf. [Sa.p], [SS11].

Remark 2. Proposition 6.1.4 shows that in the situation of 6.1.3, the Néron model and the connected Néron model are commutative group objects over S which represent subgroup functors of $\mathcal{E}xt^1(\mathbf{Z}, H')$ on $\mathcal{B}/S^\circ$.

For an object S of $\mathcal{B}(\log)$ and for an integer r, we say the log rank of S is $\le r$ if $\mathrm{rank}_{\mathbf{Z}}((M^{\mathrm{gp}}_S/\mathcal{O}^\times_S)_s) \le r$ for any $s \in S$. As an example which appears in 6.1.8 below, if S is a complex analytic manifold with a smooth divisor which gives the log structure of S, then the log rank of S is ≤ 1.

Corollary 6.1.6. *In the situation of* 6.1.3, *assume furthermore $w = -1$ and the log rank of S is ≤ 1. Then, on $\mathcal{B}/S^\circ$, the Néron model represents $\mathcal{E}xt^1(\mathbf{Z}, H')$.*

Proof. It is sufficient to prove that under the assumption of 6.1.6, W on the local system $H_{\mathbf{Q}}$ on $S^{\log}$ of any LMH H splits locally on S. We prove this.

Consider first the case where S is a point $s = \mathrm{Spec}(\mathbf{C})$ with an fs log structure. We may assume that the log structure of s is non-trivial. Let $t \in s^{\log}$. Let γ be a generator of $\pi_1(s^{\log})$ and let $N = \log(\gamma) : H_{\mathbf{Q},t} \to H_{\mathbf{Q},t}$. Let e be an element of $M(N, W)_0H_{\mathbf{Q},t}$ which lifts $1 \in \mathbf{Z} = \mathrm{gr}^W_0 H_{\mathbf{Z},t}$. Then $N(e) \in M(N, W)_{-2}\,\mathrm{gr}^W_{-1} H_{\mathbf{Q},t}$. By the admissibility of local monodromy, we have

$$M(N, W)_{-2}\,\mathrm{gr}^W_{-1} H_{\mathbf{Q},t} \subset \mathrm{Image}(N : \mathrm{gr}^W_{-1} H_{\mathbf{Q},t} \to \mathrm{gr}^W_{-1} H_{\mathbf{Q},t}).$$

Hence there is $v \in \mathrm{gr}^W_{-1} H_{\mathbf{Q},t}$ such that $N(e) = N(v)$. Replacing e by $e - v$, we find e such that $N(e) = 0$. This e gives a splitting of W on $H_{\mathbf{Q}}$.

Now we consider general S. Let $\mathcal{F}$ be the sheaf on $S^{\log}$ of splittings of W of $H_{\mathbf{Q}}$. By the proper base change theorem for the proper map $\tau : S^{\log} \to S$, we have for $s \in S$

$$\tau_*(\mathcal{F})_s \simeq \Gamma(\tau^{-1}(s), \mathcal{F}|_{\tau^{-1}(s)}).$$

As we have seen, $\Gamma(\tau^{-1}(s), \mathcal{F}|_{\tau^{-1}(s)})$ is non-empty. Hence $\tau_*(\mathcal{F})_s$ is non-empty. Thus a splitting of W on $H_{\mathbf{Q}}$ exists on an open neighborhood of s in S. □

6.1.7. We consider the relation with the classical Néron models of abelian varieties. Let S be an algebraic curve over $\mathbf{C}$, and let B be an abelian variety over the function field of S which has good reductions at any point of an open set $U =: S \smallsetminus I$ and semi-stable reduction at any point in I. Let H'_U be the variation $\{H^1(B_s)(1)\}_{s\in U}$ of Hodge structure of weight -1 on U associated to B. Over U, B is identified with $H'_{\mathbf{Z}}\backslash H'_{\mathcal{O}}/F^0H'_{\mathcal{O}}$. Since B is polarizable, H'_U is polarizable. Furthermore, by the assumption of semi-stability, H'_U has unipotent local monodromy at any point of I. Hence H'_U extends to a polarizable log Hodge structure H' of weight -1 on S endowed with the log structure defined by I by the nilpotent orbit theorem of Schmid ([Scm73]) (which is interpreted in terms of log Hodge structures as in [KMN02] Proposition 2.5, [KU09] 2.5.14).

By 6.1.3–6.1.6, our connected Néron model represents $\tau_*H'_{\mathbf{Z}}\backslash H'_{\mathcal{O}}/F^0H'_{\mathcal{O}}$ (the small Griffiths transversality is automatically satisfied), our Néron model represents $\mathcal{E}xt^1(\mathbf{Z}, H') = \mathrm{Ker}(\tau_*(H'_{\mathbf{Z}}\backslash H'_{\mathcal{O}^{\log}}/F^0) \to R^1\tau_*H'_{\mathbf{Q}})$, and we have an exact sequence

$$0 \to \tau_*(H'_{\mathbf{Z}})\backslash H'_{\mathcal{O}}/F^0H'_{\mathcal{O}} \to \mathcal{E}xt^1(\mathbf{Z}, H') \to R^1\tau_*(H'_{\mathbf{Z}})_{\mathrm{tor}} \to 0,$$

where "tor" denotes the torsion part. These tell that our Néron model (resp. our connected Néron model) coincides with the classical Néron model (resp. classical connected Néron model) of B over S.

Corollary 6.1.8. *(Cf.* [GGK10]*.) In the situation of* 6.1.6*, assume that S is a complex analytic manifold with a smooth divisor which gives the log structure of S. Then we have*

$$\begin{aligned} &(\textit{a section of the Néron model over } S) \\ =\ &(\textit{a normal function on } S^* \textit{ which is admissible with respect to } S). \end{aligned}$$

Here S^* is the complement of the divisor in S.

(Cf. [Sa96] for admissible normal functions.)

§6.2. Moduli spaces of LMH with Q-splitting local monodromy

In this subsection, we prove Theorem 6.1.1.

Let S and $(H_{(w)})_w$ be as in §6.1. Since we may work locally on S, we may and do assume that we are given the data (1)–(5) in 5.1.1, by 5.1.2.

Let $\Gamma_u = G_{\mathbf{Z},u}$ and let $\Gamma = \Gamma' \times_{G'_{\mathbf{Z}}} G_{\mathbf{Z}}$ be the corresponding group (5.1.7).

We construct certain weak fans Σ_1 and Σ_0 in the sense of §5 explicitly, and show that the Néron model J_1 in Theorem 6.1.1 is J_{Σ_1} and the connected Néron model J_0 is J_{Σ_0} (5.3.2).

Theorem 6.2.1. *Let $\sigma' = \mathrm{Hom}(P, \mathbf{R}^{\mathrm{add}}_{\geq 0}) \to \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \to \mathfrak{g}_{\mathbf{R}}$, $a \mapsto a_{\mathfrak{g}}$, be the composite map as in §5. Let Υ be a subgroup of $G_{\mathbf{Q},u}$. Let $\Sigma(\Upsilon)$ be the set of cones $\sigma_{\tau',\upsilon} := \{(x, \mathrm{Ad}(\upsilon)x_{\mathfrak{g}}) \mid x \in \tau'\} \subset \sigma' \times \mathfrak{g}_{\mathbf{R}}$, where τ' ranges over all faces of σ' and υ ranges over all elements of Υ. Then $\Sigma(\Upsilon)$ is a weak fan.*

Proof. By construction, it is easy to see the rationality and the closedness under the operation of taking a face. We examine the condition (1) in 5.1.6.

Let $\tau', \rho' \in \mathrm{face}(\sigma')$ and let $\tau_{\mathfrak{g}}, \rho_{\mathfrak{g}}$ be their images in $\mathfrak{g}_{\mathbf{R}}$, respectively. Let $\upsilon \in \Upsilon$ and let $\rho_{\mathfrak{g},\upsilon} = \mathrm{Ad}(\upsilon)(\rho_{\mathfrak{g}})$. Note that $\rho_{\mathfrak{g},\upsilon}$ coincides with the image of $\sigma_{\rho',\upsilon}$ in $\mathfrak{g}_{\mathbf{R}}$.

Assume that N is in the interior of $\sigma_{\tau',1}$ and of $\sigma_{\rho',\upsilon}$. Let $F \in \check{D}$. Assume that $(\sigma_{\tau',1}, F)$ and $(\sigma_{\rho',\upsilon}, F)$ generate nilpotent orbits.

We first claim that $N_{\mathfrak{g}}\upsilon = \upsilon N_{\mathfrak{g}}$. In fact, $\mathrm{gr}^W(N_{\mathfrak{g}}) = \mathrm{gr}^W(\mathrm{Ad}(\upsilon)^{-1}N_{\mathfrak{g}})$ because $\upsilon \in \Upsilon \subset G_{\mathbf{Q},u}$. On the other hand, $N_{\mathfrak{g}} \in \tau_{\mathfrak{g}}$ and $\mathrm{Ad}(\upsilon)^{-1}N_{\mathfrak{g}} \in \rho_{\mathfrak{g}}$ are pure of weight 0 with respect to W under the fixed splitting of W. Hence $N_{\mathfrak{g}} = \mathrm{Ad}(\upsilon)^{-1}N_{\mathfrak{g}}$, that is, $N_{\mathfrak{g}}\upsilon = \upsilon N_{\mathfrak{g}}$.

Since $\mathrm{face}(\sigma')$ is a fan and the image of N in σ' is in the interior of τ' and of ρ', we see $\tau' = \rho'$.

Since $(\sigma_{\tau',1}, F)$ generates a nilpotent orbit, the action of $\tau_{\mathfrak{g}}$ on $H_{0,\mathbf{R}}$ is admissible with respect to W. By 2.2.8, the adjoint action of $\tau_{\mathfrak{g}}$ on $\mathfrak{g}_{\mathbf{R}}$ is admissible with respect to the filtration $W\mathfrak{g}_{\mathbf{R}}$ on $\mathfrak{g}_{\mathbf{R}}$ induced by W.

Let $M = M(\mathrm{Ad}(N_{\mathfrak{g}}), W\mathfrak{g}_{\mathbf{R}})$. Since $\upsilon - 1 \in (W\mathfrak{g}_{\mathbf{R}})_{-1}$ and $\mathrm{Ad}(N_{\mathfrak{g}})(\upsilon - 1) = 0$, we see $\upsilon - 1 \in M_{-1}$ by 1.2.1.3.

Let $h \in \tau_{\mathfrak{g}}$. Then $\mathrm{Ad}(h)M_{-1} \subset M_{-3}$ by definition of M. Hence, by the above result, $h\upsilon - \upsilon h = h(\upsilon - 1) - (\upsilon - 1)h \in M_{-3}$. Applying υ^{-1} from the right, we have $h - \upsilon h\upsilon^{-1} \in M_{-3}$. Since $(\tau_{\mathfrak{g}}, F)$ and $(\tau_{\mathfrak{g},\upsilon} := \mathrm{Ad}(\upsilon)(\tau_{\mathfrak{g}}), F)$ generate nilpotent orbits in the sense of §2 (cf. 5.1.5), we have $h \in F^{-1}\mathfrak{g}_{\mathbf{C}}$ and $\upsilon h\upsilon^{-1} \in F^{-1}\mathfrak{g}_{\mathbf{C}}$, respectively (Griffiths transversality). Hence $h - \upsilon h\upsilon^{-1} \in F^{-1}\mathfrak{g}_{\mathbf{C}}$, and hence $h - \upsilon h\upsilon^{-1} \in M_{-3} \cap F^{-1}\mathfrak{g}_{\mathbf{C}} \cap \bar{F}^{-1}\mathfrak{g}_{\mathbf{C}} = 0$. Here we use the fact that $(M, F\mathfrak{g}_{\mathbf{C}})$ is an $\mathbf{R}$-mixed Hodge structure. Thus we have $\tau_{\mathfrak{g}} = \tau_{\mathfrak{g},\upsilon} = \rho_{\mathfrak{g},\upsilon}$. Here the second equality follows from $\tau' = \rho'$. Hence $\sigma_{\tau',1} = \sigma_{\rho',\upsilon}$ and the condition (1) in 5.1.6 is verified. □

[KNU10c] §2 contains an analogous result (ibid. Theorem 2.1) and its proof by the method of §2 in this paper. Theorem 6.2.1 above is its generalization by the method of §5.

6.2.2. Recall that $\Gamma_u = G_{\mathbf{Z},u}$. So Γ is a semi-direct product of Γ' in 5.1.1 (3) and $G_{\mathbf{Z},u}$.

Take a subgroup Υ of $G_{\mathbf{Q},u}$. Assume

(1) $G_{\mathbf{Z},u} \subset \Upsilon$.

(2) $\mathrm{Int}(\gamma)\Upsilon = \Upsilon$ for any $\gamma \in \Gamma_G := \mathrm{Image}(\Gamma \to G_{\mathbf{Z}})$.

$G_{\mathbf{Z},u}$ and $G_{\mathbf{Q},u}$ are examples of Υ which satisfy (1) and (2).

Lemma 6.2.3. *Under the assumption on Υ in 6.2.2, $\Sigma(\Upsilon)$ is strongly compatible with Γ.*

Proof. By Remark in 5.1.8, it is enough to show that $\Sigma := \Sigma(\Upsilon)$ is compatible with Γ. Let $\tau' \in \mathrm{face}(\sigma')$ and $\upsilon \in \Upsilon$. Let $\tau_1 := \sigma_{\tau',\upsilon}$ (6.2.1). Fix $\gamma \in \Gamma$. We have to show $\mathrm{Ad}(\gamma)(\tau_1) \in \Sigma$. Since Γ is a semi-direct product of Γ' and $G_{\mathbf{Z},u}$ (6.2.2), and since $G_{\mathbf{Z},u}$ is contained in Υ by the property (1) of Υ, we may assume that $\gamma \in \Gamma'$. By the property (2) of Υ, there exists $\upsilon' \in \Upsilon$ such that $\gamma_{G'}\upsilon = \upsilon'\gamma_{G'}$. Then, $\mathrm{Ad}(\gamma)(\tau_1) = \mathrm{Ad}(\upsilon')(\sigma_{\tau',\gamma})$, where $\sigma_{\tau',\gamma} := \{(x, \mathrm{Ad}(\gamma_{G'})x_{\mathfrak{g}}) \mid x \in \tau'\}$. Hence it is enough to show that $\sigma_{\tau',\gamma}$ belongs to Σ. But, $\sigma_{\tau',\gamma} = \sigma_{\tau',1}$, and this certainly belongs to Σ. The compatibility of Σ and Γ is proved. □

As in 5.3.2, for a weak fan Σ which is strongly compatible with Γ, let $J_\Sigma = \Gamma\backslash D_{S,\Sigma}$.

Lemma 6.2.4. *Let τ' be a face of σ' and let $\upsilon \in G_{\mathbf{Q},u}$. Let $\Gamma'(\tau')$ be the face of* $\mathrm{Hom}(P, \mathbf{N})$ *corresponding to τ'. Then, the canonical map $\Gamma(\sigma_{\tau',\upsilon}) \to \Gamma'(\tau')$ is of Kummer type, that is, it is injective and some power of any element of $\Gamma'(\tau')$ belongs to its image.*

Proof. First, let γ be an element of $\Gamma(\sigma_{\tau',\upsilon})$. Let γ_G be the image of γ in $G_{\mathbf{Z}}$. Let γ' be the image of γ in $\Gamma'(\tau')$. Let γ'_G be the image of γ' in $G'_{\mathbf{Z}}$. γ'_G is identified with its trivial extension in $G_{\mathbf{Z}}$. Since $\upsilon^{-1}\gamma_G\upsilon$ is also the trivial extension of γ'_G, it coincides with γ'_G. Hence $\gamma_G = \upsilon\gamma'_G\upsilon^{-1}$. It implies that γ_G is recovered from γ'_G, and hence γ is recovered by γ'. This means that the canonical map $\Gamma(\sigma_{\tau',\upsilon}) \to \Gamma'(\tau')$ is injective.

