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A STUDY OF MIRROR SYMMETRY THROUGH LOG MIXED HODGE THEORY

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Dedicated to Kazuya Kato and Tadashi Ashikaga on their sixtieth birthday, and Phillip Griffiths from whose works the author started his research

ABSTRACT. We hope to understand the Hodge theoretic aspect of mirror symmetry in the framework of the fundamental diagram of log mixed Hodge theory. This paper is a first step. We give a formulation of mirror theorem for quintic threefold and its mirror as the coincidence of log period maps with specified sections under the mirror map ((4) in Introduction). Since a variation of Hodge structure with unipotent monodromy on a punctured disc uniquely extends over the puncture to a log Hodge structure, we can work on and over the boundary point and we can observe clearly in high-resolution the behavior of **Z**-structure over the boundary point (3.5, (4A) and (4B) in 3.6, 3.7). This is an advantage of log Hodge theory.

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0. Introduction

Mirror symmetry for A-model of quintic 3-fold V and B-model of its mirror V° was predicted in [CDGP91], and proved in the following formulations (1)–(3), which are equivalent. Every statement is near a large radius point q_0 of the complexified Kähler moduli $\mathcal{KM}(V)$ and a maximally unipotent monodromy point p_0 of the complex moduli $\mathcal{M}(V^\circ)$. Let $t := y_1/y_0$, $u := t/2\pi i$ and $q := e^t = e^{2\pi i u}$ from 3.3 below and respective ones in 3.4 below.

(1) (*Potential.* [LLuY97], cf. [CoK99, Theorem 11.1.1]) A-model potential Φ_{GW}^V for V from 3.4 and B-model potential $\Phi_{\mathrm{GM}}^{V^\circ}$ for V° from 3.3 coincide:

$$\Phi_{\mathrm{GW}}^V(t) = \Phi_{\mathrm{GM}}^{V^\circ}(t).$$

(2) (*Solutions.* [Gi96], [Gi97p], cf. [CoK99, Example 11.2.1.3], [P98]) Let

$$J_{\mathcal{V}} = 5H\left(1 + tH + \frac{d\Phi}{dt}\frac{H^2}{5} + \left(t\frac{d\Phi}{dt} - 2\Phi\right)\frac{H^3}{5}\right)$$
$$I_{\mathcal{V}} = 5H(y_0 + y_1 H + y_2 H^2 + y_3 H^3)$$

be the solutions of A- and B-model of Picard-Fuchs equations around q_0 and p_0, respectively. (The y_i from 3.3 and $\Phi = \Phi_{\mathrm{GW}}^V$ from 3.4.) Then, $J_{\mathcal{V}} = I_{\mathcal{V}}/y_0$.

(3) (*Variation of Hodge structure.* [M97], cf. [CoK99, 8.6.1]) The isomorphism ($q_0 \in \overline{\mathcal{KM}}(V)$) $\xleftarrow{\sim}$ ($p_0 \in \overline{\mathcal{M}}(V^\circ)$) of neighborhoods of the compactifications, given by the canonical coordinate $q = \exp(2\pi i u)$, lifts to an isomorphism

$$(\mathcal{H}^V, S, \nabla^{\mathrm{middle}}, \mathcal{H}_{\mathbf{Z}}^V, \mathcal{F}, 1, [\mathrm{pt}]) \xleftarrow{\sim} (\mathcal{H}^{V^\circ}, Q, \nabla^{\mathrm{GM}}, \mathcal{H}_{\mathbf{Z}}^{V^\circ}, \mathcal{F}, \tilde{\Omega}, g_0)$$

over the punctured $\mathcal{KM}(V) \xleftarrow{\sim} \mathcal{M}(V^\circ)$ of B-variation of polarized Hodge structure with specified sections $\tilde{\Omega}$ and g_0 for V° (from 3.3 and 3.6 (2B) below) and A-variation of polarized Hodge structure with specified sections 1 and [pt] for V.

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In this paper, we give another formulation (4) below and prove that (4) is equivalent to (1)–(3).

(4) (*Log period map*) The isomorphism $(q_0 \in \overline{\mathcal{KM}}(V)) \overset{\sim}{\leftarrow} (p_0 \in \overline{\mathcal{M}}(V^\circ))$ of neighborhoods of the compactifications uniquely lifts to an isomorphism of B-log variation of polarized Hodge structure with specified sections $\tilde{\Omega}$ and g_0 for V° and A-log variation of polarized Hodge structure with specified sections 1 and [pt] for V, whose restriction over the punctured $\mathcal{KM}(V) \overset{\sim}{\leftarrow} \mathcal{M}(V^\circ)$ coincides with the isomorphism of variations of polarized Hodge structure with specified sections in (3).

This rephrases as follows. Let σ be the monodromy cone transformed by a level structure into End of a reference fiber of the local system for A- and B- models. Then, we have a commutative diagram of horizontal log period maps
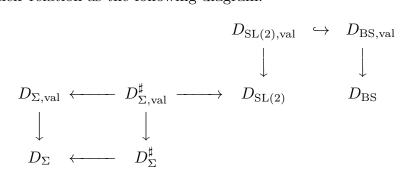
$$(q_0 \in \overline{\mathcal{KM}}(V)) \overset{\sim}{\leftarrow} (p_0 \in \overline{\mathcal{M}}(V^\circ))$$

$$\searrow \qquad \swarrow$$

$$([\sigma, \exp(\sigma_{\mathbf{C}})F_0] \in \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma)$$

with extensions of specified sections in (3), where $(\sigma, \exp(\sigma_{\mathbf{C}})F_0)$ is the nilpotent orbit and $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is the fine moduli of log Hodge structures of specified type which will be explained in Section 1 below.

Let D be a classifying space of (mixed) Hodge structures of specified type. In the series of joint works [KU99], [KU02] and [KU09] for pure case, [KNU08], [KNU09], [KNU11] and [KNU12] for mixed case, we constructed various kind of extensions of D and showed their relation as the following diagram:

$$
\begin{array}{ccccccc}
 & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS},\mathrm{val}} \\
 & & & & \downarrow & & \downarrow \\
D_{\Sigma,\mathrm{val}} & \longleftarrow & D^\sharp_{\Sigma,\mathrm{val}} & \longrightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \\
\downarrow & & \downarrow & & & & \\
D_\Sigma & \longleftarrow & D^\sharp_\Sigma & & & &
\end{array}
$$

We call this *Fundamental Diagram*, because this yields a global framework. Indeed, this diagram, taken the quotients by a discrete group Γ, shows the relation of various kind of compactifications of $\Gamma \backslash D$, i.e., shows the relation of various kind of limits of a series of points of $\Gamma \backslash D$. We hope to observe Hodge theoretic side of mirror symmetry in Fundamental Diagram. The above formulation (4), which is local, is a first step.

In Section 1, we give a brief guide to log mixed Hodge theory from [KNU12]. For the proof of above (4), local theory of log Hodge structure from [KU09] is enough, but we review here the newest results of global theory of mixed version for the communication among wider people in the future. In Section 2, we write some examples of moduli space of log mixed Hodge structures from [KU09] and [KNU12]. In Section 3, we

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review mirror symmetry of quintic 3-fold and its mirror, then we extend variations of Hodge structures from the formulation (3) over the limit points. Especially, we clarify extension of **Z**-structures by using log Hodge theory, and prove the formulation (4) is equivalent to the other formulations (1)–(3). We notice that N in Section 3 is $-N$ of N in Sections 1–2.

1. Log mixed Hodge theory

This section is a review of some results of log mixed Hodge theory from [KU09] and [KNU12]. We will explain the construction of the fine moduli $\Gamma \backslash D_\Sigma$ of log mixed Hodge structures of specified type $\Phi = (\Lambda, \Sigma, \Gamma)$, i.e., the following theorem.

Theorem. *Given a data Φ with Γ neat. Then, the functor LMH_Φ is represented by $\Gamma \backslash D_\Sigma$ in the category $\mathcal{B}(\log)$, i.e., $\mathrm{LMH}_\Phi(*) \overset{\sim}{\to} \mathrm{Map}(*, \Gamma \backslash D_\Sigma)$ as functors on $\mathcal{B}(\log)$.*

This theorem yields a generalization and refinement of period map and also normal function of Griffiths ([Gr68a], [Gr68b]).

We also add results of Néron model from [KNU12] for future use.

1.1. Category $\mathcal{B}(\log)$

Let Z be an analytic space and S be a subset of Z. The *strong topology of S in Z* is the strongest one among the topologies on S with respect to which, for any analytic space A and any morphism $f : A \to Z$ of analytic spaces satisfying $f(A) \subset S$ as sets, $f : A \to S$ is continuous.

For a local ringed space S, a *log structure* on S is a sheaf of monoids M on S together with a monoid homomorphisim $\alpha : M \to \mathcal{O}_S$ satisfying $\alpha^{-1}\mathcal{O}_S^\times \overset{\sim}{\to} \mathcal{O}_S^\times$. Here $\mathcal{O}_S$ is viewed as a sheaf of monoids by multiplication. An fs log structure is a log structure which satisfies the condition "fs", i.e., finitely generated, integral and saturated (for the precise definition, see [KU09, 2.1.5]).

An analytic space is call *log smooth* if, locally, it is isomorphic to an open set of toric variety. By a *log manifold* ([KU09, 3.5.7]), we mean a log local ringed space over **C** which has an open covering $(U_\lambda)_\lambda$ with the following property: For each λ, there exist a log smooth fs log analytic space Z_λ, a finite subset I_λ of global log differential 1-forms $\Gamma(Z_\lambda, \omega^1_{Z_\lambda})$, and an isomorphism of log local ringed spaces over **C** between U_λ and an open subset in the strong topology of

$$S_\lambda := \{z \in Z_\lambda \mid \text{the image of } I_\lambda \text{ in the stlk } \omega^1_z \text{ is zero}\}$$

in Z_λ. This S_λ becomes an analytically constructible subse of Z_λ ([KU09, 3.5.9]).

E_σ in 1.6 below and $\Gamma \backslash D_\Sigma$ in 1.7, for Γ being neat and strongly compatible with Γ, below are log manifolds. "Slits" appear because of the Griffiths tarnsversality condition imposed log point-wisely, i.e., the small Griffiths transversality condition in 1.3 below.

A typical example of slit is as follows:

Example. $S := ((\mathbf{C} \times \mathbf{C}) \setminus (\{0\} \times \mathbf{C})) \cup \{(0,0)\} \subset Z := \mathbf{C}^2$.

1.2. Ringed space $(S^{\log}, \mathcal{O}_S^{\log})$

Let S be an object of $\mathcal{B}(\log)$.

As a set, we define

$$S^{\log} := \{(s,h) \,|\, s \in S,\, h : M_s^{\mathrm{gp}} \to \mathbf{S}^1 \text{ homomorphism s.t. } h(u) = u/|u| \ (u \in \mathcal{O}_{S,s}^\times)\}.$$

Here M^{gp} is the sheaf of abelian groups associated to M (i.e., fractions of M), and $\mathbf{S}^1$ is the unit circle in $\mathbf{C}$. We endow $S^{\log}$ with the weakest topology for which the following maps (1) and (2) are continuous.

(1) $\tau : S^{\log} \to S, (s,h) \mapsto s$.

(2) For any open $U \subset S$ and any $f \in \Gamma(U, M^{\mathrm{gp}})$, $\tau^{-1}(U) \to \mathbf{S}^1$, $(s,h) \mapsto h(f_s)$.

Then, τ is proper and surjective map with fibers $\tau^{-1}(s) = (\mathbf{S}^1)^{r(s)}$, where $r(s) := \mathrm{rank}(M^{\mathrm{gp}}/\mathcal{O}_S^\times)_s$. Note that $r(s)$ varies with $s \in S$.

We define the sheaf of logarithms $\mathcal{L}$ of M^{gp} on $S^{\log}$ as the fiber product:

$$
\begin{array}{ccccc}
\mathcal{L} & \xrightarrow{\ \exp\ } & \tau^{-1}(M^{\mathrm{gp}}) & \ni & (f \text{ at } (s,h)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Cont}(*, i\mathbf{R}) & \xrightarrow{\ \exp\ } & \mathrm{Cont}(*, \mathbf{S}^1) & \ni & h(f).
\end{array}
$$

A commutative diagram

$$
\begin{array}{ccccc}
f & \in & \tau^{-1}(\mathcal{O}_S) & \xrightarrow{\ \exp\ } & \tau^{-1}(\mathcal{O}_S^\times) \subset \tau^{-1}(M^{\mathrm{gp}}) \\
\downarrow & & \downarrow & & \downarrow \\
(f - \bar{f})/2 & \in & \mathrm{Cont}(*, i\mathbf{R}) & \xrightarrow{\ \exp\ } & \mathrm{Cont}(*, \mathbf{S}^1)
\end{array}
$$

induces a homomorphism $\iota : \tau^{-1}(\mathcal{O}_S) \to \mathcal{L}$. We define

$$\mathcal{O}_S^{\log} := \frac{\tau^{-1}(\mathcal{O}_S) \otimes \mathrm{Sym}_{\mathbf{Z}}(\mathcal{L})}{(f \otimes 1 - 1 \otimes \iota(f) \,|\, f \in \tau^{-1}(\mathcal{O}_S))}.$$

Then, we have a morphism $\tau : (S^{\log}, \mathcal{O}_S^{\log}) \to (S, \mathcal{O}_S)$ of ringed spaces over $\mathbf{C}$.

