On families of Riemann surfaces with polyhedral symmetries

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Preface

The study of degenerating families of algebraic curves (Riemann surfaces) has a long history ever since Kodaira's classification [Kod] of degenerating families of elliptic curves, and which is an active research area even today. This research field is located at the crossing between algebraic geometry and topology, therefore researches from both areas are available. From the viewpoint of geography of algebraic surfaces, slopes and signatures were studied by T. Ashikaga, K. Konno [AsKo] and many others. From the viewpoint of topology, Matsumoto–Montesinos [MaMo] characterized degenerating families of Riemann surfaces in terms of their monodromy. This thesis however adopts other viewpoints to study families of Riemann surfaces, i.e., in terms of linear quotient families and polyhedral symmetries; these viewpoints reveal interesting geometric properties of families of Riemann surfaces.

(i) Linear quotient families are a special class of quotient families introduced by S. Takamura [Ta,VI]; a linear quotient family is a fibration constructed from a finite group action on a complex analytic variety together with a linear representation of the finite group. This thesis only treats linear quotient families, and "linear" is often omitted.

(ii) "Polyhedral symmetries" arise in our context as follows: Thickening of the edges of a regular polyhedron yields a cable surface with the polyhedral group action. We may assign a complex structure on this surface such that the group action is holomorphic. Then to each linear representation of the polyhedral group, a quotient family is associated.

The present work consists of two parts:

Part I: In the construction of (ii), replacing a regular polyhedron with a regular *polygon* yields a new construction of elliptic fibration. We describe the elliptic fibrations obtained in this way. We determine their fibers and the singularities on the total/base spaces; the description depends on the parity of n of the regular n-gon. We point out that our construction is different from the Weierstrass model construction of elliptic fibration in N. Nakayama [Nak], Dolgachev and Gross [DoGr]. The Weierstrass model is algebro-geometric, while our construction is topological. The advantage of ours lies in that the description of families is geometrically carried out.

Part II: To the cable surface obtained from the tetrahedron, we give a complex structure and regard it as a Riemann surface on which the tetrahedral group acts holomorphically (*caution*: the automorphism group of such a Riemann surface "contains" the tetrahedral group but does not necessarily coincide with it). This Riemann surface determines an algebraic curve with tetrahedral group action, which is called a *tetra curve*. M. Oka posed a problem: *Determine the defining equation of this curve.* We solve this problem — we actually show that a tetra curve is *not* unique: there are a sporadic one (hyperelliptic) and a 1-parameter family of non-hyperelliptic curves; this family contains the Fermat curve of degree 4 and the Klein curve (their automorphism groups "jump" and become larger than the tetrahedral group). We show that the non-hyperelliptic family of tetra curves is related to the sporadic tetra curve via a *stable reduction*: we first show that the total space of this family has eight A_1-singularities on one fiber (which is a projective line), and the stable reduction around it creates the sporadic tetra curve as the central fiber of the resulting family — the eight A_1-singularities correspond to the eight fixed points of the hyperelliptic involution.

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Contents

Chapter 1

Introduction

This thesis is composed of two parts. Part I provides a new construction of elliptic fibration and Part II describes the family of Riemann surfaces with tetrahedral symmetries.

Part I: New construction of elliptic fibrations

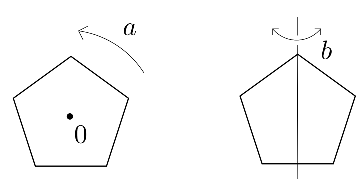
Kodaira [Kod] classified degenerating families of elliptic curves into eight types. Nakayama [Nak] and Dolgachev–Gross [DoGr] described higher-dimensional elliptic fibrations from the viewpoint of Weierstrass model. In this part, we provide a new construction and the description of the resulting higher dimensional elliptic fibrations from the viewpoint of group actions and their representations.

We briefly recall the historical background behind our construction. The study of degenerating families of Riemann surfaces was initiated by Kodaira, and subsequently the classification of degenerating families of genus 2 curves was done by Namikawa and Ueno [NaUe]. Since then, the study of degenerating families of higher genus curves has been an area of active research (e.g. see [Ta,III]). Quotient families are "equivariant quotients" of families with group actions — degenerating families of curves are examples of such families (precisely speaking they are obtained from quotient families by resolving singularities). Among quotient families, linear ones are introduced and developed in [Ta,VI]; they are constructed from finite group actions on spaces together with representations of the groups. In what follows, we only treat linear quotient families, and for simplicity, "linear" is omitted.

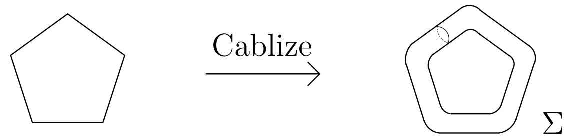
Quotient families associated with cyclic group actions on Riemann surfaces together with their 1-dimensional representations correspond to degenerating families of Riemann surfaces with periodic monodromies. The simplest groups next to cyclic groups are dihedral groups, accordingly the simplest nontrivial quotient families other than degenerating families of Riemann surfaces are those associated with dihedral group actions; they are called *dihedral quotient families*. It is natural to investigate them; among them, simple ones are those of elliptic curves. These "elliptic" dihedral quotient families are interesting enough as they are completely different from elliptic fibrations investigated by [Nak] and [DoGr]. Besides, note that dihedral groups admit double coverings (*binary* dihedral groups), and we may also construct *binary dihedral* quotient families associated with their representations. It is worthwhile investigating the difference between the dihedral and binary dihedral elliptic quotient families. They are shown to be very different.

Our construction starts from dihedral group actions on regular polygons together with representations of the dihedral groups.

Step 1 A dihedral group $D_n = \langle a, b : a^n = b^2 = 1,\ bab^{-1} = a^{-1} \rangle$ acts on the regular n-gon Δ_n as a is a $1/n$-rotation around the origin and b is a reflection along an axis:



Thickening the edges of Δ_n yields a *cable surface* Σ with D_n-action.



We may give a complex structure to Σ such that the D_n-action is holomorphic (Lemma 3.1.1).

Step 2 Given a representation $\rho : D_n \to GL_m(\mathbb{C})$, we may let D_n act on $\mathbb{C}^m$; then D_n acts on $\Sigma \times \mathbb{C}^m$ diagonally: $(z, t) \mapsto (gz, \rho(g)t)$, $g \in D_n$. The projection $\Sigma \times \mathbb{C}^m \to \mathbb{C}^m$ is D_n-equivariant, so determines a holomorphic map $\eta : (\Sigma \times \mathbb{C}^m)/D_n \to \mathbb{C}^m/D_n$. We say that η is the (*dihedral*) *quotient family of* Σ *associated with* ρ (or, quotient family of *type* D). See [HiTa1] for a similar construction for polyhedral groups.

Before proceeding, note that the dimension of an irreducible representation of D_n is either 1 or 2 [Ser]: If n is even, besides the trivial representation $\chi_1 = 1$, there are three 1-dimensional irreducible representations $\chi_i : D_n \to GL_1(\mathbb{C})$ $(i = 2, 3, 4)$ given by

$$\begin{cases} \chi_2(a) = 1, & \chi_2(b) = -1, \\ \chi_3(a) = -1, & \chi_3(b) = 1, \\ \chi_4(a) = -1, & \chi_4(b) = -1. \end{cases} \tag{1.0.1}$$

If n is odd, χ_3 and χ_4 fail to be homomorphisms, and the 1-dimensional irreducible representations of D_n are merely χ_1 and χ_2.

The 2-dimensional irreducible representations are $\rho_l : D_n \to GL_2(\mathbb{C})$ ($l = 1, 2, \ldots, \frac{n}{2} - 1$ for even n and $l = 1, 2, \ldots, \frac{n-1}{2}$ for odd n) given by

$$\rho_l(a) = \begin{pmatrix} e^{2\pi \mathrm{i} l/n} & 0 \\ 0 & e^{-2\pi \mathrm{i} l/n} \end{pmatrix}, \qquad \rho_l(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.0.2}$$

Here ρ_l is injective precisely when l is relatively prime to n.

Let $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$ be the quotient family associated with ρ_l. It is said to be *injective* if ρ_l injective. The non-injective case actually reduces to the injective case thanks to (ii) of the following:

Result 1 (Theorems 3.4.3, 3.4.11)
(i) *If l is relatively prime to n, then depending on whether n is odd or even, the singular fibers and the singular locus of $\eta_{n,l}$ are illustrated as in* (1) *or* (2) *of Figure 1.0.1.*
(ii) *If l is not relatively prime to n, then $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$ is isomorphic to $\eta_{n',l'} : (\Sigma \times \mathbb{C}^2)/D_{n'} \to \mathbb{C}^2/D_{n'}$, where $n' := n/\gcd(n,l)$ and $l' := l/\gcd(n,l)$.*

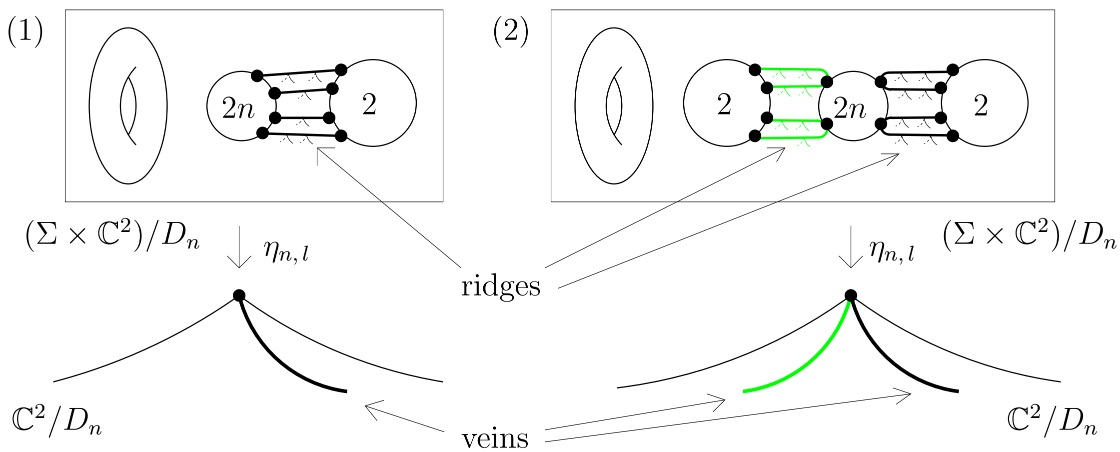


Figure 1.0.1: The numbers "2" and "$2n$" are multiplicities (see §2.1). The singular locus of $(\Sigma \times \mathbb{C}^2)/D_n$ consists of four *ridges* (see Notation 3.4.7) — each is isomorphic to a smooth complex line. The singular fibers of $\eta_{n,l}$ lie over the *veins* (the images of ridges under $\eta_{n,l}$).

Remark In degenerating families of elliptic curves, the topological monodromy of the type I_0^* singular fiber is b, while that of the type nI_0 singular fiber is a (see [Kod]). The former singular fiber appears in Figure 1.0.1 (after blowing up), while the latter does not.

The singular locus of $(\Sigma \times \mathbb{C}^2)/D_n$ is *equisingular*. In fact:

Result 2 (Theorem 3.5.11) *The singular locus of $(\Sigma \times \mathbb{C}^2)/D_n$ consists of the four disjoint complex lines — ridges — around each of which $(\Sigma \times \mathbb{C}^2)/D_n$*

is isomorphic to (complex line) $\times$ (A_1-singularity).

Convention: In the theory of quotient families, the singularity of the total and base spaces are usually remain unresolved, as higher dimensional complex analytic varieties have *no* canonical resolution except for 2-dimensional case.

Result 3 (Theorems 3.2.6, 3.4.11) *The singular fibers and the covering multiplicities* (c.m.) *are as follows:* (1) *The quotient family* $\xi_i : (\Sigma \times \mathbb{C})/D_n \to \mathbb{C}/D_n$ $(i = 1, 2, 3, 4)$ *associated with* χ_i *is as in* Table 1.0.1. (2) *The quotient family* $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$ (*where* l *is an integer such that* $1 \le l < \frac{n}{2}$) *associated with* ρ_l *is as in* Table 1.0.1.

(1)	$\xi_i^{-1}(0)$	$\xi_1^{-1}(s)$ $(s \in (\mathbb{C}/D_n) \setminus \{0\})$	$\xi_2^{-1}(s)$ $(s \in (\mathbb{C}/D_n) \setminus \{0\})$	$\xi_3^{-1}(s)$ $(s \in (\mathbb{C}/D_n) \setminus \{0\})$	$\xi_4^{-1}(s)$ $(s \in (\mathbb{C}/D_n) \setminus \{0\})$
fiber	Σ/D_n $(\mathbb{P}^1)$	Σ/D_n (elliptic curve)	$\Sigma/\langle a\rangle$ (elliptic curve)	$\Sigma/\langle a^2, b\rangle$ $(\mathbb{P}^1)$	$\Sigma/\langle a^2, ab\rangle$ $(\mathbb{P}^1)$
c.m.	$2n$	$2n$	n	n	n

(2)	$\eta_{n,l}^{-1}(0)$	$\eta_{n,l}^{-1}(s)$ $(s \in L \setminus \{0\})$	$\eta_{n,l}^{-1}(s)$ $(s \in (\mathbb{C}^2/D_n) \setminus L)$
fiber	Σ/D_n $(\mathbb{P}^1)$	$\Sigma/\langle a^{n'}, b\rangle$ $(\mathbb{P}^1)$	$\Sigma/\langle a^{n'}\rangle$ (elliptic curve)
c.m.	$2n$	$2d$	d

Table 1.0.1: $\mathbb{P}^1$ is the projective line. In (2), $L \subset \mathbb{C}^2/D_n$ denotes the locus $\mathrm{KL}_{\eta_{n,l}}$ given by (3.4.1), which consists of veins (see Figure 1.0.1).

Binary dihedral quotient families (type $\widetilde{D}$)

The binary dihedral group $\widetilde{D}_n = \langle \widetilde{a}, \widetilde{b} : \widetilde{a}^{2n} = 1, \widetilde{a}^n = \widetilde{b}^2, \widetilde{b}\widetilde{a}\widetilde{b}^{-1} = \widetilde{a}^{-1}\rangle$ is a double covering of $D_n = \langle a, b : a^n = b^2 = 1,\ bab^{-1} = a^{-1}\rangle$: the double covering homomorphism $q : \widetilde{D}_n \to D_n$ is given by $\widetilde{a} \mapsto a, \widetilde{b} \mapsto b$. Recall that the dihedral group D_n acts on the cable surface Σ of the regular n-gon. Let $\widetilde{D}_n$ also act on Σ via q, that is, $g \in \widetilde{D}_n$ acts as $q(g) \in D_n$. To each representation $\widetilde{D}_n \to GL_m(\mathbb{C})$, we may associate a quotient family $(\Sigma \times \mathbb{C}^m)/\widetilde{D}_n \to \mathbb{C}^m/\widetilde{D}_n$ of Σ, called a *binary dihedral* quotient family (or, quotient family of *type* $\widetilde{D}_n$).