Next, let γ' be an element of $\Gamma'(\tau')$. Let γ'_G be the image of γ' in $G'_{\mathbf{Z}} \subset G_{\mathbf{Z}}$. Then, γ' belongs to the image of the above map if and only if $\upsilon\gamma'_G\upsilon^{-1}$ belongs to $\Gamma_G = \mathrm{Image}(\Gamma \to G_{\mathbf{Z}})$, which is equivalent to that $\upsilon\gamma'_G\upsilon^{-1}$ belongs to $G_{\mathbf{Z}}$. Hence, to prove that the canonical map is of Kummer type, it suffices to show that there exists $n \geq 1$ such that $\upsilon(\gamma'_G)^n\upsilon^{-1}$ belongs to $G_{\mathbf{Z}}$ for any $\gamma' \in \Gamma'(\tau')$. But, we can take an n such that $n\,\mathrm{Ad}(\upsilon)\log(\gamma'_G)$ sends H_0 into H_0 for any $\gamma' \in \Gamma'(\tau')$. Multiplying n if necessary, we may assume further that $\mathrm{Int}(\upsilon)((\gamma'_G)^n)$ sends H_0 into H_0 for any $\gamma' \in \Gamma'(\tau')$. This n satisfies the desired condition. □

As in the statement of 6.2.4, we say a homomorphism $h : P \to Q$ of fs monoids is *of Kummer type* if it is injective and, for any $q \in Q$, there exists an $n \geq 1$ such that q^n belongs to the image of h. We say a morphism $f : X \to Y$ in $\mathcal{B}(\log)$ is *of Kummer type* if, for any $x \in X$, the homomorphism $(M_Y/\mathcal{O}_Y^\times)_y \to (M_X/\mathcal{O}_X^\times)_x$ of fs monoids is of Kummer type, where $y = f(x)$.

Proposition 6.2.5. (*Characterization of J_Σ for $\Sigma = \Sigma(\Upsilon)$ with Υ as in* 6.2.2.)

(i) *J_Σ is of Kummer type over S.*

(ii) *For S' of Kummer type over S, a morphism $S' \to J_\Sigma$ over S corresponds in one-to-one manner to an isomorphism class of LMH H over S' with $H(\mathrm{gr}^W_w) = H_{(w)}$ ($w \in \mathbf{Z}$) such that, locally on S', there is a splitting $H_{\mathbf{Q}}(\mathrm{gr}^W) \simeq H_{\mathbf{Q}}$ of the filtration W on the local system $H_{\mathbf{Q}}$ on $(S')^{\log}$ satisfying the following condition*: *For any local isomorphism $H_{\mathbf{Z}}(\mathrm{gr}^W) \simeq H_0(\mathrm{gr}^W)$ on $(S')^{\log}$ which belongs to the given Γ'-level structure on $H_{\mathbf{Z}}(\mathrm{gr}^W)$, the composition $H_{\mathbf{Q}} \simeq H_{\mathbf{Q}}(\mathrm{gr}^W) \simeq H_{0,\mathbf{Q}}(\mathrm{gr}^W) = H_{0,\mathbf{Q}}$ sends $H_{\mathbf{Z}}$ onto υH_0 for some $\upsilon \in \Upsilon$.*

Proof. (i) is by 6.2.4. We prove (ii). Since Γ is a semi-direct product of Γ' and $G_{\mathbf{Z},u}$ (6.2.2), any $H \in \mathrm{LMH}_Q(S')$ has automatically a unique Γ-level structure whose gr^W belongs to the given Γ'-level structure. For $t \in (S')^{\log}$, take a representative $\mu_t : H_{\mathbf{Z},t} \xrightarrow{\sim} H_0$ of such Γ-level structure, and put $\mu'_t := \mu_t(\mathrm{gr}^W)$.

The condition that H belongs to $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)}(S')$ (5.3.2), which J_Σ represents by 5.3.3, is equivalent to that, for any $s \in S'$ and any $t \in \tau^{-1}(s) \subset (S')^{\log}$, there exist a face ρ' of σ' and $\upsilon \in \Upsilon$ such that the image of the homomorphism

$$f : \pi_1^+(s^{\log}) \to \sigma' \times \mathfrak{g}_{\mathbf{R}}$$

in 5.3.2 is contained in $\sigma_{\rho',\upsilon}$. Here $\pi_1^+(s^{\log}) := \mathrm{Hom}((M_{S'}/\mathcal{O}_{S'}^\times)_s, \mathbf{N}) \subset \pi_1(s^{\log})$. If this is the case, $a := \mu_t^{-1}\upsilon^{-1}\mu'_t : H_{\mathbf{Q},t}(\mathrm{gr}^W) \xrightarrow{\sim} H_{\mathbf{Q},t}$ extends to a local section over the inverse image by $\tau : (S')^{\log} \to S'$ of some open neighborhood of s in S'. Since the composition $\mu'_t a^{-1} = \upsilon\mu_t$ sends $H_{\mathbf{Z},t}$ onto υH_0, this a satisfies the condition in (ii).

Conversely, if there is such a splitting $a : H_{\mathbf{Q}}(\mathrm{gr}^W) \xrightarrow{\sim} H_{\mathbf{Q}}$ satisfying the condition in (ii), the homomorphism $\pi_1^+(s^{\log}) \to \mathrm{Aut}(H_{\mathbf{Q},t})$ factors into

$$\pi_1^+(s^{\log}) \to \prod_w \mathrm{Aut}(H_{\mathbf{Q},t}(\mathrm{gr}^W_w)) \xrightarrow{\mathrm{Int}(a)} \mathrm{Aut}(H_{\mathbf{Q},t}).$$

Define $\upsilon := \mu'_t a^{-1}\mu_t^{-1}$. Then, υ belongs to Υ by the property of a, and the homomorphism $\pi_1^+(s^{\log}) \to \mathrm{Aut}(H_{0,\mathbf{Q}})$ is the composite of $\pi_1^+(s^{\log}) \to \prod_w \mathrm{Aut}(H_{0,\mathbf{Q}}(\mathrm{gr}^W_w))$ and $\mathrm{Int}(\upsilon) : \prod_w \mathrm{Aut}(H_{0,\mathbf{Q}}(\mathrm{gr}^W_w)) \hookrightarrow \mathrm{Aut}(H_{0,\mathbf{Q}}) \to \mathrm{Aut}(H_{0,\mathbf{Q}})$. Thus we see that f is the composite of $f' : \pi_1^+(s^{\log}) \to \sigma' \times \mathfrak{g}'_{\mathbf{R}}$, the inclusion $\sigma' \times \mathfrak{g}'_{\mathbf{R}} \hookrightarrow \sigma' \times \mathfrak{g}_{\mathbf{R}}$, and $1_{\sigma'} \times \mathrm{Ad}(\upsilon) : \sigma' \times \mathfrak{g}_{\mathbf{R}} \to \sigma' \times \mathfrak{g}_{\mathbf{R}}$. Let ρ' be the smallest cone in $\Sigma' = \mathrm{face}(\sigma')$ which contains the image of f'. (Here we identify σ' with $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}'_{\mathbf{R}}$.) Then, the image of f is contained in $\sigma_{\rho',\upsilon} \in \Sigma(\Upsilon)$. □

6.2.6. We now consider the Néron model and the connected Néron model.
Let

$$\Sigma_0 := \Sigma(G_{\mathbf{Z},u}),$$

$$\Sigma_1 := \{\sigma_{\tau',\upsilon} \in \Sigma(G_{\mathbf{Q},u}) \mid \Gamma(\sigma_{\tau',\upsilon}) \to \Gamma'(\tau') \text{ is an isomorphism}\}.$$

Then Σ_1 is a weak fan which is strongly compatible with Γ, and J_{Σ_1} coincides with the open set of $J_{\Sigma(G_{\mathbf{Q},u})}$ consisting of all points at which $J_{\Sigma(G_{\mathbf{Q},u})} \to S$ is strict.

Proposition 6.2.7. (i) $\Sigma_0 \subset \Sigma_1$. *In particular, J_{Σ_0} is an open subspace of J_{Σ_1}.*

(ii) J_{Σ_1} *(resp. J_{Σ_0}) has the property of J_1 (resp. J_0) in Theorem* 6.1.1.

Proof. (i) is easy to see. (ii) follows from 5.2.8 (i) and 6.2.5. □

This proposition completes the proof of Theorem 6.1.1.

§6.3. Case of $\mathcal{E}xt^1$

Let the notation be as in §6.2.

We discuss more on the objects in §6.2 in the case where, for some $w < 0$, $H_{(k)} = 0$ for $k \neq w, 0$, and $H_{(0)} = \mathbf{Z}$.

6.3.1. Let $H'_0 = H_0(\mathrm{gr}^W_w)$. So $H_0(\mathrm{gr}^W) = H'_0 \oplus \mathbf{Z}$.

Hence,

$$G_{\mathbf{Z},u} \simeq H'_0, \quad G_{\mathbf{Q},u} \simeq H'_{0,\mathbf{Q}},$$

and Υ satisfying (1) and (2) in 6.2.2 corresponds to a subgroup B of $H'_{0,\mathbf{Q}}$ containing H'_0 which is stable under the action of Γ' (5.1.7).

The correspondence is as follows. Let $h \in H'_{0,\mathbf{Q}}$. Then the corresponding element of $G_{\mathbf{Q},u}$ sends $e = 1 \in \mathbf{Z}$ to $e + h$.

6.3.2. In the situation 6.3.1,

(i) $\Upsilon = G_{\mathbf{Z},u}$ corresponds to $B = H'_0$.

(ii) $\Upsilon = G_{\mathbf{Q},u}$ corresponds to $B = H'_{0,\mathbf{Q}}$.

By the construction of the Néron model and the connected Néron model in 6.2.6–6.2.7, Proposition 6.1.4 follows from the following proposition.

Proposition 6.3.3. *On the full subcategory of $\mathcal{B}(\log)$ consisting of all objects of Kummer type over S, J_Σ for $\Sigma = \Sigma(\Upsilon)$ represents*

$$\mathrm{Ker}(\mathcal{E}xt^1(\mathbf{Z}, H') \to R^1\tau_* H'_{\mathbf{Z}} \to R^1\tau_*\tilde{B}),$$

where we denote by $\tilde{B}$ the sub local system of $H'_{\mathbf{Q}}$ corresponding to the subgroup $B \subset H'_{0,\mathbf{Q}}$.

Proof. Let $S' \in \mathcal{B}(\log)$ be of Kummer type over S. Then for an $H \in \mathrm{LMH}_Q(S')$, the image of H in $R^1\tau_*\tilde{B}$ vanishes if and only if locally on S', the extension $0 \to H'_{\mathbf{Z}} \to H_{\mathbf{Z}} \to \mathbf{Z} \to 0$ of local systems on $(S')^{\log}$ splits after pushed out by $H'_{\mathbf{Z}} \to \tilde{B}$. The last condition is equivalent to the condition that locally on S', there is a splitting $H_{\mathbf{Q}}(\mathrm{gr}^W) \simeq H_{\mathbf{Q}}$ of W on $H_{\mathbf{Q}}$ such that the local isomorphism $H_{\mathbf{Q}} \simeq H_{0,\mathbf{Q}}$ as in 6.2.5 (ii) sends $H_{\mathbf{Z}}$ to $\mathbf{Z}e + B$, i.e., onto υH_0 for some $\upsilon \in \Upsilon$. By 6.2.5 (ii), this is equivalent to $H \in J_\Sigma(S')$. $\square$

Proposition 6.3.4. *Assume that S is of log rank ≤ 1. Assume $\Gamma' \simeq \mathbf{Z}$, and let γ' be a generator of Γ'. Then Σ_1 in 6.2.6 coincides with $\Sigma(\Upsilon_1)$, where Υ_1 is the subgroup of $G_{\mathbf{Q},u}$ corresponding to the subgroup*

$$B_1 := \{b \in H'_{0,\mathbf{Q}} \mid \gamma' b - b \in H'_0\}$$

of $H'_{0,\mathbf{Q}}$.

Proof. Replacing γ' by $(\gamma')^{-1}$ if necessary, we may assume $\gamma' \in \mathrm{Hom}(P, \mathbf{N}) \subset \Gamma'$. Let N' be the logarithm of the image of γ' in $\mathfrak{g}'_{\mathbf{Q}}$. By abuse of notation, we denote its trivial extension in $\mathfrak{g}_{\mathbf{Q}}$ also by N'. Let $v \in G_{\mathbf{Q},u}$, and let $b = ve - e \in H'_{0,\mathbf{Q}}$. Then the homomorphism $\Gamma(\mathbf{R}_{\geq 0} \operatorname{Ad}(v)N') \to \Gamma'$ is an isomorphism if and only if $v\gamma'_G v^{-1} \in G_{\mathbf{Z}}$, where γ'_G is the trivial extension in $G_{\mathbf{Z}}$ of the image of γ' in $G'_{\mathbf{Z}}$ (cf. the proof of 6.2.4). Since $v\gamma'_G v^{-1}e - e = b - \gamma' b$, the last condition is equivalent to $b - \gamma' b \in H'_0$, i.e., to $v \in \Upsilon_1$. $\square$

6.3.5. We review some relationships with other works by summarizing [KNU10c] 4.18. (In [KNU10c] 4.18, "4.4, 4.15" should be replaced by "4.15".)

Assume that we are in the situation of the beginning of this subsection.

If S is a disk and $w = -1$, Green-Griffiths-Kerr [GGK10] constructed a Néron model, which is homeomorphic to ours (6.1.1, 6.3.4, [KNU10a] 8.2) as proved by Hayama [H11].

If S is of higher dimension, Brosnan-Pearlstein-Saito [BPS.p] constructed a generalization of the Néron model of Green-Griffiths-Kerr. In [Scn12], Schnell constructed a connected Néron model, and compared it with [BPS.p]. The relationships between theirs and the (connected) Néron model in this paper are not known.

6.3.6. Though we assumed $H_{(0)} = \mathbf{Z}$ in this §6.3, it is not essential. In fact, we can study the case where there are only two non-trivial gr^W_w, say gr^W_a and gr^W_b with $a < b$, by using its special case where $b = 0$ and $H_{(0)} = \mathbf{Z}$, and by the canonical isomorphism

$$\mathcal{E}xt^1(\operatorname{gr}^W_b, \operatorname{gr}^W_a) = \mathcal{E}xt^1(\mathbf{Z}, (\operatorname{gr}^W_b)^* \otimes \operatorname{gr}^W_a).$$

6.3.7. Normal functions of the type treated in this subsection (i.e., sections of $\mathcal{E}xt^1(\mathbf{Z}, H')$ with H' of pure weight < 0) appear in several contexts. For example, let E be a degenerating family of elliptic curves. Then, $\mathcal{E}xt^1(\mathbf{Z}, H^1(E)(2))$ is the target of the associated K_2-regulator map. See [KP11] for more examples related to the generalized Hodge conjecture.

§6.4. The proofs for the previous announcements

Here we give the proofs of the results in [KNU10c] which were omitted there.

Fix $\Lambda = (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$ as in 2.1.1. In this subsection, we use the method of §2.

6.4.1. In [KNU10c], we constructed some Néron models by the method of §2 in this paper as follows. Let Σ'_w be a weak fan on gr^W_w for each w. Let Γ'_w be a subgroup of $G_{\mathbf{Z}}(\operatorname{gr}^W_w)$ for each w, which is strongly compatible with Σ'_w. Let $\Sigma' := \prod_w \Sigma'_w$ and $\Gamma' := \prod_w \Gamma'_w$. Let Υ be a subgroup of $G_{\mathbf{Q},u}$. Let $\Sigma = \Sigma_\Upsilon$ be the weak fan consisting of all Υ-translations of the trivial extension of Σ' ([KNU10c] 2.1). Let Γ be the inverse image of Γ' in $G_{\mathbf{Z}}$.

Assume

(1) $G_{\mathbf{Z},u} \subset \Upsilon$, and

(2) $\gamma\Upsilon\gamma^{-1} = \Upsilon$ for any $\gamma \in \Gamma$.

