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Osaka J. Math.
21 (1984), 477-487

THE MINIMUM CROSSING OF 3-BRIDGE LINKS

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(Received May 16, 1983)

1. Introduction

A natural way to simplify a given projection of a link is to decrease the number of crossings of the projection. As a natural expectation, one would like to develop an algorithm to obtain a projection of a link whcih attains the minimum crossing number. Our purpose in this paper is to present such an algorithm for 3-bridge links and to argue on its utility.

A projection $p(L)$ on a plane (or a 2-sphere) of a link L divides into $2n$ arcs of two types; the first types $b_1^+, \cdots, b_n^+$, called *over bridges*, contain only overcrossings of $p(L)$ and the second types $b_1^-, \cdots, b_n^-$, called *under bridges*, contain only undercrossings of $p(L)$. Any two of them are disjoint or intersect transversely in some of the crossings, according to whether they have the same type or not. A link L is called an *n-bridge link* if it admits a projection $p(L)=b_1^+ \cup \cdots \cup b_n^+ \cup b_1^- \cup \cdots \cup b_n^-$ with such a division into $2n$ bridges.

Let ω be an arc on the plane such that for one of bridges of $p(L)$, say b_i^+, $\omega \cap p(L)=\omega \cap b_i^+=\partial\omega$. If the interior of the subarc β of b_i^+ bounded by $\partial\omega$ contains at least one crossing of $p(L)$ then ω is called a *wave* and the replacement of b_i^+ with $(b_i^+-\beta)\cup\omega$ is called a *wave move* with ω. The wave move transforms $p(L)$ into a new projection of L which has fewer crossings than $p(L)$. Note that a wave move decreases the number of crossings but does not change the number of bridges.

In general, a sequence of wave moves does not carry a given projection of a link to a minimum-crossing one. Fig. 1-(a) and (b) show two 3-bridge projections of the trefoil which have no wave. Clearly the number of crossings of the left-hand projection is minimum among *all* of projections of the trefoil. Cancelling a pair of bridges in the right-hand one, we obtain a 2-bridge projection of the trefoil which has no wave. These examples suggest that we can not find in general a *really* minimum-crossing projection of a link, fixing the number of bridges, and in particular by wave moves. Then we shall say that an n-bridge projection is *minimum-crossing*, only meaning that the number of its crossings is minimum among n-bridge projections which the link admits.

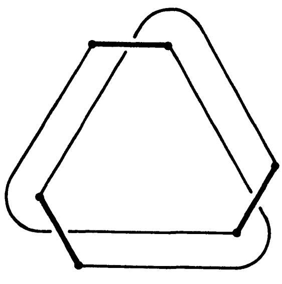


Fig. 1-(a)

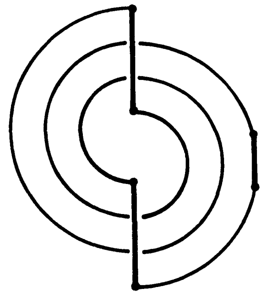
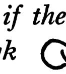


Fig. 1-(b)

Recently, Negami and Okita [4] have discussed axiomatically wave moves for 3-bridge links and have shown that every 3-bridge projection of a trivial knot or a splittable link can be transformed into a standard form by a finite sequence of wave moves; the standard form for a trivial knot is a hexagonal projection with no crossing and that for a splittable link is a disjoint union of a 1-bridge circular projection and one of Schubert's 2-bridge forms $K(p, q)$ with $2p$-2 crossings. In either case, each standard form is a minimum-crossing 3-bridge projection. The result for a trivial knot had been proved by Homma and Ochiai [2]. One may ask naturally how about other 3-bridge links. We shall give the complete answer to this question:

Theorem A. *Every* 3-*bridge projection of a link can be transformed into a minimum-crossing one by a finite sequence of wave moves if and only if the link is equivalent to one of a trivial knot, a splittable link and the Hopf link*  .