For a representation $\rho : D_n \to GL_m(\mathbb{C})$, the composition $\widetilde{\rho} := \rho \circ q : \widetilde{D}_n \to GL_m(\mathbb{C})$ is called the *lift* of ρ. A representation of $\widetilde{D}_n$ is said to be *lifted* if it is the lift of some representation of D_n, otherwise *unlifted*.

$$\begin{array}{ccc} \widetilde{D}_n & & \\ {\scriptstyle q}\downarrow & \overset{\widetilde{\rho}}{\searrow} & \\ D_n & \xrightarrow{\ \rho\ } & GL_m(\mathbb{C}). \end{array}$$

The dimension of any irreducible representation of $\widetilde{D}_n$ is either 1 or 2. Explicitly:

1-dim: The lifted ones are $\widetilde{\chi}_i : \widetilde{D}_n \to GL_1(\mathbb{C})$ $(i = 1, 2, 3, 4)$ defined by

$$\begin{array}{ll} (\widetilde{\chi}_1(\widetilde{a}),\, \widetilde{\chi}_1(\widetilde{b})) = (1, 1), & (\widetilde{\chi}_2(\widetilde{a}),\, \widetilde{\chi}_2(\widetilde{b})) = (1, -1), \\ (\widetilde{\chi}_3(\widetilde{a}),\, \widetilde{\chi}_3(\widetilde{b})) = (-1, 1), & (\widetilde{\chi}_4(\widetilde{a}),\, \widetilde{\chi}_4(\widetilde{b})) = (-1, -1). \end{array}$$

($\widetilde{\chi}_i$ is the lift of χ_i defined by (1.0.1).) The unlifted ones exist only when n is odd: they are $\sigma_k : \widetilde{D}_n \to GL_1(\mathbb{C})$ $(k = 1, 2)$ defined by $(\sigma_1(\widetilde{a}),\, \sigma_1(\widetilde{b})) = (-1, \mathrm{i})$ and $(\sigma_2(\widetilde{a}),\, \sigma_2(\widetilde{b})) = (-1, -\mathrm{i})$.

2-dim: The lifted irreducible representations of $\widetilde{D}_n$ are $\widetilde{\rho}_l : \widetilde{D}_n \to GL_2(\mathbb{C})$ (where $1 \le l < \frac{n}{2}$) defined by

$$\widetilde{\rho}_l(\widetilde{a}) = \begin{pmatrix} e^{2\pi \mathrm{i} l/n} & 0 \\ 0 & e^{-2\pi \mathrm{i} l/n} \end{pmatrix}, \qquad \widetilde{\rho}_l(\widetilde{b}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

($\widetilde{\rho}_l$ is the lift of ρ_l given by (1.0.2).) The unlifted ones are $\tau_m : \widetilde{D}_n \to GL_2(\mathbb{C})$ (where m is odd and $1 \le m < n$) defined by

$$\tau_m(\widetilde{a}) = \begin{pmatrix} e^{\pi \mathrm{i} m/n} & 0 \\ 0 & e^{-\pi \mathrm{i} m/n} \end{pmatrix}, \qquad \tau_m(\widetilde{b}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Although irreducible representations of $\widetilde{D}_n$ and D_n are similar, their associated quotient families are *very different*: unlike those associated with D_n, those associated with $\widetilde{D}_n$ have isolated singular fibers.

Result 4 (Corollary 4.1.17) *The quotient family* $\xi_{n,m} : (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n \to \mathbb{C}^2/\widetilde{D}_n$ *of* Σ *associated with any unlifted irreducible representation* $\tau_m : \widetilde{D}_n \to GL_2(\mathbb{C})$ *has a single singular fiber* (see Figure 1.0.1).

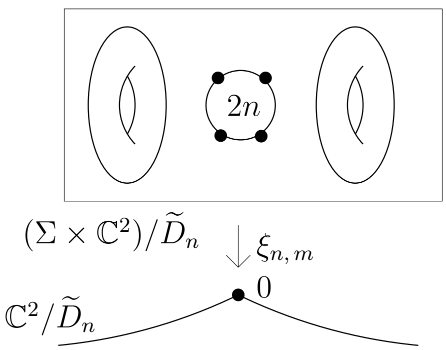


Figure 1.0.2: $\xi^{-1}_{n,m}(0)$ is the unique singular fiber. The singularity of $\mathbb{C}^2/\widetilde{D}_n$ is isolated (a D-singularity).

For a cyclic subgroup $\mathbb{Z}_m$ of order m in $GL_3(\mathbb{C})$ generated by an element of the form $\begin{pmatrix} \zeta^{n_1} & 0 & 0 \\ 0 & \zeta^{n_2} & 0 \\ 0 & 0 & \zeta^{n_3} \end{pmatrix}$ where $\zeta := e^{2\pi \mathrm{i}/m}$ and n_i $(i = 1, 2, 3)$ are integers such that $0 < n_i < m$, the quotient singularity $\mathbb{C}^3/\mathbb{Z}_m$ is called *of type* $\frac{1}{m}(n_1, n_2, n_3)$. This singularity is terminal if and only if $(n_1, n_2, n_3) = (1, \ell, -\ell)$ for some ℓ relatively prime to m (see [Ish] p.185 Theorem 8.3.17).

Result 5 (Theorem 4.1.33) *The singular locus of* $(\Sigma \times \mathbb{C}^2)/\widetilde{D}_n$ *consists of four isolated singularities, any of which is of type* $\frac{1}{4}(1, 2, 3)$ (this is not terminal).

Result 6 (Proposition 4.1.10, 4.1.13, 4.1.21) *We determine the singular fibers and the covering multiplicities of the quotient families associated with the representations* $\widetilde{\chi}_i$, $\widetilde{\rho}_l$, σ_k, τ_m:

- (Lifted case) *Let* $\widetilde{\xi}_i : (\Sigma \times \mathbb{C})/\widetilde{D}_n \to \mathbb{C}/\widetilde{D}_n$ $(i = 1, 2, 3, 4)$ *be the quotient family associated with* $\widetilde{\chi}_i$ *and* $\widetilde{\eta}_{n,l} : (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n \to \mathbb{C}^2/\widetilde{D}_n$ *(where* $1 \leq l < \frac{n}{2}$*) be the quotient family associated with* $\widetilde{\rho}_l$*. Then the singular fibers and the covering multiplicities* (c.m.) *are as in* Table 1.0.2 (1) *and* (2) *respectively.*

- (Unlifted case) *Let* $\varpi_k : (\Sigma \times \mathbb{C})/\widetilde{D}_n \to \mathbb{C}/\widetilde{D}_n$ $(k = 1, 2)$ *be the quotient family associated with* σ_k *and* $\xi_{n,m} : (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n \to \mathbb{C}^2/\widetilde{D}_n$ *(where* m *is odd and* $1 \leq m < n$*) be the quotient family associated with* τ_m*. Then the singular fibers and the covering multiplicities* (c.m.) *are as in* Table 1.0.2 (3).

(1)

	$\widetilde{\xi}_i^{-1}(0)$	$\widetilde{\xi}_1^{-1}(s)$ $(s \in (\mathbb{C}/\widetilde{D}_n) \setminus \{0\})$	$\widetilde{\xi}_2^{-1}(s)$ $(s \in (\mathbb{C}/\widetilde{D}_n) \setminus \{0\})$	$\widetilde{\xi}_3^{-1}(s)$ $(s \in (\mathbb{C}/\widetilde{D}_n) \setminus \{0\})$	$\widetilde{\xi}_4^{-1}(s)$ $(s \in (\mathbb{C}/\widetilde{D}_n) \setminus \{0\})$
fiber	Σ/D_n ($\mathbb{P}^1$)	Σ/D_n (elliptic curve)	$\Sigma/\langle a \rangle$ (elliptic curve)	$\Sigma/\langle a^2, b \rangle$ ($\mathbb{P}^1$)	$\Sigma/\langle a^2, ab \rangle$ ($\mathbb{P}^1$)
c.m.	$2n$	$2n$	n	n	n

(2)

	$\widetilde{\eta}_{n,l}^{-1}(0)$	$\widetilde{\eta}_{n,l}^{-1}(s)$ $(s \in L \setminus \{0\})$	$\widetilde{\eta}_{n,l}^{-1}(s)$ $(s \in (\mathbb{C}^2/\widetilde{D}_n) \setminus L)$
fiber	Σ/D_n ($\mathbb{P}^1$)	$\Sigma/\langle a^{n'}, b \rangle$ ($\mathbb{P}^1$)	$\Sigma/\langle a^{n'} \rangle$ (elliptic curve)
c.m.	$2n$	$2d$	d

(3)

	$\varpi_k^{-1}(0)$	$\varpi_k^{-1}(s)$ $(s \in (\mathbb{C}/\widetilde{D}_n) \setminus \{0\})$	$\xi_{n,m}^{-1}(0)$	$\xi_{n,m}^{-1}(s)$ $(s \in (\mathbb{C}^2/\widetilde{D}_n) \setminus \{0\})$
fiber	Σ/D_n ($\mathbb{P}^1$)	$\Sigma/\langle a^2 \rangle$ (elliptic curve)	Σ/D_n ($\mathbb{P}^1$)	$\Sigma/\langle a^{n'} \rangle$ (elliptic curve)
c.m.	$2n$	n	$2n$	d

Table 1.0.2: In (2), $L \subset \mathbb{C}^2/\widetilde{D}_n$ denotes the locus $\mathrm{SL}_{\widetilde{\eta}_{n,l}}$ given by (4.1.5).

We mention our further works:

Paracabling construction From a regular n-gon, we constructed a cable surface with D_n-action. More generally, as illustrated in Figure 1.0.3, we may construct *paracabling, singular paracabling* surfaces of high genera (this operation is analogous to the procedure in knot theory to produce iterated torus knots by cabling torus knots). These surfaces admit D_n-actions, thus from the representations of D_n, we may construct dihedral quotient families of surfaces of high genera. These quotient families are expected to play an important role in the theory of quotient families, as iterated torus knots did in knot theory. We plan to describe them elsewhere.

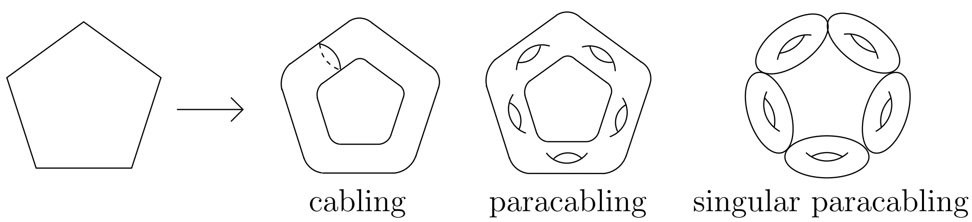


Figure 1.0.3:

Boundary fibration Suppose that a finite group G acts on a complex analytic variety Y. To each representation $\rho : G \to GL_n(\mathbb{C})$, a *quotient family* $\eta : (Y \times \mathbb{C}^n)/G \to \mathbb{C}^n/G$ is associated (§2.1). If ρ is unitary, i.e. $\rho(G) \subset U(n)$, then the action of G on $\mathbb{C}^n$ preserves both $B^{2n} = \{\mathbf{z} \in \mathbb{C}^n : |\mathbf{z}| \leq 1\}$ and $S^{2n-1} = \partial B^{2n}$. The restriction of η to S^{2n-1} is the *boundary fibration* $(Y \times S^{2n-1})/G \to S^{2n-1}/G$ of $(Y \times B^{2n})/G \to B^{2n}/G$. For example, for $\tau_m : \widetilde{D}_n \to SU(2)$, $(\Sigma \times S^3)/\widetilde{D}_n \to S^3/\widetilde{D}_n$ is such that $S^3/\widetilde{D}_n$ is a *prism manifold* [Sav] and for $\rho_l : D_n \to U(2)$, $(\Sigma \times S^3)/D_n \to S^3/D_n$ is such that S^3/D_n is the quotient of a lens space $S^3/\langle \rho(a) \rangle$ by an orientation-*reversing* involution $\rho(b)$.

Actions of different groups D_n is the semi-direct product $\mathbb{Z}_n \rtimes \mathbb{Z}_2$ of $\mathbb{Z}_n = \langle a \rangle$ and $\mathbb{Z}_2 = \langle b \rangle$. There is an elliptic curve with a periodic automorphism of order 3, 4 or 6. It moreover admits the action of $(\mathbb{Z}_n \oplus \mathbb{Z}_n) \rtimes \mathbb{Z}_l$ ($l = 3, 4$ or 6). We will describe these quotient families in our subsequent paper.

Part II: The family of Riemann surfaces with tetrahedral symmetries

Concerning our polyhedral construction of degenerations of Riemann surfaces, Mutsuo Oka raised two problems at the symposium "Contact structure, singularity, differential equation and related topics" at Kochi (2014):

I. *Globalize the above degenerations in a natural way.*

II. *What is the defining equation of such a Riemann surface?*

We solved Problem I in the joint work [HiTa1] with S. Takamura. In this paper, we solve Problem II. It however turns out that such a Riemann surface is not unique but forms a 1-parameter non-hyperelliptic family together with a sporadic hyperelliptic one. We explicitly describe this family, and in terms of stable reduction reveal the relationship between this family and the sporadic one. We moreover describe the image of this family under the moduli map.

Let Σ be an orientable real surface obtained by thickening the edges of a polyhedron (Figure 1.0.4). We say that Σ is the *cable surface* of the polyhedron — the genus of the cable surface of the n-hedron is $n-1$.

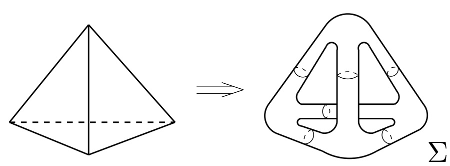


Figure 1.0.4:

The automorphism group G of the polyhedron naturally acts on Σ orientation-preservingly. Kerckhoff's theorem [Ker] ensures the existence of a complex structure on Σ such that G acts holomorphically. In this paper, we consider the cable surface of the tetrahedron (*tetra surface*); its genus is 3. We may regard this Riemann surface as an algebraic curve. Noting that any (non-hyperelliptic) curve of genus 3 is realized as a plane algebraic curve in $\mathbb{P}^2$. M. Oka asked:

Problem *Determine the defining equation of such a curve. Moreover is this* *curve hyperelliptic or not?* (The same problem may be considered for any regular polyhedron, but it is subtle — for which the cable surface, being of genus ≥ 4, is not necessarily a plane curve, so may not be defined by a *single* equation.)

The complete classification of *full* automorphism groups of genus 3 curves is known ([Bars] for non-hyperelliptic ones, [GSS] for hyperelliptic ones); this however does not give the solution of the above problem — in fact the tetrahedral group may not be the full automorphism group of a curve in question. Moreover we must take into account the topological types of group actions: the action must be topologically equivalent to the standard tetrahedral group action on the cable surface Σ.

Reformulation The tetrahedral group $\mathfrak{T}$ permutes the four vertices of the tetrahedron, which induces an isomorphism $\mathfrak{T} \cong \mathfrak{A}_4$ (alternating group of degree 4). A curve with a tetrahedral group action may be thus called an $\mathfrak{A}_4$-curve. If moreover the tetrahedral group action is topologically equivalent to the standard one, that is, the natural tetrahedral group action on Σ, then the $\mathfrak{A}_4$-curve is said to be *of tetra type*. M. Oka's problem is then reformulated as:

Problem Determine all genus 3 $\mathfrak{A}_4$-curves of tetra type.