For $s \in S$ and $t \in \tau^{-1}(s) \subset S^{\log}$, let t_j $(1 \le j \le r(s))$ be elements of $\mathcal{L}_t$ such that the images in $(M^{\mathrm{gp}}/\mathcal{O}_S^\times)_s$ of $\exp(t_j)$ form a basis. Then $\mathcal{O}_{S,t}^{\log}$ is a polynomial ring in $r(s)$ variables t_j $(1 \le j \le r(s))$ over $\mathcal{O}_{S,s}$ ([KU09, 2.2.5]).

1.3. Gradedly polarized log mixed Hodge structure

Let $S \in \mathcal{B}(\log)$.

A *pre-gradedly polarized log mixed Hodge structure* (pre-GPLMH for short) on S is a quadruple $H = (H_{\mathbf{Z}}, W, (\langle \ , \ \rangle_w)_w, H_{\mathcal{O}})$ consisting of a locally constant sheaf of $\mathbf{Z}$-free modules of finite rank on $S^{\log}$ endowed with an increasing filtration W of $H_{\mathbf{Q}} := \mathbf{Q} \otimes H_{\mathbf{Z}}$ together with a degenerate $(-1)^w$-symmetric $\mathbf{Q}$-bilinear form on each gr_w^W and a locally free $\mathcal{O}_S$-module on S which satisfy the following conditions. There exists an isomorphism $\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}} \simeq \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} H_{\mathcal{O}}$ (log Riemann-Hilbert correspondence), and there exists a decreasing filtration $FH_{\mathcal{O}}$ of $H_{\mathcal{O}}$ such that $F^p H_{\mathcal{O}}$ and $H_{\mathcal{O}}/F^p H_{\mathcal{O}}$ are locally free for all p. Put $F^p := \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} F^p H_{\mathcal{O}}$. Then it can be seen $\tau_* F^p = F^p H_{\mathcal{O}}$. Orthogonality condition $\langle F^p(\mathrm{gr}_w^W), F^q(\mathrm{gr}_w^W) \rangle_w = 0$ $(p + q > w)$ holds.

A pre-GPLMH on S is a *gradedly polarized log mixed Hodge structure* (GPLMH for short) on S if its pullback to each $s \in S$ is a GPLMH on s in the following sense. Let $(H_{\mathbf{Z}}, W, (\langle \ , \ \rangle_w)_w, H_{\mathcal{O}})$ be a pre-GPLMH on a log point s.

(1) W-relative N-filtration $M(N, W)$ exists for any logarithm N of local monodromy of the local system $(H_{\mathbf{R}}, W, (\langle \ , \ \rangle_w)_w)$ (admissibility).

(2) $\nabla F^p \subset \omega_s^{1,\log} \otimes F^{p-1}$, where $\omega_s^{1,\log}$ is the log differential 1-forms on $(s^{\log}, \mathcal{O}_s^{\log})$ and ∇ is the log Gauss-Manin connection defined by $\nabla := d \otimes 1_{H_{\mathbf{Z}}} : \mathcal{O}_s^{\log} \otimes H_{\mathbf{Z}} \to \omega_s^{1,\log} \otimes H_{\mathbf{Z}}$ (Griffiths transversality).

(3) Let $t \in s^{\log}$, and $a : \mathcal{O}_{s,t}^{\log} \to \mathbf{C}$ be a specialization (i.e., a $\mathbf{C}$-algebra homomorphism). Put $F(a) := \mathbf{C} \otimes_{\mathcal{O}_{s,t}^{\log}} F_t$ a filtration on $H_{\mathbf{C},t}$. Then $(H_{\mathbf{Z},t}(\mathrm{gr}_w^W), \langle \ , \ \rangle_w, F(a))$ is a polarized Hodge structure of weight w in the usual sense for all w and for all *sufficiently twisted* specialization a in the following sense (positivity).

Fix a subset $(q_j)_{1 \leq j \leq n}$ of the log structure M_s whose image in $M_s/\mathcal{O}_s^{\times}$ generates $M_s/\mathcal{O}_s^{\times}$ as monoid. A specialization a is *sufficiently twisted* if $\exp(a(\log q_j))$ is sufficiently near 0 for all j.

Remark. Admissibility condition of local monodromy plays an important role. The above condition (1) is the simplified version in [K86] under the existence of graded polarization. Sometimes we meet more general situation without assuming graded polarization. In this case, admissibility condition is described in [KNU12, §1] as follows.

The local monodromy of $(H_{\mathbf{R}}, W)$ is *admissible* if the local monodromy cone σ of $(H_{\mathbf{R}}, W)$ consisting of the monodromy logarithms satisfies the following condition at every point of $S^{\log}$: For every face $\tau \prec \sigma$, there exists a W-relative τ-filtration $M(\tau, W)$ satisfying the following (4)–(6).

(4) If $\tau \prec \sigma$ and $N \in \sigma$, then $N M(\tau, W)_w \subset M(\tau, W)_w$ for all $w \in \mathbf{Z}$.

(5) If $\tau \prec \sigma$ and $N \in \tau$, then $N M(\tau, W)_w \subset M(\tau, W)_{w-2}$ for all w.

(6) If $\tau \prec \sigma$ and $N \in \sigma$ and if $\tau' \prec \sigma$ is the smallest face containing τ and N, then $M(\tau', W) = M(N, M(\tau, W))$, where the right-hand side is $M(\tau, W)$-relative N-filtration.

When S is log smooth (1.1), a gradedly polarized log mixed Hodge structure H on S is called a *log variation of gradedly polarized mixed Hodge structure* if it satisfies the

big Griffiths transversality condition:

$$(7) \qquad\qquad (d \otimes 1_{H_{\mathbf{C}}})F^p \subset \omega_S^{1,\log} \otimes F^{p-1} \quad \text{for all } p,$$

where $d \otimes 1_{H_{\mathbf{C}}} : \mathcal{O}_S^{\log} \otimes_{\mathbf{C}} H_{\mathbf{C}} \to \omega_S^{1,\log} \otimes_{\mathbf{C}} H_{\mathbf{C}}$.

Let S be as above. The condition (7) is stronger than the condition (2) for each log point $s \in S$ endowed with the pullback log structure. In contrast with the former, we call the latter the *small Griffiths transversality* (cf. [KU09, 2.4.9] for pure case).

1.4. Nilpotent orbit and period map

Fix a data $\Lambda := (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$, where H_0 is a free $\mathbf{Z}$-module of finite rank, W is an increasing filtration on $H_{0,\mathbf{Q}} := \mathbf{Q} \otimes H_0$, $\langle\ ,\ \rangle_w$ is a non-degenerate $(-1)^w$-symmetric bilinear form on gr_w^W, and $(h^{p,q})_{p,q}$ is a set of Hodge numbers. Let D be the classifying space of gradely polarized mixed Hodge structures for the data Λ, and let $\check{D}$ be the "compact dual" ([U84], [KNU12]). For $A = \mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}$, let $G_A := \mathrm{Aut}(H_{0,A}, W, (\langle\ ,\ \rangle_w)_w)$, and, for $A = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, let $\mathfrak{g}_A := \mathrm{End}(H_{0,A}, W, (\langle\ ,\ \rangle_w)_w)$.

Let σ be a *nilpotent cone in* $\mathfrak{g}_{\mathbf{R}}$, i.e., a cone in $\mathfrak{g}_{\mathbf{R}}$ not containing a line and generated over $\mathbf{R}_{\geq 0}$ by a finite number of mutually commutative nilpotent elements of $\mathfrak{g}_{\mathbf{R}}$. A subset $Z \subset \check{D}$ is a *σ-nilpotent orbit* if the following four conditions are satisfied for $F \in Z$.

(1) $Z = \exp(\sigma_{\mathbf{C}})F$.
(2) σ is admissible with respect to W, i.e., there exists W-relative N-filtration $M(N, W)$ for any $N \in \sigma$.
(3) $NF^p \subset F^{p-1}$ for any $N \in \sigma$ any p.
(4) If $N, \ldots, N_n$ generate σ, then $\exp(\sum_j iy_j N_j)F \in D$ for sufficiently large y_j ($\forall j$).

In this situation, we say that (σ, Z) is a *nilpotent orbit*. We say also that (σ, F) or even $(N, \ldots, N_n, F)$ *generates a nilpotent orbit*.

A *weak fan Σ in* $\mathfrak{g}_{\mathbf{Q}}$ is a set of nilpotent cones in $\mathfrak{g}_{\mathbf{R}}$, defined over $\mathbf{Q}$, such that

(5) Every $\sigma \in \Sigma$ is admissible.
(6) If $\sigma \in \Sigma$ and $\tau \prec \sigma$, then $\tau \in \Sigma$.
(7) If $\sigma, \sigma' \in \Sigma$ have a common interior point and if there exists $F \in \check{D}$ such that (σ, F) and (σ', F) generate nilpotent orbits, then $\sigma = \sigma'$.

Note that this notion is wider than the familiar notion of fan. Weak fan is necessary to construct even a connected Néron model (2.4 below). See [KNU12, 7.2] for an example which shows that fan is too narrow and weak fan is necessary.

Let Σ be a weak fan and Γ be a subgroup of $G_{\mathbf{Z}}$. Σ and Γ are *strongly compatible* if the following (8) and (9) hold:

(8) If $\sigma \in \Sigma$ and $\gamma \in \Gamma$, then $\mathrm{Ad}(\gamma)\sigma \in \Sigma$.
(9) For any $\sigma \in \Sigma$, σ is generated over $\mathbf{R}_{\geq 0}$ by $\log \Gamma(\sigma)$, where $\Gamma(\sigma) := \Gamma \cap \exp(\sigma)$.

Using the argument in [KU09, Ch. 2], we will briefly explain that the notion of a gradedly polarized log mixed Hodge structure on an fs log point is equivalent to the notion of a nilpotent orbit in mixed case. The basic idea is that, on the ringed space $(S^{\log}, \mathcal{O}_S^{\log})$ ($S \in \mathcal{B}(\log)$), any local system with unipotent monodromy becomes isomorphic to the constant sheaf after shrinking S and tensoring $\mathcal{O}_S^{\log}$ (Proposition 2.3.2 loc.cit., see Appendix to Section 1 below).

Let $S \in \mathcal{B}(\log)$. Let $H = (H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w, H_{\mathcal{O}})$ be a pre-gradedly polarized log mixed Hodge structure on S of Hodge type $(h^{p,q})_{p,q}$. Fix a base point $t_0 \in S^{\log}$ and let $H_0 = H_{\mathbf{Z},t_0}$, the stalk of $H_{\mathbf{Z}}$ at t_0, endowed with the induced W and $(\langle\ ,\ \rangle_w)_w$. Let Γ be a subgroup of $G_{\mathbf{Z}}$ containing the monodromy group of $(H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w)$. Let μ be a Γ-level structure, i.e., a section of the sheaf $\Gamma \backslash \mathcal{I}som((H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w), (H_0, W, (\langle\ ,\ \rangle_w)_w))$ on $S^{\log}$. We can consider the set-theoretic period map associated to H, analogously to Section 2.5.3 in [KU09].

Let

$$\check{D}_{\mathrm{orb}} := \{(\sigma, Z) \mid \sigma \text{ is a nilpotent cone in } \mathfrak{g}_{\mathbf{R}}, \ Z = \exp(\sigma_{\mathbf{C}})F \text{ for some } F \in Z\}.$$

Define a map

$$\check{\varphi} : S \to \Gamma \backslash \check{D}_{\mathrm{orb}}$$

as follows. Let $s \in S$, and $t \in S^{\log}$ lying over s, and let $\tilde{\mu}_t : (H_{\mathbf{Z},t}, W, (\langle\ ,\ \rangle_w)_w) \xrightarrow{\sim} (H_0, W, (\langle\ ,\ \rangle_w)_w)$ be a lifting of the germ μ_t of μ at t. Then, via $\tilde{\mu}_t$, the action of $\pi_1(s^{\log})$ on $H_{\mathbf{Z},t}$ defines a nilpotent action on H_0 preserving $(W, (\langle\ ,\ \rangle_w)_w)$. Let σ be the cone in $\mathfrak{g}_{\mathbf{R}}$ generated by the logarithms of the actions of elements of $\pi_1^+(s^{\log})$ on H_0 via $\tilde{\mu}_t$. Take generators N_j ($1 \le j \le n$) of the cone σ, and let $h_j \in \mathrm{Hom}(M_s/\mathcal{O}_s^\times, \mathbf{N}) \simeq \pi_1^+(s^{\log})$ be the element corresponding to $\exp N_j$. Take elements $q_k \in M_s \setminus \mathcal{O}_s^\times$ ($1 \le k \le m$) such that their images in $M_s/\mathcal{O}_s^\times$ generate this monoid. Let $a_0, a \in \mathrm{sp}(t)$ be specializations, and take $z_j \in \mathbf{C}$ ($1 \le j \le n$) such that

$$\frac{1}{2\pi i}(a(\log q_k) - a_0(\log q_k)) = \sum_{j=1}^{n} z_j h_j(q_k) \quad (1 \le k \le m).$$

Then, by Proposition 2.5.1 loc.cit., we have

$$F(a) = \exp\Big(\sum_{j=1}^{n} z_j N_j\Big) F(a_0).$$

Hence $Z := \{\tilde{\mu}_t(F(a)) \mid a \in \mathrm{sp}(t)\} = \exp(\sigma_{\mathbf{C}})F(a_0)$ in $\check{D}$. We define

$$\check{\varphi}(s) := ((\sigma, Z) \bmod \Gamma) \in \Gamma \backslash \check{D}_{\mathrm{orb}}.$$

The point $\check{\varphi}(s)$ is independent of the choices of t and $\tilde{\mu}_t$. If $S' \to S$ in $\mathcal{B}(\log)$ is strict (i.e., the log structure of S' coincides with the pullback of that of S), then by using the

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pullback of H to S' we have a commutative diagram

$$
\begin{array}{ccc}
S' & \xrightarrow{\ \check{\varphi}'\ } & \Gamma\backslash\check{D}_{\mathrm{orb}} \\
\downarrow & & \| \\
S & \xrightarrow{\ \check{\varphi}\ } & \Gamma\backslash\check{D}_{\mathrm{orb}}
\end{array}
$$

(Lemma 2.5.4 loc.cit.).