As is shown in [4], theorems on wave moves for 3-bridge projections of a link can be translated into those on wave moves for genus 2 Heegaard diagrams of an orientable closed 3-manifold as the 2-fold branched covering of a 3-sphere S^3 branched over the link. The translations of the results for a trivial knot and a splittable link have been presented in [3] and [4], respectively. Notice that the 2-fold branched covering of S^3 branched over the Hopf link is homeomorphic to a real projective space P^3. Then we have:

Theorem B. *Every genus* 2 *Heegaard diagram of* P^3 *can be transformed into one of the two standard forms shown in Fig.* 2-(a) *and* (b) *by a finite sequence*

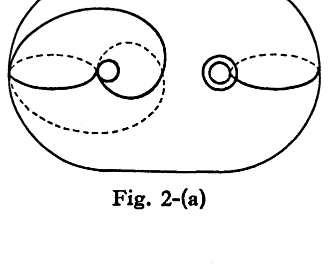


Fig. 2-(a)

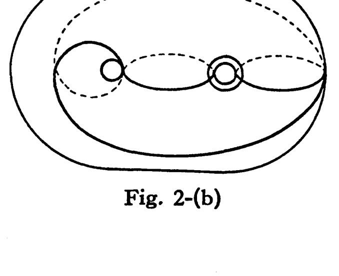


Fig. 2-(b)

of wave moves.

Our proofs of Theorem A and B will be obtained as the total of the following three sections.

2. Jump moves for 3-bridge projections

First of all, we shall introduce Negami and Okita's arguments and show what we should do to prove Theorem A. See their paper [4] for details.

An n-bridge projection of a link L in S^3 represents a decomposition of (S^3, L) into two n-component trivial tangles which attach to each other along their boundary 2-sphere $S^2(L)$. The separating 2-sphere $S^2(L)$ is called an *n-bridge decomposing sphere* of L and it indicates an *n-bridge decomposition* of L. There are a number of n-bridge projections which represent a common n-bridge decomposition of L with a fixed decomposing sphere $S^2(L)$. Then Negami and Okita defined a *jump move* to relate such n-bridge projections to one another.

Let $p(L)$ be an n-bridge projection of a link L with decomposing sphere $S^2(L)$ $(\supset p(L))$ and let $b_1^\pm, \cdots, b_n^\pm$ be its over and under bridges. For one of these bridges, say b_1^+, choose an arc β on $S^2(L)$ so that $\beta \cap (b_1^+ \cup \cdots \cup b_n^+) = \partial b_1^+ = \partial\beta$ and that the circle $b_1^+ \cup \beta$ separates $b_2^+, \cdots, b_n^+$ in $S^2(L)$, and replace b_1^+ with β, then a new n-bridge projection $p'(L) = \beta \cup b_2^+ \cup \cdots \cup b_n^+ \cup b_1^- \cup \cdots \cup b_n^-$ which represents the same n-bridge decomposition of L will be obtained. This transformation of $p(L)$ into $p'(L)$ is called a *jump move* and we say that b_1^+ *jumps* to β and also that $p(L)$ *jumps* to $p'(L)$.

An n-bridge projection $p(L)$ on $S^2(L)$ is said to be *normalized* if no region of $S^2(L)$ divided by $p(L)$ is bounded by precisely two edges. It has been proved that *any two n-bridge normalized projections which represent a common n-bridge decomposition of a link can be connected by a finite sequence of jump moves.*

A property Q of 3-bridge projections of a link with decomposing sphere is said to be *W-admissible* if the following two conditions hold:

(i) If a 3-bridge projection $p(L)$ which has the property Q jumps, by only one step, to a projection $p'(L)$ then there is a finite sequence of wave moves which transforms $p'(L)$ into a projection $p''(L)$ having the property Q.

(ii) Normalization of 3-bridge projections does not break the property Q.

Negami and Okita's main result, Proposition 4-2 in [4], states that *every 3-bridge projection of a link L with a fixed decomposing sphere $S^2(L)$ can be transformed into one which has a property Q by a finite sequence of wave moves if and only if the property Q is a W-admissible one such that the n-bridge decomposition of L with $S^2(L)$ admits at least one n-bridge projection having the property Q.*

Therefore what we should do is to ask when the minimality of the crossing number of a link is W-admissible. The second condition (ii) is vacuously true for the minimality, so we shall analyze the minimum-crossing 3-bridge projections $p(L)$ which satisfy the first condition (i) with the minimality of the crossing number substituted for the property Q. Since it has been done completely for a trivial knot and a splittable link in [4], we shall assume throughout the remaining part that a link L is equivalent to neither a trivial knot nor a splittable link. Then each 3-bridge projection of L has at least two crossings and is connected.