We will show that:

Solution (Theorem 6.2.9 (1)) *The genus* 3 *$\mathfrak{A}_4$-curves of tetra type are as follows:*

(H) *The hyperelliptic curve B defined by $y^2 = x^8+14x^4+1$ in $\mathbb{C}^2$* (more precisely, compactify this curve in $\mathbb{P}^1 \times \mathbb{P}^1$ and then resolve its singularities, which yields B; refer to [GrHa] p.254 for this procedure).

(NH) *The non-hyperelliptic curves C_t ($t \in \mathbb{C} \setminus \{\pm 2, -1\}$) in $\mathbb{P}^2$ given by*

$$x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0.$$

(Note: All degree 4 curves are non-hyperelliptic ([Har] p.315, Exercise 3.2 (c)).)

We will actually show much more. Observe first that the $\mathfrak{A}_4$-actions on B and C_t are *a priori* 'independent' and moreover these curves are unrelated (as seen from their defining equations). This is however *not* the case; there exists an analytic deformation from B to C_s ($s = (t-2)^2$) that is compatible with $\mathfrak{A}_4$-action (we say an "$\mathfrak{A}_4$-deformation"). The construction of this deformation is carried out by stable reduction (so B and C_t are said to be *stably connected*). We will also show that the singularities of the complex surface $S = \{C_t\}_{t\in\mathbb{C}}$ are eight A_1-singularities and they arise as the quotient under a hyperelliptic involution.

In the theory of algebraic curves, the classification of automorphism groups of curves (of fixed genus) is usually carried out separately for hyperelliptic curves or non-hyperelliptic curves; then there often appears a pair of a hyperelliptic G-curve X and a family of non-hyperelliptic G-curves Y_t (where G is a finite group) such that these G-actions are topologically equivalent (examples of such pairs indeed appear in the list of S. Hirose in his talk at the symposium "Algebraic topology around transformation groups" at RIMS, 2017). Based on our results, we pose the following:

Stably-connectedness problem *Are X and Y_t connected via a G-deformation? Are they related via stable reduction?*

We plan to discuss this in our subsequent paper.

Main results

We state our main results explicitly:

Main theorem (1) *The genus* 3 *$\mathfrak{A}_4$-curves of tetra type are exhausted by* :

(i) *a hyperelliptic curve* $B : y^2 = x^8 + 14x^4 + 1$ *and*

(ii) *non-hyperelliptic curves* $C_t : x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0$ *in* $\mathbb{P}^2$*, where* $t \in \mathbb{C} \setminus \{\pm 2, -1\}$ (Theorem 6.2.9 (1)). *Here* $C_{\pm 2}$, C_{-1}

are excluded, because they are singular (Lemma 7.1.4; see also Figure 1.0.5):

- C_2 *is* $\mathbb{P}^1$ *of multiplicity* 2.
- C_{-2} *consists of four* $\mathbb{P}^1$*'s and any two of them intersect at one point.*
- C_{-1} *consists of two* $\mathbb{P}^1$*'s intersecting at four points.*

(2) *Let* S *be a complex surface defined by*

$$S := \left\{(x, y, z, t) \in \mathbb{P}^2 \times \mathbb{C} : x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0\right\}$$

and $p : S \to \mathbb{C}$ *be the projection* $(x, y, z, t) \mapsto t$*; so* $C_t = p^{-1}(t)$*. Then the singularities of* S *are eight* A_1*-singularities and they lie on* C_2 (Theorem 6.2.9 (2)).
(3) *Take a sufficiently small disk* Δ *centered at* $t = 2$ *in* $\mathbb{C}$ *and set* $W := p^{-1}(\Delta)$*. Let* $\mathfrak{r} : M \to W$ *be the minimal resolution of the singularities. Then* $\pi := p \circ \mathfrak{r} : M \to \Delta$ *is a degeneration of smooth curves whose monodromy is a hyperelliptic involution* (Proposition 6.2.11).
(4) *Let* $p'' : N \to \Delta$ *be the* $\mathbb{Z}_2$*-stable reduction of* $p : W \to \Delta$ *via the base change* $\Delta \to \Delta$, $t - 2 \mapsto (t - 2)^2$*. Then the central fiber of* p'' *is* B (Theorem 6.2.9 (3)) *and the natural* $\mathbb{Z}_2$*-action on* B *is a hyperelliptic involution with* $B/\mathbb{Z}_2 = C_2$ (see Corollary 6.2.5).

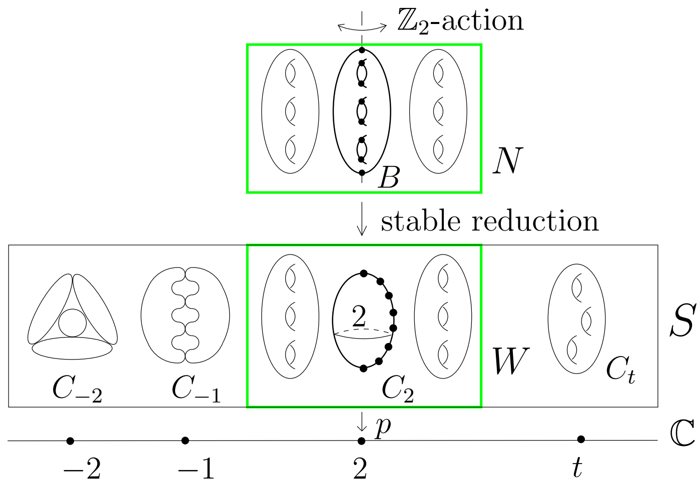


Figure 1.0.5: The eight bold points on C_2 are A_1-singularities.

Remark 1.0.1. The family of curves C_t is also studied by other researchers: Kuribayashi–Sekita [KuSe], which is subsequently used in our discussion, and Alwaleed and Sakai [AlSa], which classified the 2-Weierstrass points on C_t and determined the numbers of flexes and sextactic points.

Description of the moduli map Let $\mathcal{M}_3$ be the moduli space of Riemann surfaces of genus 3 and $\overline{\mathcal{M}}_3$ be its Deligne–Mumford compactification. Consider the moduli map $f : \mathbb{C} \setminus \{2\} \to \overline{\mathcal{M}}_3$ of the family $\{C_t\}_{t \in \mathbb{C} \setminus \{2\}}$. As $t \to 2$, $f(t) = [B]$, so f is bounded, thus naturally extends to a holomorphic map $f : \mathbb{C} \to \overline{\mathcal{M}}_3$. Set $\mathrm{Im} f := f(\mathbb{C})$. Then:

(1) f is injective except for two values $t = \frac{-3 \pm 3\sqrt{-7}}{4}$, for which C_t are the Klein curve ([KuSe] Theorem 2 p.121). Moreover $\mathrm{Im} f$ intersects *transversally* at the point corresponding to the Klein curve (this is shown by using linear quotient family; see [SaTa] for details).

(2) $\mathrm{Im} f$ intersects the hyperelliptic locus in $\mathcal{M}_3$ at one point $f(2) = [B]$ (from Main theorem (4)).

(3) $\mathrm{Im} f$ intersects the boundary of $\mathcal{M}_3$ at $f(-2)$ and $f(-1)$, which correspond to the stable curves C_{-2} and C_{-1}.

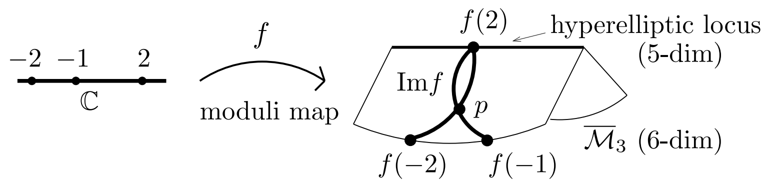


Figure 1.0.6: The point p corresponds to the Klein curve.

Exotic $\mathfrak{S}_4$-action Each C_t actually admits a larger group action than $\mathfrak{A}_4$. Indeed the symmetric group $\mathfrak{S}_4$ acts on it (see [Bars] Table p.10). Since C_t is homeomorphic to the cable surface Σ of the tetrahedron, this $\mathfrak{S}_4$-action is transformed to Σ. On the other hand, besides the automorphism group $\mathfrak{T}$ of the tetrahedron, the full automorphism group $\widehat{\mathfrak{T}}$ (which contains orientation-reversing automorphisms) also acts on Σ, and this group is isomorphic to $\mathfrak{S}_4$. It is thus plausible that the previous $\mathfrak{S}_4$-action coincides with this $\mathfrak{S}_4$-action. However this is *not* the case, because the former contains *no* orientation-reversing automorphisms (as it is holomorphic). Thus Σ has two distinct $\mathfrak{S}_4$-actions: the standard one by $\widehat{\mathfrak{T}}$ and the *exotic* one from the $\mathfrak{S}_4$-action on C_t.

Part I

New Construction of Elliptic Fibrations

Chapter 2

Preparation

2.1 Quotient families in general

S. Takamura [Ta,VI] developed the theory of quotient families of complex analytic varieties. For quotient families of Riemann surfaces, he introduced four series in terms of group action: *dihedral, polyhedral, modular, triangular series* (the first is the target of this paper).

We briefly review [Ta,VI]. Suppose that a finite group G acts on a complex analytic variety Y holomorphically. (Unless otherwise mentioned, the G-action is assumed to be *effective*.) Let $\rho : G \to GL_m(\mathbb{C})$ be a representation, via which G acts on $\mathbb{C}^m$, and accordingly on $Y \times \mathbb{C}^m$ diagonally. The projection map $\mathrm{pr} : Y \times \mathbb{C}^m \to \mathbb{C}^m$, being G-equivariant, determines a holomorphic map $\eta := \overline{\mathrm{pr}} : (Y \times \mathbb{C}^m)/G \to \mathbb{C}^m/G$. This is called the *quotient family of* Y *associated with* ρ.

Theorem 2.1.1 (**Quotient fiber theorem [Ta,VI]**). *A fiber* $\eta^{-1}(s)$ $(s \in \mathbb{C}^m/G)$ *of a quotient family* $\eta : (Y \times \mathbb{C}^m)/G \to \mathbb{C}^m/G$ *is described as follows: Let* $q : \mathbb{C}^m \to \mathbb{C}^m/G$ *be the quotient map. Take a lift* $t \in q^{-1}(s)$ *and let* $H_t := \{g \in G : \rho(g)\, t = t\}$ *be its stabilizer. Then* $\eta^{-1}(s) = Y/H_t$. (This is, up to isomorphism, independent of the choice of t: if t' is another lift, $Y/H_t \cong Y/H_{t'}$.)

Proof. Consider the following commutative diagram (q' and q are quotient maps):

$$\begin{array}{ccc} Y \times \mathbb{C}^m & \xrightarrow{\ q'\ } & (Y \times \mathbb{C}^m)/G \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow {\scriptstyle \eta} \\ \mathbb{C}^m & \xrightarrow{\ q\ } & \mathbb{C}^m/G. \end{array} \tag{2.1.1}$$

The commutativity of this diagram implies $\eta^{-1}(s) = q'\mathrm{pr}^{-1}q^{-1}(s)$. Write $q^{-1}(s) = \{t_1, t_2, \ldots, t_l\}$ where $t_1 = t$, then $\mathrm{pr}^{-1}\big(q^{-1}(s)\big) = \mathrm{pr}^{-1}(t_1)\amalg\mathrm{pr}^{-1}(t_2)\amalg \cdots \amalg \mathrm{pr}^{-1}(t_l)$ (disjoint union). Here $\mathrm{pr}^{-1}(t_i) = Y \times \{t_i\}$. For brevity write it as Y_i, then

$$\eta^{-1}(s) = q'(Y_1 \amalg Y_2 \amalg \cdots \amalg Y_l). \tag{2.1.2}$$

Now G acts transitively on the set $\{Y_1, Y_2, \ldots, Y_l\}$ while H_{t_1} stabilizes Y_1. Thus $(Y_1 \amalg Y_2 \amalg \cdots \amalg Y_l)/G = Y_1/H_{t_1}$, that is, $q'(Y_1 \amalg Y_2 \amalg \cdots \amalg Y_l) = Y_1/H_{t_1}$. From this and (2.1.2), $\eta^{-1}(s) = Y_1/H_{t_1}(= Y/H_t)$. □

Example 2.1.2. If $s = \overline{0}$, i.e. $t = 0$, then $H_0 = G$, so $\eta^{-1}(0) = Y/G$.

Lemma 2.1.3. *Let $\eta : (Y \times \mathbb{C})/G \to \mathbb{C}/G$ be the quotient family associated with a 1-dimensional representation $\rho : G \to GL_1(\mathbb{C})$. Then the following hold*:

(1) $$H_t = \begin{cases} G & t = 0, \\ \mathrm{Ker}(\rho) & t \neq 0. \end{cases}$$ (2) $$\eta^{-1}(s) = \begin{cases} Y/G & s = 0, \\ Y/\mathrm{Ker}(\rho) & s \neq 0. \end{cases}$$

Proof. We show (1). $H_0 = G$ is trivial. We show $H_t = \mathrm{Ker}(\rho)$ for $t \neq 0$. First $\mathrm{Ker}(\rho) \subset H_t$ (because if $g \in \mathrm{Ker}(\rho)$, then $\rho(g) = 1$, so $\rho(g)t = t$). Next $\mathrm{Ker}(\rho) \supset H_t$ (because if $\rho(g)t = t$, then $\rho(g) = 1$, so $g \in \mathrm{Ker}(\rho)$). (2) is immediate from (1) by Theorem 2.1.1. □

Let $\eta : (Y \times \mathbb{C}^m)/G \to \mathbb{C}^m/G$ be the quotient family associated with a representation $\rho : G \to GL_m(\mathbb{C})$. By Theorem 2.1.1, $\eta^{-1}(s) = Y/H_t$, where $t \in \mathbb{C}^m$ is a lift of $s \in \mathbb{C}^m/G$. The *covering multiplicity* of $\eta^{-1}(s)$ is defined as the covering degree of the quotient map $Y \to Y/H_t$, which is equal to the order $|H_t|$ of H_t.

Definition 2.1.4. In Theorem 2.1.1, if $H_t \neq \{1\}$, $\eta^{-1}(s)$ is called a *kaleido fiber* and otherwise a *pure fiber* (in the former $\eta^{-1}(s) \neq Y$ and in the latter $\eta^{-1}(s) = Y$). The locus of $\mathbb{C}^m/G$ over which kaleido fibers lie is called the *kaleido locus* of η and denoted by KL_η.

Proposition 2.1.5. (1) *If $\rho : G \to GL_m(\mathbb{C})$ is not injective, KL_η is the whole of $\mathbb{C}^m/G$.*
(2) *If $\rho : G \to GL_m(\mathbb{C})$ is injective, KL_η is a proper subset of $\mathbb{C}^m/G$ consisting of the image of a finite union of (proper) linear subspaces of $\mathbb{C}^m$ under the quotient map $\mathbb{C}^m \to \mathbb{C}^m/G$.*

Proof. We show (1). $K := \mathrm{Ker}(\rho)$ acts on $\mathbb{C}^m$ trivially, so $K \subset H_t$ for any t. If ρ is not injective, then $K \neq \{1\}$, so $H_t \neq \{1\}$, and any fiber of η is kaleido, thus $\mathrm{KL}_\eta = \mathbb{C}^m/G$.