Positivity conditions 1.3 (3) and 1.4 (4) coincide (Proposition 2.5.5 (ii) loc.cit.).

Small Griffiths transversality conditions 1.3 (2) and 1.4 (3) coincide, because the composition $H^0(s^{\log}, \mathcal{O}_s^{\log} \otimes H_{\mathbf{Z}}) \to \mathcal{O}_s^{\log} \otimes H_{\mathbf{Z}} \xrightarrow{a} H_{\mathbf{C},t}$ is an isomorphism (Proposition 2.5.5 (i) loc.cit.).

1.5. Toric variety

Let σ be a nilpotent cone in $\mathfrak{g}_{\mathbf{R}}$ and Γ be a subgroup of $G_{\mathbf{Z}}$, and consider the monoid $\Gamma(\sigma) = \Gamma \cap \exp(\sigma)$ (1.4). Assume that the cone generators of σ can be taken from $\mathfrak{g}_{\mathbf{Q}}$ and that σ is generated over $\mathbf{R}_{\geq 0}$ by $\log \Gamma(\sigma)$ (1.4 (9)). Let $P(\sigma) := \Gamma(\sigma)^{\vee} = \mathrm{Hom}(\Gamma(\sigma), \mathbf{N})$ be the dual monoid. Define a toric variety and a torus by

$$
\mathrm{toric}_{\sigma} := \mathrm{Hom}(P(\sigma), \mathbf{C}^{\mathrm{mult}}) \supset \mathrm{torus}_{\sigma} := \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{C}^{\times}),
$$

where $\mathbf{C}^{\mathrm{mult}}$ is $\mathbf{C}$ regarded as a monoid with multiplication and $P(\sigma)^{\mathrm{gp}}$ is the group generated by the monoid $P(\sigma)$. The exponential sequence $0 \to \mathbf{Z} \to \mathbf{C} \to \mathbf{C}^{\times} \to 1$ induces the universal covering of the torus

$$
0 \to \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{Z}) \to \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{C}) \xrightarrow{\mathbf{e}} \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{C}^{\times}) \to 1,
$$

where $\mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{Z})$ is considered as the fundamental group of torus_{σ}, and $\mathbf{e}(z \otimes \log \gamma) := e^{2\pi i z} \otimes \gamma$ ($z \in \mathbf{C}$, $\gamma \in \Gamma(\sigma)^{\mathrm{gp}} = \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{Z})$).

A face ρ of σ induces a surjective homomorphism $P(\rho) \leftarrow P(\sigma)$ and hence an open immersion $\mathrm{toric}_{\rho} \hookrightarrow \mathrm{toric}_{\sigma}$. The origin $0_{\rho} \in \mathrm{toric}_{\rho}$ is defined to be the monoid homomorphism $P(\rho) \to \mathbf{C}^{\mathrm{mult}}$ sending 1 to 1 and all the other elements of $P(\rho)$ to 0. This point $0_{\rho} \in \mathrm{toric}_{\rho}$ is viewed as a point of toric_{σ} by the above immersion. Then, as a set, we have

$$
\mathrm{toric}_{\sigma} = \{\mathbf{e}(z)0_{\rho} \mid \rho \prec \sigma,\ z \in \sigma_{\mathbf{C}}/(\rho_{\mathbf{C}} + \log \Gamma(\sigma)^{\mathrm{gp}})\}.
$$

For $S := \mathrm{toric}_{\sigma}$, polar coordinate $\mathbf{R}_{\geq 0} \times \mathbf{S}^1 \to \mathbf{R}_{\geq 0} \cdot \mathbf{S}^1 = \mathbf{C}$ induces $\tau : S^{\log} \to S$ as

$$
\begin{aligned}
\tau : S^{\log} &= \mathrm{Hom}(P(\sigma), \mathbf{R}_{\geq 0}^{\mathrm{mult}}) \times \mathrm{Hom}(P(\sigma), \mathbf{S}^1) \\
&= \{(\mathbf{e}(iy)0_{\rho}, \mathbf{e}(x)) \mid \rho \prec \sigma,\ x \in \sigma_{\mathbf{R}}/(\rho_{\mathbf{R}} + \log \Gamma(\sigma)^{\mathrm{gp}}),\ y \in \sigma_{\mathbf{R}}/\rho_{\mathbf{R}}\} \\
&\to S = \mathrm{Hom}(P(\sigma), \mathbf{C}^{\mathrm{mult}}), \\
\tau(\mathbf{e}(iy)0_{\rho}, \mathbf{e}(x)) &= \mathbf{e}(x + iy)0_{\rho}.
\end{aligned}
$$

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Since $0 \to \rho_{\mathbf{R}}/\log\Gamma(\rho)^{\mathrm{gp}} \to \sigma_{\mathbf{R}}/\log\Gamma(\sigma)^{\mathrm{gp}} \to \sigma_{\mathbf{R}}/(\rho_{\mathbf{R}} + \log\Gamma(\sigma)^{\mathrm{gp}}) \to 0$ is exact, the fiber of τ, as a set, is described as

$$\tau^{-1}(\mathbf{e}(a+ib)0_\rho) = \{(\mathbf{e}(ib)0_\rho, \mathbf{e}(a+x)) \mid x \in \rho_{\mathbf{R}}/\log\Gamma(\rho)^{\mathrm{gp}}\} \simeq (\mathbf{S}^1)^r,$$

where $r = r(\rho) := \mathrm{rank}\,\rho$ varies with $\rho \prec \sigma$.

Let $H_\sigma = (H_{\sigma,\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w)$ be the canonical locally constant sheaf on $S^{\log}$ given by the representation of $\pi_1(S^{\log}) = \Gamma(\sigma)^{\mathrm{gp}} \subset G_{\mathbf{Z}} = \mathrm{Aut}(H_0, W, (\langle\ ,\ \rangle_w)_w)$. We recall [KU09, 2.3.7] in the present situation.

In the general setting as in Proposition 2.3.2 loc.cit. (Appendix to Section 1), the isomorphism ν appears locally on X depending on a local choice of $(q_j)_j$. But in the case $S = \mathrm{toric}_\sigma$, a canonical ν exists globally on S.

Indeed, let A be a subring of $\mathbf{C}$, let $L := H_{\sigma,A}$ be the locally constant sheaf on $S^{\log}$ of free A-modules of finite rank with unipotent local monodromy, and let L_1 (denoted by L_0 in 2.3.2 loc.cit.) be the stalk of L at the unit point $1 = 0_{\{0\}} \in \mathrm{torus}_\sigma = \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{C}^\times)$ regarded as a constant sheaf on $S^{\log}$. We identify L_1 with H_0 in 1.4. Then there is a unique isomorphism of $\mathcal{O}_S^{\log}$-modules

$$\nu : \mathcal{O}_S^{\log} \otimes_A L \overset{\sim}{\to} \mathcal{O}_S^{\log} \otimes_A L_1$$

satisfying the following (1) and (2) for *any* $\mathbf{Z}$-basis $(q_j)_{1 \le j \le n}$ of $P(\sigma)^{\mathrm{gp}}$. Let $(\gamma_j)_{1 \le j \le n}$ be the $\mathbf{Z}$-basis of $\pi_1(\mathrm{torus}_\sigma, 1) = \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{Z})$ which is dual to $(q_j)_{1 \le j \le n}$, and let $N_j : L_{1,A'} \to L_{1,A'}$ be the logarithm of γ_j, where A' is the subring of $\mathbf{C}$ generated by A and $\mathbf{Q}$.

(1) $\qquad \nu(1 \otimes L) = \xi^{-1}(1 \otimes L_1) \quad$ with $\ \xi = \exp(\sum_{j=1}^n (2\pi i)^{-1}\log(q_j) \otimes N_j)$.

(2) Let $\log(q_j)_{1,0}$ be the branch in $\mathcal{O}_{S,1}^{\log}$ of the germ of $\log(q_j)$ at $1 \in \mathrm{torus}_\sigma$ which has the value 0 at 1, and let $\xi_{1,0} = \exp(\sum_{j=1}^n (2\pi i)^{-1}\log(q_j)_{1,0} \otimes N_j)$. Then the map $\xi_{1,0} \circ \nu_1 : 1 \otimes L_1 \to 1 \otimes L_1$ is the identity map.

The proof is similar to that of 2.3.2 loc.cit. First fix $(q_j)_{1 \le j \le n}$. For the locally constant subsheaf $L' := \xi^{-1}(1 \otimes L_1)$ of $\mathcal{O}_S^{\log} \otimes L_1$, the isomorphism $\xi_{1,0}^{-1} : L_1 \to L'_1$ of stalks preserves the actions of $\pi_1(S^{\log}, 1) \simeq \pi_1(\mathrm{torus}_\sigma, 1)$, and it is extended uniquely to an isomorphism $\nu : L \overset{\sim}{\to} L'$ on $S^{\log}$. This induces an isomorphism of $\mathcal{O}_S^{\log}$-modules $\nu : \mathcal{O}_S^{\log} \otimes_A L \overset{\sim}{\to} \mathcal{O}_S^{\log} \otimes_A L' = \mathcal{O}_S^{\log} \otimes_A L_1$. It is easy to check that ν is independent of the choice of $(q_j)_{1 \le j \le n}$.

1.6. Moduli of log mixed Hodge structures of type Φ

Let $\Phi = (\Lambda, \Sigma, \Gamma)$, where Λ is from 1.4, Σ is a weak fan and Γ is a subgroup of $G_{\mathbf{Z}}$ such that Σ and Γ are strongly compatible.

Let $\sigma \in \Sigma$. Let $S = \mathrm{toric}_\sigma$ and let $H_\sigma = (H_{\sigma,\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w)$ be the canonical locally constant sheaf on $S^{\log}$ from 1.5.. The universal pre-GPLMH H on $\check{E}_\sigma := \mathrm{toric}_\sigma \times \check{D}$

is given by H_σ together with the isomorphism $\mathcal{O}^{\log}_{\check{E}_\sigma} \otimes_{\mathbf{Z}} H_{\sigma,\mathbf{Z}} = \mathcal{O}^{\log}_{\check{E}_\sigma} \otimes_{\mathcal{O}_{\check{E}_\sigma}} H_{\mathcal{O}}$, where $H_{\mathcal{O}} := \mathcal{O}_{\check{E}_\sigma} \otimes H_0$ is the free $\mathcal{O}_{\check{E}_\sigma}$-module coming from that on $\check{D}$ endowed with the universal Hodge filtration F. Define

$$E_\sigma := \{x \in \check{E}_\sigma \mid H(x) \text{ is a GPLMH}\}.$$

Note that slits appear in E_σ because of the small Griffiths transversality 1.3 (2), or equivalently 1.4 (3).

As a set, define

$$D_\Sigma := \{(\sigma, Z) \in \check{D}_{\mathrm{orb}} \mid \text{nilpotent orbit}, \sigma \in \Sigma, Z \subset \check{D}\}.$$

Let $\sigma \in \Sigma$. Assume Γ is neat, i.e., for each $\gamma \in \Gamma$, the subgroup of $\mathbf{C}$ generated by all the eigenvalues of γ is torsion-free. A structure as an object of $\mathcal{B}(\log)$ on $\Gamma \backslash D_\Sigma$ is introduced by the diagram:

$$
\begin{array}{ccc}
E_\sigma & \overset{\mathrm{GPLMH}}{\subset} & \check{E} := \mathrm{toric}_\sigma \times \check{D} \\
\downarrow{\scriptstyle \sigma_{\mathbf{C}}\text{-torsor}} & & \\
\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma & & \\
\downarrow{\scriptstyle \text{loc. isom.}} & & \\
\Gamma \backslash D_\Sigma & &
\end{array}
$$

Here the action of $h \in \sigma_{\mathbf{C}}$ on $(\mathbf{e}(a)0_\rho, F) \in E_\sigma$ is $(\mathbf{e}(h+a)0_\rho, \exp(-h)F)$, and the projection to the quotient is described as $(\mathbf{e}(a)0_\rho, F) \mapsto (\rho, \exp(\rho_{\mathbf{C}} + a)F)$. By using the fundamental diagram from Introduction, we transported good properties of the Borel-Serre type compactification $\Gamma \backslash D_{\mathrm{BS}}$, such as Hausdorffness, to the $\sigma_{\mathbf{C}}$-action on E_σ and showed that the projection $E_\sigma \to E_\sigma/\sigma_{\mathbf{C}} = \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is a $\sigma_{\mathbf{C}}$-torsor in the category $\mathcal{B}(\log)$. This is the hardest part of the proof (see [KU09], [KNU12]).