A 3-bridge projection $p(L)$ will be often illustrated by three circles and arcs joining them; each circle stands for a unspecified situation of a neighborhood of each over (or under) bridge of $p(L)$ and several parallel subarcs of the under (or over) bridges of $p(L)$ are replaced with a heavy line. Then each normalized 3-bridge projection has one of the two types shown in Fig. 3-(a) and (b). Type (b) however admits a wave joining the central over bridge. In particular, a minimum-crossing 3-bridge projection is of the *triangular type* (a) with a, b, $c>0$, where a, b and c denote the number of parallel arcs in each part.

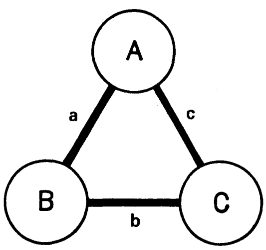


Fig. 3-(a)

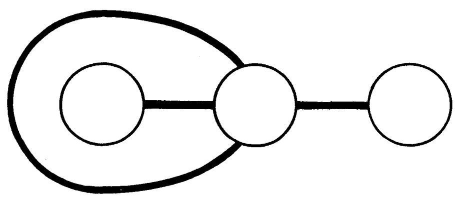


Fig. 3-(b)

Now we shall classify the jump moves for 3-bridge projections of triangular type. There are two ways for under bridges to pass through the over bridge surrounded by circle A. We say that circle A or the over bridge has *type* 1^+ (or 2^+) if two arcs starting from two ends of the over bridge go to the different circles (or not) as illustrated in the middle of Fig. 4 (or Fig. 5). The

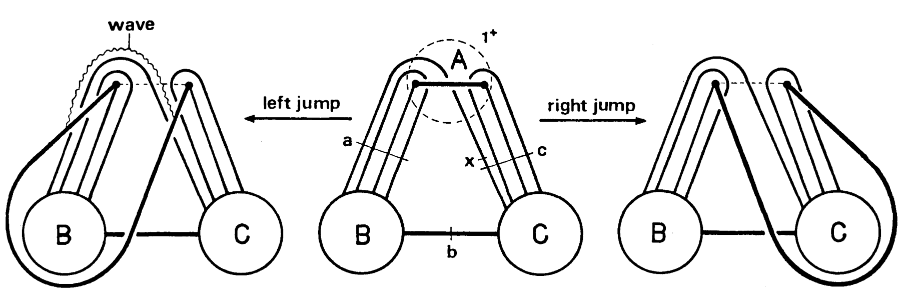


Fig. 4

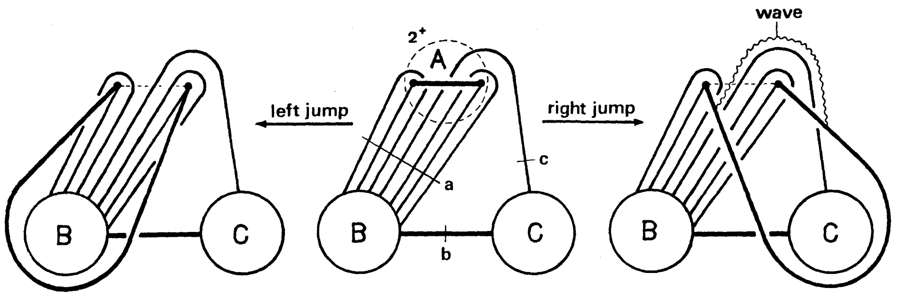


Fig. 5

mirror images of type 1^+ and 2^+ are called type 1^- and 2^-, respectively. The *left jump move* and the *right jump move* for the over bridge are defined as the jump moves from the middle to the left and the right, respectively, in each figure.