Then, Σ is strongly compatible with Γ ([KNU10c] 4.2). This is shown in the same way as 6.2.3. For any $\sigma' \in \Sigma'$ and any $\upsilon \in G_{\mathbf{Q},u}$, the map $\Gamma(\mathrm{Ad}(\upsilon)(\sigma)) \to \Gamma'(\sigma')$ is of Kummer type, where σ is the trivial extension of σ' (ibid. 4.3). This is shown in the same way as 6.2.4. (The statement after ibid. 4.2 on its proof is misleading. In fact, ibid. 4.3 is not necessary for the proof of ibid. 4.2.)

6.4.2. Let $D' := D(\mathrm{gr}^W)$. Let S be an object of $\mathcal{B}(\log)$. Assume that a morphism $S \to \Gamma' \backslash D'_{\Sigma'}$ is given. Then, we can obtain a moduli space of LMH with the given gr^W on S as the fiber product J_Σ of $S \to \Gamma' \backslash D'_{\Sigma'} \leftarrow \Gamma \backslash D_\Sigma$.

6.4.3. More precisely, J_Σ represents the functor associating with $T \in \mathcal{B}(\log)/S$ the set of isomorphism classes of an LMH H on T endowed with a Γ-level structure μ with the given gr^W satisfying the following conditions (1) and (2).

(1) Locally on T, there is a splitting of the local system $a : H_{\mathbf{Q}}(\mathrm{gr}^W) \simeq H_{\mathbf{Q}}$ such that for any isomorphism $b : H_{\mathbf{Z}} \to H_0 = H_0(\mathrm{gr}^W)$ belonging to μ, we have $ba\,\mathrm{gr}^W(b)^{-1} \in \Upsilon$.

(2) For any $t \in T^{\log}$, any element of the smallest $\sigma' \in \Sigma'$ whose exp contains the image of the induced map $\pi_1^+(\tau^{-1}\tau(t)) \to \prod_w \mathrm{Aut}(H_0(\mathrm{gr}^W_w))$ satisfies Griffiths transversality with respect to $\mathrm{gr}^W(b_t)a_t^{-1}(\mathbf{C} \otimes_{\mathcal{O}^{\log}_{T,t}} F_t)$ on H_0, which is independent of the choice of a specialization $\mathcal{O}^{\log}_{T,t} \to \mathbf{C}$ at t (1.3.1). Here F is the Hodge filtration of H.

We do not use this fact in this paper and omit the proof.

6.4.4. In the following, we assume that $\Sigma' = \mathrm{face}(\prod \sigma'_w)$ for some $\sigma'_w \in \Sigma'_w$. Let σ be the trivial extension of $\sigma' := \prod \sigma'_w$. Let Υ_1 be the subset of $G_{\mathbf{Q},u}$ consisting of all elements υ such that $\Gamma(\mathrm{Ad}(\upsilon)(\sigma)) \to \Gamma'(\sigma')$ is an isomorphism. Assume further that there are only two non-trivial gr^W_k. Then, as [KNU10c] 4.6 (i) said, Υ_1 forms a subgroup and satisfies (1) and (2) in 6.4.1. This is seen as follows.

Let a, b be integers with $a < b$, and assume $\mathrm{gr}^W_w = 0$ unless $w = a, b$. In the following, we use the same symbol for an element of $G_{\mathbf{R}}(\mathrm{gr}^W)$ and its trivial extension to $G_{\mathbf{R}}$ by abuse of notation. Let $\upsilon \in G_{\mathbf{Q},u}$. As seen in the proof of 6.2.4, υ belongs Υ_1 if and only if $\upsilon\gamma'\upsilon^{-1}$ belongs to $G_{\mathbf{Z}}$ for any $\gamma' \in \Gamma'(\sigma')$. Since $\gamma' \in G_{\mathbf{Z}}$, the latter condition is satisfied by any $\upsilon \in G_{\mathbf{Z},u}$, that is, $G_{\mathbf{Z},u} \subset \Upsilon_1$.

Next, to see that Υ_1 forms a subgroup of $G_{\mathbf{Q},u}$, we describe the above condition as follows. Let q be the $\mathrm{Hom}(\mathrm{gr}^W_b, \mathrm{gr}^W_a)$-component of υ. Let γ'_a and γ'_b be the $G_{\mathbf{Z}}(\mathrm{gr}^W_a)$-component and $G_{\mathbf{Z}}(\mathrm{gr}^W_b)$-component of γ' respectively. Then, the $\mathrm{Hom}(\mathrm{gr}^W_b, \mathrm{gr}^W_a)$-component of $\upsilon\gamma'\upsilon^{-1}$ is $q\gamma'_b - \gamma'_a q$. Hence, the above condition is equivalent to the condition that the element $q\gamma'_b - \gamma'_a q$ of $\mathrm{Hom}(\mathrm{gr}^W_b, \mathrm{gr}^W_a)$ sends $(H_0 \cap W_b)/(H_0 \cap W_{b-1})$ into $(H_0 \cap W_a)/(H_0 \cap W_{a-1})$. If $q_1, q_2 \in G_{\mathbf{Q},u}$ satisfy this last condition, then, $q_1 - q_2$ also satisfies it. Since $G_{\mathbf{Q},u}$ is naturally isomorphic to the additive group $\mathrm{Hom}(\mathrm{gr}^W_b, \mathrm{gr}^W_a)$, our Υ_1 is a subgroup.

Finally, let $\gamma \in \Gamma$, and we prove $\gamma\Upsilon_1\gamma^{-1} = \Upsilon_1$. Let $\upsilon \in \Upsilon_1$. Let $\gamma' \in \Gamma'(\sigma')$. We have to prove $(\gamma\upsilon\gamma^{-1})\gamma'(\gamma\upsilon^{-1}\gamma^{-1}) \in G_{\mathbf{Z}}$. Since Γ is a semi-direct product of Γ' and $G_{\mathbf{Z},u}$ and since any element of $G_{\mathbf{Z},u}$ commutes with υ (here we use the assumption that $\mathrm{gr}^W_w = 0$ unless $w = a, b$ again), we may assume that γ is in Γ'. Then, $\gamma^{-1}\gamma'\gamma \in \Gamma'$ and $\log(\gamma^{-1}\gamma'\gamma) = \mathrm{Ad}(\gamma^{-1})\log\gamma' \in \mathrm{Ad}(\gamma^{-1})(\sigma') = \sigma'$, where the last equality is by the fact that $\Sigma' = \text{face}(\sigma')$ and Γ' are compatible. Hence $\gamma^{-1}\gamma'\gamma \in \Gamma'(\sigma')$. Since $v \in \Upsilon_1$, this implies that $v(\gamma^{-1}\gamma'\gamma)v^{-1} \in G_{\mathbf{Z}}$, and hence $\gamma v \gamma^{-1}\gamma'\gamma v^{-1}\gamma^{-1} \in G_{\mathbf{Z}}$, which completes the proof.

6.4.5. Assume that we are in the situation of 6.4.4. Let Υ be as in 6.4.1. If $\Upsilon \subset \Upsilon_1$, then, $\Gamma\backslash D_\Sigma \to \Gamma'\backslash D'_{\Sigma'}$ is strict. As is said in [KNU10c] 4.6 (iii), this projection is a relative manifold with slits in the sense of ibid. In fact, by 5.5.2, this is reduced to 5.2.8 (i) and the fact that any strict relative log manifold is a relative manifold with slits.

6.4.6. We continue to assume that we are in the situation of 6.4.4. Assume further that there exists a strict morphism $S \to \Gamma'\backslash D'_{\Sigma'}$. Let Σ_{Υ_1} be the weak fan corresponding to Υ_1 (6.4.1). By 6.4.5, $J_{\Sigma_{\Upsilon_1}} \to S$ is strict and a relative manifold with slits.

As is said in [KNU10c] 4.8, the following (i)–(iii) hold.

(i) For an object T of $\mathcal{B}/S^\circ$, there is a natural functorial injection $\iota(T) : \text{Mor}_{S^\circ}(T, J_{\Sigma_{\Upsilon_1}}) \to \{H \mid \text{LMH on } T \text{ with the given } \text{gr}^W \text{ such that } W \text{ on } H_{\mathbf{Q}} \text{ splits locally on } T\}$.

(ii) If $\dim(\sigma') = 1$, then ι is bijective.

(iii) Assume that $s \in S$ is given such that $\pi_1^+(s^{\log}) \to \Gamma'(\sigma')$ is bijective. Let $T \to S$ be a strict morphism, and let $t \in T$ be a point lying over s. Then, the stalk of ι at t is bijective.

(i) is proved similarly as 6.2.5.

We prove (ii) and (iii). Let $T \to S$ be a strict morphism from an object T in $\mathcal{B}(\log)$. Let (H, b) be an LMH over T with polarized graded quotients endowed with a Γ-level structure. Assume that its gr^W is the given one and that W splits locally on T. To prove (ii) (resp. (iii)), it is enough to show that, under the assumption of (ii) (resp. (iii)), (H, b) satisfies the conditions (1) and (2) in 2.6.2 (resp. after replacing T with an open neighborhood of t).

First, 2.6.2 (1) is always satisfied.

We consider the condition 2.6.2 (2). Let $t \in T$ lying over $s \in S$ and $t' \in T^{\log}$ lying over t. Let $\tilde{b}_{t'}$ be a representative of the germ of b at t'. Since $T \to S \to \Gamma'\backslash D'_{\Sigma'}$ is strict, the map $f' : \pi_1^+(t^{\log}) \xrightarrow{\sim} \pi_1^+(s^{\log}) \to \Gamma'(\sigma')$ is injective and its image is a face. On the other hand, since the W on the local system splits over $\mathbf{Q}$, there exists $v \in G_{\mathbf{Q},u}$ such that the image of the map $f : \pi_1^+(t^{\log}) \to \text{Aut}(H_0)$ is contained in $\exp(\text{Ad}(v)(\sigma'))$. We observe that if f' is surjective, then $v \in \Upsilon_1$. This is because f' factors as $\pi_1^+(t^{\log}) \xrightarrow{f} \Gamma(\text{Ad}(v)(\sigma')) \hookrightarrow \Gamma'(\sigma')$ so that if f' is surjective, then $\Gamma(\text{Ad}(v)(\sigma')) \hookrightarrow \Gamma'(\sigma')$ becomes an isomorphism.

Now we assume that S is of log rank ≤ 1. Then, the map f' is either trivial or surjective. Hence, in both cases, the image of f is contained in $\exp(\text{Ad}(v)(\sigma'))$ for some $v \in \Upsilon_1$. Thus the former half of 2.6.2 (2) is satisfied. Further, $\tilde{b}_{t'}(\mathbf{C} \otimes_{\mathcal{O}^{\log}_{T,t'}} F_{t'})$ satisfies the Griffiths transversality with respect to each element of the image of f. This implies that it generates a τ-nilpotent orbit, where τ is the smallest cone of Σ_{Υ_1} such that $\exp(\tau)$ contains the image of f (that is, τ is $\{0\}$ or $\text{Ad}(v)(\sigma')$). Therefore, the latter half of 2.6.2 (2) is also satisfied, which completes the proof of (ii).

Finally, suppose that we are in the situation of (iii). Then, at least, the image of f at the prescribed t is contained in $\exp(\text{Ad}(v)(\sigma'))$ for some $v \in \Upsilon_1$ because f' at t

is bijective by the assumption. Since, for any point u of T which is sufficiently near t, the map f at u naturally factors as $\pi_1^+(u^{\log}) \to \pi_1^+(t^{\log}) \to \mathrm{Aut}(H_0)$, the image of f at u is contained in $\exp(\mathrm{Ad}(\upsilon)(\tau'))$, where τ' is the smallest face of σ' such that $\exp(\tau')$ contains the image of f' at u. Hence, after replacing T, the former part of 2.6.2 (2) is satisfied. In fact, $\mathrm{Ad}(\upsilon)(\tau')$ is the smallest cone which has the above property that its exp contains the image of f at u. Since the Griffiths transversality is satisfied with respect to each element of this cone, the latter half of 2.6.2 (2) is also satisfied, which completes the proof of (iii).

6.4.7. Let the assumption be as in 6.4.6. Let Υ_1 be as in 6.4.4. From now on, as in 6.3, assume further $\mathrm{gr}_0^W = \mathbf{Z}$ with the standard polarization, and there is $w < 0$ such that $\mathrm{gr}_k^W = 0$ for $k \neq 0, w$.

Then, Υ_1 corresponds to the subgroup $B_1 = \{x \in H'_{0,\mathbf{Q}} \mid \gamma x - x \in H'_0$ for all $\gamma \in \Gamma'(\sigma')\}$ of $H'_{0,\mathbf{Q}}$. This is seen similarly as in the proof of 6.3.4.

Let Υ be as in 6.4.1 and assume $\Upsilon \subset \Upsilon_1$. Let B be the subgroup of $H'_{0,\mathbf{Q}}$ corresponding to Υ. Then, on $\mathcal{B}/S^{\circ}$, J_Σ represents the functor {LMH with the given gr^W whose image in $R^1\tau_*H'_{\mathbf{Z}}$ belongs to the kernel of $R^1\tau_*H'_{\mathbf{Z}} \to R^1\tau_*B$} ([KNU10c] 4.11). This is easily seen in the same way as 6.3.3.

6.4.8. Let the assumption be as in 6.4.6. In [KNU10c] 4.4, we called J_Σ the connected Néron model when $\Upsilon = G_{\mathbf{Z},u}$. Also, in [KNU10c] 4.7, we called $J_{\Sigma_{\Upsilon_1}}$ the Néron model.

As seen easily, the connected Néron model in this sense is the connected Néron model in this paper (6.1.1). However, the Néron model in this sense is not necessarily the Néron model in this paper (6.1.1). Their relationship is as follows. Consider the weak fan $\Sigma_1 := \{\mathrm{Ad}(\upsilon)(\tau') \mid \upsilon \in G_{\mathbf{Q},u}, \tau' \in \Sigma'$, such that $\Gamma(\mathrm{Ad}(\upsilon)(\tau')) \simeq \Gamma'(\tau')\}$. Then Σ_{Υ_1} is a subfan of Σ_1. Let J_{Σ_1} be the fiber product of $S \to \Gamma'\backslash D'_{\Sigma'} \leftarrow \Gamma\backslash D_{\Sigma_1}$. Then, this J_{Σ_1} is the Néron model in this paper, and $J_{\Sigma_{\Upsilon_1}}$ is an open subspace of J_{Σ_1} having a group structure.

6.4.9. Finally, we remark that, though the assumption of the existence of a strict morphism $S \to \Gamma'\backslash D'_{\Sigma'}$ in 6.4.6 is rather restrictive, the following holds. Let S be an object of $\mathcal{B}(\log)$ whose log rank is ≤ 1. We assume that we are given a family $Q = (H_{(w)})_{w\in\mathbf{Z}}$ of PLH's over S. Then, as was stated in [KNU10c] 4.14, locally on S, we can always take Σ', Γ', and a strict morphism $S \to \Gamma'\backslash D'_{\Sigma'}$ such that the pullback of the canonical family of PLH's on $\Gamma'\backslash D'_{\Sigma'}$ is isomorphic to Q. (The assumption in [KNU10c] 4.14 was inaccurate.) This is shown in [KU09] 4.3.

We remark also that, for a general S, the conclusion of 6.4.7 holds even if the strictness is weakened into the condition that $S \to \Gamma'\backslash D'_{\Sigma'}$ is log injective. Here a morphism $f : X \to Y$ in $\mathcal{B}(\log)$ is called *log injective* if the induced map $f^{-1}(M_Y/\mathcal{O}_Y^\times) \to M_X/\mathcal{O}_X^\times$ is injective.

§7. Examples and complements

In 7.1, basic examples, including log abelian varieties and log complex tori, are described. In 7.2, we discuss by examples why weak fans are essential. In 7.3, we propose some problems on the existence of complete weak fans. A partial affirmative result for one of them is in 7.4. In 7.5, we gather some statements on the existences of period maps.

§7.1. Basic examples

7.1.1. $0 \to \mathbf{Z}(1) \to H \to \mathbf{Z} \to 0$.