We next show (2). The preimage $\widetilde{\mathrm{KL}}_\eta$ of KL_η under the quotient map $\mathbb{C}^m \to \mathbb{C}^m/G$ is given by $\widetilde{\mathrm{KL}}_\eta = \big\{t \in \mathbb{C}^m : H_t \neq \{1\}\big\} = \bigcup_{g \in G \setminus \{1\}} \mathrm{Fix}(g)$, where $\mathrm{Fix}(g) := \{t \in \mathbb{C}^m : \rho(g)t = t\}$ is a linear subspace of $\mathbb{C}^m$. Here note that this union is finite (as G is finite) and that if ρ is injective, $\mathrm{Fix}(g)$ is proper. Thus the assertion holds. □

Let $\eta : (Y \times \mathbb{C}^m)/G \to \mathbb{C}^m/G$ be the quotient family associated with a representation $\rho : G \to GL_m(\mathbb{C})$. Here ρ is generally not injective. Set $K := \mathrm{Ker}(\rho)$. The quotient group $\overline{G} := G/K$ naturally acts on $\overline{Y} := Y/K$ and ρ induces an injective representation $\overline{\rho} : \overline{G} \to GL_m(\mathbb{C})$. We shall show that η is isomorphic to the quotient family $\overline{\eta} : (\overline{Y} \times \mathbb{C}^m)/\overline{G} \to \mathbb{C}^m/\overline{G}$ associated with $\overline{\rho}$. We refer to $\overline{\eta}$ as the *injectivization* of η.

Note first that $(Y \times \mathbb{C}^m)/G \cong (\overline{Y} \times \mathbb{C}^m)/\overline{G}$ and $\mathbb{C}^m/G \cong \mathbb{C}^m/\overline{G}$. Indeed

$$\begin{aligned}(Y \times \mathbb{C}^m)/G &\cong (Y \times \mathbb{C}^m)/K \Big/ G/K \\ &= (Y/K \times \mathbb{C}^m) \Big/ G/K \quad \text{as } K \text{ acts on } \mathbb{C}^m \text{ trivially} \\ &= (\overline{Y} \times \mathbb{C}^m)/\overline{G}.\end{aligned}$$

Similarly we can confirm that $\mathbb{C}^m/G \cong \mathbb{C}^m/\overline{G}$. Moreover the following diagram commutes:

$$\begin{array}{ccc} (Y \times \mathbb{C}^m)/G & \xrightarrow{\cong} & (\overline{Y} \times \mathbb{C}^m)/\overline{G} \\ \eta \downarrow & & \downarrow \overline{\eta} \\ \mathbb{C}^m/G & \xrightarrow{\cong} & \mathbb{C}^m/\overline{G}. \end{array}$$

Therefore the following is obtained:

Lemma 2.1.6. *$\eta : (Y \times \mathbb{C}^m)/G \to \mathbb{C}^m/G$ and $\overline{\eta} : (\overline{Y} \times \mathbb{C}^m)/\overline{G} \to \mathbb{C}^m/\overline{G}$ are isomorphic.*

Caution: Since η and $\overline{\eta}$ are isomorphic, fibers $\eta^{-1}(s)$ and $\overline{\eta}^{-1}(s)$ are isomorphic. However their covering multiplicities are generally *distinct*, that is, $|H_t| \neq |\overline{H}_t|$. In fact the following holds:

Lemma 2.1.7. *$|H_t| = |K||\overline{H}_t|$, that is,*

$$\text{(covering multiplicity of } \eta^{-1}(s)) = |K| \times \text{(covering multiplicity of } \overline{\eta}^{-1}(s)).$$

Proof. Recall that $H_t := \{g \in G : \rho(g)t = t\}$ and $\overline{H}_t := \{\overline{g} \in \overline{G} : \overline{\rho}(\overline{g})t = t\}$. Here since $\overline{H}_t \cong H_t/K$, we have $|H_t| = |K||\overline{H}_t|$. □

Remark 2.1.8. For "kaleido/pure fiber" in Definition 2.1.4, there are similar notions "special/generic fiber": Noting $K \subset H_t$, we call a fiber $\eta^{-1}(s) = Y/H_t$ *generic* if $H_t = K$ and *special* otherwise. (If $K = \{1\}$, that is, ρ is injective, then special/generic coincides with kaleido/pure.) The locus of $\mathbb{C}^m/G$ over which special fibers lie is called the *special locus* and denoted by SL_η.

The following is shown by S. Takamura [Ta,VI].

Theorem 2.1.9. (1) *The quotient family $\eta : (Y \times \mathbb{C}^m)/G \to \mathbb{C}^m/G$ associated with $\rho : G \to GL_m(\mathbb{C})$ is canonically isomorphic to the quotient family $\overline{\eta} : (\overline{Y} \times \mathbb{C}^m)/\overline{G} \to \mathbb{C}^m/\overline{G}$ associated with $\overline{\rho} : \overline{G} \to GL_m(\mathbb{C})$. Here set $K := \mathrm{Ker}(\rho)$, then the following holds*:

$$\text{(covering multiplicity of } \eta^{-1}(s)) = |K| \times \text{(covering multiplicity of } \overline{\eta}^{-1}(s)).$$

(2) *Let* SL_η *be the special locus of* η *and* $\mathrm{KL}_{\overline{\eta}}$ *be the kaleido locus of* $\overline{\eta}$ *Under the isomorphism in* (1), $\mathrm{SL}_\eta = \mathrm{KL}_{\overline{\eta}}$.

Chapter 3

Dihedral quotient families

3.1 Dihedral quotient families

Given a regular n-gon, thickening its edges yields a cable surface Σ, on which the dihedral group $D_n := \langle a, b : a^n = b^2 = 1,\ bab^{-1} = a^{-1} \rangle$ acts as illustrated in Figure 3.1.1: a is a $1/n$-rotation, while b a $1/2$-rotation fixing four points.

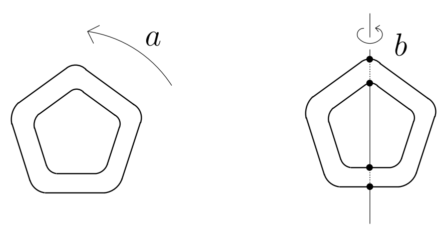


Figure 3.1.1:

Let us make this action *holomorphic*. First express Σ as a *complex* torus $\mathbb{C}/L$, the quotient of $\mathbb{C}$ under the additive action of a lattice $L := \mathbb{Z} \oplus \lambda\mathbb{Z}$ ($\lambda \in \mathbb{C}, \mathrm{Im}\lambda > 0$). Define two automorphisms A, B of $\mathbb{C}$ by $A : z \mapsto z + \dfrac{1}{n}$ and $B : z \mapsto -z$. Then

(i) $A^n(z) = z + 1$ (so $A^n(z) \equiv z \bmod \mathbb{Z} \oplus \lambda\mathbb{Z}$), $B^2 = 1$, $BAB^{-1} = A^{-1}$.

(ii) For any $T \in L$ (say, T acts on $\mathbb{C}$ as a translation $z \mapsto z + m + \lambda n$ $(m, n \in \mathbb{Z})$), we have $AT(z) = TA(z)$ and $BT(z) \equiv TB(z) \bmod \mathbb{Z} \oplus \lambda\mathbb{Z}$ (indeed $BT(z) = B(z+m+\lambda n) = -z-m-\lambda n$ and $TB(z) = T(-z) = -z + m + \lambda n$).

(ii) ensures that A and B descend to automorphisms $\overline{A}$ and $\overline{B}$ of $\mathbb{C}/L$. By (i), $D_n = \langle \overline{A}, \overline{B} \rangle$, where $\overline{A}$ and $\overline{B}$ correspond to a and b. See Figure 3.1.2.

The above construction is independent of λ (in $L = \mathbb{Z} \oplus \lambda\mathbb{Z}$), so we obtain the following:

Lemma 3.1.1. *For any complex structure on the cable surface Σ, we may let D_n act on Σ holomorphically.*

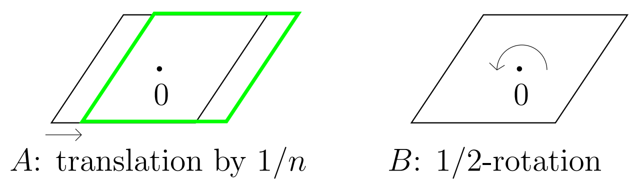


Figure 3.1.2: Actions of A and B on a fundamental domain of $\mathbb{C}/L$.

Note next the following:

Lemma 3.1.2. *Any element of D_n is expressed as a^k or $a^k b$ $(k = 0, 1, \ldots, n-1)$.*

Proof. This is immediate from the fact that the generator a, b of D_n satisfy relations $a^n = b^2 = 1$ and $bab^{-1} = a^{-1}$. □

The following plays a fundamental role in our later discussion:

Lemma 3.1.3. (1) *a^k acts on Σ as a k/n-rotation (see Figure 3.1.1).*

(2) *Any $a^k b$ $(k = 0, 1, \ldots, n-1)$ is an involution fixing four points — it locally acts as a half rotation around each fixed point.*

(3) *Let ℓ be the axis of the involution b, and $p_i \in \ell$ $(i = 1, 2, 3, 4)$ be the fixed points of b. Then in (2), the axis of the involution $a^k b$ is $a^{k/2}\ell$, and $a^{k/2}p_i \in a^{k/2}\ell$ $(i = 1, 2, 3, 4)$ are the fixed points of $a^k b$.* (Here $a^{k/2}$ is a $k/2n$-rotation, and $a^{k/2} \in \mathrm{Aut}(\Sigma)$ but in general $a^{k/2} \notin D_n$.)

Proof. (1) is obvious. We show (2). For $k = 0$, that is, for b, this is obvious. For other $a^k b$, this follows from the fact that $a^k b$ is conjugate to b in $\mathrm{Aut}(\Sigma)$, indeed $a^k b = a^{k/2} b a^{-k/2}$ (from $a^{k/2} b = b a^{-k/2}$). (3) is clear from (2). □

In what follows, regard the cable surface Σ as a Riemann surface on which D_n acts *holomorphically*. To a representation $\rho : D_n \to GL_m(\mathbb{C})$, a quotient family $\eta : (\Sigma \times \mathbb{C}^m)/D_n \to \mathbb{C}^m/D_n$ is then associated. In case ρ is irreducible, we shall describe the quotient family η. We separate into two cases depending on the dimension (1 or 2) of ρ.

3.2 1-dimensional quotient families

The 1-dimensional representations of D_n are as follows (see [Ser] §5.3 p.36):

- If n is even, D_n has four 1-dimensional (necessarily irreducible) representations. They are $\chi_i : D_n \to GL_1(\mathbb{C})$ $(i = 1, 2, 3, 4)$ given by

$$(\chi_i(a), \chi_i(b)) = \begin{cases} (1, 1) & i = 1, \\ (1, -1) & i = 2, \\ (-1, 1) & i = 3, \\ (-1, -1) & i = 4. \end{cases} \tag{3.2.1}$$

- If n is odd, the representations χ_3 and χ_4 fail to be homomorphisms, and the 1-dimensional representations of D_n are merely the representations χ_1 and χ_2.

None of $\chi_1, \chi_2, \chi_3, \chi_4$ are injective, indeed:

Lemma 3.2.1.

$$\mathrm{Ker}(\chi_i) = \begin{cases} D_n & i = 1, \\ \langle a \rangle & i = 2, \\ \langle a^2, b \rangle & i = 3, \\ \langle a^2, ab \rangle & i = 4. \end{cases} \tag{3.2.2}$$

Proof. The case $i = 1$ is trivial. The other cases are confirmed as follows:

Case $i = 2$: Note that $a \in \mathrm{Ker}(\chi_2)$ and $b \notin \mathrm{Ker}(\chi_2)$. So $\langle a \rangle \subset \mathrm{Ker}(\chi_2) \subsetneqq D_n \,(= \langle a, b \rangle)$. Here $|\langle a \rangle| = n$ and $|D_n| = 2n$, thus necessarily $|\mathrm{Ker}(\chi_2)| = n$ and $\mathrm{Ker}(\chi_2) = \langle a \rangle$.

Case $i = 3$: Note that $a^2, b \in \mathrm{Ker}(\chi_3)$ and $a \notin \mathrm{Ker}(\chi_3)$. So $\langle a^2, b \rangle \subset \mathrm{Ker}(\chi_3) \subsetneqq D_n \,(= \langle a, b \rangle)$. Here $|\langle a^2, b \rangle| = n$ and $|D_n| = 2n$, thus necessarily $|\mathrm{Ker}(\chi_3)| = n$ and $\mathrm{Ker}(\chi_3) = \langle a^2, b \rangle$.

Case $i = 4$: Note that $a^2, ab \in \mathrm{Ker}(\chi_4)$ and $a \notin \mathrm{Ker}(\chi_4)$. So $\langle a^2, ab \rangle \subset \mathrm{Ker}(\chi_4) \subsetneqq D_n \,(= \langle a, b \rangle)$. Here $|\langle a^2, ab \rangle| = n$ and $|D_n| = 2n$, thus necessarily $|\mathrm{Ker}(\chi_4)| = n$ and $\mathrm{Ker}(\chi_4) = \langle a^2, ab \rangle$. □

	Ker	|Ker|
χ_1	D_n	$2n$
χ_2	$\langle a \rangle \; (\cong \mathbb{Z}_n)$	n
χ_3	$\langle a^2, b \rangle \; (\cong \mathbb{Z}_{n/2} \rtimes \mathbb{Z}_2)$	n
χ_4	$\langle a^2, ab \rangle \; (\cong \mathbb{Z}_{n/2} \rtimes \mathbb{Z}_2)$	n

Table 3.2.1:

We describe the quotient family $\xi_i : (\Sigma \times \mathbb{C})/D_n \to \mathbb{C}/D_n$ associated with χ_i. By Lemma 2.1.3, $\xi_i^{-1}(0) = \Sigma/D_n$ (c.m. $|D_n| = 2n$) and for $s \neq 0$, $\xi_i^{-1}(s) = \Sigma/\mathrm{Ker}(\chi_i)$ (c.m. $|\mathrm{Ker}(\chi_i)|$), explicitly:

$$\xi_i^{-1}(s) = \begin{cases} \Sigma/D_n & \text{(c.m. } 2n) & i = 1, \\ \Sigma/\langle a \rangle & \text{(c.m. } n) & i = 2, \\ \Sigma/\langle a^2, b \rangle & \text{(c.m. } n) & i = 3, \\ \Sigma/\langle a^2, ab \rangle & \text{(c.m. } n) & i = 4. \end{cases} \tag{3.2.3}$$

Note that $\xi_1 : \Sigma/D_n \times \mathbb{C} \to \mathbb{C}$ is a projection.

We will explicitly describe the four quotient spaces in (3.2.3) after some technical preparation.

Lemma 3.2.2. *Suppose that a group G acts on a Riemann surface X. Let H be a normal subgroup of G and let $q : X \to X/H$ be the quotient map. Then for each $g \in G$, the following hold:*

(1) *The automorphism $g : X \to X$ descends to an automorphism $\overline{g} : X/H \to X/H$, that is, $q \circ g = \overline{g} \circ q$ holds. If the order of g is n, then $\overline{g}^n = \mathrm{id}$; so the order of $\overline{g}$ is at most n.*

(2) *If there exists a point $x \in X$ such that $gx \notin Hx$, then $\overline{g} \neq \mathrm{id}$.* (Caution: in general, even if $g \notin H$, possibly $\overline{g} = \mathrm{id}$.)