Claim 1. *Let $p'(L)$ be a normalized 3-bridge projection obtained from a 3-bridge projection $p(L)$ of triangular type by a single jump move for an over bridge of type 1^+. Then there is a wave for $p'(L)$ such that the wave move transforms $p'(L)$ into a projection equivalent to $p(L)$ if the jump move is not the right jump move.*

It is easy to find such a wave for $p'(L)$ if the jump move is the left one. Other jump moves send the over bridge to the result of the right or left jump move with a number of twists around circle A, and there is the desired wave like Fig. 6. By similar argument, we have:

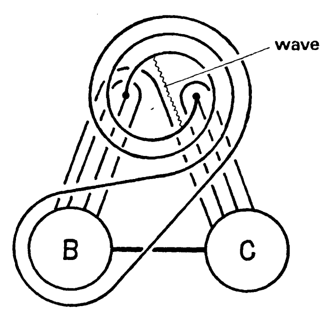


Fig. 6

Claim 2. *Let $p'(L)$ be a normalized 3-bridge projection obtained from a 3-bridge projection $p(L)$ of triangular type by a single jump move for an over bridge of type 2^+. Then there is a wave for $p'(L)$ such that the wave move transforms $p'(L)$ into a projection equivalent to $p(L)$ if the jump move is not the left jump move.*

Let $p_0(L)$ denote a minimum-crossing 3-bridge projection of a link L hereafter. We say that a bridge of $p_0(L)$ has *jump-up-wave-back property* if the result of each jump move for the bridge can be transformed into a minimum-crossing one by a finite sequence of wave moves. The two claims below are needed for our later arguments. To prove them, it is sufficient to take only account of the right or left jump move by Claim 1 and 2.

Claim 3. *Suppose that circle A of $p_0(L)$ has type 1^+ and let x denote the number of arcs which join circle B and C, passing through circle A. Then the over bridge in circle A has jump-up-wave-back property if and only if*

(I) $x=0$,

or (II) $x=b$ (*and* a, $c<b$ *necessarily.*)

The right jump move increases the crossing number by exactly $b-x$. Since it cannot decrease that of $p(L)$, x does not exceed b. In case of $x=b$, $p_0(L)$ is transformed into also a minimum-crossing one by the right jump move. In case of $x<b$, the result $p'(L)$ of the right jump move is not minimum-crossing. If $x=0$ then a wave move which transforms $p'(L)$ back into $p(L)$ can be found easily. Conversely suppose that $p'(L)$ has a wave. If $x<0$ then the wave could be chosen not to meet the original over bridge in circle A, and it could be regarded as a wave for $p_0(L)$, which is contrary to $p_0(L)$ being minimum-crossing. Thus $x=0$.

Discuss it similarly for circle A of type 2^+. The left jump move increases the crossing number by precisely $b-c$. Since $c>0$ now, there is not a case corresponding to (I) above.

Claim 4. *Suppose that circle A of $p_0(L)$ has type 2^+. Then the over bridge in circle A has jump-up-wave-back property if and only if $b=c$ (and $a>b$, c necessarily.)*

3. The 3-bridge Hopf projection

Fig. 7 shows a unique minimum-crossing 3-bridge projection of the Hopf link. We call it *the 3-bridge Hopf projection*, or simply *the Hopf projection.*

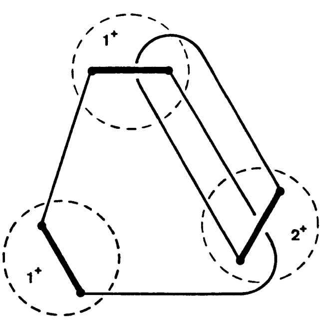


Fig. 7

The Hopf projection seen from the reverse side, that is, the one with the role of its over and under bridges interchanged is also the Hopf projection. Its two circles have type 1^+ and the other type 2^+, and they satisfy the conditions in Claim 3 and 4, respectively. So it follows that:

Claim 5. *Each bridge of the* 3*-bridge Hopf projection has jump-up-wave-back property. Therefore, being the* 3*-bridge Hopf projection is a W-admissible property.*

Notice that the Hopf projection is only a presentation of *a certain* 3*-bridge decomposition* of the Hopf link. We are not sure in this stage that there is no other 3-bridge decomposition of the Hopf link, which will be proved in the next section.