We take $\Lambda = (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$ in 2.1.1 as follows. Let

$$H_0 = \mathbf{Z}^2 = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2,$$

$$W_{-3} = 0 \subset W_{-2} = W_{-1} = \mathbf{R}e_1 \subset W_0 = H_{0,\mathbf{R}},$$
$$\langle e_1', e_1' \rangle_{-2} = 1, \quad \langle e_2', e_2' \rangle_0 = 1,$$

where for $j = 1$ (resp. $j = 2$), e_j' denotes the image of e_j in gr^W_{-2} (resp. gr^W_0),

$$h^{0,0} = h^{-1,-1} = 1, \quad h^{p,q} = 0 \text{ for all the other } (p,q).$$

Then $D = \mathbf{C}$, where $z \in \mathbf{C}$ corresponds to $F = F(z) \in D$ defined by

$$F^1 = 0 \subset F^0 = \mathbf{C} \cdot (ze_1 + e_2) \subset F^{-1} = H_{0,\mathbf{C}}.$$

This D appeared in [KNU09] and [KNU11] as Example I.

Let

$$\Gamma = G_{\mathbf{Z},u} = \begin{pmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{pmatrix} \subset \mathrm{Aut}(H_0, W, (\langle\ ,\ \rangle_w)_w),$$

$$\Sigma = \{\sigma, -\sigma, \{0\}\} \quad \text{with} \quad \pm\sigma = \begin{pmatrix} 0 & \pm\mathbf{R}_{\geq 0} \\ 0 & 0 \end{pmatrix} \subset \mathfrak{g}_{\mathbf{R}}.$$

Then Σ is a fan and is strongly compatible with Γ. We have

$$D = \mathbf{C} \to \Gamma\backslash D = \mathbf{C}^\times \subset \Gamma\backslash D_\Sigma = \mathbf{P}^1(\mathbf{C}),$$

where $\mathbf{P}^1(\mathbf{C})$ is endowed with the log structure corresponding to the divisor $\{0, \infty\}$. Here $\mathbf{C} \to \mathbf{C}^\times$ corresponding to $D \to \Gamma\backslash D$ is $z \mapsto e^{2\pi i z}$, and 0 (resp. ∞) $\in \mathbf{P}^1(\mathbf{C})$ corresponds to the class of the nilpotent orbit (σ, Z) (resp. $(-\sigma, Z)$), where $Z = D$. We have $D_{\Sigma,\mathrm{val}} = D_\Sigma$. We have

$$D^\sharp_\Sigma = D^\sharp_{\Sigma,\mathrm{val}} = D^I_{\mathrm{SL}(2)} = D^{II}_{\mathrm{SL}(2)} = \mathbf{R} \times [-\infty, \infty]$$

as topological spaces, where the second equality is given by the CKS map (§3.3), the last equality is given in [KNU11] 3.6.1 Example I, $D = \mathbf{C}$ is embedded in $\mathbf{R} \times [-\infty, \infty]$ by $x + iy \mapsto (x, y)$ $(x, y \in \mathbf{R})$, and the projection $D^\sharp_\Sigma \to \Gamma\backslash D_\Sigma$ sends $(x, y) \in \mathbf{R} \times [-\infty, \infty]$ to $e^{2\pi i(x+iy)} \in \mathbf{P}^1(\mathbf{C})$.

Next we treat this example by the formulation in §5. Let S be an object of $\mathcal{B}(\log)$ and let $Q = (H_{(w)})_w$ with

$$H_{(-2)} = \mathbf{Z}(1), \quad H_{(0)} = \mathbf{Z}, \quad H_{(w)} = 0 \text{ for } w \neq -2, 0.$$

Then $\mathrm{LMH}_Q = \mathcal{E}xt^1_{\mathrm{LMH}}(\mathbf{Z}, \mathbf{Z}(1))$ is identified with the following functor. For an object T of $\mathcal{B}(\log)$ over S, $\mathrm{LMH}_Q(T) = \Gamma(T, M_T^{\mathrm{gp}})$. Here $a \in \Gamma(T, M_T^{\mathrm{gp}})$ corresponds to the class of the following LMH H on T. Define the local system $H_{\mathbf{Z}}$ of $\mathbf{Z}$-modules on $T^{\log}$ by the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{Z}(1) & \to & H_{\mathbf{Z}} & \to & \mathbf{Z} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbf{Z}(1) & \to & \mathcal{L}_T & \xrightarrow{\exp} & \tau^{-1}(M_T^{\mathrm{gp}}) & \to & 0 \end{array}$$

of sheaves on $T^{\log}$, where τ is the canonical map $T^{\log} \to T$, the bottom row is the exact sequence of the sheaf of logarithms $\mathcal{L}_T$ (cf. [KN99], [KU09] 2.2.4), and the right vertical arrow $\mathbf{Z} \to \tau^{-1}(M_T^{\mathrm{gp}})$ is the homomorphism which sends 1 to a^{-1}. Let H be the LMH $(H_{\mathbf{Z}}, W, F)$, where

$$W_{-3} = 0 \subset W_{-2} = W_{-1} = \mathbf{R}(1) \subset W_0 = H_{\mathbf{R}}$$

and F is the decreasing filtration on $\mathcal{O}_T^{\log} \otimes H_{\mathbf{Z}}$ defined by

$$F^1 = 0 \subset F^0 = \mathrm{Ker}(c : \mathcal{O}_T^{\log} \otimes H_{\mathbf{Z}} \to \mathcal{O}_T^{\log}) \subset F^{-1} = \mathcal{O}_T^{\log} \otimes H_{\mathbf{Z}}$$

with c being the $\mathcal{O}_T^{\log}$-homomorphism induced by $H_{\mathbf{Z}} \to \mathcal{L}_T \subset \mathcal{O}_T^{\log}$. Take $\sigma' = \{0\}$ in §5.1. Then, the above Σ is regarded as a fan in §5, and the above Γ is regarded as Γ in §5. $\Gamma\backslash D_{S,\Sigma}$ in §5 is identified with $S \times \Gamma\backslash D_\Sigma \simeq S \times \mathbf{P}^1(\mathbf{C})$ with $\Gamma\backslash D_\Sigma$ as above. The inclusion map $\mathrm{LMH}_Q^{(\Sigma)}(T) = \mathrm{Mor}_{\mathcal{B}(\log)}(T, \mathbf{P}^1(\mathbf{C})) \subset \mathrm{LMH}_Q(T) = \Gamma(T, M_T^{\mathrm{gp}})$ is understood by the fact that $\mathbf{P}^1(\mathbf{C})$ with the above log structure represents the subfunctor $T \mapsto \Gamma(T, M_T \cup M_T^{-1})$ of $T \mapsto \Gamma(T, M_T^{\mathrm{gp}})$.

7.1.2. $0 \to H^1(E)(1) \to H \to \mathbf{Z} \to 0$ (E a degenerating elliptic curve).

We take $\Lambda = (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$ in 2.1.1 as follows. Let

$$H_0 = \mathbf{Z}^3 = \bigoplus_{j=1}^3 \mathbf{Z}e_j,$$

$$W_{-2} = 0 \subset W_{-1} = \mathbf{R}e_1 + \mathbf{R}e_2 \subset W_0 = H_{0,\mathbf{R}},$$

$$\langle e'_2, e'_1 \rangle_{-1} = 1, \quad \langle e'_3, e'_3 \rangle_0 = 1,$$

where for $j = 1, 2$ (resp. $j = 3$), e'_j denotes the image of e_j in gr^W_{-1} (resp. gr^W_0),

$$h^{0,0} = 1, \quad h^{0,-1} = h^{-1,0} = 1, \quad h^{p,q} = 0 \text{ for all the other } (p, q).$$

Then $D = \mathfrak{h} \times \mathbf{C}$, where $\mathfrak{h}$ is the upper half plane, and $(\tau, z) \in \mathfrak{h} \times \mathbf{C}$ corresponds to the following $F(\tau, z) \in D$. For $\tau, z \in \mathbf{C}$, we define $F = F(\tau, z) \in \check{D}$ by

$$F^1 = 0 \subset F^0 = \mathbf{C} \cdot (\tau e_1 + e_2) + \mathbf{C} \cdot (z e_1 + e_3) \subset F^{-1} = H_{0,\mathbf{C}}.$$

This D appeared in [KNU09] and [KNU11] as Example II.

Consider

$$G_{\mathbf{Z},u} = \begin{pmatrix} 1 & 0 & \mathbf{Z} \\ 0 & 1 & \mathbf{Z} \\ 0 & 0 & 1 \end{pmatrix} \subset \Gamma := \begin{pmatrix} 1 & \mathbf{Z} & \mathbf{Z} \\ 0 & 1 & \mathbf{Z} \\ 0 & 0 & 1 \end{pmatrix} \subset G_{\mathbf{Z}}.$$

The quotients $G_{\mathbf{Z},u} \backslash D$ and $\Gamma \backslash D$ are "universal elliptic curves" over $\mathfrak{h}$ and over $\mathbf{Z} \backslash \mathfrak{h} \simeq \Delta^* = \Delta \smallsetminus \{0\}$ ($\Delta := \{q \in \mathbf{C} \mid |q| < 1\}$), respectively, and described as

$$\begin{array}{ccc} G_{\mathbf{Z},u} \backslash D = (\mathfrak{h} \times \mathbf{C})/\sim & \longrightarrow & \Gamma \backslash D = \bigcup_{q \in \Delta^*} \mathbf{C}^\times / q^{\mathbf{Z}} \\ \downarrow & & \downarrow \\ D(\mathrm{gr}^W_{-1}) = \mathfrak{h} & \longrightarrow & \Gamma' \backslash D(\mathrm{gr}^W_{-1}) = \Delta^*. \end{array}$$

Here, in the above diagram, the notation is as follows. $\mathbf{Z} \backslash \mathfrak{h} \simeq \Delta^*$ is given by $\tau \mapsto e^{2\pi i \tau}$. The equivalence relation $\sim$ on $\mathfrak{h} \times \mathbf{C}$ is defined as follows. For $\tau, \tau' \in \mathfrak{h}$ and $z, z' \in \mathbf{C}$, $(\tau, z) \sim (\tau', z')$ if and only if $\tau = \tau'$ and $z \equiv z' \mod \mathbf{Z}\tau + \mathbf{Z}$. $D(\mathrm{gr}^W_{-1})$ is the D for $\Lambda(\mathrm{gr}^W_{-1})$. Γ' denotes the image of Γ in $\mathrm{Aut}(\mathrm{gr}^W_{-1})$ and is identified with $\mathbf{Z}$. The fiber $\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$ of $(\mathfrak{h} \times \mathbf{C})/\sim \to \mathfrak{h}$ over $\tau \in \mathfrak{h}$ is identified with the fiber $\mathbf{C}^\times / q^{\mathbf{Z}}$ of $\bigcup_q \mathbf{C}^\times / q^{\mathbf{Z}} \to \Delta^*$ over $q = e^{2\pi i \tau} \in \Delta^*$, via the isomorphism $\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z}) \simeq \mathbf{C}^\times / q^{\mathbf{Z}}$, $z \bmod (\mathbf{Z}\tau + \mathbf{Z}) \mapsto e^{2\pi i z} \bmod q^{\mathbf{Z}}$.

For $n \in \mathbf{Z}$, define $N_n \in \mathfrak{g}_{\mathbf{Q}}$ by

$$N_n(e_1) = 0,\ N_n(e_2) = e_1,\ N_n(e_3) = n e_1.$$

Define the fans Σ and Σ_0 by

$$\Sigma = \{\sigma_{n,n+1},\ \sigma_n\ \ (n \in \mathbf{Z}),\ \{0\}\} \quad \text{with } \sigma_{n,n+1} = \mathbf{R}_{\geq 0} N_n + \mathbf{R}_{\geq 0} N_{n+1},\ \ \sigma_n = \mathbf{R}_{\geq 0} N_n,$$

$$\Sigma_0 = \{\sigma_n\ (n \in \mathbf{Z}),\ \{0\}\}.$$

Then (Σ, Γ) is strongly compatible and (Σ_0, Γ) is also strongly compatible. Let $\sigma' \subset \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_{-1})$ be the cone $\mathbf{R}_{\geq 0} N'$, where N' is the linear map $\mathrm{gr}^W_{-1} \to \mathrm{gr}^W_{-1}$ which sends e'_1 to 0 and e'_2 to e'_1. We have a commutative diagram of fs log analytic spaces

$$\begin{array}{ccccc} \Gamma \backslash D = \bigcup_{q \in \Delta^*} \mathbf{C}^\times / q^{\mathbf{Z}} & \subset & \Gamma \backslash D_{\Sigma_0} = (\Gamma \backslash D) \cup \mathbf{C}^\times & \subset & \begin{array}{c} \Gamma \backslash D_\Sigma = \\ (\Gamma \backslash D) \cup (\mathbf{P}^1(\mathbf{C})/(0 \sim \infty)) \end{array} \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma' \backslash D(\mathrm{gr}^W_{-1}) = \Delta^* & \subset & \Gamma' \backslash D(\mathrm{gr}^W_{-1})_{\sigma'} = \Delta & = & \Gamma' \backslash D(\mathrm{gr}^W_{-1})_{\sigma'}. \end{array}$$

Here the fiber of the middle (resp. right) vertical arrow over $0 \in \Delta$ is $\mathbf{C}^\times$ (resp. the quotient $\mathbf{P}^1(\mathbf{C})/(0 \sim \infty)$ of $\mathbf{P}^1(\mathbf{C})$ obtained by identifying 0 and ∞). The log structures of $\Gamma\backslash D_{\Sigma_0}$ and of $\Gamma\backslash D_\Sigma$ are given by the inverse images of $0 \in \Delta$ which are divisors with normal crossings. The element $t \in \mathbf{C}^\times$ in the fiber over $0 \in \Delta$ corresponds to the class of the nilpotent orbit (σ_0, Z) with $Z = F(\mathbf{C}, z)$ for $z \in \mathbf{C}$ such that $t = e^{2\pi i z}$. The element $0 = \infty \in \mathbf{P}^1(\mathbf{C})/(0 \sim \infty)$ in the fiber over $0 \in \Delta$ corresponds to the class of the nilpotent orbit $(\sigma_{0,1}, Z)$ with $Z = F(\mathbf{C}, \mathbf{C})$.

The infinity of the topological spaces $D^\sharp_{\Sigma_0}$, $D^\sharp_\Sigma$, $D^\sharp_{\Sigma_0,\mathrm{val}}$, $D^\sharp_{\Sigma,\mathrm{val}}$, the projection $D^\sharp_\Sigma \to \Gamma\backslash D_\Sigma$, and the CKS map $D^\sharp_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ are described as follows. We have a homeomorphism $\mathbf{Z} \times \mathbf{R} \times \mathbf{C} \xrightarrow{\sim} D^\sharp_{\Sigma_0} \smallsetminus D$, $(n, x, z) \mapsto p_n(x, z)$, where

$$p_n(x, z) = \lim_{y \to \infty} F(x + iy, z + iny).$$

We have a homeomorphism $\mathbf{Z} \times \mathbf{R} \times \mathbf{R} \xrightarrow{\sim} D^\sharp_\Sigma \smallsetminus D^\sharp_{\Sigma_0}$, $(n, x, a) \mapsto p_{n,n+1}(x, a)$, where

$$p_{n,n+1}(x, a) = \lim_{y,\, y' \to \infty} F(x + i(y + y'), a + in(y + y') + iy').$$

We have $D^\sharp_{\Sigma_0} = D^\sharp_{\Sigma_0,\mathrm{val}}$. The points of $D^\sharp_{\Sigma,\mathrm{val}} \smallsetminus D^\sharp_{\Sigma_0,\mathrm{val}}$ are written uniquely as either $p_r(x, z)$ with $r \in \mathbf{Q} \smallsetminus \mathbf{Z}$, $x \in \mathbf{R}$ and $z \in \mathbf{C}$, or $p_r(x, a)$ with $r \in \mathbf{R} \smallsetminus \mathbf{Q}$ and $x, a \in \mathbf{R}$, or $p_{r,+}(x, a)$ with $r \in \mathbf{Q}$ and $x, a \in \mathbf{R}$, or $p_{r,-}(x, a)$ with $r \in \mathbf{Q}$ and $x, a \in \mathbf{R}$. Here

$$p_r(x, z) = \lim_{y \to \infty} F(x + iy, z + iry) \quad (r \in \mathbf{R}, x \in \mathbf{R}, z \in \mathbf{C})$$

(in the case $r \notin \mathbf{Q}$, we have $p_r(x, z) = p_r(x, \mathrm{Re}(z))$),

$$p_{r,\pm}(x, a) = \lim_{y',\, y/y' \to \infty} F(x + iy, a + iry \pm iy').$$

For $n \in \mathbf{Z}$, the projection $D^\sharp_{\Sigma,\mathrm{val}} \to D^\sharp_\Sigma$ sends $p_r(x, z)$ ($r \in \mathbf{R}$, $n < r < n + 1$, $x \in \mathbf{R}$, $z \in \mathbf{C}$) to $p_{n,n+1}(x, \mathrm{Re}(z))$, and sends $p_{r,+}(x, a)$ ($n \le r < n+1$, $x, a \in \mathbf{R}$) and $p_{r,-}(x, a)$ ($n < r \le n + 1$, $x, a \in \mathbf{R}$) to $p_{n,n+1}(x, a)$. The projection $D^\sharp_\Sigma \to \Gamma\backslash D_\Sigma$ sends $p_n(x, z)$ ($n \in \mathbf{Z}, x \in \mathbf{R}, z \in \mathbf{C}$) to $e^{2\pi i z} \in \mathbf{C}^\times$ in the fiber over $0 \in \Delta$, and sends $p_{n,n+1}(x, a)$ to $0 = \infty \in \mathbf{P}^1(\mathbf{C})/(0 \sim \infty)$.