Let $S \in \mathcal{B}(\log)$. A *log mixed Hodge structure of type* Φ (*LMH of type* Φ for short) *on* S is a pre-GPLMH $H = (H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w, H_{\mathcal{O}})$ (1.3) endowed with a Γ-level structure

$$\mu \in H^0(S^{\log}, \Gamma \backslash \mathrm{Isom}((H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w), (H_0, W, (\langle\ ,\ \rangle_w)_w)))$$

satisfying the following condition: For any $s \in S$, any $t \in \tau^{-1}(s) = s^{\log}$, any representative $\tilde{\mu}_t : H_{\mathbf{Z},t} \overset{\sim}{\to} H_0$, and the local monodromy monoid P_s of $H_{\mathbf{Z}}$ at s, i.e., $P_s := \mathrm{Image}(\mathrm{Hom}((M_S/\mathcal{O}_S^\times)_s, \mathbf{N}) \hookrightarrow \pi_1(s^{\log}) \to \mathrm{Aut}(H_{\mathbf{Z},t}))$, there exists $\sigma \in \Sigma$ such that σ contains $\tilde{\mu}_t P_s \tilde{\mu}_t^{-1}$ and that $(\sigma, \tilde{\mu}_t(\mathbf{C} \otimes_{\mathcal{O}^{\log}_{S,t}} F_t))$ generates a nilpotent orbit (1.4). (Then, the smallest such σ exists.)

As explained in 1.4, a log mixed Hodge structure of type Φ is a gradely polarized log mixed Hodge structure in 1.3 (cf. [KNU12, 2.2.2]).

Theorem. (i) $\Gamma \backslash D_\Sigma \in \mathcal{B}(\log)$, *which is Hausdorff. If* Γ *is neat,* $\Gamma \backslash D_\Sigma$ *is a log manifold.*
(ii) *On* $\mathcal{B}(\log)$, $\Gamma \backslash D_\Sigma$ *represents the functor* LMH_Φ *of LMH of type* Φ.

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Log period map. *Let $S \in \mathcal{B}(\log)$. Then we have an isomorphism*

$$\mathrm{LMH}_\Phi(S) \overset{\sim}{\to} \mathrm{Map}(S, \Gamma \backslash D_\Sigma), \quad H \mapsto \left(S \ni s \mapsto (\sigma, \exp(\sigma_{\mathbf{C}}) \tilde{\mu}_t(\mathbf{C} \otimes_{\mathcal{O}^{\log}_{S,t}} F_t)) \bmod \Gamma \right)$$

which is functorial in S.

This is the log period map mentioned at the beginning of Section 1.

1.7. Infinitesimal calculus

Let $S \in \mathcal{B}(\log)$ be a log manifold. Define $\theta_S := \mathcal{H}om_{\mathcal{O}_S}(\omega^1_S, \mathcal{O}_S)$.

Theorem. *Assume Γ is neat. Put $S := \Gamma \backslash D_\Sigma$. Then we have a commutative diagram*

$$
\begin{array}{ccc}
\theta_S & \overset{\sim}{\longrightarrow} & \mathcal{E}nd\,(H_{\mathcal{O}}, W, (\langle\ ,\ \rangle_w)_w)/F^0 \mathcal{E}nd\,(\cdots) \\[4pt]
\cup & & \cup \\[4pt]
\theta^h_S & \overset{\sim}{\longrightarrow} & \mathrm{gr}^{-1}_F \mathcal{E}nd\,(H_{\mathcal{O}}, W, (\langle\ ,\ \rangle_w)_w)
\end{array}
$$

where θ^h_S is the horizontal subbundle of θ_S.

Idea of Proof. Taking τ_* of $d \otimes 1_{H_{\mathbf{C}}} : \mathcal{O}^{\log}_S \otimes H_{\mathbf{C}} \to \omega^{1,\log}_S \otimes H_{\mathbf{C}}$, we have the Gauss-Manin connection in the log sense $\nabla : H_{\mathcal{O}} \to \omega^1_S \otimes H_{\mathcal{O}}$. Define $\theta_S \to \mathcal{E}nd\,(H_{\mathcal{O}})$, by $\delta \mapsto \nabla_\delta$. $\square$

In the algebraic context, by the analogous argument of [KO68], the above construction of ∇ is globalized to get an algebraic connection. Note that the above ∇ is nothing but the canonical connection in the sense of [D97] [CaK99, (8.50)].

Example. Let $S := \Delta^{n+t}$ be a polydisc and $f : X \to S$ be a projective morphisim of complex manifolds such that the restriction of f over $S^* := (\Delta^*)^n \times \Delta^t$ is smooth. Put $X^* := f^{-1}(S^*)$. Let $A \subset X$ be a normal crossing divisor such that $A \cap X^*$ is relative normal crossing over S^*. Let $h : X^* \smallsetminus A \to S^*$ be the restriction of f. Assume that the local monodromy of $H_{\mathbf{R}} := R^m h_* \mathbf{R}$ over S^* is unipotent. Then, the variation of mixed Hodge structure with graded quotients on $H_{\mathbf{Z}} := R^m h_* \mathbf{Z}$ over S^* canonically extends to a log mixed Hodge structure H on S of type Φ with log structures $M_S :=$ (the one corresponding to the divisor $S \smallsetminus S^*$), $M_X :=$ (the one corresponding to A), a fan $\Sigma :=$ (faces of the local monodromy cone at $0 \in S$), and a monodromy group $\Gamma := \pi_1(S^*) = \pi_1(S^{\log})$. Let $\varphi = \varphi_H : S \to \Gamma \backslash D_\Sigma$ be the corresponding period map. We have a commutative diagram

$$
\begin{array}{ccc}
\theta_S & \overset{d\varphi}{\longrightarrow} & \varphi^* \theta^h_{\Gamma \backslash D_\Sigma} = \mathrm{gr}^{-1}_F \mathcal{E}nd(H_{\mathcal{O}}, W, (\langle\ ,\ \rangle_w)_w) \\[4pt]
\text{\footnotesize Kodara-Spencer} \downarrow & & \downarrow \text{\footnotesize inclusion} \\[4pt]
R^1 f_* \theta_{X/S} & \overset{\text{coupling}}{\longrightarrow} & \bigoplus_p \mathcal{H}om_{\mathcal{O}_S}(R^{m-p} f_* \omega^p_{X/S}, R^{m-p+1} f_* \omega^{p-1}_{X/S}).
\end{array}
$$

This is a log mixed Hodge theoretic generalization of Griffiths' formula of infinitesimal period map.

Appendix to Section 1. Typographical correction of [KU09, 2.3.2]

There are only typographical errors and no other problems in Proposition 2.3.2 of [KU09]. But since they make this fundamental part uncomfortable to read, we add the corrected version here for readers' convenience. The notation and the indications in this appendix are all those loc.cit.

Proposition 2.3.2. *Let X be an object of $\overline{\mathcal{A}}_1(\log)$ (which contains $\mathcal{B}(\log)$), let A be a subring of $\mathbf{C}$, and let L be a locally constant sheaf on $X^{\log}$ of free A-modules of finite rank. Let $x \in X$, let y be a point of $X^{\log}$ lying over x, and assume that the local monodromy of L at y is unipotent. Let $(q_j)_{1 \le j \le n}$ be a finite family of elements of $M_{X,x}^{\mathrm{gp}}$ whose image in $(M_X^{\mathrm{gp}}/\mathcal{O}_X^{\times})_x$ is a $\mathbf{Z}$-basis, and let $(\gamma_j)_{1 \le j \le n}$ be the dual $\mathbf{Z}$-basis of $\pi_1(x^{\log})$ in the duality in 2.2.9. Then if we replace X by some open neighborhood of x, we have an isomorphism of $\mathcal{O}_X^{\log}$-modules*

$$\nu : \mathcal{O}_X^{\log} \otimes_A L \xrightarrow{\sim} \mathcal{O}_X^{\log} \otimes_A L_0, \quad L_0 := \text{the stalk } L_y$$

where L_0 is regarded as a constant sheaf, satisfying the condition (1) below. Let A' be the subring of $\mathbf{C}$ generated by A and $\mathbf{Q}$, let $L_{0,A'} = A' \otimes_A L_0$, and let

$$N_j : L_{0,A'} \to L_{0,A'}$$

be the endmorphism of constant sheaf which is induced by the logarithm of the monodromy action of γ_j on the stalk L_y of the locally constant sheaf L. Lift q_j in $\Gamma(X, M_X^{\mathrm{gp}})$ (by replacing X by an open neighborhood of x), and let

$$\xi = \exp(\textstyle\sum_{j=1}^n (2\pi i)^{-1} \log(q_j) \otimes N_j) : \mathcal{O}_X^{\log} \otimes_{A'} L_{0,A'} \xrightarrow{\sim} \mathcal{O}_X^{\log} \otimes_{A'} L_{0,A'}.$$

Note that the operator ξ depends on the choices of the branches of $\log(q_j)$ in $\mathcal{O}_X^{\log}$ locally on $X^{\log}$, but that the subsheaf $\xi^{-1}(1 \otimes L_0)$ of $\mathcal{O}_X^{\log} \otimes_A L_0$ is independent of the choices and hence is defined globally on $X^{\log}$.

(1) The restriction of ν to $L = 1 \otimes L$ induces an isomorphism of locally constant sheaves

$$\nu : L \xrightarrow{\sim} \xi^{-1}(1 \otimes L_0).$$

If we fix branches $\log(q_j)_{y,0}$ in $\mathcal{O}_{X,y}^{\log}$ of the germs $\log(q_j)_y$ at y ($1 \le j \le n$), we can take an isomorphism ν which satisfies above (1) and also the following (2).

(2) The branch $\xi_{y,0}$ of the germ ξ_y, defined by the fixed branches $\log(q_j)_{y,0}$ of the germs $\log(q_j)_y$, satisfies

$$\nu(1 \otimes v) = \xi_{y,0}^{-1}(1 \otimes v) \quad \text{for any } v \in L_y = L_0.$$

Proof. Let L' be the locally constant subsheaf $\xi^{-1}(1 \otimes L_0)$ of $\mathcal{O}_X^{\log} \otimes L_0$. Fix a branch $\log(q_j)_{y,0}$ of the germ $\log(q_j)_y$ at y for $1 \le j \le n$, and let $\nu : L_y \to (L')_y$ be the

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isomorphism of A-modules $v \mapsto \xi_{y,0}^{-1}(1 \otimes v)$, where $\xi_{y,0}$ is defined by the fixed branches $\log(q_j)_{y,0}$ of $\log(q_j)_y$. Then ν preserves the local monodromy actions of $\pi_1(x^{\log})$ on these stalks of the locally constant sheaves L and L'. In fact, for $v \in L_0$ and for $1 \le k \le n$,

$$\begin{aligned}
\gamma_k(\nu(v)) &= \gamma_k(\xi_{y,0}^{-1}(1 \otimes v)) = \gamma_k(\xi_{y,0})^{-1} \cdot (1 \otimes v) \\
&= \exp(-(\textstyle\sum_{j=1}^{n}((2\pi i)^{-1} \log(q_j)_{y,0} - \delta_{jk}) \otimes N_j) \cdot (1 \otimes v) \\
&= \xi_{y,0}^{-1} \exp(1 \otimes N_k)(1 \otimes v) = \xi_{y,0}^{-1}(1 \otimes \gamma_k(v)) = \nu(\gamma_k(v))
\end{aligned}$$

(δ_{jk} is the Kronecker symbol, and for the signature before δ_{jk}, see Appendix A1). Here the second equality follows from the monodromy action of γ_k on the locally constant sheaf L', the fifth equality follows from the endmorphism N_k of the constant sheaf L_0, and $\gamma_k(v)$ in the second last and in the last is the image of the element $v \in L_0 = L_y$ by the monodromy action of γ_k on the locally constant sheaf L at y. Hence there is a unique isomorphism $\nu : L|_{x^{\log}} \to L'|_{x^{\log}}$ between the pullbacks of L and L' to $x^{\log}$ which induces the above isomorphism ν on the stalks at y.

By the proper base change theorem (Appendix A2) applied to the proper map $\tau : X^{\log} \to X$ and to the sheaf $\mathcal{F}$ of isomorphisms from L to L' on $X^{\log}$, the isomorphism ν extends to an isomorphism $\nu : L \overset{\sim}{\to} L'$ if we replace X by some open neighborhood of x in X. This isomorphism ν induces an isomorphism of $\mathcal{O}_X^{\log}$-modules

$$\nu : \mathcal{O}_X^{\log} \otimes_A L \overset{\sim}{\to} \mathcal{O}_X^{\log} \otimes_A L' = \mathcal{O}_X^{\log} \otimes_A L_0. \quad \square$$

2. Examples

We see some examples of moduli space of log mixed Hodge structures from [KU09] and [KNU12].

2.1. Elliptic curve ([KU09, Ch. 0])

Let $H_0 := \mathbf{Z}^2 = \mathbf{Z}e_1 + \mathbf{Z}e_2$, weight $= -1$, $\langle \ , \ \rangle$ anti-symmetric form defined by $\langle e_2, e_1 \rangle = 1$, D the classifying space of polarized Hodge structures of weight -1, and $\check{D}$ the compact dual. D is identified with upper-half plane $\mathfrak{h}$ via $\tau \leftrightarrow F^0 := \mathbf{C}(\tau e_1 + e_2)$ ($\tau \in \mathfrak{h}$), and $\check{D} = \mathbf{P}^1(\mathbf{C})$.