Conversely, we shall show that:

Claim 6. *If each over bridge of* $p_0(L)$ *has jump-up-wave-back property and if its two circles, say circle A and B, have type* 1^+ *or* 1^-*, then* $p_0(L)$ *is equivalent to the Hopf projection.*

Both circle A and B do not satisfy the condition (II) in Claim 3 simultaneously; otherwise, it would imply two inconsistent inequalities a, $c>b$ and a, $b>c$. Suppose that the different situations of (I) and (II) arise for circle A and B, say (I) for circle A and (II) for B. (The other case is its mirror image.) Then $p_0(L)$ has a form shown in Fig. 8-(a) (or Fig. 8-(b)) if circle B is of type 1^+ (or 1^-). In either case, the number y of arcs running from circle A to C via B would be equal to c and we would count arcs around circle C by an odd number, which contradicts the fact that each circle is filled with the parts like Fig. 9.

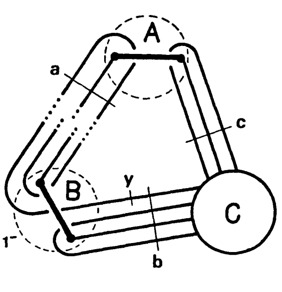


Fig. 8-(a)

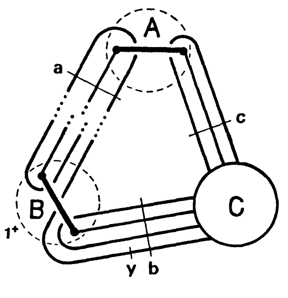


Fig. 8-(b)

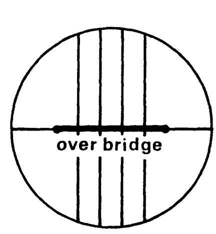


Fig. 9

Thus both circle A and B lie in the situation of (I). Since any loop in a projection is divided into bridges, circle A and B are connected by only one arc which attaches to ends of the over bridges in them, that is, $a=1$. If (I) holds for circle C in addition, then $a=b=c=1$ symmetrically and hence $p_0(L)$ becomes a hexagonal projection. This case must be however excluded by

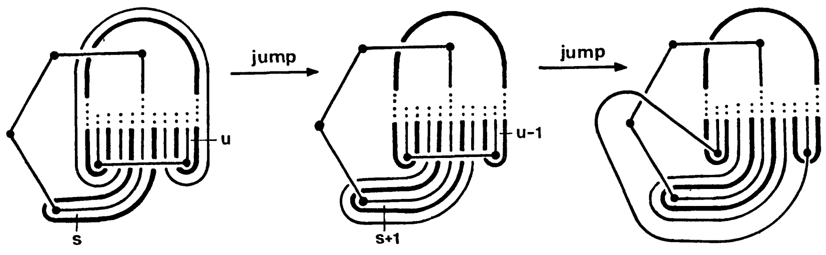


Fig. 10

the non-triviality of L, so (II) holds for circle C. Then $p(L)$ has a form like the left-hand of Fig. 10, where u and s denote the number of parallel arcs in each part.

If $u>0$ then $p_0(L)$ could jump to the middle projection in Fig. 10 by a jump move for an under bridge corresponding to the outermost arc among the u parallel ones, and the middle could jump to the right-hand by a jump move for the over bridge in circle C. The crossing number of the third projection would be less than that of $p_0(L)$ by one, which is contrary to $p_0(L)$ being minimum-crossing. Thus $u=0$ and necessarily $s=0$. This implies that $p_0(L)$ is nothing but the Hopf projection.

Now we shall observe that:

Claim 7. *If two circles of $p_0(L)$ have type 2^+ or 2^-, then jump-up-wave-back property breaks for some over bridge of $p_0(L)$.*

Suppose that each over bridge of $p_0(L)$ has jump-up-wave-back property. Without loss of generality, we may assume that circle A has type 2^+. Then $a>b=c$, which implies first that circle C has type $1^\pm$ with (I) and next that circle B has type 2^-. Thus $p_0(L)$ is described as Fig. 11. Since $p_0(L)$ has no wave, two arcs joined by the dotted line belong to a common under bridge of $p_0(L)$. The under bridge could jump to a curve along the dotted line so that the result of the jump move would have fewer crossings than $p_0(L)$. It is a contradiction.