(3) *If the automorphism $g : X \to X$ has a fixed point, then the automorphism $\overline{g} : X/H \to X/H$ also has a fixed point. In fact under the quotient map q, the fixed points of g descend to fixed points of $\overline{g}$.* (Caution: there may be other fixed points of $\overline{g}$.)

Proof. We show (1). A point of X/H is denoted by $p \bmod H$ for some $p \in X$. Define $\overline{g} : X/H \to X/H$ by $p \bmod H \mapsto gp \bmod H$. Then $q \circ g = \overline{g} \circ q$ holds from the normality condition $gH = Hg$. Thus $\overline{g}$ is the descent of g. If $g^n = \mathrm{id}$, then from the construction of $\overline{g}$, we have $\overline{g}^n = \mathrm{id}$. We show (2). Set $\overline{x} := q(x)$. The condition $gx \notin Hx$ implies $\overline{g}\,\overline{x} \neq \overline{x}$, so $\overline{g} \neq \mathrm{id}$. We show (3). From $q \circ g = \overline{g} \circ q$, we have $q \circ g(x) = \overline{g} \circ q(x)$ for any $x \in X$. If x is a fixed point of g, then $g(x) = x$, so $q(x) = \overline{g} \circ q(x)$. Set $\overline{x} := q(x)$, then we have $\overline{x} = \overline{g}(\overline{x})$, that is, $\overline{x}$ is a fixed point of $\overline{g}$. □

We return to the action of D_n on Σ. Recall that $a^k b$ ($k = 0, 1, 2, \ldots, n-1$) is an involution of the elliptic curve Σ fixing four points (Lemma 3.1.3 (2)), that is, $a^k b$ is an *elliptic involution*. The following holds:

Lemma 3.2.3. *Let Σ be the cable surface of a regular n-gon, which is an elliptic curve with D_n-action. Write $D_n = \langle a, b \rangle$, where a is the $1/n$-rotation* *and b is the elliptic involution in Figure 3.1.1. Let l be an integer and H be the cyclic group generated by a^l. If $x \notin \bigcup_{k=0}^{n-1} \mathrm{Fix}(a^k b)$, then the following hold:*

(1) *The D_n-orbit $D_n x$ of x is written as a disjoint union $D_n = A \amalg B$, where*

$$A = \{a^i x \,:\, i = 0, 1, \ldots, n-1\}, \qquad B = \{a^j bx \,:\, j = 0, 1, \ldots, n-1\}.$$

(2) $a^k bx \notin Hx$ *for any* $k = 0, 1, \ldots, n-1$.

(3) *Under the quotient map $q : \Sigma \to \Sigma/H$, any elliptic involution $a^k b$ of Σ descends to an automorphism $\overline{a^k b}$ of Σ/H such that $\overline{a^k b}^2 = \mathrm{id}$ and $\overline{a^k b} \neq \mathrm{id}$, that is, $\overline{a^k b}$ is an involution.*

Proof. We show (1). Since $D_n = \{a^i, a^j b \,:\, i, j = 0, 1, \ldots, n-1\}$ (see Lemma 3.1.2), we have $D_n = A \cup B$. It suffices to show $A \cap B \neq \emptyset$. An element of $A \cap B$, if any, is written as $a^i x = a^j bx$, that is, $a^{j-i} bx = x$, so $x \in \mathrm{Fix}(a^{j-i} b)$. This contradicts the assumption that $x \notin \bigcup_{k=0}^{n-1} \mathrm{Fix}(a^k b)$. We show (2). Since

$$Hx = \{a^{ml} x \,:\, m = 0, 1, \ldots\} \subset A,$$

we have $Hx \cap B = \emptyset$ by (1), thus $a^k bx \notin Hx$ for any $k = 0, 1, \ldots, n-1$. We show (3). Note that $H = \langle a^l \rangle$ is normal in D_n (as $bab^{-1} = a^{-1}$). Applying Lemma 3.2.2 (1) to the case $X = \Sigma$, $G = D_n$ and $g = a^k b$ shows that $a^k b$ descends to an automorphism $\overline{a^k b}$ of Σ/H such that $\overline{a^k b}^2 = \mathrm{id}$. Here by Lemma 3.2.2 (2), $\overline{a^k b} \neq \mathrm{id}$ (note that $a^k bx \notin Hx$ by (2), so the assumption of that lemma is satisfied). □

We next show the following:

Lemma 3.2.4. *Let Σ be the cable surface of a regular n-gon, which is an elliptic curve with D_n-action. Write $D_n = \langle a, b \rangle$, where a is the $1/n$-rotation and b is the elliptic involution in Figure 3.1.1. Then the following hold:*

(1) *For any integer l, the quotient map $q : \Sigma \to \Sigma/\langle a^l \rangle$ is an unramified covering and the quotient $\Sigma/\langle a^l \rangle$ is an elliptic curve.*

(2) *Under the quotient map q in* (1)*, any elliptic involution $a^k b$ of Σ descends to an elliptic involution of $\Sigma/\langle a^l \rangle$, that is, the involution $\overline{a^k b}$ in Lemma 3.2.3* (3) *is an elliptic involution.*

Proof. We show (1). As the action of $\langle a \rangle$ on Σ is free (see Figure 3.1.1), so is the action of $\langle a^l \rangle$ on Σ, thus $q : \Sigma \to \Sigma/\langle a^l \rangle$ is unramified, that is, $\Sigma/\langle a^l \rangle$ has no branch points. Then by the Riemann–Hurwitz formula, $\chi(\Sigma) = |\langle a^l \rangle| \, \chi(\Sigma/\langle a^l \rangle)$. Here Σ is an elliptic curve, so $\chi(\Sigma) = 0$, thus $\chi(\Sigma/\langle a^l \rangle) = 0$, in turn $\Sigma/\langle a^l \rangle$ is an elliptic curve. We next show (2). As we saw in (1), $\Sigma/\langle a^l \rangle$ is an elliptic curve. It is well-known that an involution of an elliptic curve is either an elliptic involution or a translation of order 2 (the latter is fixed point free). Now since $a^k b$ has a fixed point (Lemma 3.1.3 (2)), $\overline{a^k b}$ also has a fixed point (Lemma 3.2.2 (3)), so $\overline{a^k b}$ must be an elliptic involution. □

Note that when a group G acts on Σ and a subgroup N of G is normal, the induced quotient map $\Sigma/N \to \Sigma/G$ is a Galois covering with covering transformation group G/N.

Lemma 3.2.5. *Fix arbitrary integers l and k and consider a subgroup $\langle a^l, a^k b \rangle$ of $D_n = \langle a, b \rangle$. Then the following hold:*

(1) *$\langle a^l \rangle$ is a normal subgroup of $\langle a^l, a^k b \rangle$ and the quotient group $\langle a^l, a^k b \rangle / \langle a^l \rangle$ is cyclic group of order* 2.

(2) *$\Sigma/\langle a^l \rangle \to \Sigma/\langle a^l, a^k b \rangle$ is the quotient of the elliptic curve $\Sigma/\langle a^l \rangle$ by the elliptic involution $\overline{a^k b}$ in Lemma 3.2.4* (2)*; thus this covering is two-fold with four branch points, and $\Sigma/\langle a^l, a^k b \rangle$ is a projective line.*

(3) *The quotient map $p : \Sigma \to \Sigma/\langle a^l, a^k b \rangle$ is a ramified covering with four branch points.*

Proof. We show (1), first, that $\langle a^l \rangle$ is normal in $\langle a^l, a^k b\rangle$, that is, $ga^l g^{-1} \in \langle a^l \rangle$ for $g = a^l$ and $a^k b$. For $g = a^l$, this is trivial and for $g = a^k b$, this follows from $(a^k b)a^l(a^k b)^{-1} = a^{-l}$, which is confirmed as follows:

$$\begin{aligned}(a^k b)a^l(a^k b)^{-1} &= a^k(ba^l b^{-1})a^{-k} = a^k a^{-l} a^{-k} \quad \text{from } bab^{-1} = a^{-1} \\ &= a^{-l}.\end{aligned}$$

We next show that the quotient group $\langle a^l, a^k b\rangle/\langle a^l\rangle$ is a cyclic group of order 2. Consider the short exact sequence of groups:

$$1 \longrightarrow \langle a^l \rangle \longrightarrow \langle a^l, a^k b\rangle \longrightarrow \langle a^l, a^k b\rangle/\langle a^l\rangle \longrightarrow 1.$$

Here $\langle a^l, a^k b\rangle/\langle a^l\rangle \cong \langle a^k b\rangle$ and $(a^k b)^2 = 1$, so the quotient group $\langle a^l, a^k b\rangle/\langle a^l\rangle$ is a cyclic group of order 2. We show (2). From (1), $\Sigma/\langle a^l\rangle \to \Sigma/\langle a^l, a^k b\rangle$ is a Galois covering with covering transformation group $\langle a^l, a^k b\rangle/\langle a^l\rangle \cong \langle a^k b\rangle$. Here $a^k b$ acts on $\Sigma/\langle a^l\rangle$ as $\overline{a^k b}$, which is an elliptic involution of $\Sigma/\langle a^l\rangle$ (Lemma 3.2.4 (2)). Thus the Galois covering $\Sigma/\langle a^l\rangle \to \Sigma/\langle a^l, a^k b\rangle$ is the quotient of $\Sigma/\langle a^l\rangle$ by the involution $\overline{a^k b}$. Therefore it is a two-fold covering with four branch points, and $\Sigma/\langle a^l, a^k b\rangle$ is a projective line. We show (3). Write $p : \Sigma \to \Sigma/\langle a^l, a^k b\rangle$ as the composition $p = r \circ q$ of quotient maps $r : \Sigma/\langle a^l\rangle \to \Sigma/\langle a^l, a^k b\rangle$ and $q : \Sigma \to \Sigma/\langle a^l\rangle$. Here r is a ramified covering with four branch points by (2) and q is an unramified covering by Lemma 3.2.4 (1), thus the assertion holds. Figure 3.2.1 illustrates the case $(l, k) = (1, 0)$. $\square$

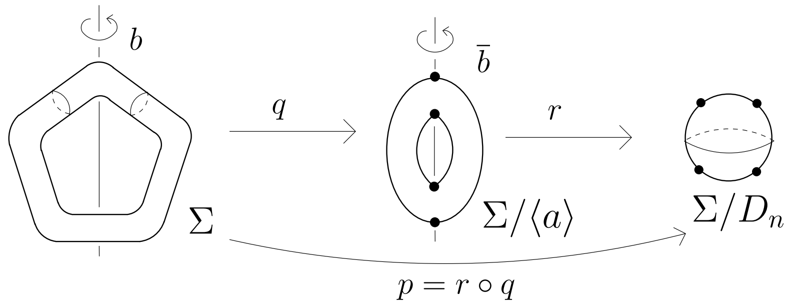


Figure 3.2.1: $(l, k) = (1, 0)$: q is the quotient map under the $\langle a\rangle$-action and r is the quotient map under the $\langle \bar{b}\rangle$-action, where the descent $\bar{b}$ of b is an elliptic involution.

We may now explicitly determine the fibers $\xi_i^{-1}(s)$ in (3.2.3). By Lemma 3.2.5 (2), any of Σ/D_n, $\Sigma/\langle a^2, b\rangle$ and $\Sigma/\langle a^2, ab\rangle$ is a projective line (note: $D_n = \langle a, b\rangle$). On the other hand, by Lemma 3.2.4 (1), $\Sigma/\langle a\rangle$ is an elliptic curve. The results obtained so far are summarized as follows:

Theorem 3.2.6. *Let $\xi_i : (\Sigma \times \mathbb{C})/D_n \to \mathbb{C}/D_n$ $(i = 1, 2, 3, 4)$ be the quotient family of Σ associated with $\chi_i : D_n \to GL_1(\mathbb{C})$. Then the following hold*:

(i) $\xi_1 : (\Sigma/D_n) \times \mathbb{C} \to \mathbb{C}$ *is a projection, so for any s, $\xi_1^{-1}(s) = \Sigma/D_n$; its covering multiplicity is $|D_n| = 2n$.*

(ii) *For ξ_2, ξ_3 and ξ_4, the following hold* (c.m. means covering multiplicity):

$$\xi_2^{-1}(s) = \begin{cases} \text{projective line } \Sigma/D_n \quad (\text{c.m. } 2n) & \text{if } s = 0, \\ \text{elliptic curve } \Sigma/\langle a\rangle \quad (\text{c.m. } n) & \text{if } s \neq 0, \end{cases}$$

$$\xi_3^{-1}(s) = \begin{cases} \text{projective line } \Sigma/D_n \quad (\text{c.m. } 2n) & \text{if } s = 0, \\ \text{projective line } \Sigma/\langle a^2, b\rangle \quad (\text{c.m. } n) & \text{if } s \neq 0, \end{cases}$$

$$\xi_4^{-1}(s) = \begin{cases} \text{projective line } \Sigma/D_n \quad (\text{c.m. } 2n) & \text{if } s = 0, \\ \text{projective line } \Sigma/\langle a^2, ab\rangle \quad (\text{c.m. } n) & \text{if } s \neq 0. \end{cases}$$

3.3 Properties of representations of D_n

The 2-dimensional irreducible representations of D_n are given by the representations $\rho_l : D_n \to GL_2(\mathbb{C})$ (where l is an integer such that $1 \le l < \frac{n}{2}$) defined by

$$\rho_l(a) = \begin{pmatrix} \zeta^l & 0 \\ 0 & \zeta^{-l} \end{pmatrix}, \text{where } \zeta := e^{2\pi \mathrm{i}/n}, \qquad \rho_l(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.3.1}$$

In what follows, via ρ_l let D_n act on $\mathbb{C}^2$. Recall that any element of D_n is expressed as a^k or $a^k b$ ($k = 0, 1, \dots, n-1$); see Lemma 3.1.2.

Lemma 3.3.1. *Set* $\mathrm{Fix}(g) := \{t \in \mathbb{C}^2 : \rho_l(g)t = t\}$. *Then* $\mathrm{Fix}(a^k) = \{0\}$ *for* $k \neq 0$ (*while* $\mathrm{Fix}(a^0) = \mathbb{C}^2$) *and* $\mathrm{Fix}(a^k b) = \left\{ \lambda \begin{pmatrix} \zeta^{lk} \\ 1 \end{pmatrix} \in \mathbb{C}^2 \,:\, \lambda \in \mathbb{C} \right\}$ *for any* k.