As in [KNU11] 3.6.1 Example II, we have $D^I_{\mathrm{SL}(2)} = D^{II}_{\mathrm{SL}(2)}$ as topological spaces. For the admissible set Ψ of weight filtrations on $H_{0,\mathbf{R}}$ (3.1.2) consisting of single filtration

$$W'_{-3} = 0 \subset W'_{-2} = W'_{-1} = \mathbf{R}e_1 \subset W'_0 = H_{0,\mathbf{R}},$$

the open set $D^I_{\mathrm{SL}(2)}(\Psi)$ (3.1.2) of $D^I_{\mathrm{SL}(2)}$ has a description as a topological space

$$D^I_{\mathrm{SL}(2)}(\Psi) = \mathrm{spl}(W) \times \bar{\mathfrak{h}} = \mathbf{R}^2 \times \bar{\mathfrak{h}},$$

where $\bar{\mathfrak{h}} = \{x + iy \mid x \in \mathbf{R}, 0 < y \le \infty\}$, and we identify $s \in \mathrm{spl}(W)$ with $(a, b) \in \mathbf{R}^2$ such that $s(e'_3) = ae_1 + be_2 + e_3$. The embedding $D \to D_{\mathrm{SL}(2)}$ sends $F(\tau, z)$ ($\tau \in \mathfrak{h}$,

$z \in \mathbf{C}$) to $(\mathrm{Re}(z) - \mathrm{Re}(\tau)\,\mathrm{Im}(z)/\,\mathrm{Im}(\tau), -\,\mathrm{Im}(z)/\,\mathrm{Im}(\tau), \tau) \in \mathbf{R}^2 \times \mathfrak{h}$. The CKS map $D^{\sharp}_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ has image in $D^{I}_{\mathrm{SL}(2)}(\Psi)$ and sends $p_r(x,z)$ ($r \in \mathbf{R}, x \in \mathbf{R}, z \in \mathbf{C}$) to $(\mathrm{Re}(z) - rx, -r, x + i\infty) \in \mathbf{R}^2 \times \bar{\mathfrak{h}}$, and sends $p_{r,+}(x,a)$ and $p_{r,-}(x,a)$ ($r \in \mathbf{Q}$, $x, a \in \mathbf{R}$) to $(a - rx, -r, x + i\infty) \in \mathbf{R}^2 \times \bar{\mathfrak{h}}$.

Using the formulation in §5, let $S = \Delta = \Gamma'\backslash D(\mathrm{gr}^W_{-1})_{\sigma'}$ with the log structure at $0 \in \Delta$, and consider $Q = (H_{(w)})_w$, where $H_{(-1)}$ is the universal log Hodge structure of weight -1 on S (so $H_{(-1)}$ is of rank 2), $H_{(0)} = \mathbf{Z}$, and $H_{(w)} = 0$ for $w \neq -1, 0$. Then the above σ' is regarded as σ' in §5. Since $\sigma' \to \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_{-1})$ is injective, the above Σ and Σ_0 are regarded as fans in §5, and the above Γ is regarded as Γ in §5. (Σ, Γ) and also (Σ_0, Γ) are strongly compatible. We have $\Gamma\backslash D_{S,\Sigma} = \Gamma\backslash D_{\Sigma}$, $\Gamma\backslash D_{S,\Sigma_0} = \Gamma\backslash D_{\Sigma_0}$. Furthermore, Σ_0 is identified with Σ_j in §6 for $j = 0, 1$ (Σ_0 and Σ_1 in §6 coincide in this case). That is, $\Gamma\backslash D_{\Sigma_0} \to S$ is the connected Néron model and is also the Néron model.

7.1.3. $0 \to H^1(E)(b) \to H \to \mathbf{Z} \to 0$ ($b \geq 2$) (E a degenerating elliptic curve).

Let $b \geq 2$. We take $\Lambda = (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$ in 2.1.1 as follows. Let

$$H_0 = \mathbf{Z}^3 = \textstyle\bigoplus_{j=1}^3 \mathbf{Z}e_j,$$

$$W_{-2b} = 0 \subset W_{1-2b} = W_{-1} = \mathbf{R}e_1 + \mathbf{R}e_2 \subset W_0 = H_{0,\mathbf{R}},$$

$$\langle e'_2, e'_1\rangle_{1-2b} = 1, \quad \langle e'_3, e'_3\rangle_0 = 1,$$

where for $j = 1, 2$ (resp. $j = 3$), e'_j denotes the image of e_j in gr^W_{1-2b} (resp. gr^W_0),

$$h^{0,0} = 1, \quad h^{1-b,-b} = h^{-b,1-b} = 1, \quad h^{p,q} = 0 \text{ for all the other } (p,q).$$

Then $D = \mathfrak{h} \times \mathbf{C}^2$, where $(\tau, z, w) \in \mathfrak{h} \times \mathbf{C}^2$ corresponds to the following $F(\tau, z, w) \in D$. For $\tau, z, w \in \mathbf{C}$, we define $F = F(\tau, z, w) \in \check{D}$ by

$$F^1 = 0 \subset F^0 = F^{2-b} = \mathbf{C}\cdot(ze_1 + we_2 + e_3) \subset F^{1-b} = F^{2-b} + \mathbf{C}\cdot(\tau e_1 + e_2) \subset F^{-b} = H_{0,\mathbf{C}}.$$

This D in the case $b = 2$ appeared in [KNU09] and [KNU11] as Example III.

Like in 7.1.2, consider

$$\Gamma = \begin{pmatrix} 1 & \mathbf{Z} & \mathbf{Z} \\ 0 & 1 & \mathbf{Z} \\ 0 & 0 & 1 \end{pmatrix} \subset G_{\mathbf{Z}}, \quad N_n = \begin{pmatrix} 0 & 1 & n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_{\mathbf{Q}} \quad (n \in \mathbf{Z}),$$

$$\Sigma_0 = \{\sigma_n\ (n \in \mathbf{Z}), \{0\}\} \quad \text{with } \sigma_n = \mathbf{R}_{\geq 0} N_n.$$

Then (Σ_0, Γ) is strongly compatible. We have $\Gamma(\sigma_0) = \exp(\mathbf{Z}N_0) \simeq \mathbf{Z}$. We have a commutative diagram in $\mathcal{B}(\log)$

$$\begin{array}{ccc} \Gamma(\sigma_0)^{\mathrm{gp}}\backslash D = \Delta^* \times \mathbf{C}^2 & \subset & \Gamma(\sigma_0)^{\mathrm{gp}}\backslash D_{\sigma_0} = \{(q, z, w) \in \Delta \times \mathbf{C}^2 \mid w = 0 \text{ if } q = 0\} \\ \downarrow & & \downarrow \\ \Gamma'\backslash D(\mathrm{gr}^W_{1-2b}) = \Delta^* & \subset & \Gamma'\backslash D(\mathrm{gr}^W_{1-2b})_{\sigma'} = \Delta, \end{array}$$

where the isomorphism $\Gamma(\sigma_0)^{\mathrm{gp}}\backslash D \simeq \Delta^* \times \mathbf{C}^2$ sends the class of $F(\tau, z, w)$ to $(e^{2\pi i\tau}, z-\tau w, w)$. Here Γ' denotes the image of Γ in $\mathrm{Aut}(\mathrm{gr}^W_{1-2b})$, which is isomorphic to $\mathbf{Z}$, and σ' denotes the image of σ_0 in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_{1-2b})$. The element $(0, z, 0) \in \Delta \times \mathbf{C}^2$ corresponds to the class of the nilpotent orbit (σ_0, Z), where $Z = F(\mathbf{C}, z, 0)$.

Note that $\Gamma(\sigma_0)^{\mathrm{gp}}\backslash D_{\sigma_0} \to \Gamma\backslash D_{\Sigma_0}$ is locally an isomorphism (Theorem 2.5.4), and is surjective. Hence, by the above, we have a local description of $\Gamma\backslash D_{\Sigma_0}$. The fiber of $\Gamma\backslash D_{\Sigma_0} \to \Gamma'\backslash D(\mathrm{gr}^W_{1-2b})_{\sigma'} = \Delta$ over $q \in \Delta$ is described, as a group object, as follows. If $q \neq 0$, the fiber is isomorphic to $(\mathbf{C}^\times)^2/\mathbf{Z}$, where $a \in \mathbf{Z}$ acts on $(\mathbf{C}^\times)^2$ as $(u, v) \mapsto (uv^a, v)$. Here, for any fixed τ with $q = e^{2\pi i\tau}$, the isomorphism is given by sending the class of $F(\tau, z, w)$ in the fiber to the class of $(e^{2\pi iz}, e^{2\pi iw}) \in (\mathbf{C}^\times)^2$. The fiber over $0 \in \Delta$ is isomorphic to $\mathbf{C}^\times$. Here the class of the nilpotent orbit $(\sigma_0, F(\mathbf{C}, z, 0))$ in the fiber is sent to $e^{2\pi iz} \in \mathbf{C}^\times$.

If we define $\Sigma = \{\sigma_{n,n+1},\ \sigma_n\ \ (n \in \mathbf{Z}),\ \{0\}\}$ with $\sigma_{n,n+1} = \mathbf{R}_{\geq 0}N_n + \mathbf{R}_{\geq 0}N_{n+1}$ just as in 7.1.2, we have $D_\Sigma = D_{\Sigma_0}$ in this case.

We have a homeomorphism $\mathbf{Z} \times \mathbf{R} \times \mathbf{C} \xrightarrow{\sim} D^\sharp_{\Sigma_0} \smallsetminus D$, $(n, x, z) \mapsto p_n(x, z)$, where

$$p_n(x, z) = \lim_{y\to\infty} F(x + iy, z, -n).$$

The projection $D^\sharp_{\sigma_0} = D \cup \{p_0(x, z) \mid x \in \mathbf{R}, z \in \mathbf{C}\} \to \Gamma(\sigma_0)^{\mathrm{gp}}\backslash D_{\sigma_0} \subset \Delta \times \mathbf{C} \times \mathbf{C}$ sends $p_0(x, z)$ to $(0, z, 0)$. We have $D^\sharp_{\Sigma_0} = D^\sharp_{\Sigma_0,\mathrm{val}}$.

We have $D^I_{\mathrm{SL}(2)} = D^{II}_{\mathrm{SL}(2)}$. As in [KNU11] 3.6.1 Example III, if $b = 2$, for the admissible set Ψ of weight filtrations on $H_{0,\mathbf{R}}$ (3.1.2) consisting of single filtration

$$W'_{-5} = 0 \subset W'_{-4} = W'_{-3} = \mathbf{R}e_1 \subset W'_{-2} = H_{0,\mathbf{R}},$$

the open set $D^I_{\mathrm{SL}(2)}(\Psi)$ (3.1.2) of $D^I_{\mathrm{SL}(2)}$ has a description as a topological space

$$D^I_{\mathrm{SL}(2)}(\Psi) = \mathrm{spl}(W) \times \bar{\mathfrak{h}} \times \bar{L} = \mathbf{R}^2 \times \bar{\mathfrak{h}} \times \bar{L},$$

where $L = \mathbf{R}^2$ and $\bar{L} = L \cup \{\infty v \mid v \in L \smallsetminus \{0\}\}$ with $\infty v := \lim_{t\to\infty} tv$. In this $b = 2$ case, the CKS map $D^\sharp_{\Sigma_0,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ has the image in $D^I_{\mathrm{SL}(2)}(\Psi)$ and sends $p_n(x, z)$ to

$$(\mathrm{Re}(z),\ -n\ , x + i\infty\ , \mathrm{Im}(z)\ , 0) \in \mathbf{R}^2 \times \bar{\mathfrak{h}} \times L.$$

Using the formulation in §5, let $S = \Gamma'\backslash D(\mathrm{gr}^W_{1-2b})_{\sigma'} = \Delta$ with the log structure at $0 \in \Delta$, and consider $Q = (H_{(w)})_w$, where $H_{(1-2b)}$ is the universal log Hodge structure of weight $1 - 2b$ on S (so $H_{(1-2b)}$ is of rank 2), $H_{(0)} = \mathbf{Z}$, and $H_{(w)} = 0$ for $w \neq 1 - 2b, 0$. Then the above σ' is regarded as σ' in §5. Since $\sigma' \to \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_{1-2b})$ is injective, the above Σ_0 is regarded as a fan in §5, and the above Γ is regarded as Γ in §5. We have $\Gamma\backslash D_{S,\Sigma_0} = \Gamma\backslash D_{\Sigma_0}$. Furthermore, Σ_0 is identified with Σ_j in §6 for $j = 0, 1$ (Σ_0 and Σ_1 in §6 coincide also in this case). That is, $\Gamma\backslash D_{\Sigma_0} \to S$ is the connected Néron model and is also the Néron model.

7.1.4. Log abelian varieties and log complex tori.

This is a short review of [KKN08] from the point of view of this paper.

Let S be an object of $\mathcal{B}(\log)$. A log complex torus A over S is a sheaf of abelian groups on the category $\mathcal{B}(\log)/S$ which is written in the form $A = \mathcal{E}xt^1_{\mathrm{LMH}}(\mathbf{Z}, H')$ for some log Hodge structure H' on S of weight -1 whose Hodge filtration F satisfies $F^1 H'_{\mathcal{O}} = 0$ and $F^{-1} H'_{\mathcal{O}} = H'_{\mathcal{O}}$. Thus, $A = \mathrm{LMH}_Q$ for $Q = (H_{(w)})_w$, where $H_{(-1)} = H'$, $H_{(0)} = \mathbf{Z}$, and $H_{(w)} = 0$ for $w \neq -1, 0$. This H' is determined by A (see [KKN08]). Note that we do not assume that H' is polarizable. (The definition of LMH_Q in §5 did not use the polarization.) We call a log complex torus A a log abelian variety over S if the pullback of H' to the log point s is polarizable for any $s \in S$. For example, LMH_Q for the Q in the end of 7.1.2 is a log abelian variety over $S = \Delta$.

In [KKN08], for a cone decomposition Σ in the sense of ibid. 5.1.2, a representable subfunctor $A^{(\Sigma)}$ of a log complex torus A is constructed. If S is an fs log analytic space, $A^{(\Sigma)}$ is also an fs log analytic space which is log smooth over S. We explain that this $A^{(\Sigma)}$ is a special case of $\mathrm{LMH}_Q^{(\Sigma)}$ of §5. In the following, for simplicity, we assume that H' is polarizable and hence A is a log abelian variety. For the general case, we have to extend the formulation in §5 a little.