For a fan $\Sigma := \{\mathbf{R}_{\ge 0} N \mid N \in \mathfrak{sl}(2, \mathbf{Q})\}$, we have $D_\Sigma = D \cup \mathbf{P}^1(\mathbf{Q})$. In fact, for $a \in \mathbf{P}^1(\mathbf{Q})$, let $V_a := \mathbf{R}(a e_1 + e_2)$, and define

$$\sigma_a := \{N \in \mathfrak{sl}(2, \mathbf{R}) \mid N(H_{0,\mathbf{R}}) \subset V_a, \ N(V_a) = 0, \ \langle x, Nx \rangle \ge 0 \ (\forall x \in H_{0,\mathbf{R}})\},$$

$$Z_a := \{F \in \check{D} \mid F^1 \ne V_{a,\mathbf{C}}\}.$$

Then, $\mathbf{P}^1(\mathbf{Q}) \ni a \leftrightarrow (\sigma_a, Z_a) \in D_\Sigma \smallsetminus D$.

For example, $V_\infty = \mathbf{R}e_1$, $\sigma := \sigma_\infty = \begin{pmatrix} 0 & \mathbf{R}_{\ge 0} \\ 0 & 0 \end{pmatrix}$, $Z_\infty = \mathbf{C} \subset \mathbf{P}^1(\mathbf{C}) = \check{D}$. Let $\Gamma := \mathrm{Ker}(\mathrm{SL}(2, \mathbf{Z}) \to \mathrm{SL}(2, \mathbf{Z}/n\mathbf{Z}))$. It is well-known that Γ is neat if and only if $n \ge 3$.

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We assume this. Let $\Gamma(\sigma)^{\mathrm{gp}} := \Gamma \cap \exp(\sigma_{\mathbf{R}}) = \begin{pmatrix} 1 & n\mathbf{Z} \\ 0 & 1 \end{pmatrix}$. Then we have a period map and its extension

$$\begin{array}{ccc} \Delta^* & \longrightarrow & \Gamma(\sigma)^{\mathrm{gp}}\backslash D \\ \cap & & \cap \\ \Delta & \longrightarrow \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \longrightarrow \Gamma\backslash D_\Sigma \end{array} \qquad \begin{array}{ccc} e^{2\pi i \tau/n} & \longrightarrow & \tau \bmod \Gamma(\sigma)^{\mathrm{gp}} \\ \\ 0 & \longrightarrow & (\sigma, \mathbf{C}) \bmod \Gamma. \end{array}$$

2.2. Tate type ([KNU12, 7.1.1])

Let $H_0 := \mathbf{Z}^2 = \mathbf{Z}e_1 + \mathbf{Z}e_2$,

$$W_{-3} = 0 \subset W_{-2} = W_{-1} = \mathbf{R}e_1 \subset W_0 = W_{-1} + \mathbf{R}e_2 = H_{0,\mathbf{R}},$$

and D the classifying space of mixed Hodge structures with weight filtration W. Then, $D = \mathbf{C}$ via $F = F(z) \leftrightarrow z$ where

$$F^1 := 0 \subset F^0 := \mathbf{C}(ze_1 + e_2) \subset F^{-1} := F^0 + \mathbf{C}e_1 = H_{0,\mathbf{C}}.$$

Let $\Gamma := \begin{pmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{pmatrix}$, $\Sigma = \{\sigma, -\sigma, \{0\}\}$ where $\pm\sigma := \begin{pmatrix} 1 & \pm\mathbf{R}_{\geq 0} \\ 0 & 1 \end{pmatrix}$. Then, $\Gamma\backslash D = \mathbf{C}^\times \subset \Gamma\backslash D_\Sigma = \mathbf{P}^1(\mathbf{C})$ via $(\sigma, D) \leftrightarrow 0$ and $(-\sigma, D) \leftrightarrow \infty$.

$\mathbf{Z}$-*structure and model.*

Let $S \in \mathcal{B}(\log)$. Then, $\mathcal{E}xt^1_{\mathrm{LMH}/S}(\mathbf{Z}, \mathbf{Z}(1)) = M_S^{\mathrm{gp}}$ as functors on $\mathcal{B}(\log)$. In fact, for $a \in \Gamma(T, M_T^{\mathrm{gp}})$ ($T \in \mathcal{B}(\log)/S$), the corresponding object H on T of the left side functor is defined as follows. Let $\tau : T^{\log} \to T$ be as in 1.2 and let $0 \to \mathbf{Z}(1) \to \mathcal{L}_T \to \tau^{-1}M_T^{\mathrm{gp}} \to 0$ be the exact sequence of the sheaf of logarithm from 1.2. Pulling back this sequence by $\mathbf{Z} \to \tau^{-1}M_T^{\mathrm{gp}}$, $1 \mapsto a^{-1}$, we have the extension $0 \to \mathbf{Z}(1) \to H_{\mathbf{Z}} \to \mathbf{Z} \to 0$ of $\mathbf{Z}$-structures. The Hodge filtration is defined by $F^0 := \mathrm{Ker}(\mathcal{O}_T^{\log} \otimes H_{\mathbf{Z}} \xrightarrow{c} \mathcal{O}_T^{\log})$, where c is the morphism induced by $H_{\mathbf{Z}} \to \mathcal{L}_T \subset \mathcal{O}_T^{\log}$.

This functor $\mathcal{E}xt^1_{\mathrm{LMH}/S}(\mathbf{Z}, \mathbf{Z}(1)) = M_S^{\mathrm{gp}}$ is *not* representable. But the subfunctor $T \mapsto \Gamma(T, M_T \cup M_T^{-1})$ of $T \mapsto \Gamma(T, M_T^{\mathrm{gp}})$ is represented by $\Gamma\backslash D_\Sigma = \mathbf{P}^1(\mathbf{C})$ (Σ as above).

2.3. Mirror quintic

Let $w = 3$, $h^{p,q} = 1$ ($p + q = 3$, $p, q \geq 0$), and $h^{p,q} = 0$ otherwise. Let $H_0 = \bigoplus_{j=1}^4 \mathbf{Z}e_j = \mathbf{Z}^4$, and $\langle e_3, e_1 \rangle_0 = \langle e_4, e_2 \rangle_0 = 1$. Let D be the corresponding classifying space of polarized Hodge structures, and $\check{D}$ the "compact dual".

Let $S = $ (square free positive integers), and let $m \in S$. Define $N_\alpha, N_\beta, N_m \in \mathrm{End}(H_0, \langle \ , \ \rangle_0)$ as follows.

$$N_\alpha(e_3) = e_1, \quad N_\alpha(e_j) = 0 \ (j \neq 3);$$
$$N_\beta(e_4) = e_3, \quad N_\beta(e_3) = -e_1, \quad N_\beta(e_1) = -e_2, \quad N_\beta(e_2) = 0;$$
$$N_m(e_1) = e_3, \quad N_m(e_4) = -me_2, \quad N_m(e_2) = N_m(e_3) = 0.$$

Let $\sigma_\alpha = \mathbf{R}_{\geq 0}N_\alpha$, $\sigma_\beta = \mathbf{R}_{\geq 0}N_\beta$, and $\sigma_m = \mathbf{R}_{\geq 0}N_m$ ($m \in S$).

Proposition ([KU09, 12.3]). *Let $\Sigma = $ (rational nilpotent cones in $\mathfrak{g}_{\mathbf{R}}$ of rank ≤ 1). Then $\Sigma = \{\mathrm{Ad}(g)\sigma \mid \sigma = \{0\}, \sigma_\alpha, \sigma_\beta, \sigma_m \ (m \in S), g \in G_{\mathbf{Q}}\}$. This is a complete fan, i.e., if there exists $Z \subset \check{D}$ such that (σ, Z) is a nilpotent orbit, then $\sigma \in \Sigma$.*

$\Gamma \backslash D_\Sigma$ is the complete fine moduli of log Hodge structures of specified type $\Phi = (H_0, w, \langle \ , \ \rangle_0, (h^{p,q})_{p,q}, \Sigma, \Gamma)$ which receives the log period map for H^3 of mirror quintic 3-folds. Indeed, the period map of mirror quintic 3-folds is uniquely extended over the compactification of the moduli and a generic global Torelli theorem for the period map of mirror quintic 3-folds is proved by using the boundary points in [U08].

2.4. Néron model

We write here a relative version of the moduli space of LMH after [KNU12, 5–6].

Let S be an object of $\mathcal{B}(\log)$, and assume that, for each $w \in \mathbf{Z}$, we are given a polarized log Hodge structure $H_{(w)}$ of weight w on S. Assume $H_{(w)} = 0$ for almost all w. Let S° be S regarded as an object of $\mathcal{B}$ by forgetting the log structure, and $S' \in \mathcal{B}/S^\circ$ is endowed with the pullback of the log structure of S.

Theorem ([KNU12, 6.1.1]). *In the above situation, there is a relative log manifold J_1 (resp. J_0) over S which is strict over S, i.e., the log structure is the pullback of that of S, and which represents the following functor on $\mathcal{B}/S^\circ$:*

$\mathcal{B}/S^\circ \ni S' \mapsto \big($the set of all isomorphism classes of LMH H on S' with $H(\mathrm{gr}_w^W) = H_{(w)}$ $(w \in \mathbf{Z})$ satisfying the following condition: Locally on S', there is a splitting $H_{\mathbf{Q}}(\mathrm{gr}^W) \simeq H_{\mathbf{Q}}$ (resp. $H_{\mathbf{Z}}(\mathrm{gr}^W) \simeq H_{\mathbf{Z}}$) of the weight filtration W on the local system $H_{\mathbf{Q}}$ (resp. $H_{\mathbf{Z}}$) on $(S')^{\log}\big)$.

We specialize the case. Let w be a negative integer. Assume $H_{(k)} = 0$ for $k \neq w, 0$, and that $H_{(0)} = \mathbf{Z}$. Let $H' = H_{(w)}$. Consider the exact sequence on $S^{\log}$

$$0 \to H'_{\mathbf{Z}} \to H'_{\mathcal{O}^{\log}}/F^0 \to H'_{\mathbf{Z}} \backslash H'_{\mathcal{O}^{\log}}/F^0 \to 0.$$

Descending this by $\tau : S^{\log} \to S$, we have an exact sequence of sheaves of abelian groups on $\mathcal{B}(\log)/S$

$$0 \to \tau_* H'_{\mathbf{Z}} \backslash (H'_{\mathcal{O}}/F^0 H'_{\mathcal{O}})^{sG} \to \mathcal{E}xt^1(\mathbf{Z}, H') \to R^1\tau_* H'_{\mathbf{Z}}.$$

Here $(\ )^{sG}$ is the part restricted by the small Griffiths transversality (1.3). $\mathcal{E}xt^1(\mathbf{Z}, H') = \mathcal{E}xt^1_{\mathrm{LMH}/S}(\mathbf{Z}, H')$ is the sheaf functor

$$\mathcal{B}(\log)/S \ni S' \mapsto \mathrm{Ext}^1_{\mathrm{LMH}/S'}(\mathbf{Z}, H'|_{S'}),$$

where the image is the set of isomorphism classes of log mixed Hodge structure H on S' such that $0 \to H'|_{S'} \to H \to \mathbf{Z} \to 0$ is exact in the category of log mixed Hodge structures, i.e., exact of lattice, of weight filtration, and of Hodge filtration. $\mathcal{E}xt^1(\mathbf{Z}, H')$ is identified, via $H \mapsto e_F - e_{\mathbf{Z}}$, with the subgroup of $\tau_*(H'_{\mathbf{Z}} \backslash H'_{\mathcal{O}^{\log}}/F^0)^{sG}$ restricted by the condition of the admissibility of local monodromy.

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Proposition ([KNU12, 6.1.4]). *In the above situation:*
(i) *On $\mathcal{B}/S^\circ$, the Néron model represents*

$$\mathrm{Ker}(\mathcal{E}xt^1(\mathbf{Z}, H') \to R^1\tau_* H'_{\mathbf{Z}} \to R^1\tau_* H'_{\mathbf{Q}}).$$

(ii) *On $\mathcal{B}/S^\circ$, the connected Néron model represents*

$$\mathrm{Ker}(\mathcal{E}xt^1(\mathbf{Z}, H') \to R^1\tau_* H'_{\mathbf{Z}}) = \tau_* H'_{\mathbf{Z}} \backslash (H'_{\mathcal{O}}/F^0 H'_{\mathcal{O}})^{sG}.$$

We further specialize the case. Assume that S is a complex analytic manifold with a smooth divisor which gives the log structure of S, and denote by S^* the complement of the divisor in S. Assume $w = -1$.

Corollary ([KNU12, 6.1.6, 6.1.8]). *In the above situation, the Néron model J_1 represents $\mathcal{E}xt^1(\mathbf{Z}, H')$ on $\mathcal{B}/S^\circ$, and we have*

$$(\textit{a section of the Néron model over } S)$$
$$= (\textit{a normal function on } S^* \textit{ which is admissible with respect to } S).$$

3. Quintic threefolds

We recall some results for A-model of quintic threefold and B-model of mirror quintic mainly using the notation from [CoK99]. We introduce $\mathbf{Z}$-structure on $S^{\log}$, and prove the equivalence of the formulation 0 (4) with the other formulations (1)–(3) in Introduction. We notice that N in Section 3 is $-N$ of N in Sections 1–2.