Proof. The assertion follows from $\rho_l(a^k) = \begin{pmatrix} \zeta^{lk} & 0 \\ 0 & \zeta^{-lk} \end{pmatrix}$ and $\rho_l(a^k b) = \begin{pmatrix} 0 & \zeta^{lk} \\ \zeta^{-lk} & 0 \end{pmatrix}$. □

Let $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$ be the quotient family associated with ρ_l. Its *prekaleido locus* is given by $\widetilde{\mathrm{KL}}_{\eta_{n,l}} = \{t \in \mathbb{C}^2 \,:\, H_t \neq 1\}$, where H_t denotes the stabilizer of $t \in \mathbb{C}^2$ for the D_n-action on $\mathbb{C}^2$. The kaleido locus is then given by $\mathrm{KL}_{\eta_{n,l}} = \widetilde{\mathrm{KL}}_{\eta_{n,l}}/D_n$. Note that $\widetilde{\mathrm{KL}}_{\eta_{n,l}}$ is expressed as $\widetilde{\mathrm{KL}}_{\eta_{n,l}} = \bigcup_{x \in D_n \setminus \{1\}} \mathrm{Fix}(x)$. Here $x = a^k$ or $a^k b$ for some $k \in \{0, 1, \dots, n-1\}$ (Lemma 3.1.2) and note that $\mathrm{Fix}(a^k) = \{0\}$ and $0 \in \mathrm{Fix}(a^k b)$ (Lemma 3.3.1). We may thus write $\widetilde{\mathrm{KL}}_{\eta_{n,l}} = \bigcup_{k=0}^{n-1} \mathrm{Fix}(a^k b)$. Accordingly we have $\mathrm{KL}_{\eta_{n,l}} = \left(\bigcup_{k=0}^{n-1} \mathrm{Fix}(a^k b) \right) \Big/ D_n$. Here the D_n-action is given as follows: $g \in D_n$ maps $\mathrm{Fix}(a^k b)$ to $\mathrm{Fix}(g a^k b g^{-1})$. We explicitly describe $\mathrm{Fix}(g a^k b g^{-1})$, for which we shall rewrite $g a^k b g^{-1}$ (see (2) below).

Lemma 3.3.2. (1) *For any* i, k, $a^i (a^k b) a^{-i} = a^{k+2i} b$ *and* $(a^i b)(a^k b)(a^i b)^{-1} = a^{-k+2i} b$.

(2) *For any* $g \in D_n$, $g(a^k b)g^{-1} = a^{k+2i}b$ *or* $a^{-k+2i}b$ *for some* $i \in \{0, 1, \dots, n-1\}$.

(3) $\{ga^k bg^{-1} : g \in D_n\} = \{a^{k+2i}b, a^{-k+2i}b : i = 0, 1, \dots, n-1\}$.

Proof. (1) Recall that $D_n := \langle a, b : a^n = b^2 = 1,\ bab^{-1} = a^{-1}\rangle$. The first equation is confirmed as follows:

$$\begin{aligned} a^i(a^k b)a^{-i} &= a^i(a^k b)ba^i b^{-1} && \text{as } a^{-i} = ba^i b^{-1} \text{ (from } a^{-1} = bab^{-1}) \\ &= a^{i+k}b^2 a^i b^{-1} = a^{k+2i}b && \text{as } b^2 = 1 \text{ and } b^{-1} = b. \end{aligned}$$

The second equation is confirmed as follows:

$$\begin{aligned} (a^i b)(a^k b)(a^i b)^{-1} &= a^i(ba^{k-i}) = a^i(a^{-(k-i)}b) && \text{as } ba^{k-i} = a^{-(k-i)}b \\ &= a^{-k+2i}b. \end{aligned}$$

(2) By Lemma 3.1.2, $g = a^i$ or $a^i b$ for some $i \in \{0, 1, \dots, n-1\}$. Accordingly we obtain

$$g(a^k b)g^{-1} = \begin{cases} a^i(a^k b)a^{-i} = a^{k+2i}b \text{ (by (1))} \quad \text{or} \\ (a^i b)(a^k b)(a^i b)^{-1} = a^{-k+2i}b \text{ (by (1))}. \end{cases}$$

(3) Set $S := \{ga^k bg^{-1} : g \in D_n\}$ and $T := \{a^{k+2i}b, a^{-k+2i}b : i = 0, 1, \dots, n-1\}$. By (1), we have $S \supset T$ and by (2), we have $S \subset T$. Thus we obtain $S = T$. □

Corollary 3.3.3. *If* $k \equiv k' \bmod 2$*, then* $a^k b$ *and* $a^{k'} b$ *are conjugate in* D_n.

Proof. Write $k = k' + 2l$, where $l := \dfrac{k - k'}{2}$ is an integer by assumption. Then by Lemma 3.3.2 (1), we have $a^l(a^{k'}b)a^{-l} = a^{k'+2l}b = a^k b$. □

For even n, the converse of Corollary 3.3.3 holds:

Corollary 3.3.4. (1) *If* n *is even, then* $a^k b$ *and* $a^{k'} b$ *are conjugate if and only if* $k \equiv k' \bmod 2$.

(2) *If n is odd, then $a^k b$ ($k = 0, 1, \ldots, n-1$) are mutually conjugate.*

Proof. (1) It suffices to show that if $a^k b$ and $a^{k'} b$ are conjugate then $k \equiv k' \bmod 2$. The set of conjugates of $a^k b$ is equal to $T = \{a^{k+2i} b, a^{-k+2i} b : i = 0, 1, \ldots, n-1\}$ (Lemma 3.3.2 (3)). If $a^k b$ and $a^{k'} b$ are conjugate then $a^{k'} b \in T$, so $k' \equiv k + 2i$ or $-k + 2i \bmod n$ for some i. Here n is even, thus $k \equiv k' \bmod 2$.

(2) Since $a^k b$ and $a^{k'} b$ ($k \equiv k' \bmod 2$) are conjugate (Corollary 3.3.3), elements of $A := \{a^k b : k \text{ is even}\}$ are mutually conjugate and elements of $B := \{a^k b : k \text{ is odd}\}$ are mutually conjugate. Next since $n - 1 \equiv n + 1 \bmod 2$, $a^{n-1} b$ and $a^{n+1} b$ are conjugate. Here $a^n = 1$, so $a^{n+1} b = ab$. Thus $a^{n-1} b$ and ab are conjugate. Note that $a^{n-1} b \in A$ (as $n-1$ is even) and $ab \in B$. Thus elements of A and B are mutually conjugate. □

Corollary 3.3.4 combined with the fact that $g \in D_n$ maps $\mathrm{Fix}(x)$ to $\mathrm{Fix}(gxg^{-1})$ yields: *If n is odd, then for any k, k' there exists an element of D_n that maps $\mathrm{Fix}(a^k b)$ to $\mathrm{Fix}(a^{k'} b)$. If n is even, this is the case only when* $k \equiv k' \bmod 2$. Therefore the following is obtained:

Lemma 3.3.5. *Set $L_k := \mathrm{Fix}(a^k b)$. The D_n-action on $\mathbb{C}^2$ permutes (distinct) complex lines $\{L_0, L_1, \ldots, L_{n-1}\}$: if n is odd, this action is transitive and if n is even, this action has two orbits: $\{L_0, L_2, \ldots, L_{n-2}\}$ and $\{L_1, L_3, \ldots, L_{n-1}\}$.*

3.4 2-dimensional quotient families

To each representation $\rho_l : D_n \to GL_2(\mathbb{C})$ where l is an integer such that $1 \leq l < \frac{n}{2}$, we shall describe the associated quotient family $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$. Note first that:

Lemma 3.4.1. $\mathrm{Ker}(\rho_l) = \langle a^{n'} \rangle$, *where we set $n' := n/\gcd(l, n)$.* (Thus the order of $\mathrm{Ker}(\rho_l)$ is $\gcd(l, n)$. In particular ρ_l is injective if and only if $\gcd(l, n) = 1$.)

Proof. Note first that $\mathrm{Ker}(\rho_l) \subset \langle a \rangle$, as $\rho_l(a^k b) = \begin{pmatrix} 0 & \zeta^{lk} \\ \zeta^{-lk} & 0 \end{pmatrix} \neq I$ for any k. The assertion is then immediate from the following equivalence:

$$\rho_l(a^k) = \begin{pmatrix} \zeta^{lk} & 0 \\ 0 & \zeta^{-lk} \end{pmatrix} = I \iff k \text{ is a multiple of } n' \text{ (in other words } n', 2n', \ldots, \gcd(l,n)n').$$

□

We separate into two cases depending on whether ρ_l is injective.

3.4.1 Injective case

Let $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$ be the quotient family associated with an *injective* representation ρ_l (in this case $\gcd(l,n) = 1$). We first determine its kaleido locus $\mathrm{KL}_{\eta_{n,l}}$ (over which kaleido fibers lie; see Definition 2.1.4). This is the image of the prekaleido locus $\widetilde{\mathrm{KL}}_{\eta_{n,l}} = \bigcup_{g \in D_n \setminus \{1\}} \mathrm{Fix}(g)$ under the quotient map $\mathbb{C}^2 \to \mathbb{C}^2/D_n$. Here note that

$$\begin{aligned}
\widetilde{\mathrm{KL}}_{\eta_{n,l}} &= \bigcup_{k=1}^{n-1} \mathrm{Fix}(a^k) \cup \bigcup_{k=0}^{n-1} \mathrm{Fix}(a^k b) = \bigcup_{k=0}^{n-1} \mathrm{Fix}(a^k b) \\
&= \bigcup_{k=0}^{n-1} \left\{ \lambda \begin{pmatrix} \zeta^{lk} \\ 1 \end{pmatrix} \in \mathbb{C}^2 \,:\, \lambda \in \mathbb{C} \right\} \quad \text{by Lemma 3.3.1.}
\end{aligned}$$

This confirms the following:

Lemma 3.4.2. *The prekaleido locus of $\eta_{n,l}$ is given by $\widetilde{\mathrm{KL}}_{\eta_{n,l}} = \bigcup_{k=0}^{n-1} L_k$, where we set*

$$L_k := \mathrm{Fix}(a^k b) = \left\{ \lambda \begin{pmatrix} \zeta^{lk} \\ 1 \end{pmatrix} \in \mathbb{C}^2 \,:\, \lambda \in \mathbb{C} \right\}.$$

We consequently obtain $\mathrm{KL}_{\eta_{n,l}} = \bigcup_{k=0}^{n-1} \overline{L}_k$, where $\overline{L}_k$ is the image of L_k under the quotient map $\mathbb{C}^2 \to \mathbb{C}^2/D_n$. Here the lines $\overline{L}_0, \overline{L}_1, \ldots, \overline{L}_{n-1}$ are *not* distinct: From Lemma 3.3.5, if n is odd, $\overline{L}_0 = \overline{L}_1 = \cdots = \overline{L}_{n-1}$ and if n is even, $\overline{L}_0 = \overline{L}_2 = \cdots = \overline{L}_{n-2}$ and $\overline{L}_1 = \overline{L}_3 = \cdots = \overline{L}_{n-1}$. So

$$\mathrm{KL}_{\eta_{n,l}} = \begin{cases} \overline{L}_0 & \text{if } n \text{ is odd,} \\ \overline{L}_0 \cup \overline{L}_1 & \text{if } n \text{ is even.} \end{cases} \tag{3.4.1}$$

Moonsault of elliptic curves Let $\Sigma \times \mathbb{C}^2 \to \mathbb{C}^2$ be the projection, on which D_n acts equivariantly (via ρ_l on $\mathbb{C}^2$). (Its quotient is the quotient family $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$.) Every fiber of $\Sigma \times \mathbb{C}^2 \to \mathbb{C}^2$ is of course the elliptic curve Σ. As a consequence of Lemma 3.3.5:

- Odd n: D_n permutes elliptic curves over $L_0, L_1, \ldots, L_{n-1}$ (Figure 3.4.1).
- Even n: D_n permutes elliptic curves over $L_0, L_2, \ldots, L_{n-2}$ as well as elliptic curves over $L_1, L_3, \ldots, L_{n-1}$ (Figure 3.4.2).

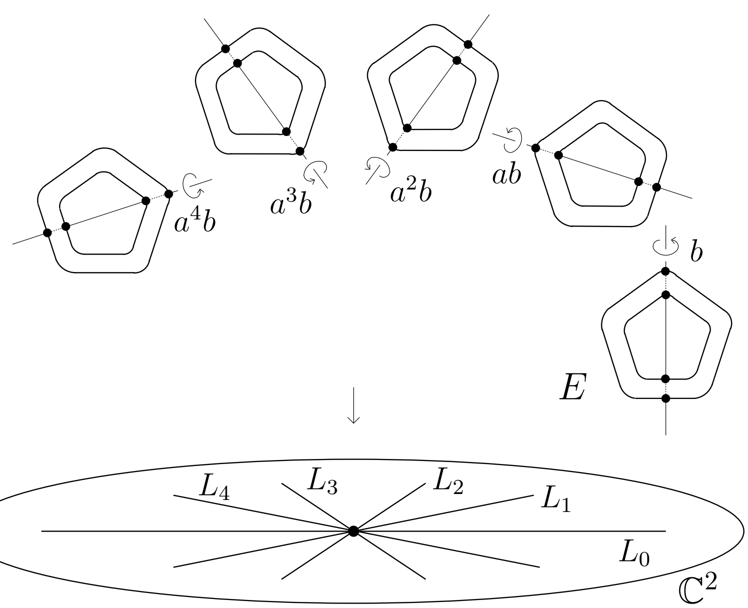


Figure 3.4.1: Odd case ($n = 5$): *Moonsault* (gymnastics skill) of an elliptic curve E in $\Sigma \times \mathbb{C}^2$. Each $a^k b$ ($k = 0, 1, \ldots$) is an involution (Lemma 3.1.3).

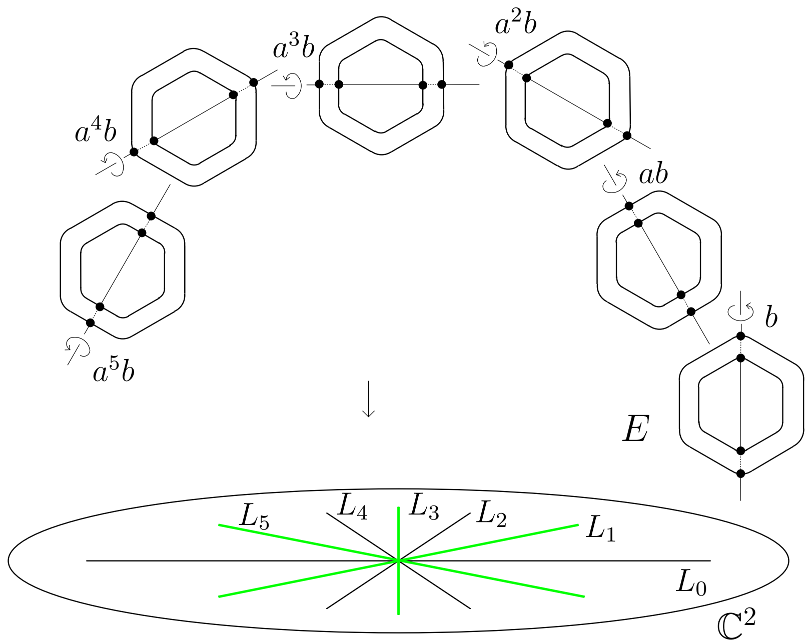


Figure 3.4.2: Even case ($n = 6$): *Moonsault* of an elliptic curve E in $\Sigma \times \mathbb{C}^2$.