Let H', Q, $A = \mathcal{E}xt^1(\mathbf{Z}, H') = \mathrm{LMH}_Q$ be as above. Then, locally on S, there are finitely generated free $\mathbf{Z}$-modules X and Y and an exact sequence

$$0 \to \mathrm{Hom}(X, \mathbf{Z}) \to H'_{\mathbf{Z}} \to Y \to 0 \tag{1}$$

of sheaves on $S^{\log}$ such that the canonical map

$$F^0 H'_{\mathcal{O}} \oplus (\mathcal{O}_S \otimes_{\mathbf{Z}} \mathrm{Hom}(X, \mathbf{Z})) \to H'_{\mathcal{O}}$$

is an isomorphism. Locally on S, we can take $H_{0,(-1)}$ in 5.1.1 and an exact sequence $0 \to \mathrm{Hom}(X, \mathbf{Z}) \to H_{0,(-1)} \to Y \to 0$ which corresponds to the above exact sequence (1). Let σ' be as in §5.1. Then a cone decomposition Σ in the sense of [KNN08] 5.1.2 is naturally regarded as a rational fan Σ in $\sigma' \times \mathrm{Hom}(X, \mathbf{R})$, and then regarded as a fan Σ in §5 by identifying $\sigma' \times \mathrm{Hom}(X, \mathbf{R})$ with a part of $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$, by the following embedding. We send $(a, l) \in \sigma' \times \mathrm{Hom}(X, \mathbf{R})$ to an element (a, N) of $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$, where N sends $1 \in \mathbf{Z} = H_{0,(0)}$ to $l \in \mathrm{Hom}(X, \mathbf{R}) \subset H_{0,(-1),\mathbf{R}}$ and N induces the linear map $H_{0,(-1),\mathbf{R}} \to H_{0,(-1),\mathbf{R}}$ given by the action of a. Then, we have $A^{(\Sigma)} = \mathrm{LMH}_Q^{(\Sigma)}$.

We consider the relation with Néron models. Let the assumption and notation be as in 6.1.7. Since H' there is polarizable, we have the log abelian variety A over S corresponding to H'. The usual (analytic) connected Néron model of B over S is the subgroup functor G of A in [KKN08]. As explained in 6.1.7, this also coincides with the connected Néron model in the sense of §6. The usual (analytic) Néron model of B over S is the fiber product of $A \to A/G \leftarrow \Gamma(S, A/G)$. As explained in 6.1.7, this also coincides with the Néron model in §6.

7.1.5. We consider an example of degeneration of an intermediate Jacobian. Let

$$E = \bigcup_{q \in \Delta^*} \mathbf{C}^\times / q^{\mathbf{Z}} \to \Delta^*$$

be the family of elliptic curves which appeared in 7.1.2, and consider the product family

$$X := E^3 \to S^* := (\Delta^*)^3.$$

Let $S = \Delta^3$ and endow S with the log structure associated to the divisor $S \smallsetminus S^*$. Consider the connected Néron model J_0 over S (§6.1) of the second intermediate Jacobian $J^2(X/S^*)$ of X over S^*. It is the connected Néron model for the case $Q = (H_{(w)})_w$ with $H_{(0)} = \mathbf{Z}$, $H_{(-1)}$ is the unique extension on S as a log Hodge structure of weight -1 (obtained by the nilpotent orbit theorem of Schmid [Scm73]) of the variation of Hodge structure $H^3(X/S^*)(2)$ endowed with a polarization on S^*, and $H_{(w)} = 0$ for $w \neq -1, 0$. Here $H^3(X/S^*)$ is the third higher direct image of the constant Hodge structure $\mathbf{Z}$ on X under $X \to S^*$.

As a group object, the fiber of $J_0 \to S$ over $(q_1, q_2, q_3) \in \Delta^3$ has the following structure. If $q_1q_2q_3 \neq 0$, the fiber (which is the second intermediate Jacobian of a product of three elliptic curves) is a quotient of $\mathbf{C}^{10}$ by a discrete group which is isomorphic to $\mathbf{Z}^{20}$. If one of q_j is zero and two of them are non-zero, the fiber is isomorphic to a quotient of $\mathbf{C}^9$ by a discrete subgroup which is isomorphic to $\mathbf{Z}^{14}$. If two of q_j are zero and one of them is non-zero, the fiber is isomorphic to a quotient of $\mathbf{C}^8$ by a discrete subgroup which is isomorphic to $\mathbf{Z}^{10}$. If $q_1 = q_2 = q_3 = 0$, the fiber is isomorphic to a quotient of $\mathbf{C}^7$ by a discrete subgroup which is isomorphic to $\mathbf{Z}^7$.

This can be seen as follows. Since the connected Néron model is the slit Zucker model, the fiber has the form V/L, where V is the small Griffiths part of the fiber of the vector bundle $H_{(-1),\mathcal{O}}/F^0H_{(-1),\mathcal{O}}$ and L is the stalk of $\tau_*(H_{(-1),\mathbf{Z}})$ ($\tau : S^{\log} \to S$). Since the fiber is Hausdorff (Theorem 5.2.8 (iii)), L must be discrete in V. Let H' be the unique extension on Δ of $H^1(E/\Delta^*)(1)$ on Δ^*, as a log Hodge structure of weight -1. Then we have

$$H_{(-1)} \simeq (\textstyle\bigotimes_{j=1}^3 \mathrm{pr}_j^*(H'))(-1) \oplus \mathrm{pr}_1^*(H')^{\oplus 2} \oplus \mathrm{pr}_2^*(H')^{\oplus 2} \oplus \mathrm{pr}_3^*(H')^{\oplus 2}.$$

From this, we have that if $q_1 = 0$ and $q_2q_3 \neq 0$ (resp. $q_1 = q_2 = 0$ and $q_3 \neq 0$, resp. $q_1 = q_2 = q_3 = 0$), the dimension of V is $3+2+2+2 = 9$ (resp. $2+2+2+2 = 8$, resp. $1+2+2+2 = 7$), and the rank of L is $4+2+4+4 = 14$ (resp. $2+2+2+4 = 10$, resp. $1+2+2+2 = 7$).

The Néron model coincides with the connected Néron model in this case. To construct this Néron model, we need weak fan (not necessarily fan). See 7.2.2.

§7.2. The necessity of weak fan

We explain by examples that a weak fan (not a fan) is necessary to construct the connected Néron model.

7.2.1. We give an example in which the weak fan Σ_0 in §6 used for the construction of the connected Néron model, is not a fan.

In §6, assume $H_{0,(-2)} = L \otimes L$ with $L = \mathbf{Z}^2 = \mathbf{Z}e_1 + \mathbf{Z}e_2$, $H_{0,(0)} = \mathbf{Z}$, $H_{0,(w)} = 0$ for $w \neq -2, 0$, $\sigma' = \mathbf{R}_{\geq 0}N'_1 + \mathbf{R}_{\geq 0}N'_2$, where $N'_j : H_{0,(-2),\mathbf{Q}} \to H_{0,(-2),\mathbf{Q}}$ ($j = 1, 2$) are as follows: $N'_1 = N' \otimes 1$, $N'_2 = 1 \otimes N'$ with $N' : L \to L$ being the homomorphism defined by

$N'(e_1)=0$, $N'(e_2)=e_1$. Let $H_0=\bigoplus_w H_0(\mathrm{gr}^W_w)$ be the fixed splitting of the filtration W on H_0 as in 5.1.3. Let e be the base 1 of $H_{0,(0)}\subset H_0$, let $\tau=\mathbf{R}_{\geq 0}N_1+\mathbf{R}_{\geq 0}N_2$ with $N_j\in\mathfrak{g}_{\mathbf{Q}}$ being the extension of N'_j such that $N_j(e)=0$. Then the weak fan $\Sigma_0=\{\mathrm{Ad}(\gamma)(\alpha)\mid\gamma\in G_{\mathbf{Z},u},\ \alpha \text{ is a face of } \tau\}$ in §6 is not a fan. To see this, let $\gamma_{m,n}\in G_{\mathbf{Z},u}$ for integers $m,n>0$ be the element which fixes all elements of $H_{0,(-2)}$ and which sends e to $e+me_1\otimes e_2-ne_2\otimes e_1$. Then $\tau\cap\mathrm{Ad}(\gamma_{m,n})(\tau)=\mathbf{R}_{\geq 0}\cdot(mN_1+nN_2)$, because $\mathrm{Ad}(\gamma_{m,n})(mN_1+nN_2)=mN_1+nN_2$, and this intersection is not a face of τ.

For a later use in 7.3.3, we remark that if τ_1 is a subcone of τ of rank 2, there is no fan Σ which is stable under the adjoint action of $G_{\mathbf{Z},u}$ such that $\tau_1\subset\sigma$ for some $\sigma\in\Sigma$. In fact, the interior of τ_1 contains mN_1+nN_2 and $m'N_1+n'N_2$ for some integers $m,n,m',n'>0$ such that $mn'-m'n\neq 0$. If such a fan Σ exists, the smallest $\sigma\in\Sigma$ such that $\tau_1\subset\sigma$ exists, and any interior point of τ_1 is an interior point of σ. Hence σ and $\mathrm{Ad}(\gamma_{m,n})(\sigma)$ contain a common interior point mN_1+nN_2 and hence we should have $\mathrm{Ad}(\gamma_{m,n})(\sigma)=\sigma$. Thus $\mathrm{Ad}(\gamma_{m,n})^k(\sigma)=\sigma$ for all $k\in\mathbf{Z}$. Let $N_0\in\mathfrak{g}_{\mathbf{Q}}$ be the element which annihilates $H_{0,(-2)}$ and which sends e to $e_1\otimes e_1\in H_{0,(-2)}$. Then $\mathrm{Ad}(\gamma_{m,n})^k(m'N_1+n'N_2)=m'N_1+n'N_2+k(m'n-mn')N_0$. But since σ is a finitely generated sharp cone, it can not contain the set $\{m'N_1+n'N_2+k(m'n-mn')N_0\mid k\in\mathbf{Z}\}$.

This $((H_{0,(w)})_w,\sigma')$ appears in §5 in the case S and $(H_{(w)})_w$ are as follows: $S=\Delta^2$ with the log structure associated to the divisor $S\smallsetminus S^*$ with normal crossings, where $S^*=(\Delta^*)^2$. $H_{(0)}=\mathbf{Z}$. $H_{(-2)}$ is the extension of $\mathrm{pr}_1^*H^1(E/\Delta^*)(1)\otimes\mathrm{pr}_2^*H^1(E/\Delta^*)(1)$ on S^* to S as a log Hodge structure of weight -2, where $E=\bigcup_{q\in\Delta^*}\mathbf{C}^\times/q^{\mathbf{Z}}$ as in 7.1.5 and $H^1(E/\Delta^*)$ is the VHS on Δ^* obtained as the first higher direct image of the constant Hodge structure $\mathbf{Z}$ on E under $E\to\Delta^*$. $H_{(w)}=0$ for $w\neq -2,0$.

7.2.2. We show that, in the example 7.1.5, we need a weak fan, not a fan, to construct the connected Néron model of the second intermediate Jacobian. Let $X=E^3\to S^*=(\Delta^*)^3\subset S=\Delta^3$ be as in 7.1.5. Let $L=\mathbf{Z}^2=\mathbf{Z}e_1+\mathbf{Z}e_2$ and $N':L\to L$ be as in 7.2.1. We take $H_{0,(0)}=\mathbf{Z}$, $H_{0,(-1)}=L\otimes L\otimes L\oplus L^{\oplus 2}\oplus L^{\oplus 2}\oplus L^{\oplus 2}$, $H_{0,(w)}=0$ for $w\neq -1,0$, and regard L as a stalk of $H^1(E/\Delta^*)(1)_{\mathbf{Z}}$. Define $N'_j\in\mathfrak{g}'_{\mathbf{Q}}$ $(j=1,2,3)$ as $N'_1=(N'\otimes 1\otimes 1)\oplus(N')^{\oplus 2}\oplus 0\oplus 0$, $N'_2=(1\otimes N'\otimes 1)\oplus 0\oplus(N')^{\oplus 2}\oplus 0$, $N'_3=(1\otimes 1\otimes N')\oplus 0\oplus 0\oplus(N')^{\oplus 2}$. Extend N'_j to $N_j\in\mathfrak{g}_{\mathbf{Q}}$ by $N_j(e)=0$, where e is $1\in\mathbf{Z}=H_{0,(0)}\subset H_0$. In the situation in 5.1.1, take $P=\mathbf{N}^3$, let $\sigma'=\mathrm{Hom}(P,\mathbf{R}^{\mathrm{add}}_{\geq 0})$, take the inclusion morphism $S\to\mathrm{toric}_{\sigma'}=\mathrm{Spec}(\mathbf{C}[P])_{\mathrm{an}}=\mathbf{C}^3$, and take the homomorphism $\Gamma':=\mathrm{Hom}(P^{\mathrm{gp}},\mathbf{Z})=\mathbf{Z}^3\to\mathfrak{g}'_{\mathbf{R}}$ given by (N'_1,N'_2,N'_3). We show that the weak fan Σ_0 in 6.2.6 is not a fan. We show more strongly that if τ_1 is a finitely generated subcone of $\tau:=\mathbf{R}_{\geq 0}N_1+\mathbf{R}_{\geq 0}N_2+\mathbf{R}_{\geq 0}N_3$ of rank 3, there is no fan Σ which is stable under the adjoint action of $G_{\mathbf{Z},u}$ such that $\tau_1\subset\sigma$ for some $\sigma\in\Sigma$. The interior of τ_1 has elements of the form $mN_1+nN_2+\ell N_3$ and $m'N_1+n'N_2+\ell N_3$ with $m,n,m',n',\ell\in\mathbf{Z}$ such that $mn'-m'n\neq 0$. Let γ be the element of $G_{\mathbf{Z},u}$ which fixes every element of $H_{0,(-1)}$ and which sends $e\in H_{0,(0)}$ to $e+(me_1\otimes e_2\otimes e_1-ne_2\otimes e_1\otimes e_1,\ 0,0,0)$. Then $\mathrm{Ad}(\gamma)(mN_1+nN_2+\ell N_3)=mN_1+nN_2+\ell N_3$, $\mathrm{Ad}(\gamma)^k(m'N_1+n'N_2+\ell N_3)=m'N_1+n'N_2+\ell N_3+k(m'n-mn')N_0$, where $N_0(H_{0,(-1)})=0$ and $N_0(e)=(e_1\otimes e_1\otimes e_1,\ 0,0,0)$. Hence, by the same argument as in 7.2.1, we can deduce that there is no such fan Σ.

7.2.3. The necessity of weak fans was first observed in [KNU10c]. For example, in ibid. 4.13, we showed the following. Let the assumption be as in 6.4.7 and assume that $\sigma' = \sigma'_w$ is generated by two elements N'_1 and N'_2. Assume that $\Sigma_{G_{\mathbf{Z},u}}$ is a fan. Then, $\mathrm{Ker}(N'_1 + N'_2)$ has to be contained in $\mathrm{Ker}(N'_1)$, which imposes a strong constraint on σ'.

§7.3. Completeness of weak fan

It is desirable to find a bigger Σ so that the moduli space $\Gamma\backslash D_\Sigma$ have more points at infinity. As is explained below in 7.3.2, in the pure and the classical situation, [AMRT75] gives an ideal Σ. In general case, the first problem is what is the bigness, i.e., how to define the completeness of Σ. The trial in [KU09] was rather not enough. In particular, the conjecture in [KU09] "Added in the proof" after 12.7.7 is false. In this subsection, we reformulate such a kind of definition and problems on the existence of a big Σ by replacing fan with weak fan (generalizing it to the mixed Hodge case).

7.3.1. In the situation of §2 (resp. §5), we say a weak fan Σ is *complete* (resp. *relatively complete*) if the following condition (1) is satisfied.

(1) For any rational nilpotent cone τ such that a τ-nilpotent orbit exists, there is a finite rational subdivision $\{\tau_j\}_{1\le j\le n}$ of τ having the following property. For each $1 \le j \le n$, there is an element σ of Σ such that $\tau_j \subset \sigma$.

7.3.2. Assume that we are in the situation of §2. In the pure case, for a fan, this completeness is weaker than the completeness defined in [KU09] 12.6.1.