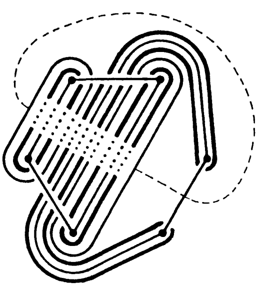


Fig. 11

Combining Claim 5, 6 and 7 with Negami and Okita's proposition in [4], we have got the conclusion that:

Claim 8. *Every 3-bridge projection of a link with 3-bridge decomposition can be transformed into a minimum-crossing one by a finite sequence of wave moves if and only if the Hopf projection represents its 3-bridge decomposition.*

Notice that the statement of Theorem A refers to the link type of a link but does not to its 3-bridge decomposition. That is a gap between Theorem A and the above claim.

4. Genus 2 Heegaard splittings of a real projective space

As is mentioned in introduction, genus 2 Heegaard splittings of orientable closed 3-manifolds are closely related to 3-bridge decompositions of links. In fact, Birman and Hilden [1] proved that there is a bijective correspondence between equivalence classes of 3-bridge decompositions and those of genus 2 Heegaard splittings, as follows: Let $S^2(L)$ be a 3-bridge decomposing sphere of a link L in S^3 and let M^3 be the 2-fold branched covering of S^3 branched over L with projection map $q: M^3 \to S^3$. Then M^3 admits a genus 2 Heegaard splitting with splitting surface $F^2 = q^{-1}(S^2(L))$, that is, M^3 decomposes into two genus 2 handlebodies $U^{\pm}$ which meet F^2 along their boundaries. (See Theorem 8 in [1].)

Furthermore, there is a correspondence between 3-bridge projections and genus 2 Heegaard diagrams. A genus 2 handlebody contains three pairwise disjoint, non-parallel, proper disks which split it into two balls. Let $u_1^{\pm}$, $u_3^{\pm}$, $u_2^{\pm}$ $(\subset F^2)$ be the boundary loops of such three disks in $U^{\pm}$, respectively. Any quadruple $H = (u_i^+, u_j^+; u_k^-, u_l^-)$ $(i \neq j,\ k \neq l)$, called a *genus* 2 *Heegaard diagram*, represents the Heegaard splitting (M^3, F^2) and the hexad $\tilde{H} = (u_1^+, u_2^+, u_3^+; u_1^-, u_2^-, u_3^-)$ is called an *extension* or an *extended Heegaard diagram* of H. Let $p(L)$ is a 3-bridge projection of L on $S^2(L)$ with over bridges b_1^+, b_2^+, b_3^+ and under bridges b_1^-, b_2^-, b_3^-. Then each $q^{-1}(b_i^{\pm})$ is a loop $u_i^{\pm}$ on F^2 so that $(u_1^+, u_2^+, u_3^+; u_1^-, u_2^-, u_3^-)$ is an extended Heegaard diagram of the splitting (M^3, F^2). For example, Fig. 12 illustrates the extended Heegaard diagram associated with the Hopf projection.

To fill the gap between Theorem A and Claim 8, we need Otal's result [5] that every lens space $L(p, q)$ admits a unique Heegaard splitting surface of any genus$\geqq 1$, up to ambient isotopy. In case of genus 2, his result for a real projective space $P^3 = L(2, 1)$ is translated, by the above correspondence, into the uniqueness of 3-bridge decompositions of the Hopf link $K(2, 1)$. That is, any 3-bridge decomposition of the Hopf link is represented by the Hopf projection. Therefore, Claim 8 is equivalent to Theorem A.

Since the Hopf projection is a unique minimum-crossing 3-bridge projection of the Hopf link, we have especially:

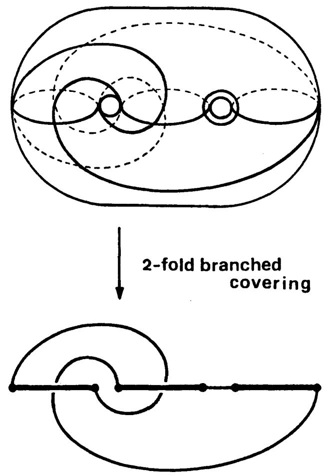


Fig. 12

Corollary C. *Every 3-bridge projection of the Hopf link, different from the Hopf projection, has a wave.*