3.1. Picard-Fuchs ideal

We review here [CoK99, 5.1.2]. Let $X \to S$ be a proper smooth family of Kähler manifolds of dimension n over a smooth base S, and let H be the variation of Hodge structure over S arising from this family on the n-th relative cohomology sheaf. Let $\mathcal{D}$ be the sheaf of rings of holomorphic differential operators on S. The Gauss-Manin connection ∇ on H gives the $\mathcal{D}$-module structure on H. Let Ω be a fixed local section of the Hodge filter F^n at a point $p \in S$. Let $z_1, \ldots, z_r$ be local coordinates of S at p. Then, locally on S, $\mathcal{D} = \mathbf{C}\{z_1, \ldots, z_r\}[\partial/\partial z_1, \ldots, \partial/\partial z_r]$ and the Gauss-Manin connection ∇ define an $\mathcal{O}_S$-homomorphism

$$\phi: \mathcal{D} \to F^0 \quad \text{by} \quad \phi(X_1 \cdots X_m) := \nabla_{X_1} \cdots \nabla_{X_m} \Omega$$

for vector fields X_i on S. The ideal $I := \mathrm{Ker}(\phi)$ is called the *Picard-Fuchs ideal*. Since

$$\frac{\partial}{\partial z_j} \int_\gamma \Omega = \int_\gamma \nabla_{\partial/\partial z_j} \Omega$$

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for a locally constant homology n-cycle γ, the periods $y := \int_\gamma \Omega$ of Ω are solutions of the equations $Dy = 0$ for all $D \in I$.

By analytic continuation, the above things make sense as algebraic differential equations over all S if we restrict our attention to the periods of algebraic differentials.

3.2. Quintic threefold and its mirror

Let V be a generic quintic threefold in $\mathbf{P}^4$. Let V_ψ be a hypersurface in $\mathbf{P}^4$ defined by $f := x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \psi x_1 x_2 x_3 x_4 x_5 = 0$ ($\psi \in \mathbf{P}^1$). Let $G := \{(a_j) \in (\mu_5)^5 \mid a_1 \ldots a_5 = 1\}$. Under the coordinate-wise action, $x_j \mapsto a_j x_j$, take the quotient V_ψ/G. Then a mirror to V is given by a resolution V_ψ° of the quotient singularity of V_ψ/G (see [M93], [CoK99]).

3.3. Picard-Fuchs equation on the mirror V°

Let Ω be the holomorphic 3-form on the mirror V_ψ° induced from the one

$$\mathrm{Res}_{V_\psi} \left(\frac{\psi}{f} \sum_{j=1}^5 (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge (dx_j)^\wedge \wedge \cdots \wedge dx_5 \right)$$

on V_ψ. Let $z := 1/\psi^5$ and $\delta := z d/dz$. For Ω, the Picard-Fuchs equation near $z = 0$ is

$$L := \delta^4 + 5z(5\delta + 1)(5\delta + 2)(5\delta + 3)(5\delta + 4),$$

$$Ly = \left(\delta^4 + \frac{2 \cdot 5^5 z}{1 + 5^5 z} \delta^3 + \frac{7 \cdot 5^4 z}{1 + 5^5 z} \delta^2 + \frac{2 \cdot 5^4 z}{1 + 5^5 z} \delta + \frac{24 \cdot 5z}{1 + 5^5 z} \right) y = 0$$

(cf. [CoK99, 5.4]), and its indicial equation at $z = 0$ is $\rho^4 = 0$, i.e., maximally unipotent. By the Frobenius method, we have a basis of solutions $y_i(z)$ ($0 \le i \le 3$) as follows. Let

$$\tilde{y}(-z; \rho) := \sum_{n=0}^\infty \frac{\prod_{m=1}^{5n}(5\rho + m)}{\prod_{m=1}^n (\rho + m)^5} (-z)^{n+\rho}$$

be a solution of $L(\tilde{y}(-z; \rho)) = \rho^4 (-z)^\rho$, and let

$$\tilde{y}(-z; \rho) = y_0(z) + y_1(z)\rho + y_2(z)\rho^2 + y_3(z)\rho^3 + \cdots, \quad y_i(z) := \frac{1}{i!} \frac{\partial^i \tilde{y}(-z; \rho)}{\partial \rho^i} \Big|_{\rho=0}$$

be the Taylor expansion at $\rho = 0$. The first two functions of the basis are computed as

$$y_0 = \sum_{n=0}^\infty \frac{(5n)!}{(n!)^5} (-z)^n,$$

$$y_1 = y_0 \log(-z) + 5 \sum_{n=1}^\infty \frac{(5n)!}{(n!)^5} \left(\sum_{j=n+1}^{5n} \frac{1}{j} \right) (-z)^n.$$

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Let $t := y_1/y_0$ and $u := t/2\pi i$. The canonical coordinate is given by $q := e^t = e^{2\pi iu}$ which is a specific chart of the log structure and gives the mirror map.

The Gauss-Manin potential of the mirror V_ψ° is given by

$$\Phi_{\mathrm{GM}}^{V^\circ} = \frac{5}{2}\left(\frac{y_1}{y_0}\frac{y_2}{y_0} - \frac{y_3}{y_0}\right)$$

(cf. [CoK99, Theorem 8.6.5]).

[CDGP91] computed the above complete solutions of the Picard-Fuchs equation by using Gamma function, their analytic continuations, and local monodoromies around $z = 0, \infty$, and $-1/5^5$. The Yukawa coupling at the maximally unipotent point $z = 0$ is given by

$$Y := -\int_{V^\circ} \tilde{\Omega} \wedge \nabla_\delta \nabla_\delta \nabla_\delta \tilde{\Omega} = \frac{5}{(1+5^5z)y_0(z)^2}\left(\frac{q}{z}\frac{dz}{dq}\right)^3,$$

where $\tilde{\Omega} := \Omega/y_0$ is the normalized holomorphic 3-form (cf. [CoK99, Example 5.6.4.1]).

3.4. A-model of the quintic V

Let V be a generic quintic threefold. Then $\mathrm{Aut}(V) = \{1\}$. Let $T_1 := H$ be the hyperplane section of V in $\mathbf{P}^4$ and let $K(V) = \mathbf{R}_{>0}T_1$ be the Kähler cone of V. Let u be the coordinate of $\mathbf{C}T_1$. The complexified Kähler moduli $\mathcal{KM}(V)$ of V is isomorphic to the punctured disc:

$$\mathcal{KM}(V) = (H^2(V,\mathbf{R}) + iK(V))/\mathrm{Image}\, H^2(V,\mathbf{Z}) \xrightarrow{\sim} \Delta^* \subset \Delta, \quad uT_1 \mapsto q := \exp(2\pi iu).$$

The basis of $H_2(V,\mathbf{Z}) \simeq \mathbf{Z}$ is given by the homology class C of the line on V. Let T^1 be the cohomology class of the line, i.e., the Poincaré dual of C. For $\beta = dC \in H_2(V,\mathbf{Z})$ $(d \in \mathbf{Z})$, define $q^\beta := q^{\int_\beta T^1} = q^d$. The Gromov-Witten potential of V is defined by

$$\Phi_{\mathrm{GW}}^V := \frac{1}{6}\int_V (tT_1)^3 + \sum_{0 \neq \beta \in H_2(V,\mathbf{Z})} \langle I_{0,0,\beta}\rangle q^\beta$$

$$= \frac{5t^3}{6} + \sum_{d>0}\langle I_{0,0,d}\rangle q^d = \frac{5(2\pi iu)^3}{6} + \sum_{d>0}\langle I_{0,0,d}\rangle q^d$$

([CoK99, (8.51)]), where $t = 2\pi iu$. Here $N_d := \langle I_{0,0,d}\rangle$ is the Gomov-Witten invariant defined as follows. Fix a positive integer d and consider the diagram of moduli spaces of stable curves

$$\overline{M}_{0,0}(\mathbf{P}^4,d) \xleftarrow{\pi_1} \overline{M}_{0,1}(\mathbf{P}^4,d) \xrightarrow{e_1} \mathbf{P}^4$$

where π_1 forgets the marked point and e_1 is the evaluation map. Define

$$\langle I_{0,0,d}\rangle := \int_{\overline{M}_{0,0}(\mathbf{P}^4,d)} c_{5d+1}(\pi_{1*}e_1^*\mathcal{O}_{\mathbf{P}^4}(5)).$$

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We have $\langle I_{0,0,d} \rangle = \sum_{k|d} n_{d/k} k^{-3}$, where $n_{d/k}$ is the instanton number (see, e.g., [CoK99, 11.1.1]). $N_d = 0$ if $d \leq 0$.

3.5. Z-structure

We first consider B-model $\mathcal{H}^{V^{\circ}}$ of quintic-mirror from (3) in Introduction.

Let $f : X \to S^*$ be a smooth projective family of quintic-mirrors over a punctured neighborhood of the maximally unipotent point. Let $\mathcal{H}_{\mathbf{Z}}^{V^{\circ}} = R^3 f_* \mathbf{Z}$, and extend this to the local system of $\mathbf{Z}$-modules over $S^{\log}$ (cf. [KU09, 2.3.5], [O03], [U01]), still denoted $\mathcal{H}_{\mathbf{Z}}^{V^{\circ}}$. Let N be the monodromy logarithm at the maximally unipotent point, and $W = W(N)$ be the monodromy weight filtration. Define $W_{k,\mathbf{Z}} := W_k \cap \mathcal{H}_{\mathbf{Z}}^{V^{\circ}}$ for all k.

Take a point $b \in S^{\log}$ and fix it. We claim that there is a symplectic $\mathbf{Z}$-basis g_0, g_1, g_3, g_2 of $\mathcal{H}_{\mathbf{Z}}^{V^{\circ}}(b)$ for the cup product $\langle \ | \ \rangle$ which gives a splitting of the filtration W, i.e., $g_0, \ldots, g_k$ generate W_{2k} for all k, and $\langle g_0 | g_3 \rangle = 1$, $\langle g_1 | g_2 \rangle = 1$, $\langle g_0 | g_j \rangle = 0$ for $j \neq 3$, $\langle g_1 | g_j \rangle = 0$ for $j \neq 2$. This claim follows from [M93]. In fact, let g_0, g_1 be a good integral basis of $W_2(b)$ in the sense of [M93, 2], i.e., g_0 is a generator of $W_{0,\mathbf{Z}}(b)$, and g_1 is an indivisible element of $\mathcal{H}_{\mathbf{Z}}^{V^{\circ}}(b)$ which can be written as $g_1 = (1/\lambda)N^2 g$ for some $\lambda > 0$ and some $g \in \mathcal{H}_{\mathbf{Z}}^{V^{\circ}}(b)$ such that $\langle g_0 | g \rangle = 1$. Put $g_3 := g$. Since the cup product is skew-symmetric unimodular, we can take $g_2' \in \mathcal{H}_{\mathbf{Z}}^{V^{\circ}}(b)$ such that $\langle g_1 | g_2' \rangle = 1$. Let $g_2 := g_2' - \langle g_0 | g_2' \rangle g_3 + \langle g_3 | g_2' \rangle g_0$. Then, g_0, g_1, g_2, g_3 is a desired basis.

Log Hodge theoretic description of $\mathcal{H}_{\mathbf{Z}}^{V^{\circ}}$ is as follows. Let $(S^{\log}, \mathcal{O}_S^{\log})$ be the ringed space from 1.2. Then, for a local section $s \in \mathcal{O}_S^{\log} \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O}}^{V^{\circ}}$, the following statements are equivalent:

(1) s belongs to $\mathcal{H}_{\mathbf{Z}}^{V^{\circ}}$.
(2) $\nabla s = 0$ $(\nabla = \nabla^{\mathrm{GM}})$ and $s(b) \in \mathcal{H}_{\mathbf{Z}}^{V^{\circ}}(b)$ for some $b \in S^{\log}$.
(3) $\nabla s = 0$ and $s(\mathrm{gr}_k^W) \in \mathrm{gr}_{k,\mathbf{Z}}^W$ for $k := \min\{l \ | \ s \in \mathcal{O}_S^{\log} \otimes W_l\}$.

Now we consider A-model $\mathcal{H}^V$ of quintic threefold from (3) in Introduction.

Let $\nabla = \nabla^{\mathrm{middle}}$ be the A-model connection from 3.6 (3A) below. For $s \in \mathcal{O}_S^{\log} \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O}}^V$, define $s \in \mathcal{H}_{\mathbf{Z}}^V$ if $\nabla s = 0$ and $s(\mathrm{gr}_{2p}^W) \in H^{3-p,3-p}(V, \mathbf{Z})$ for $p := \min\{q \ | \ s \in \mathcal{O}_S^{\log} \otimes W_{2q}\}$. Here $W_{2q} := \bigoplus_{l \leq q} H^{3-l,3-l}(V, \mathbf{R})$.

For log Hodge theoretic description, we recall the following. Let $0 \in S = \Delta$ and $b \in \tau^{-1}(0) \subset S^{\log}$. From 1.2, the stalk over b is $\mathcal{O}_{S,b}^{\log} = \mathcal{O}_{S,0}[t] = \mathbf{C}\{q\}[t]$ which is a polynomial ring in t over the convergent power series in q, where $q = e^{2\pi i u} = e^t$ from 3.4. Let $u = x + iy$ with x, y real.

For $s \in \mathcal{O}_S^{\log} \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O}}^V$, the following statements are equivalent:

(4) s belongs to $\mathcal{H}_{\mathbf{Z}}^V$.
(5) $\nabla s = 0$ and $s(b) \in \mathcal{H}_{\mathbf{Z}}^V(b)$ for some $b \in S^{\log}$.
(6) $\nabla s = 0$ and, for fixed $x = 0$, the limit of $\exp(iy(-N))s$ over $S^{\log}$ belongs to $\bigoplus_p H^{p,p}(V, \mathbf{Z})$.
(7) $\nabla s = 0$ and the specialization $x \mapsto 0$ of the limit of $\exp(iy(-N))s$ over $S^{\log}$ with x fixed and $y \to \infty$ belongs to $\bigoplus_p H^{p,p}(V, \mathbf{Z})$.