In [KU09] 12.6.8, the following relation of completeness and compactness was proved for the completeness of a fan in the sense of [KU09] 12.6.1. The same proof works for the present completeness of a weak fan. Note that if Γ is a subgroup of finite index of $G_{\mathbf{Z}}$ and Σ is a weak fan which is compatible with Γ, then Σ is strongly compatible with Γ.

Theorem. *Assume that we are in the pure case and that we are in the classical situation* [KU09] 0.4.14*, that is, we are in one of the following cases.*

Case (1) $w = 2t+1$, $h^{t+1,t} = h^{t,t+1} \ge 0$, $h^{p,q} = 0$ *for the other* (p,q)*.*

Case (2) $w = 2t$, $h^{t+1,t-1} = h^{t-1,t+1} \le 1$, $h^{t,t} \ge 0$, $h^{p,q} = 0$ *for the other* (p,q)*.*

Let Σ *be a weak fan in* $\mathfrak{g}_{\mathbf{Q}}$*, let* Γ *be a subgroup of* $G_{\mathbf{Z}}$ *of finite index, and assume that* Γ *is compatible with* Σ*. Then,* $\Gamma\backslash D_\Sigma$ *is compact if and only if* Σ *is complete.*

In the pure case and in the classical situation [KU09] 0.4.14, a complete fan was constructed in [AMRT75].

In general, we ask if the following statement is valid.

(P). There is a complete weak fan which is compatible with $G_{\mathbf{Z}}$ (and hence with any subgroup of $G_{\mathbf{Z}}$).

7.3.3. Assume that we are in the situation of §2. Even in the pure case, the statement (P) in 7.3.2 becomes not true if we replace weak fan by fan. We show this.

Consider the case $\Lambda = (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$, where

$$H_0 = (L \otimes L) \oplus \mathrm{Sym}^2(L) \quad \text{with } L = \mathbf{Z}^2 = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2,$$

$$W_1 = 0 \subset W_2 = H_{0,\mathbf{R}},$$

$\langle\ ,\ \rangle_2$ is induced from the anti-symmetric $\mathbf{Z}$-bilinear form $L \times L \to \mathbf{Z}$ which sends (e_2, e_1) to 1, and

$$h^{2,0} = h^{0,2} = 2, \quad h^{1,1} = 3, \quad h^{p,q} = 0 \text{ for all the other } (p,q).$$

Let $N : L \to L$ be the homomorphism defined by $N(e_1) = 0$, $N(e_2) = e_1$. Define $N_1, N_2, N_3 \in \mathfrak{g}_{\mathbf{Q}}$ by

$$N_1 = (N \otimes 1) \oplus 0, \quad N_2 = (1 \otimes N) \oplus 0, \quad N_3 = 0 \oplus \mathrm{Sym}^2(N),$$

and let $\tau = \mathbf{R}_{\geq 0}N_1 + \mathbf{R}_{\geq 0}N_2 + \mathbf{R}_{\geq 0}N_3$.

Note that there exists a τ-nilpotent orbit. In fact, for any decreasing filtration F on $L_{\mathbf{C}}$ such that $F^0 = L_{\mathbf{C}}$, $\dim_{\mathbf{C}}(F^1) = 1$, and $F^2 = 0$, $(N_1, N_2, N_3, (F \otimes F) \oplus \mathrm{Sym}^2(F))$ generates a nilpotent orbit. (This nilpotent orbit appears in the degeneration of $H^1(E_1) \otimes H^1(E_2) \oplus \mathrm{Sym}^2(H^1(E_3))$, where E_j are degenerating elliptic curves.) We show that if τ_1 is a finitely generated subcone of τ of rank 3, there is no fan Σ which is compatible with $G_{\mathbf{Z}}$ such that $\tau_1 \subset \sigma$ for some $\sigma \in \Sigma$.

Assume that such a τ_1 and such a fan Σ exist. Then the interior of the cone τ_1 contains elements of the form $mN_1 + nN_2 + \ell N_3$ and $m'N_1 + n'N_2 + \ell N_3$ with $m, n, m', n', \ell \in \mathbf{Z}$ such that $mn' - m'n \neq 0$. Let γ be the element of $G_{\mathbf{Z}}$ which does not change $(e_1 \otimes e_1, 0)$, $(e_2 \otimes e_2, 0)$, $(0, e_1e_2)$, $(0, e_1^2)$ and for which

$$\begin{aligned}
\gamma(0, e_2^2) &= (0, e_2^2) + (me_1 \otimes e_2 - ne_2 \otimes e_1, 0),\\
\gamma(e_1 \otimes e_2, 0) &= (e_1 \otimes e_2, 0) + (0, ne_1^2),\\
\gamma(e_2 \otimes e_1, 0) &= (e_2 \otimes e_1, 0) + (0, me_1^2).
\end{aligned}$$

Then $\mathrm{Ad}(\gamma)(mN_1 + nN_2 + \ell N_3) = mN_1 + nN_2 + \ell N_3$, $\mathrm{Ad}(\gamma)^k(m'N_1 + n'N_2 + \ell N_3) = m'N_1 + n'N_2 + \ell N_3 + k(m'n - mn')N_0$, where $N_0(0, e_2^2) = (e_1 \otimes e_1, 0)$ and N_0 annihilates $(0, e_1^2)$, $(0, e_1e_2)$ and all $(e_j \otimes e_k, 0)$. By the argument in 7.2.1, we see that Σ as above does not exist.

7.3.4. We explain variants of the statement (P) in 7.3.2. Let I be the set of all rational nilpotent cones τ for which a τ-nilpotent orbit exists. We consider two more statements.

(P_a). There is a weak fan Σ which is compatible with $G_{\mathbf{Z}}$ such that $\bigcup_{\tau \in I}\ \tau = \bigcup_{\sigma \in \Sigma}\ \sigma$.

(P_b). There is a weak fan Σ which is compatible with $G_{\mathbf{Z}}$ such that $\bigcup_{\tau \in I}\ \tau \subset \bigcup_{\sigma \in \Sigma}\ \sigma$.

Clearly we have the implications

$$\text{(P) in 7.3.2} \Rightarrow (\mathrm{P}_b) \Leftarrow (\mathrm{P}_a).$$

If weak fan in (P_a) is replaced by fan, it becomes the conjecture stated after [KU09] 12.7.7 (in the pure case). We show that the last conjecture fails. We show more strongly that (P_b) becomes not true if weak fan in (P_b) is replaced by fan (even in the pure case).

Consider the example in 7.3.3. Let a_1, a_2, a_3 be positive real numbers which are linearly independent over $\mathbf{Q}$. If a fan Σ as in (P_b) exists, there is a $\sigma \in \Sigma$ such that $a_1N_1 + a_2N_2 + a_3N_3 \in \sigma$. Let $\tau_1 = \tau \cap \sigma$, where $\tau = \mathbf{R}_{\geq 0}N_1 + \mathbf{R}_{\geq 0}N_2 + \mathbf{R}_{\geq 0}N_3$. Since τ_1 is rational and contains $a_1N_1 + a_2N_2 + a_3N_3$ with a_1, a_2, a_3 linearly independent over $\mathbf{Q}$, τ_1 is of rank 3. But this contradicts what we have seen in 7.3.3.

Question. In case where the statement (P) in 7.3.2 is valid, we may ask if we can further require that, for any cone σ in the weak fan in (P), a σ-nilpotent orbit exists. Note that a delicate example constructed by Watanabe ([W08]) is harmless to this question.

7.3.5. Next, assume that we are in the situation of §5.

Recall that, as was remarked in 5.1.8, if Γ_u is of finite index in $G_{\mathbf{Z},u}$ and Σ is compatible with Γ, then Σ is strongly compatible with Γ (here Γ_u and Γ are as in 5.1.7).

If Γ_u is of finite index in $G_{\mathbf{Z},u}$ and Σ is compatible with Γ, then the relative completeness of Σ is related to the properness of $\Gamma\backslash D_{S,\Sigma} \to S$ in certain cases ([KNU10a] §5 Proposition). We ask:

Is there a relatively complete weak fan which is compatible with any Γ as in 5.1.7?

This is not valid if we replace weak fan by fan. In the cases 7.2.1 and 7.2.2, there is no relatively complete fan which is stable under the adjoint action of $G_{\mathbf{Z},u}$, as is seen from the arguments in 7.2.1 and 7.2.2.

§7.4. Existence of a relatively complete fan

In this subsection, we give a partial affirmative answer to the problem in 7.3.5.

7.4.1. We show that if the following conditions (1) and (2) are satisfied, then the problem in 7.3.5 is affirmative even if we replace weak fan by fan.

(1) There are only two w such that $H_{(w)} \neq 0$.

(2) $\Gamma' \simeq \mathbf{Z}$ (cf. 5.1.1 (3)).

The following construction is a variant of that in [KNU10a] §3.

Assume (1) and (2), and assume $H_{(a)} \neq 0$, $H_{(b)} \neq 0$, $a < b$. Note that the fs monoids P and $\mathrm{Hom}(P, \mathbf{N})$ are isomorphic to $\mathbf{N}$. Let γ' be a generator of $\mathrm{Hom}(P, \mathbf{N}) \subset \Gamma'$, and let $N'_a : H_{0,(a),\mathbf{Q}} \to H_{0,(a),\mathbf{Q}}$ and $N'_b : H_{0,(b),\mathbf{Q}} \to H_{0,(b),\mathbf{Q}}$ be the actions of $\log(\gamma')$ on

$H_{0,(a),\mathbf{Q}}$ and on $H_{0,(b),\mathbf{Q}}$, respectively. Let $V = \mathrm{Hom}_{\mathbf{Q}}(H_{0,(b),\mathbf{Q}}, H_{0,(a),\mathbf{Q}})$, and define $\mathbf{Q}$-subspaces X and Y of V as follows:

$$X := \{h \in V \mid hM(N'_b, W(\mathrm{gr}_b^W))_w \subset M(N'_a, W(\mathrm{gr}_a^W))_{w-2} \text{ for all } w\} + \{hN'_b - N'_a h \mid h \in V\},$$
$$Y := \{h \in X \mid hN'_b = N'_a h\} \subset X.$$

Let $\gamma_a := \mathrm{gr}_a^W(\gamma')$ and $\gamma_b := \mathrm{gr}_b^W(\gamma')$. Fix a finitely generated $\mathbf{Z}$-submodule L of V which is γ_a-stable and γ_b-stable such that

$$L \supset \mathrm{Hom}_{\mathbf{Z}}(H_{0,(b)}, H_{0,(a)}) + \{hN'_b - N'_a h \mid h \in \mathrm{Hom}_{\mathbf{Z}}(H_{0,(b)}, H_{0,(a)})\}.$$

Fix a $\mathbf{Z}$-homomorphism $s : (X \cap L)/(Y \cap L) \to X \cap L$ which is a section of the projection $X \cap L \to (X \cap L)/(Y \cap L)$. For an element x of X/Y, let $d(x)$ be the order of the image of x in $X/((X \cap L) + Y)$ which is a torsion element. Fix a $\mathbf{Z}$-basis $(e_j)_{1 \le j \le m}$ of $Y \cap L$. Let v be the image of γ' under the canonical injection $\mathrm{Hom}(P, \mathbf{N}) \to \sigma'$, which is a generator of $\sigma' \simeq \mathbf{R}_{\ge 0}$.

For $x \in X/Y$ and $n = (n_j)_{1 \le j \le m} \in \mathbf{Z}^m$, let $\sigma(x, n)$ be the nilpotent cone in $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$ generated by all elements of the form (v, N), where N is an element of $\mathfrak{g}_{\mathbf{R}}$ satisfying the following (3)–(5).

(3) The $\mathrm{Hom}_{\mathbf{Q}}(H_{0,(a),\mathbf{Q}}, H_{0,(a),\mathbf{Q}})$-component of N coincides with N'_a.

(4) The $\mathrm{Hom}_{\mathbf{Q}}(H_{0,(b),\mathbf{Q}}, H_{0,(b),\mathbf{Q}})$-component of N coincides with N'_b.

(5) The $\mathrm{Hom}_{\mathbf{Q}}(H_{0,(b),\mathbf{Q}}, H_{0,(a),\mathbf{Q}})$-component of N has the form

$$s(x) + \frac{1}{d(x)} \sum_{j=1}^m c_j e_j \quad \text{with } n_j \le c_j \le n_j + 1 \text{ for all } j.$$

Let

$$\Sigma = \{\text{face of } \sigma(x, n) \mid x \in X/Y, n \in \mathbf{Z}^m\}.$$

Then we have:

Σ is a relatively complete fan which is compatible with any Γ as in 5.1.7.

First we prove the compatibility. For simplicity, we assume that either N'_a or N'_b is non-trivial and regard $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}} \subset \mathfrak{g}_{\mathbf{R}}$ via the second projection. The case $N'_a = N'_b = 0$ is similar and easier.

Let γ be any element of Γ. It is sufficient to prove $\mathrm{Ad}(\gamma)\sigma(x, n) \in \Sigma$ for any $x \in X/Y$ and $n \in \mathbf{Z}^m$. Write the $\mathrm{Hom}_{\mathbf{Q}}(H_{0,(b),\mathbf{Q}}, H_{0,(a),\mathbf{Q}})$-component of γ^{-1} by h.

Let $N \in \sigma(x, n)$ be an element satisfying (3)–(5). Then, since γ_b and N'_b commute, the $\mathrm{Hom}_{\mathbf{Q}}(H_{0,(b),\mathbf{Q}}, H_{0,(a),\mathbf{Q}})$-component of $\gamma N \gamma^{-1}$ is

$$(6) \qquad \gamma_a s(x) \gamma_b^{-1} + \gamma_a(N'_a h - hN'_b) + d(x)^{-1} \sum_{j=1}^m c_j e_j.$$

Write $\gamma_a s(x) \gamma_b^{-1} + \gamma_a(N'_a h - hN'_b) = s(y) + z$ with $y \in X/Y$ and $z \in Y$. Since L is γ_a-stable and γ_b-stable, we have $d(y) = d(x)$ and $d(x)z \in Y \cap L$. Write $d(x)z = \sum_{j=1}^m m_j e_j$ with $m_j \in \mathbf{Z}$, and let $m = (m_j)_j \in \mathbf{Z}^m$. Then, (6) equals to

$$s(y) + d(y)^{-1} \sum_{j=1}^m (c_j + m_j) e_j.$$

This means

$$\mathrm{Ad}(\gamma)\sigma(x,n)=\sigma(y,n+m).$$

Next, the relative completeness can be proved by noticing the following two points (7) and (8). Let R be the set of all elements N of $\mathfrak{g}_{\mathbf{R}}$ such that the $\mathrm{Hom}_{\mathbf{R}}(H_{0,(a),\mathbf{R}},H_{0,(a),\mathbf{R}})$-component of N coincides with N'_a and the $\mathrm{Hom}_{\mathbf{R}}(H_{0,(b),\mathbf{R}},H_{0,(b),\mathbf{R}})$-component of N coincides with N'_b. For $N\in R$, let N_u be the $\mathrm{Hom}_{\mathbf{R}}(H_{0,(b),\mathbf{R}},H_{0,(a),\mathbf{R}})$-component of N. Then:

(7) Let $N\in R$. Then $M(N,W)$ exists if and only if N_u belongs to the $\mathbf{R}$-subspace $\mathbf{R}\otimes_{\mathbf{Q}}X$ of $\mathrm{Hom}_{\mathbf{R}}(H_{0,(b),\mathbf{R}},H_{0,(a),\mathbf{R}})$.

(8) Let $N_1,N_2\in R$ and assume $(N_1)_u,(N_2)_u\in\mathbf{R}\otimes_{\mathbf{Q}}X$ in $\mathrm{Hom}_{\mathbf{R}}(H_{0,(b),\mathbf{R}},H_{0,(a),\mathbf{R}})$. Then, $N_1N_2=N_2N_1$ if and only if $(N_1)_u-(N_2)_u$ belongs to the $\mathbf{R}$-subspace $\mathbf{R}\otimes_{\mathbf{Q}}Y$ of $\mathrm{Hom}_{\mathbf{R}}(H_{0,(b),\mathbf{R}},H_{0,(a),\mathbf{R}})$.