This suggests an algorithm to determine if a given link with a 3-bridge projection is equivalent to the Hopf link; decrease the crossing number of the projection as much as possible by wave moves and compare it with the Hopf projection. Replace "the Hopf projection" with "a hexagonal one" or "a disconnected one" for a trivial knot or a splittable link, respectively. Theorem A however implies that no other 3-bridge link type can be recognized by such a treatment of only minimum-crossing projections. For every 3-bridge link different from a trivial knot, a splittable link and the Hopf link admits a 3-bridge projection which has no wave but which is not minimum-crossing.

Let $H=(u_1, u_2; v_1, v_2)$ be a genus 2 Heegaard diagram on a splitting surface F^2. Choose an arc ω on F^2 so that $\omega\cap(u_1\cup u_2\cup v_1\cup v_2)=\partial\omega\subset u_1$ and two ends of ω start from the same side of u_1. Then for one of two circles in $\omega\cup u_1$ different from u_1, say u_3, the quadruple $H=(u_3, u_2; v_1, v_2)$ is a new Heegaard diagram of the same splitting. We call ω a *wave* for H and the replacement of u_1 with u_3 a *wave move* if the number of crossings of H' is less than that of H.

To prove Theorem B, we shall translate Corollary C by Proposition 5-2 in [4], which states that *any genus 2 Heegaard diagram H of an orientable closed 3-manifold M^3 admits either a wave or an extension $\tilde{H}$ associated with a 3-bridge projection $p_0(L)$ which has no wave if M^3 is not homeomorphic to $S^2\times S^1\# L(p, q)$.* Recall that P^3 is the 2-fold branched covering of S^3 branched over the Hopf link. Then we take the Hopf link as L in the above.

By Corollary C, the 3-bridge projection $p_0(L)$ is the Hopf projection. When H has no wave, then H admits an extension $\tilde{H}$ equivalent to the diagram

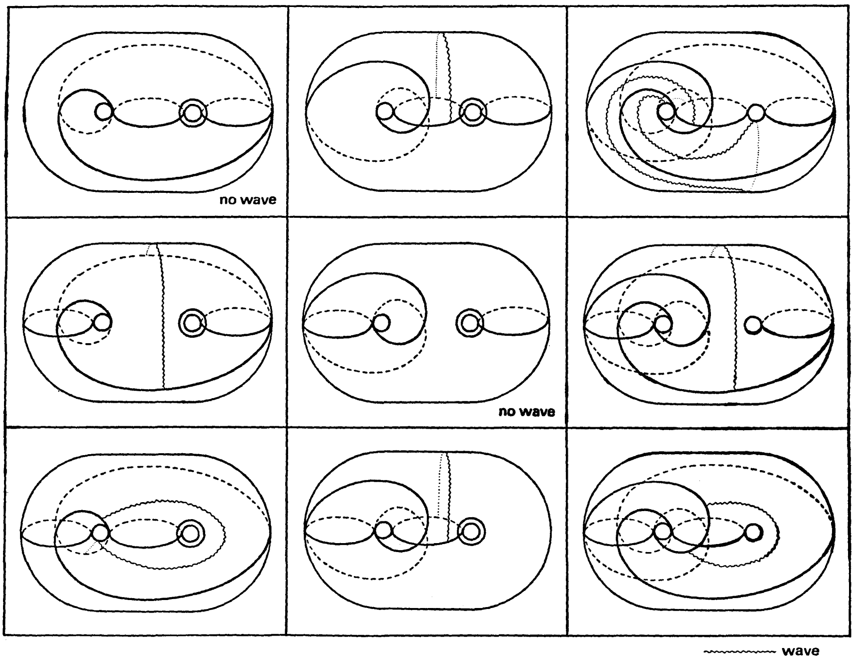


Fig. 13

shown in Fig. 12. The list of Fig. 13 presents all nine subdiagrams of the extension associated with the Hopf projection. Pick up those with no wave out of the nine, then H must be equivalent to one of them. They are nothing but the two diagrams in Fig. 2-(a) and (b), and hence Theorem B follows.

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