We describe the kaleido fibers of $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$. For $s \in \mathbb{C}^2/D_n$, let $H_{\widetilde{s}}$ be the stabilizer of $\widetilde{s} \in \mathbb{C}^2$ for the D_n-action ($\widetilde{s}$ is a *lift* of s), then $\eta_{n,l}^{-1}(s) = \Sigma/H_{\widetilde{s}}$ by Theorem 2.1.1. This is essentially independent of the choice of a lift $\widetilde{s}$, as, for any other lift $\widetilde{s}'$, $H_{\widetilde{s}} \cong H_{\widetilde{s}'}$ and $\Sigma/H_{\widetilde{s}} \cong \Sigma/H_{\widetilde{s}'}$ *canonically*. We shall explicitly determine $\Sigma/H_{\widetilde{s}}$. Note that

$$H_{\widetilde{s}} = \begin{cases} D_n & \text{if } \widetilde{s} = 0, \\ \langle a^k b \rangle & \text{if } \widetilde{s} \in L_k \setminus \{0\} \ (k = 0, 1, \ldots, n-1), \ \text{where } L_k := \mathrm{Fix}(a^k b), \\ \{1\} & \text{if } \widetilde{s} \in \mathbb{C}^2 \setminus \bigcup_{k=0}^{n-1} L_k. \end{cases} \tag{3.4.2}$$

Here the second case "$H_{\widetilde{s}} = \langle a^k b \rangle$ if $\widetilde{s} \in L_k \setminus \{0\}$ ($k = 0, 1, \ldots, n-1$)" may be rewritten depending on the parity of n as follows:

- Odd n: $H_{\widetilde{s}} = \langle b \rangle$ if $\widetilde{s} \in L_0 \setminus \{0\}$,
  because the D_n-action on $\{L_0, L_1, \ldots, L_{n-1}\}$ is transitive (Lemma 3.3.5).

- Even n: $H_{\widetilde{s}} = \begin{cases} \langle b \rangle & \widetilde{s} \in L_0 \setminus \{0\} \text{ if } k \text{ is even,} \\ \langle ab \rangle & \widetilde{s} \in L_1 \setminus \{0\} \text{ if } k \text{ is odd,} \end{cases}$
because the D_n-action on $\{L_0, L_1, \dots, L_{n-1}\}$ has two orbits $\{L_0, L_2, \dots, L_{n-2}\}$ and $\{L_1, L_3, \dots, L_{n-1}\}$ (Lemma 3.3.5).

Hence the following is obtained:

- Odd n:

$$H_{\widetilde{s}} = \begin{cases} D_n & s = 0, \\ \langle b \rangle & s \in \overline{L}_0 \setminus \{0\}, \\ \{1\} & s \in (\mathbb{C}^2/D_n) \setminus \overline{L}_0. \end{cases}$$

- Even n:

$$H_{\widetilde{s}} = \begin{cases} D_n & s = 0, \\ \langle b \rangle & s \in \overline{L}_0 \setminus \{0\}, \\ \langle ab \rangle & s \in \overline{L}_1 \setminus \{0\}, \\ \{1\} & s \in (\mathbb{C}^2/D_n) \setminus (\overline{L}_0 \cup \overline{L}_1). \end{cases}$$

Now recall that the covering multiplicity (c.m.) of $\eta_{n,l}^{-1}(s) = \Sigma/H_{\widetilde{s}}$ is $|H_{\widetilde{s}}|$. The following holds:

(i) Odd n:

$$\eta_{n,l}^{-1}(s) = \begin{cases} \Sigma/D_n & s = 0 & (\text{c.m. } |D_n| = 2n), \\ \Sigma/\langle b \rangle & s \in \overline{L}_0 \setminus \{0\} & (\text{c.m. } |\langle b \rangle| = 2), \\ \Sigma & s \in (\mathbb{C}^2/D_n) \setminus \overline{L}_0 & (\text{c.m. } 1), \end{cases}$$

where note that $\mathrm{KL}_{\eta_{n,l}} = \overline{L}_0$.

(ii) Even n:

$$\eta_{n,l}^{-1}(s) = \begin{cases} \Sigma/D_n & s = 0 & (\text{c.m. } |D_n| = 2n), \\ \Sigma/\langle b \rangle & s \in \overline{L}_0 \setminus \{0\} & (\text{c.m. } |\langle b \rangle| = 2), \\ \Sigma/\langle ab \rangle & s \in \overline{L}_1 \setminus \{0\} & (\text{c.m. } |\langle ab \rangle| = 2), \\ \Sigma & s \in (\mathbb{C}^2/D_n) \setminus (\overline{L}_0 \cup \overline{L}_1) & (\text{c.m. } 1), \end{cases}$$

where note that $\mathrm{KL}_{\eta_{n,l}} = \overline{L}_0 \cup \overline{L}_1$.

In (i) and (ii), we have the following (see Lemma 3.2.5 (2)):

- Σ/D_n is $\mathbb{P}^1$, and $\Sigma \to \Sigma/D_n$ has four branch points.
- $\Sigma/\langle b\rangle$ is $\mathbb{P}^1$, and $\Sigma \to \Sigma/\langle b\rangle$ has four branch points.
- $\Sigma/\langle ab\rangle$ is $\mathbb{P}^1$, and $\Sigma \to \Sigma/\langle ab\rangle$ has four branch points.

The above results are summarized as follows:

Theorem 3.4.3. *Let* $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$ $(1 \le l < \frac{n}{2})$ *be the quotient family of* Σ *associated with* $\rho_l : D_n \to GL_2(\mathbb{C})$*. If* ρ_l *is injective* (*equivalently* $\gcd(l,n) = 1$)*, then the following hold* (c.m. means covering multiplicity):

$$\eta_{n,l}^{-1}(s) = \begin{cases} \text{projective line } \Sigma/D_n \quad (\text{c.m.}\, 2n) & \text{if } s = 0, \\ \text{projective line } \Sigma/\langle b\rangle \quad (\text{c.m.}\, 2) & \text{if } s \in \mathrm{KL}_{\eta_{n,l}} \setminus \{0\}, \\ \text{elliptic curve } \Sigma \quad (\text{c.m.}\, 1) & \text{if } s \in (\mathbb{C}^2/D_n) \setminus \mathrm{KL}_{\eta_{n,l}}. \end{cases}$$

$$\mathrm{KL}_{\eta_{n,l}} = \begin{cases} \overline{L}_0 & (n : \text{odd}), \\ \overline{L}_0 \cup \overline{L}_1 & (n : \text{even}). \end{cases}$$

Remark 3.4.4. While $\Sigma/\langle b\rangle$ appears as a fiber of $\eta_{n,l}$, for any $a^k \ne 1$, $\Sigma/\langle a^k\rangle$ does *not*. Reason: By the quotient fiber theorem, $\eta_{n,l}^{-1}(s) = \Sigma/H_t$, where H_t is the stabilizer of $t \in \mathbb{C}^2$. We claim that $H_t \ne \langle a^k\rangle$ for any $t \in \mathbb{C}^2$. Indeed since $\mathrm{Fix}(a^k) = \{0\}$, if $a^k \in H_t$ then $t = 0$, but $H_0 = D_n$.

To determine $\mathbb{C}^2/D_n$, we need the following:

Lemma 3.4.5. $\rho_l(b)$ *is a reflection.*

Proof. $\rho_l(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in $GL_2(\mathbb{C})$ via $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. □

Now we can determine $\mathbb{C}^2/D_n$:

Lemma 3.4.6. $\mathbb{C}^2/D_n \,(= \mathbb{C}^2/\langle \rho_l(a), \rho_l(b)\rangle)$ *is isomorphic to* $\mathbb{C}^2$*.*

Proof. By computation, the invariant ring of $\mathbb{C}[x, y]$ under the D_n-action is isomorphic to $\mathbb{C}[x, y]$ itself, thus the assertion holds. □

The total space $(\Sigma \times \mathbb{C}^2)/D_n$ of the quotient family $\eta_{n,l}$ is singular:

Odd Case If n is odd, the ramification points of the quotient map $\Sigma \to \Sigma/D_n$ consists of four D_n-orbits as illustrated in Figure 3.4.3. Each consists of n points: $\{p_0, \dots, p_{n-1}\}$, $\{q_0, \dots, q_{n-1}\}$, $\{r_0, \dots, r_{n-1}\}$, $\{s_0, \dots, s_{n-1}\}$, where for each $k = 0, \dots, n-1$, p_k, q_k, r_k, s_k are the fixed points of $a^k b$.

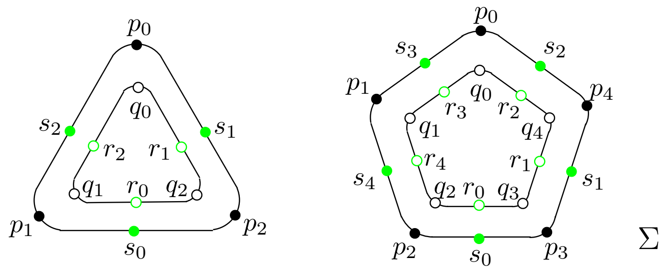


Figure 3.4.3: The D_n-orbits of the ramification points for $n = 3, 5$

Notation 3.4.7. Let x_k denote p_k, q_k, r_k, or s_k. Each line $\{x_k\} \times L_k$ in $\Sigma \times \mathbb{C}^2$ is mapped to a line $\overline{\{x_k\} \times L_k}$ in $(\Sigma \times \mathbb{C}^2)/D_n$, which lies over $\overline{L}_0$. Noting that $\overline{\{x_k\} \times L_k}$ does not depend on k, write this as R_x $(x = p, q, r, s)$. See Figure 3.4.4. As we will show in Theorem 3.5.11, the total space $(\Sigma \times \mathbb{C}^2)/D_n$ is singular along R_p, R_q, R_r, R_s. They are called the *ridges* of $(\Sigma \times \mathbb{C}^2)/D_n$.

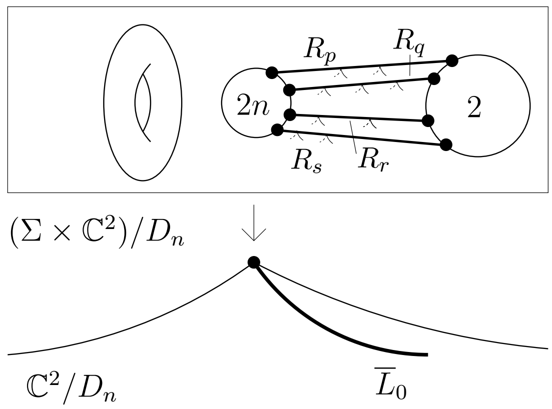


Figure 3.4.4: The ridges of the quotient family $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$ (n: odd)

Even Case If n is even, the ramification points of the quotient map $\Sigma \to \Sigma/D_n$ consists of four D_n-orbits as illustrated in Figure 3.4.5. Each consists of n points: $\{p_0, \ldots, p_{n-1}\}$, $\{q_0, \ldots, q_{n-1}\}$, $\{r_0, \ldots, r_{n-1}\}$, $\{s_0, \ldots, s_{n-1}\}$. Here $p_k, q_k, p_{k+n/2}, q_{k+n/2}$ $(k = 0, \ldots, n-1)$ are the fixed points of $a^{2k}b$, while $r_k, s_k, r_{k+n/2}, s_{k+n/2}$ $(k = 0, \ldots, n-1)$ are the fixed points of $a^{2k+1}b$.

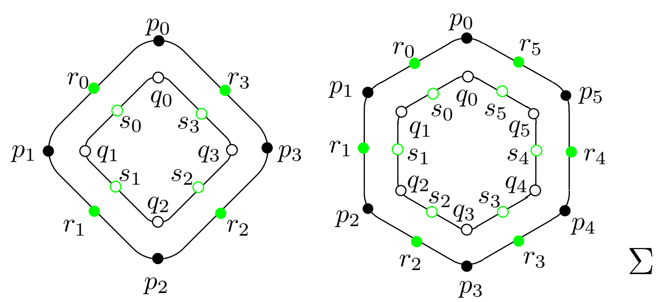


Figure 3.4.5: The D_n-orbits of the ramification points for $n = 4, 6$

Notation 3.4.8. Let x_k denote p_k, q_k, r_k, or s_k. Each line $\{x_k\} \times L_k$ in $\Sigma \times \mathbb{C}^2$ is mapped to a line $\overline{\{x_k\} \times L_k}$ in $(\Sigma \times \mathbb{C}^2)/D_n$. Noting that $\overline{\{x_k\} \times L_k}$ does not depend on k, write this as R_x. The lines R_p and R_q lie over $\overline{L}_0$ and the lines R_r and R_s lie over $\overline{L}_1$. See Figure 3.4.6. (As we will show in

Theorem 3.5.11, the total space $(\Sigma \times \mathbb{C}^2)/D_n$ is singular along the ridges R_p, R_q, R_r, R_s.)

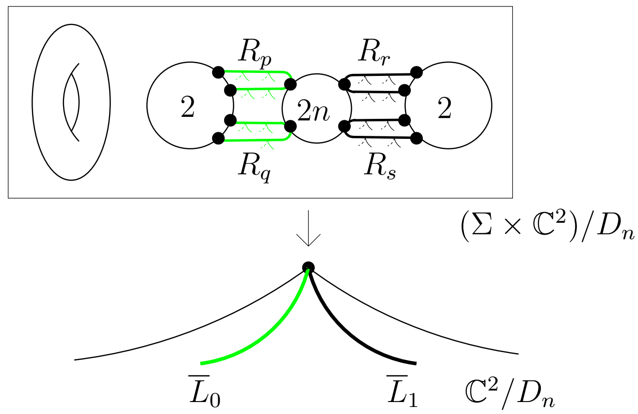


Figure 3.4.6: The ridges of the quotient family $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$ (n: even)

3.4.2 Non-injective case

Let $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$ be the quotient family associated with the irreducible representation $\rho_l : D_n \to GL_2(\mathbb{C})$ given by

$$\rho_l(a) = \begin{pmatrix} e^{2\pi i l/n} & 0 \\ 0 & e^{-2\pi i l/n} \end{pmatrix}, \qquad \rho_l(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In what follows, consider the case that ρ_l is *not* injective. The kaleido locus $\mathrm{KL}_{\eta_{n,l}}$ of $\eta_{n,l}$ is then the whole of $\mathbb{C}^2/D_n$, i.e. every fiber of $\eta_{n,l}$ is kaleido (Proposition 2.1.5 (1)). By Theorem 2.1.9 (1), $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$ is canonically isomorphic to the quotient family $\overline{\eta}_{n,l} : \big(\Sigma/\mathrm{Ker}(\rho_l) \times \mathbb{C}^2\big)/\big(D_n/\mathrm{Ker}(\rho_l)\big) \to \mathbb{C}^2/\big(D_n/\mathrm{Ker}(\rho_l)\big)$ associated with $\overline{\rho}_l : D_n/\mathrm{Ker}(\rho_l) \to GL_2(\mathbb{C})$. Now set $d := \gcd(n, l)$, $n' := n/d$, and $l' := l/d$. By Lemma 3.4.1, $\gcd(n, l) \geq 2$ and $\mathrm{Ker}(\rho_l) = \langle a^{n'} \rangle$, so $D_n/\mathrm{Ker}(\rho_l) \cong D_{n'}$. Note that $\Sigma' := \Sigma/\langle a^{n'} \rangle$ is an elliptic curve (Lemma 3.2.4 (1)).