7.4.2. *Remark* 1. 7.4.1 gives a proof of [KNU10a] §3, and hence a proof of the theorem in [KNU10a] §2. (In fact, for the proofs, we have to show that the fan Σ defined above satisfies the admissibility in the sense of [KNU10a], i.e., the condition (2) in 1.2.6. But this can be checked easily.)

Remark 2. Here is a correction to [KNU10a]. The submodule L in ibid. §3 should have been imposed to be γ'-stable as in 7.4.1 above. Without this condition, the resulting Σ was not necessarily compatible with Γ.

7.4.3. In the part of 7.1.1 concerning the situation of §5, the fan Σ is relatively complete. In the part of 7.1.2 concerning the situation of §5, the fan Σ is relatively complete. In the part of 7.1.3 concerning the situation of §5, the fan Ξ (5.1.6) is relatively complete. In 7.1.5, there is no relatively complete fan as is seen by the argument in 7.2.2.

§7.5. Extended period maps

In this subsection, we see extensions of period maps.

Theorem 7.5.1. *Let S be a connected, log smooth, fs log analytic space, and let U be the open subspace of S consisting of all points of S at which the log structure of S is trivial. Let H be a variation of mixed Hodge structure on U with polarized graded quotients for the weight filtration, and with unipotent local monodromy along $S\smallsetminus U$. Assume that H extends to a log mixed Hodge structure on S (that is, H is admissible along $S\smallsetminus U$ as a variation of mixed Hodge structure). Fix a base point $u\in U$ and let $\Lambda=(H_0,W,(\langle\ ,\ \rangle_w)_w,(h^{p,q})_{p,q})$ be $(H_{\mathbf{Z},u},W,(\langle\ ,\ \rangle_{w,u})_w,$(the Hodge numbers of H)). Let Γ be a subgroup of $G_{\mathbf{Z}}$ which contains the global monodromy group* $\mathrm{Image}(\pi_1(U,u)\to G_{\mathbf{Z}})$ *and assume that Γ is neat. Let $\varphi:U\to\Gamma\backslash D$ be the associated period map. Then:*

(i) *Let $S^{\log}_{\mathrm{val}}$ be the topological space over $S^{\log}$ defined as in* [KU09] 3.6.26*, which contains U as a dense open subspace. Then the map $\varphi:U\to\Gamma\backslash D$ extends uniquely to a continuous map*

$$S^{\log}_{\mathrm{val}}\to\Gamma\backslash D^I_{\mathrm{SL}(2)}.$$

(ii) *For any point* $s \in S$, *there exist an open neighborhood* V *of* s, *a log modification* V' *of* V ([KU09] 3.6.12), *a commutative subgroup* Γ' *of* Γ, *and a fan* Σ *in* $\mathfrak{g}_{\mathbf{Q}}$ *which is strongly compatible with* Γ' *such that the period map* $\varphi|_{U\cap V}$ *lifts to a morphism* $U \cap V \to \Gamma'\backslash D$ *which extends uniquely to a morphism* $V' \to \Gamma'\backslash D_\Sigma$ *of log manifolds.*

$$\begin{array}{ccccc} U & \supset & U\cap V & \subset & V' \\ {}^{\varphi}\downarrow & & \downarrow & & \downarrow \\ \Gamma\backslash D & \leftarrow & \Gamma'\backslash D & \subset & \Gamma'\backslash D_\Sigma. \end{array}$$

Furthermore, we have:

(ii-1) *Assume* $S \smallsetminus U$ *is a smooth divisor. Then we can take* $V = V' = S$ *and* $\Gamma' = \Gamma$. *That is, we have a commutative diagram*

$$\begin{array}{ccc} U & \subset & S \\ {}^{\varphi}\downarrow & & \downarrow \\ \Gamma\backslash D & \subset & \Gamma\backslash D_\Sigma. \end{array}$$

(ii-2) *Assume that* Γ *is commutative. Then we can take* $\Gamma' = \Gamma$.

(ii-3) *Assume that* Γ *is commutative and that the following condition* (1) *is satisfied.*

(1) *There is a finite family* $(S_j)_{1\le j\le n}$ *of connected locally closed analytic subspaces of* S *such that* $S = \bigcup_{j=1}^n S_j$ *as a set and such that, for each* j, *the inverse image of the sheaf* $M_S/\mathcal{O}_S^\times$ *on* S_j *is locally constant.*

Then we can take $\Gamma' = \Gamma$ *and* $V = S$.

Note that in (ii), we can take a fan Σ (we do not need a weak fan, because the situation is local on S and Γ is replaced with Γ').

Proof. (i) (resp. (ii)) is the mixed Hodge version of Theorem 0.5.29 (resp. Theorem 4.3.1) of [KU09]. By using the results in §2 and §3, the proof goes exactly in the same way as in the pure case treated in [KU09]. □

We deduce the following Theorem 7.5.2 from Theorem 7.5.1 (i).

Theorem 7.5.2. *Let the assumption be as in* 7.5.1. *Let* $\tilde{H}_{\mathbf{R}}$ *be the unique extension of* $H_{\mathbf{R}}$ *as a local system on* $S^{\log}_{\mathrm{val}}$. *Then the canonical continuous splitting of the weight filtration* W *of* $H_{\mathbf{R}}$ (3.1.1) *extends uniquely to a continuous splitting of the weight filtration of* $\tilde{H}_{\mathbf{R}}$.

Theorem 7.5.2 was proved also by Brosnan and Pearlstein ([BP.p] Theorem 2.21) by another method.

Proof of Theorem 7.5.2. As in [KNU11], the map $\mathrm{spl}_W : D \to \mathrm{spl}(W)$ in 3.1.1, where $\mathrm{spl}(W)$ denotes the set of all splittings of W, extends to a continuous map

$D^I_{\mathrm{SL}(2)} \to \mathrm{spl}(W)$. Hence the period map $S^{\log}_{\mathrm{val}} \to \Gamma\backslash D^I_{\mathrm{SL}(2)}$ in Theorem 7.5.1 (i) induces a continuous map $u : S^{\log}_{\mathrm{val}} \to \Gamma\backslash \mathrm{spl}(W)$. Since the map $\mathrm{spl}(W) \to \Gamma\backslash \mathrm{spl}(W)$ is a local homeomorphism, u lifts locally on $S^{\log}_{\mathrm{val}}$ to a continuous map to $\mathrm{spl}(W)$. This shows that locally on $S^{\log}_{\mathrm{val}}$, the canonical continuous splitting of W of $H_{\mathbf{R}}$ given on U extends to a continuous splitting of W of $\tilde{H}_{\mathbf{R}}$. Since the local extensions are unique, this proves Theorem 7.5.2. □

7.5.3. Finally, we give the details of the proof of the proposition in [KNU10a] §7.

First recall the statement. Let the situation be as in §2. Assume that $h^{p,q} = 0$ unless $p+q = -1$ or $(p,q) = (0,0)$, and $h^{0,0} = 1$. Let $H'_0 := \mathrm{gr}^W_{-1}(H_0)$. Let Γ' be a subgroup of $G_{\mathbf{Z}}(\mathrm{gr}^W_{-1})$ generated by one element γ', which is isomorphic to $\mathbf{Z}$. Let $N' = \log(\gamma')$. Let Γ be the subgroup of $G_{\mathbf{Z}}$ consisting of all elements whose restrictions to H'_0 are contained in Γ' and which induce 1 on $\mathrm{gr}^W_0(H_0) = \mathbf{Z}$. Let Σ be a fan in $\mathfrak{g}_{\mathbf{Q}}$. Suppose that the following conditions (1) and (2) are satisfied.

(1) Any $\sigma \in \Sigma$ is admissible and the image of σ to $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_{-1})$ is contained in $(\mathbf{R}_{\geq 0})N'$. Furthermore, Σ is strongly compatible with Γ (2.2.6).

(2) For any rational admissible nilpotent cone σ in $\mathfrak{g}_{\mathbf{Q}}$ (2.2.1) whose image in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_{-1})$ is contained in $\mathbf{R}_{\geq 0}N'$, there is a finite subdivision $\{\sigma_j\}$ of σ such that each σ_j is contained in some element of Σ.

(Note that the admissibility in [KNU10a] (cf. 1.2.6) is equivalent to the admissibility in this paper in this simple situation.)

Let H' be a PLH of the concerned type on the log pointed unit disk S, and $S \to \Gamma'\backslash D_{\Sigma_{-1}}$ the associated period map, where Σ_{-1} is the fan in $\mathfrak{g}_{\mathbf{Q}}(\mathrm{gr}^W_{-1})$ consisting of the two elements $\mathbf{R}_{\geq 0}N'$ and $\{0\}$.

Let J_Σ be the fiber product of $S \to \Gamma'\backslash D_{\Sigma_{-1}} \leftarrow \Gamma\backslash D_\Sigma$.

Then, the proposition in [KNU10a] §7 is contained in the following statement:

(3) For any fs log analytic space S' over S and any $a \in \mathrm{Ext}^1_{S'}(\mathbf{Z}, H')$, locally on S', there is a log modification $S'' \to S'$ ([KU09] 3.6) and a subdivision Σ' of Σ satisfying (1) such that $a|_{S''}$ belongs to $\mathrm{Mor}(S'', J_{\Sigma'})$.

Actually, [KNU10a] §7 states it only for an H' of geometric origin.

As is said in loc. cit., the proof of this fact (3) is similar to [KU09] 4.3, but, actually, as is seen as follows, it is easier and the conclusion is stronger because the situation is simpler.

First, we prove that (3) is reduced to the following fact:

(4) For any fs log analytic space S' over S and any $a \in \mathrm{Ext}^1_{S'}(\mathbf{Z}, H')$, locally on S', there is a log modification $S'' \to S'$ and a fan Σ_1 satisfying (1) and (2) such that $a|_{S''}$ belongs to $\mathrm{Mor}(S'', J_{\Sigma_1})$.

In fact, assume (4). Then the fan $\Sigma' := \Sigma \sqcap \Sigma_1 := \{\tau \cap \tau_1 \mid \tau \in \Sigma, \tau_1 \in \Sigma_1\}$ is a subdivision of Σ by (2) for Σ_1, $S''(\Sigma') \to S''$ is a log modification by (2) for Σ, and $a|_{S''(\Sigma')}$ belongs to $\mathrm{Mor}(S''(\Sigma), J_{\Sigma'})$. Hence (3) follows.

We prove (4). Let C be the image of a local monodromy cone of S' in $\mathfrak{g}_{\mathbf{R}}$. It is enough to show the existence of Σ_1 satisfying (1) and (2) which contains a finite subfan supported by C.

As in [KNU10a] §3, let

$$P := \mathrm{Im}(N' : H'_{0,\mathbf{Q}} \to H'_{0,\mathbf{Q}}),$$
$$Q := \mathrm{Ker}(N' : H'_{0,\mathbf{Q}} \to H'_{0,\mathbf{Q}}) \cap P.$$

We describe the action of Γ on $\mathfrak{g}_{\mathbf{R}}$. Let $\gamma \in \Gamma$ be an element whose restriction to H'_0 is γ'^n ($n \in \mathbf{Z}$). Let a be the element of H'_0 such that $\gamma e = e + a$, where e is 1 in $\mathrm{gr}^W_0(H_0) = \mathbf{Z}$. Let $N \in \mathfrak{g}_{\mathbf{R}}$ be an element whose restriction to H'_0 is N'. Then, we have $(\mathrm{Ad}(\gamma)(N))(e) = \gamma'^n N(e) - N'(a)$ in $H'_{0,\mathbf{R}}$. Thus, if we identify N with $N(e)$, the action is identified with $h \mapsto \gamma'^n h - N'(a)$ on $H'_{0,\mathbf{R}}$. The last action of Γ on $H'_{0,\mathbf{R}}$ preserves $P_{\mathbf{R}}$ and $Q_{\mathbf{R}}$, and induces the action on P/Q. For $x \in P/Q$, we denote by Γ_x the stabilizer of x.

Let $X \subset P/Q$ be a complete set of representatives with respect to this action of Γ on P/Q. Then, to give a fan Σ_1 in $\mathfrak{g}_{\mathbf{Q}}$ satisfying the following (5) is equivalent to give a fan Σ_x for each $x \in X$ which is compatible with Γ_x and which is supported by the real cone spanned by x, that is, the smallest $\mathbf{R}_{\geq 0}$-monoid containing the set $x \subset P$.

(5) Σ_1 is compatible with Γ and is supported by the union of all admissible cones (= the union of the real cones spanned by y ($y \in P/Q$)).

Note that the latter half of (5) implies that Σ_1 satisfies (2). Further, Σ_1 clearly satisfies (1).

Thus, it is enough to show the existence of a fan Σ_x which is compatible with Γ_x and which is supported by the real cone spanned by x, where x is a (unique, unless C is trivial) element of P/Q such that the real cone spanned by x contains C.

Fix an element $p_0 \in x$. Then, the action of Γ_x on the convex hull of x is described as $p_0 + q \mapsto p_0 + q + (\gamma'^n - 1)p_0 - N'a$, where n and a are determined by $\gamma \in \Gamma_x$ as above. Since $\gamma'^n - 1 = nN' + \dfrac{(nN')^2}{2!} + \dfrac{(nN')^3}{3!} + \cdots$, the elements $(\gamma'^n - 1)p_0 - N'a$ are contained in a lattice in $Q_{\mathbf{R}}$ when $n \in \mathbf{Z}$ and $a \in H'_0$ run.

Thus the problem is reduced to the following lemma, which completes the proof of the proposition in [KNU08a] §7. In the rest, we work over $\mathbf{Q}$. In particular, a cone means a $\mathbf{Q}_{\geq 0}$-monoid.

Lemma 7.5.4. *Let $G = \mathbf{Z}^r$ be a free $\mathbf{Z}$-module of finite rank. Let $V = \mathbf{Q}^r \times \mathbf{Q}$. Let $g \in G$ act linearly on $V; (v, 1) \mapsto (v + g, 1)$. Let C be the cone in V spanned by a bounded polytope in $\mathbf{Q}^r \times \{1\}$. Then, there is a G-stable fan in V containing a finite subdivision of C which is supported by the cone $(\mathbf{Q}^r \times \mathbf{Q}_{>0}) \cup \{0\}$ spanned by $\mathbf{Q}^r \times \{1\}$.*

Proof. We use [KKN08] 5.2.14. For $n = (n_1, \ldots, n_r) \in (\frac{1}{2}\mathbf{Z})^r$, let σ_n be the cone spanned by $\left(\prod_{1 \leq j \leq r}([n_j, n_j + \frac{1}{2}] \cap \mathbf{Q})\right) \times \{1\}$. Let J be the union of the set $\{\sigma_n \cap C \mid n \in (\frac{1}{2}\mathbf{Z})^r\}$ and the set $\{\sigma_n \mid n \in \{0, \frac{1}{2}\}^r\}$. Then, J is a finite set and it is easy to see the following (6) and (7).

(6) For any $\sigma \in J$ and $g \in G$, the intersection of σ and $g\sigma$ is a face of σ, on which g acts trivially.

(7) For any $\sigma, \tau \in J$, the set of cones $\{\sigma \cap g\tau \mid g \in G\}$ is finite.

In fact, if $g \neq 1 \in G$, then $\sigma \cap g\sigma$ is $\{0\}$. Then (6) follows. Further, for $\sigma, \tau \in J$, there is at most one g such that $\sigma \cap g\tau$ is not trivial. Then (7) follows.

By [KKN08] 5.2.14, there is a G-stable fan containing a finite subdivision of σ for each $\sigma \in J$ which is supported by the union of all $g\sigma$ $(g \in G, \sigma \in J)$. This fan is the desired one. □

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Kazuya KATO
Department of mathematics
University of Chicago
Chicago, Illinois, 60637, USA
kkato@math.uchicago.edu

Chikara NAKAYAMA
Graduate School of Science
Tokyo Institute of Technology
Meguro-ku, Tokyo, 152-8551, Japan
cnakayam@math.titech.ac.jp

Sampei USUI
Graduate School of Science

Osaka University
Toyonaka, Osaka, 560-0043, Japan
usui@math.sci.osaka-u.ac.jp