For computation of specific examples, see 3.7 below.

3.6. Correspondence table

The following is the correspondence table expanded the one from [CoK99, 8.5–8.6].

(1A) *Polarization of A-model of V.*

$$S(\alpha, \beta) := (-1)^p \int_V \alpha \cup \beta \quad (\alpha \in H^{p,p}(V), \beta \in H^{3-p,3-p}(V)).$$

(1B) *Polarization of B-model of V°.*

$$Q(\alpha, \beta) := (-1)^{3(3-1)/2} \int_{V^\circ} \alpha \cup \beta = -\int_{V^\circ} \alpha \cup \beta \quad (\alpha, \beta \in H^3(V^\circ)).$$

(2A) *Specified sections inducing $\mathbf{Z}$-basis of graded quotients of A-model of V.*
We take

$$T_0 := 1 \in H^0(V, \mathbf{Z}), T_1 := H \in H^2(V, \mathbf{Z}), T^1 := C \in H^4(V, \mathbf{Z}), T^0 := [pt] \in H^6(V, \mathbf{Z}),$$

Then $S(T_0, T^0) = 1$ and $S(T_1, T^1) = -1$. Hence T_0, T_1, $-T^0$, T^1 form a symplectic base for S.

(2B) *Specified sections inducing $\mathbf{Z}$-basis of graded quotients of B-model of V°.*
After [D97], we use the decomposition of the associated mixed Hodge structure of Hodge-Tate type $(\mathcal{W}, \mathcal{F})$:

$$\mathcal{H}_\mathcal{O} = \bigoplus_p I^{p,p}, \quad \text{where} \quad I^{p,p} := \mathcal{W}_{2p} \cap \mathcal{F}^p \overset{\sim}{\to} \mathrm{gr}_{2p}^\mathcal{W}.$$

Since $N(\mathrm{gr}_{2p}^W) = 0$, gr_{2p}^W is a constant sheaf and hence

$$\mathrm{gr}_{2p}^\mathcal{W} \supset \mathrm{gr}_{2p}^W \supset (\mathrm{gr}_{2p}^W)_\mathbf{Z} := W_{2p,\mathbf{Z}}/W_{2p-1,\mathbf{Z}}.$$

Now we can take

$$e_0 := \tilde{\Omega} \in I^{3,3}, e_1 \in I^{2,2}, e^1 \in I^{1,1}, e^0 = g_0 \in I^{0,0}$$

inducing generators of $(\mathrm{gr}_{2p}^W)_\mathbf{Z}$ and satisfying $Q(e_0, e^0) = 1$ and $Q(e_1, e^1) = -1$. Hence e_0, e_1, $-e^0$, e^1 form a symplectic base for Q.

(3A) *A-model connection $\nabla = \nabla^{\mathrm{middle}}$ of V.*
For $\delta := d/du$, A-model connection is defined by

$$\nabla_\delta T^0 := 0, \quad \nabla_\delta T^1 := T^0,$$
$$\nabla_\delta T_1 := \frac{1}{(2\pi i)^3} \frac{d^3\Phi}{du^3} T^1 = \left(5 + \frac{1}{(2\pi i)^3} \frac{d^3\Phi_{\mathrm{hol}}}{du^3}\right) T^1,$$
$$\nabla_\delta T_0 := T_1.$$

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∇ is flat, i.e., $\nabla^2 = 0$ (cf. [CoK99, Prop. 8.5.2]).

(3B) *B-model connection* $\nabla = \nabla^{\mathrm{GM}}$ *of* V°.
For $\delta := d/du$, the Gauss-Manin connection is computed in [CDGP91] as

$$\nabla_\delta e^0 = 0, \quad \nabla_\delta e^1 = e^0,$$

$$\nabla_\delta e_1 = \frac{1}{(2\pi i)^3} \frac{d^3 \Phi}{du^3} e^1 = Y e^1 = \frac{5}{(1 + 5^5) y_0(z)^2} \left(\frac{q}{z} \frac{dz}{dq} \right)^3 e^1,$$

$$\nabla_\delta e_0 = e_1.$$

(4A) ∇-*flat* **Z**-*basis for* $\mathcal{H}_{\mathbf{Z}}^V$.
It is computed as

$$s^0 := T^0,$$

$$s^1 := T^1 - uT^0 = \exp(-uH)T^1,$$

$$s_1 := T_1 - \frac{1}{(2\pi i)^3} \frac{d^2 \Phi}{du^2} T^1 + \frac{1}{(2\pi i)^3} \frac{d\Phi}{du} T^0$$

$$= \exp(-uH)T_1 - \left(\sum_{d>0} \frac{N_d d^2}{2\pi i} q^d \right) T^1 + \left(\sum_{d>0} \frac{N_d d}{(2\pi i)^2} q^d \right) T^0,$$

$$s_0 := T_0 - uT_1 + \frac{1}{(2\pi i)^3} \left(u \frac{d^2 \Phi}{du^2} - \frac{d\Phi}{du} \right) T^1 - \frac{1}{(2\pi i)^3} \left(u \frac{d\Phi}{du} - 2\Phi \right) T^0$$

$$= \exp(-uH)T_0 + \left(\sum_{d>0} \frac{N_d d^2}{2\pi i} u q^d - \sum_{d>0} \frac{N_d d}{(2\pi i)^2} q^d \right) T^1$$

$$- \left(\sum_{d>0} \frac{N_d d}{(2\pi i)^2} u q^d - \sum_{d>0} \frac{2N_d}{(2\pi i)^3} q^d \right) T^0$$

(4B) ∇-*flat* **Z**-*basis for* $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$.
It is computed as

$$s^0 := e^0,$$

$$s^1 := e^1 - ue^0,$$

$$s_1 := e_1 - \frac{1}{(2\pi i)^3} \frac{d^2 \Phi}{du^2} e^1 + \frac{1}{(2\pi i)^3} \frac{d\Phi}{du} e^0,$$

$$s_0 := e_0 - ue_1 + \frac{1}{(2\pi i)^3} \left(u \frac{d^2 \Phi}{du^2} - \frac{d\Phi}{du} \right) e^1 - \frac{1}{(2\pi i)^3} \left(u \frac{d\Phi}{du} - 2\Phi \right) e^0.$$

This is an explicit description of the **Z**-basis of $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$ (3.5) by using the basis from (2B).

(5A) *Monodromy logarithm for A-model of V around the large radius limit point.*

It is computed as

$$Ns^0 = 0, \quad Ns^1 = -s^0, \quad Ns_1 = -5s^1, \quad Ns_0 = -s_1.$$

Note that the matrix of monodromy logarithm N via the basis s^0, s^1, s_1, s_0 coincides with the matrix of cup product with $-H$ via the basis T^0, T^1, T_1, T_0 (2A).

(5B) *Monodromy logarithm for B-model of V° around the maximally unipotent monodromy point.*

It is computed as

$$Ns^0 = 0, \quad Ns^1 = -s^0, \quad Ns_1 = -5s^1, \quad Ns_0 = -s_1.$$

(6A) *Expression of T^i, T_i by s^i, s_i.*

It is computed that T^i, T_i are written by the ∇-flat $\mathbf{Z}$-basis s^i, s_i of $\mathcal{H}^V_{\mathbf{Z}}$ as follows.

$$T^0 = s^0,$$
$$T^1 = s^1 + us^0,$$
$$T_1 = s_1 + \frac{1}{(2\pi i)^3}\frac{d^2\Phi}{du^2}s^1 + \frac{1}{(2\pi i)^3}\left(u\frac{d^2\Phi}{du^2} - \frac{d\Phi}{du}\right)s^0,$$
$$= \left(s_1 + 5us^1 + \frac{5}{2}u^2 s^0\right) + \left(\sum_{d>0}\frac{N_d d^2}{2\pi i}q^d\right)s^1 + \left(\sum_{d>0}\frac{N_d d^2}{2\pi i}uq^d - \sum_{d>0}\frac{N_d d}{(2\pi i)^2}q^d\right)s^0$$
$$T_0 = s_0 + us_1 + \frac{1}{(2\pi i)^3}\frac{d\Phi}{du}s^1 + \frac{1}{(2\pi i)^3}\left(u\frac{d\Phi}{du} - 2\Phi\right)s^0$$
$$= \left(s_0 + us_1 + \frac{5}{2}u^2 s^1 + \frac{5}{6}u^3 s^0\right) + \left(\sum_{d>0}\frac{N_d d}{(2\pi i)^2}q^d\right)s^1$$
$$+ \left(\sum_{d>0}\frac{N_d d}{(2\pi i)^2}uq^d - 2\sum_{d>0}\frac{N_d}{(2\pi i)^3}q^d\right)s^0.$$

Note that the section $1 = T_0$ varies with respect to the the lattice $\mathcal{H}^V_{\mathbf{Z}}$ as above while the section $[\mathrm{pt}] = T^0 = s^0$ does not.

(6B) *Expression of e^i, e_i by s^i, s_i.*

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It is computed that e^i, e_i are written by the ∇-flat $\mathbf{Z}$-basis s^i, s_i of $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$ as follows.

$$e^0 = s^0,$$
$$e^1 = s^1 + us^0,$$
$$e_1 = s_1 + \frac{1}{(2\pi i)^3}\frac{d^2\Phi}{du^2}s^1 + \frac{1}{(2\pi i)^3}\left(u\frac{d^2\Phi}{du^2} - \frac{d\Phi}{du}\right)s^0,$$
$$= \left(s_1 + 5us^1 + \frac{5}{2}u^2 s^0\right) + \left(\sum_{d>0}\frac{N_d d^2}{2\pi i}q^d\right)s^1 + \left(\sum_{d>0}\frac{N_d d^2}{2\pi i}uq^d - \sum_{d>0}\frac{N_d d}{(2\pi i)^2}q^d\right)s^0$$
$$e_0 = s_0 + us_1 + \frac{1}{(2\pi i)^3}\frac{d\Phi}{du}s^1 + \frac{1}{(2\pi i)^3}\left(u\frac{d\Phi}{du} - 2\Phi\right)s^0$$
$$= \left(s_0 + us_1 + \frac{5}{2}u^2 s^1 + \frac{5}{6}u^3 s^0\right) + \left(\sum_{d>0}\frac{N_d d}{(2\pi i)^2}q^d\right)s^1$$
$$+ \left(\sum_{d>0}\frac{N_d d}{(2\pi i)^2}uq^d - 2\sum_{d>0}\frac{N_d}{(2\pi i)^3}q^d\right)s^0$$
$$= s_0 + \frac{1}{2\pi i}\frac{y_1}{y_0}s_1 + \frac{5}{(2\pi i)^2}\frac{y_2}{y_0}s^1 + \frac{5}{(2\pi i)^3}\frac{y_3}{y_0}s^0.$$

Note that the normalized holomorphic 3-form $\tilde{\Omega} := \Omega/y_0 = e_0$ varies with respect to the the the lattice $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$ as above, while the section $g_0 = e^0 = s^0$ does not.

(7A) *A-model monodromy weight filtration of $\mathcal{H}^V$.*
It is computed as follows, which does not vary.

$$W_0 = \mathbf{R}[\mathrm{pt}] = \mathbf{R}T^0 = \mathbf{R}s^0 = W_1 \subset W_2 = W_1 + \mathbf{R}C = W_1 + \mathbf{R}T^1 = W_3$$
$$\subset W_4 = W_3 + \mathbf{R}H = W_3 + \mathbf{R}T_1 = W_5 \subset W_6 = W_5 + \mathbf{R}\cdot 1 = W_5 + \mathbf{R}T_0.$$

(8A) *A-model Hodge filtration of $\mathcal{H}^V$.*
Let $\mathcal{F}^p := F^p \times \mathcal{KM}(V) \subset \mathcal{H}_{\mathcal{O}}^V$. This is computed as follows.

$$\mathcal{F}^3 = H^{0,0}(V) = \mathbf{C}\cdot 1 = \mathbf{C}T_0 \subset \mathcal{F}^2 = \mathcal{F}^3 + H^{1,1}(V) = \mathcal{F}^3 + \mathbf{C}H = \mathcal{F}^3 + \mathbf{C}T_1$$
$$\subset \mathcal{F}^1 = \mathcal{F}^2 + H^{2,2}(V) = \mathcal{F}^2 + \mathbf{C}C = \mathcal{F}^2 + \mathbf{C}T^1$$
$$\subset \mathcal{F}^0 = \mathcal{F}^1 + H^{3,3}(V) = \mathcal{F}^1 + \mathbf{C}[\mathrm{pt}] = \mathcal{F}^1 + \mathbf{C}T^0.$$

Note that the mixed Hodge structure for A-model of V from (7A) and (8A) is opposite to the original one from (2A).

3.7. Examples of computation

Let $u = x + iy$ be as in 3.5.

First we compute for s_0 from 3.6 (4A) the specialization $x \mapsto 0$ of the limit of $\exp(iy(-N))s_0$ over $S^{\log}$ with x fixed and $y \to \infty$.

$$\exp(iy(-N))s_0 = \left(T_0 - xT_1 + \frac{5}{2}x^2T^1 - \frac{5}{6}x^3T^0\right)$$
$$+ \left(\sum_{d>0} \frac{N_d d^2}{2\pi i}xq^d - \sum_{d>0} \frac{N_d d}{(2\pi i)^2}q^d\right)T^1 + \left(2\sum_{d>0} \frac{N_d}{(2\pi i)^3}q^d - \sum_{d>0} \frac{N_d d}{(2\pi i)^2}xq^d\right)T^0$$
$$\xrightarrow{(x \text{ fixed, } y\to\infty)} T_0 - xT_1 + \frac{5}{2}x^2T^1 - \frac{5}{6}x^3T^0 \xrightarrow{(x \mapsto 0)} T_0.$$

We notice that $-N$ here in Section 3 is N in Sections 1–2.

If we use $\Phi_{\text{GM}}^{V^\circ} = \Phi_{\text{GW}}^V$, we have an analogous computation for s_0 from 3.6 (4B).

$$\exp(iy(-N))s_0 = \left(e_0 - xe_1 + \frac{5}{2}x^2e^1 - \frac{5}{6}x^3e^0\right)$$
$$+ \left(\sum_{d>0} \frac{N_d d^2}{2\pi i}xq^d - \sum_{d>0} \frac{N_d d}{(2\pi i)^2}q^d\right)e^1 + \left(2\sum_{d>0} \frac{N_d}{(2\pi i)^3}q^d - \sum_{d>0} \frac{N_d d}{(2\pi i)^2}xq^d\right)e^0$$
$$\xrightarrow{(x \text{ fixed, } y\to\infty)} e_0 - xe_1 + \frac{5}{2}x^2e^1 - \frac{5}{6}x^3e^0 \xrightarrow{(x \mapsto 0)} e_0.$$

We make a comment on the "value at $q = 0$ of the canonical extension", e.g. $\exp(-uN)s_0$ for s_0 from 3.6 (4A). It would be computed as

$$\exp(-uN)s_0 = s_0 - (uN)s_0 + \frac{1}{2}(uN)^2 s_0 - \frac{1}{6}(uN)^3 s_0$$
$$= s_0 - uNs_0 + \frac{1}{2}u^2 N^2 s_0 - \frac{1}{6}u^3 N^3 s_0 = s_0 + us_1 + \frac{5}{2}u^2 s^1 + \frac{5}{6}u^3 s^0$$
$$= T_0 + \left(\frac{5}{2}u^2 - \frac{1}{(2\pi i)^3}\frac{d\Phi}{du}\right)T^1 + \left(-\frac{5}{3}u^3 + \frac{2}{(2\pi i)^3}\Phi\right)T^0$$
$$= T_0 - \sum_{d>0} \frac{N_d d}{(2\pi i)^2}q^d T^1 + \sum_{d>0} \frac{2N_d}{(2\pi i)^3}q^d T^0 \xrightarrow{(q \to 0)} T_0.$$

In this computation, it is allowed only to take a limit over the space $S^{\log}$ with fixed x and $y \to \infty$. Then the $(uN)^j = u^j N^j$ hold in the second equality. If u goes around the origin, it does not hold. This point is not clear in [CoK99] Proposition 5.6.1 and Proposition 8.5.4, their proofs, and the arguments after them.

3.8. On equivalent statements (1)–(3) in Introduction

The formulation 0 (1), i.e., coincidence of the potentials, is proved in [LLuY97], cf. [CoK99, 11.1, especially Theorem 11.1.1].

The formulation 0 (2), i.e., coincidence of the complete set of solutions, is announced and proved in [Gi96], [Gi97p], cf. [CoK99, 11.2, especially Example 11.2.1.3].

The formulation 0 (3), i.e., coincidence of polarized variation of Hodge structure with specified sections, is proved in [M97], cf. [CoK99, 8.6.1–8.6.2, especially Theorem 8.6.2].

Assuming $\Phi_{\text{GW}}^V = \Phi_{\text{GM}}^{V^\circ}$, we made the table in 3.6. The equations 3.6 (6A) and (6B) explain 0 (2), and the equation 3.6 (6B) explains $\Phi_{\text{GM}}^{V^\circ} = \frac{5}{2}\left(\frac{y_1}{y_0}\frac{y_2}{y_0} - \frac{y_3}{y_0}\right)$.

We explain a little about the A-variation of polarized Hodge structure of the quintic V. $\mathcal{H}^V = (\mathcal{H}_\mathcal{O}^V, \nabla^{\text{middle}}, \mathcal{H}_{\mathbf{Z}}^V, S, \mathcal{F})$, constructed in 3.6, is indeed a polarized variation of Hodge structure on the punctured disc. The idea of proof is to verify the following two properties.

(1) Let $\nabla = \nabla^{\text{middle}}$. Then $\nabla \mathcal{F}^p \subset \Omega_{\mathcal{KM}(V)}^1 \otimes \mathcal{F}^{p-1}$ for all p.

(2) $\mathcal{F}^p \oplus \overline{\mathcal{F}^{3-p+1}} = \mathcal{H}_\mathcal{O}$ for all p.

(1) is easy, because the small quantum product $*_{\text{small}}$, from which A-model connection actually defined, is compatible with Hodge structure (see, e.g., [CoK99, 8.5.4]). (2) is hard. One uses the results of [CaKS87] and [CaKS86] (see, e.g., [CoK99, 8.5.4]).

3.9. Proof of (4) in Introduction

Proof. We prove the equivalence of (3) and (4) in Introduction.

We show 0 (3) $\Rightarrow$ 0 (4). Let $S^* := \mathcal{KM}(V)$ and $S := \overline{\mathcal{KM}}(V)$ for A-model, and $S^* := \mathcal{M}(V^\circ)$ and $S := \overline{\mathcal{M}}(V^\circ)$ for B-model. S is endowed with the log structure associated to the divisor $S \smallsetminus S^*$. In both cases, the variation of polarized Hodge structure on S^* extends uniquely to a log variation of polarized Hodge structure on S by the log Hodge theoretic interpretation [KU09, 2.5.14] of nilpotent orbit theorem of Schmid [S73]. The specified sections $1 = T_0$ for A-model and $\tilde{\Omega} = e_0$ for B-model extend over S as the canonical extensions as mentioned above, and the specified sections $[\text{pt}] = T^0$ for A-model and $g_0 = e^0$ (3.5, 3.6) for B-model extend over S as invariant sections.

The converse implication 0 (4) $\Rightarrow$ 0 (3) is obvious. $\square$

Another Direct Proof. If we use the correspondence table in 3.6, the implication 0 (3) $\Rightarrow$ 0 (4) is verified directly as follows.

Let S^* and S be as in the above Proof. We use u and q from 3.4 for A-model and 3.3 for B-model, respectively. Let $\tilde{S}^* := \mathbf{R} \times i(0, \infty)$ and $\tilde{S}^{\log} := \mathbf{R} \times i(0, \infty]$ be the universal covering of S^* and $S^{\log}$, respectively. The coordinate u of $\tilde{S}^*$ extends over $\tilde{S}^{\log}$. Denote $0 + i\infty \in \tilde{S}^{\log}$ by u_0 and its image $\bar{0} + i\infty \in S^{\log}$ by b. These points u_0 and b lie over the point $q = 0 \in S$ which corresponds to q_0 for A-model and p_0 for B-model, respectively.

Let $H_{\mathbf{Z}} := \mathcal{H}_{\mathbf{Z}}^V$ for A-model, and $H_{\mathbf{Z}} := \mathcal{H}_{\mathbf{Z}}^{V^\circ}$ for B-model. The local system $H_{\mathbf{Z}}$ over S^*, endowed with the respective symplectic basis $s_0, s_1, -s^0, s^1$ from (4A) and (4B) in 3.6, extends as a local system over $S^{\log}$ with the extended symplectic basis (cf. [KU09, 2.3.5], [O03], [U01]). Note that to fix a base point $u = u_0$ on $\tilde{S}^{\log}$ is equivalent to fix a base point b on $S^{\log}$ and also a branch of $(2\pi i)^{-1} \log q$. Regarding the stalk $H_0 := H_{\mathbf{Z},u_0} = H_{\mathbf{Z},b}$ as a constant sheaf on $S^{\log}$, we have an isomorphism $\mathcal{O}_S^{\log} \otimes H_{\mathbf{Z}} \simeq \mathcal{O}_S^{\log} \otimes H_0$ of $\mathcal{O}_S^{\log}$-modules whose restriction induces the identity $1 \otimes H_{\mathbf{Z},b} = 1 \otimes H_0$ ([KU09, 2.3.2], see Appendix to Section 1).

The direct image $\tau_*(\mathcal{O}_S^{\log} \otimes H_{\mathbf{Z}})$, via $\tau : S^{\log} \to S$ from 1.2, yields the canonical extension $\mathcal{O}_S$-module of Deligne over S of $H_\mathcal{O}$ over S^*. The bases T_0, T_1, T^1, T^0 from

(2A) and e_0, e_1, e^1, e^0 from (2B) in 3.6 yield monodromy invariant bases of $\mathcal{O}_S$-modules respecting the Hodge filtration for each case. These bases extend over $q = 0$ and yield the canonical extensions of the respective Hodge filtrations. This is because, by (6A) and (6B) in 3.6, T_j, T^j and e_j, e^j are linear combinations of respective s_j, s^j with coefficients in $\mathcal{O}_{S,b}^{\log} = \mathbf{C}\{q\}[u]$ (u is considered as a multi-values function on a neighborhood of $q = 0$ in S^*), and s_j, s^j are $\mathcal{O}_{S,b}^{\log}$-linear combination of the elements $s_j(b), s^j(b)$, and hence T_j, T^j and e_j, e^j are $\mathcal{O}_{S,b}^{\log}$-linear combination of the constant sections $s_j(b), s^j(b) \in H_{\mathbf{Z},b} = H_0$. For $h \in \mathcal{O}_{S,b}^{\log}$ appeared as the last coefficients, h is monodromy invariant holomorphic function on S^* which satisfies $\lim_{q \to 0} qh = 0$, and hence $q = 0$ is a removable singularity of h. Thus the value of h at $q = 0$ is determined. (Using the table in 3.6, we can also examine the above argument by a direct computation.)

Now we verify that the polarized variation of Hodge structure H of type (Λ, Γ) over S^* extends uniquely to a log variation of polarized Hodge structure of type $\Phi = (\Lambda, \Sigma, \Gamma)$ over S, where $\Gamma = \exp(\mathbf{Z}N)$ and $\Sigma = \{\{0\}, \sigma\}$ with N and σ from 0 (4). (Note that N here is $-N$ of N in Sections 1–2.) In fact, we have already seen the canonical extensions of Hodge filters as $\mathcal{O}_S$-modules. Admissibility 1.3 (1) is obvious in pure case. Big Griffiths transversality 1.3 (7) follows from the above choice of the bases T_0, T_1, T^1, T^0 and e_0, e_1, e^1, e^0. Positivity 1.3 (3) is verified by direct computation using these bases and the polarizations S from (1A) and Q from (1B) in 3.6 in the following way. We do this for B-model case. A-model case is analogous. For $u = u_0$, let $F(0) \in \check{D}$ be the fixed Hodge filtration corresponding to the basis $e_0(u_0), e_1(u_0), e^1(u_0), e^0(u_0)$. Let $F_y := \exp(iy(-N))F(0) \in \check{D}$. Then, $v_3(y) := \exp(iy(-N))e_0(u_0)$ and $\exp(iy(-N))e_1(u_0)$ form a basis of F_y^2 respecting F_y^3. From this, we can compute a basis $v_2(y)$ of $F_y^2 \cap \overline{F_y^1} = F_y^2 \cap (\overline{F_y^3})^\perp$, where $\perp$ is for Q. We can check that the coefficients of the highest terms in the variable y of the Hodge norms $i^3 Q(v_3(y), \overline{v_3(y)})$ and $iQ(v_2(y), \overline{v_2(y)})$ are both positive.

The extension of the specific sections has already seen. $\square$

3.10. Comments

(i) We want to see the relation of N_β from 2.3 and the monodromy logarithm N from 0 (4). It is enough to see the relation of the following A and B.

$$A := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We want to find $g \in \mathrm{Sp}(4, \mathbf{Q})$ and a non-zero real number r such that $B = r^{-1} g A g^{-1}$.

Such g and r are given by $g = \begin{pmatrix} a & 0 & 0 & 0 \\ b & ra & 0 & 0 \\ e & 5rb & 5r^2 a & 0 \\ h & re & 5r^2 b & 5r^3 a \end{pmatrix}$ such that $2ae = 5b^2$ and

$5r^3a^2 = 1$. For example, for $a = 5$, $b = e = 2$ and $r = 1/5$, $g = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ h & 2/5 & 2/5 & 1/5 \end{pmatrix}$.

We cannot find g in $\mathrm{Sp}(4, \mathbf{Z})$, because of the second condition for g.

(ii) For general Calabi-Yau threefolds satisfying the Integrality Conjecture, we can show the equivalence of the formulation (4) and formulations (1)–(3) in Introduction. The result will appear somewhere.

(iii) Open mirror symmetry started by Walcher [W07]. [MW09], [PSW08], [LiLY12] etc. are related works. After [MW09], in B-model side, a main subject is a normal function associated with a family of quintic-mirrors and specific algebraic cycle homologous to 0 on it, especially its values for the normalized holomorphic 3-forms, which is called domainwall tension in physics. This normal function extends over the Néron model from Corollary in 2.4. This extended normal function is an interesting topic to study.

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