Lemma 3.4.9. *Two representations $\overline{\rho}_l : D_{n'} \to GL_2(\mathbb{C})$ and $\rho_{l'} : D_{n'} \to GL_2(\mathbb{C})$ coincide.*

Proof. The images $\overline{a}, \overline{b}$ of a, b under the quotient map $D_n \to D_{n'}$ ($\cong D_n/\langle a^{n'}\rangle$) generate $D_{n'}$, and

$$\overline{\rho}_l(\overline{a}) = \rho_{l'}(\overline{a}) = \begin{pmatrix} e^{2\pi i l'/n'} & 0 \\ 0 & e^{-2\pi i l'/n'} \end{pmatrix}, \qquad \overline{\rho}_l(\overline{b}) = \rho_{l'}(\overline{b}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

□

The above results combined with Theorem 2.1.9 yield the following:

Lemma 3.4.10. (1) *The quotient family $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$ associated with $\rho_l : D_n \to GL_2(\mathbb{C})$ is canonically isomorphic to the quotient family $\eta_{n',l'} : (\Sigma' \times \mathbb{C}^2)/D_{n'} \to \mathbb{C}^2/D_{n'}$ associated with the injective representation $\rho_{l'} : D_{n'} \to GL_2(\mathbb{C})$. Here $\Sigma' := \Sigma/\mathrm{Ker}(\rho_l)$ is an elliptic curve and $|\mathrm{Ker}(\rho_l)| = d\,(= \gcd(n,l))$,* so

$$\text{(covering multiplicity of } \eta_{n,l}^{-1}(s)) = d\times(\text{covering multiplicity of } \eta_{n',l'}^{-1}(s)).$$

(2) *Let $\mathrm{SL}_{\eta_{n,l}}$ be the special locus of $\eta_{n,l}$ (Remark 2.1.8) and $\mathrm{KL}_{\eta_{n',l'}}$ be the kaleido locus of $\eta_{n',l'}$. Under the isomorphism in (1), $\mathrm{SL}_{\eta_{n,l}} = \mathrm{KL}_{\eta_{n',l'}}$.*

Here by Theorem 3.4.3,

$$\eta_{n',l'}^{-1}(s) = \begin{cases} \text{projective line } \Sigma'/D_{n'}\ (\cong \Sigma/D_n) \quad (\text{c.m. } 2n') & \text{if } s = 0, \\ \text{projective line } \Sigma'/\langle\overline{b}\rangle\ (\cong \Sigma/\langle a^{n'}, b\rangle) \quad (\text{c.m. } 2) & \text{if } s \in \mathrm{KL}_{\eta_{n',l'}} \setminus \{0\}, \\ \text{elliptic curve } \Sigma'\ (:= \Sigma/\langle a^{n'}\rangle) \quad (\text{c.m. } 1) & \text{if } s \in (\mathbb{C}^2/D_n) \setminus \mathrm{KL}_{\eta_{n',l'}}. \end{cases}$$

Consequently the following holds:

Theorem 3.4.11. *Let $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$ be the quotient family of Σ associated with $\rho_l : D_n \to GL_2(\mathbb{C})$ and set $n' := n/\gcd(n,l)$. Then the following holds*:

$$\eta_{n,l}^{-1}(s) = \begin{cases} \text{projective line } \Sigma/D_n \quad (\text{c.m. } 2n'd\,(= 2n)) & \text{if } s = 0, \\ \text{projective line } \Sigma/\langle a^{n'}, b\rangle \quad (\text{c.m. } 2d) & \text{if } s \in \mathrm{SL}_{\eta_{n,l}} \setminus \{0\}, \\ \text{elliptic curve } \Sigma/\langle a^{n'}\rangle \quad (\text{c.m. } d) & \text{if } s \in (\mathbb{C}^2/D_n) \setminus \mathrm{SL}_{\eta_{n,l}}. \end{cases}$$

3.5 Singular loci of total spaces

Let $\rho_l : D_n \to GL_2(\mathbb{C})$ be the representation given by (3.3.1) and $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$ be its associated quotient family. We determine the type of the singular locus of $(\Sigma \times \mathbb{C}^2)/D_n$. Note first the following:

Lemma 3.5.1. *If $y \in (\Sigma \times \mathbb{C}^2)/D_n$ is a singularity, then $H_{\widetilde{y}} \neq \{1\}$, where $H_{\widetilde{y}}$ is the stabilizer of a lift $\widetilde{y} \in \Sigma \times \mathbb{C}^2$ of y.* (If moreover ρ_l is injective, the converse holds (Lemma 3.5.5 below).)

Proof. Write $\widetilde{y} = (z, t) \in \Sigma \times \mathbb{C}^2$, and take a sufficiently small $H_{\widetilde{y}}$-stable disk $\Delta \subset \Sigma$ centered at z. Then $(\Sigma \times \mathbb{C}^2)/D_n$ is around y isomorphic to $(\Delta \times \mathbb{C}^2)/H_{\widetilde{y}}$. If y is a singularity of $(\Sigma \times \mathbb{C}^2)/D_n$, then y is a singularity of $(\Delta \times \mathbb{C}^2)/H_{\widetilde{y}}$, so necessarily $H_{\widetilde{y}} \neq \{1\}$. □

Since the D_n-action on $\Sigma \times \mathbb{C}^2$ is diagonal $\big((z,t) \mapsto (gz, \rho_l(g)t)\big)$, the following holds:

Lemma 3.5.2. *For any $(z, t) \in \Sigma \times \mathbb{C}^2$, $H_{(z,t)} = H_z \cap H_t$.* (Thus $H_{(z,t)} \neq \{1\}$ is restated as $H_z \cap H_t \neq \{1\}$.)

To determine the singularities of $(\Sigma \times \mathbb{C}^2)/D_n$, we shall determine (z, t) such that $H_{(z,t)} \neq \{1\}$. Recall that $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$ is the quotient family associated with the representation $\rho_l : D_n \to GL_2(\mathbb{C})$. Thanks to Lemma 3.4.10 (1), *we may assume that ρ_l is injective.*

Recall that $D_n = \{a^k, a^k b \ : \ k = 0, 1, \ldots, n-1\}$. Here a^k, being a translation of Σ (Figure 3.1.1), fixes *no* point of Σ. Thus $a^k \notin H_z$ for any z. Next $a^k b$, being a involution of Σ (Figure 3.1.1), fixes four points. If z is a fixed point of $a^k b$, then $\langle a^k b \rangle \subset H_z$ and $a^{k'} b \notin H_z$ for any $k' \neq k$, so $\langle a^k b \rangle = H_z$. This confirms (i) of the following (while (ii) is nothing but (3.4.2); note ρ_l is injective):

Lemma 3.5.3.

(i) $$H_z = \begin{cases} \langle a^k b \rangle & \textit{if } z \in \mathrm{Fix}_\Sigma(a^k b), \\ \{1\} & \textit{otherwise}, \end{cases}$$ (ii) $$H_t = \begin{cases} D_n & \textit{if } t = 0, \\ \langle a^k b \rangle & \textit{if } t \in \mathrm{Fix}_{\mathbb{C}^2}(a^k b) \setminus \{0\}, \\ \{1\} & \textit{otherwise}, \end{cases}$$

where $\mathrm{Fix}_S(a^k b)$ $(S = \Sigma, \mathbb{C}^2)$ *denotes the fixed point set of the action of* $a^k b$ *on* S.

From $H_{(z,t)} = H_z \cap H_t$ and Lemma 3.5.3, we have:

Corollary 3.5.4. *Suppose that* ρ_l *is injective. Then the following conditions are equivalent:*

(i) $H_{(z,t)} \neq \{1\}$.

(ii) $H_{(z,t)} = \langle a^k b \rangle$ *for some* k.

(iii) $(z, t) \in \mathrm{Fix}_\Sigma(a^k b) \times \mathrm{Fix}_{\mathbb{C}^2}(a^k b)$.

The image of $(z, t) \in \Sigma \times \mathbb{C}^2$ under the quotient map $\Sigma \times \mathbb{C}^2 \to (\Sigma \times \mathbb{C}^2)/D_n$ is denoted by $[z, t] \in (\Sigma \times \mathbb{C}^2)/D_n$. If $[z, t] \in (\Sigma \times \mathbb{C}^2)/D_n$ is a singularity, then $H_{(z,t)} \neq \{1\}$ (Lemma 3.5.1). Conversely the following holds:

Lemma 3.5.5. *Suppose that* ρ_l *is injective. If* $H_{(z,t)} \neq \{1\}$*, then* $[z, t] \in (\Sigma \times \mathbb{C}^2)/D_n$ *is a singularity.*

Proof. If $H_{(z,t)} \neq \{1\}$, then $H_{(z,t)} = \langle a^k b \rangle$ for some k (Corollary 3.5.4). Noting that $a^k b$ is *not* a pseudo-reflection (Lemma 3.5.10 below), $H_{(z,t)} = \langle a^k b \rangle$ is a small group, so $[z, t] \in (\Sigma \times \mathbb{C}^2)/D_n$ is a singularity. □

Combining Lemma 3.5.1, Corollary 3.5.4, and Lemma 3.5.5 yield the following:

Proposition 3.5.6. *Suppose that* ρ_l *is injective. Then the following are equivalent:*

(i) $[z,t] \in (\Sigma \times \mathbb{C}^2)/D_n$ *is a singularity.*

(ii) $H_{(z,t)} \neq \{1\}$.

(iii) $H_{(z,t)} = \langle a^k b\rangle$ *for some* k.

(iv) $(z,t) \in \mathrm{Fix}_\Sigma(a^k b) \times \mathrm{Fix}_{\mathbb{C}^2}(a^k b)$.

We determine the singular locus of $(\Sigma \times \mathbb{C}^2)/D_n$.

Proposition 3.5.7. *Suppose that* ρ_l *is injective. Then the singular locus of* $(\Sigma \times \mathbb{C}^2)/D_n$ *consists of the ridges* R_p, R_q, R_r, R_s *(illustrated in Figure 3.4.4 for odd* n *and Figure 3.4.6 for even* n*;* R_p, R_q, R_r, R_s *are disjoint).*

Proof. By Proposition 3.5.6, it suffices to show that

$$(z,t) \in \mathrm{Fix}_\Sigma(a^k b) \times \mathrm{Fix}_{\mathbb{C}^2}(a^k b) \iff [z,t] \in R_p \cup R_q \cup R_r \cup R_s.$$

Note first that

$$\begin{cases} \mathrm{Fix}_\Sigma(a^k b) = \{p_k, q_k, r_k, s_k\} \text{ (see Figure 3.4.3 for odd } n\text{, Figure 3.4.5 for even } n\text{)}, \\ \mathrm{Fix}_{\mathbb{C}^2}(a^k b) =: L_k \text{ (see Lemma 3.4.2)}. \end{cases}$$

Thus $(z,t) \in \mathrm{Fix}_\Sigma(a^k b) \times \mathrm{Fix}_{\mathbb{C}^2}(a^k b)$ is restated as $(z,t) \in \{x_k\} \times L_k$ ($x_k = p_k, q_k, r_k, s_k$), that is, $[z,t] \in R_x$ as $R_x := \overline{\{x_k\} \times L_k}$ (see Notation 3.4.7 for odd n and Notation 3.4.8 for even n). □

Consider next the case that ρ_l is *not* injective. We reduce this to the injective case: Set $n' := n/\gcd(n,l)$ and $l' := l/\gcd(n,l)$ and $\Sigma' := \Sigma/\langle a^{n'}\rangle$. Then note that the quotient family $\eta_{n,l} : (\Sigma \times \mathbb{C}^2)/D_n \to \mathbb{C}^2/D_n$ associated with $\rho_l : D_n \to GL_2(\mathbb{C})$ is naturally identified with the quotient family $\eta_{n',l'} : (\Sigma' \times \mathbb{C}^2)/D_{n'} \to \mathbb{C}^2/D_{n'}$ associated with the injective representation $\rho_{l'} (= \overline{\rho}_l) : D_{n'} \to GL_2(\mathbb{C})$. In fact,

(i) $$\begin{aligned}(\Sigma \times \mathbb{C}^2)/D_n &= (\Sigma \times \mathbb{C}^2)/\langle a^{n'}\rangle/(D_n/\langle a^{n'}\rangle) \\ &\cong \big(\Sigma/\langle a^{n'}\rangle \times \mathbb{C}^2\big)/(D_n/\langle a^{n'}\rangle) \quad \text{as } \langle a^{n'}\rangle\text{-action on } \mathbb{C}^2 \text{ is trivial} \\ &= (\Sigma' \times \mathbb{C}^2)/D_{n'},\end{aligned}$$

$$\begin{aligned}\text{(ii)}\quad \mathbb{C}^2/D_n &= (\mathbb{C}^2/\langle a^{n'}\rangle)\big/(D_n/\langle a^{n'}\rangle)\\ &\cong \mathbb{C}^2\big/(D_n/\langle a^{n'}\rangle) \quad \text{as } \langle a^{n'}\rangle\text{-action on } \mathbb{C}^2 \text{ is trivial}\\ &= \mathbb{C}^2/D_{n'},\end{aligned}$$

and the following diagram commutes:

$$\begin{array}{ccc} (\Sigma\times\mathbb{C}^2)/D_n & \xrightarrow{\cong} & (\Sigma'\times\mathbb{C}^2)/D_{n'} \\ \eta_{n,l}\downarrow & & \downarrow\eta_{n',l'} \\ \mathbb{C}^2/D_n & \xrightarrow{\cong} & \mathbb{C}^2/D_{n'}. \end{array} \tag{3.5.1}$$

Consequently the assumption "ρ_l is injective" in Proposition 3.5.7 may be omitted:

Proposition 3.5.8. *The singular locus of* $(\Sigma\times\mathbb{C}^2)/D_n$ *consists of the ridges* R_p, R_q, R_r, R_s.

Remark 3.5.9. The isomorphism $(\Sigma\times\mathbb{C}^2)/D_n \xrightarrow{\cong} (\Sigma'\times\mathbb{C}^2)/D_{n'}$ in (3.5.1) is explicitly given by $[y,t]\mapsto[y \bmod \langle a^{n'}\rangle, t]$.

We next determine the type of each singularity of $(\Sigma\times\mathbb{C}^2)/D_n$. The following is needed:

Lemma 3.5.10. *The action of* a^kb *on* $\Sigma\times\mathbb{C}^2$ *is given by* $M := \begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$ *up to conjugation.* (Note: M is *not* a pseudo-reflection while $-M$ is a pseudo-reflection.)

Proof. For $z\in \mathrm{Fix}_\Sigma(a^kb)$, take an $\langle a^kb\rangle$-invariant small disk Δ in Σ centered at z. Then the action of a^kb on Δ is given by $x\mapsto -x$ (Lemma 3.1.3 (2)) and that on $\mathbb{C}^2$ is given by $\begin{pmatrix}0&\zeta^{lk}\\ \zeta^{-lk}&0\end{pmatrix}$ (see (3.3.1)), where recall that $\zeta := e^{2\pi \mathrm{i}/n}$. So the action of a^kb on $\Delta\times\mathbb{C}^2$ is given by $\begin{pmatrix}-1&0&0\\0&0&\zeta^{lk}\\0&\zeta^{-lk}&0\end{pmatrix}$,

which is diagonalized to $M := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ in $GL_3(\mathbb{C})$. □

Write the matrix M in Lemma 3.5.10 as $M = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}$, where $N := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\mathbb{C}^2/\langle N \rangle$ is an A_1-singularity and $\mathbb{C}^3/\langle M \rangle \cong \mathbb{C} \times (\mathbb{C}^2/\langle N \rangle)$. This with Proposition 3.5.8 yields the following:

Theorem 3.5.11. *The singular locus of* $(\Sigma \times \mathbb{C}^2)/D_n$ *consists of the ridges* R_p, R_q, R_r, R_s, *around each of which* $(\Sigma \times \mathbb{C}^2)/D_n$ *is isomorphic to* (complex line)$\times$ (A_1-singularity). (Note that the types of singularities do *not* depend on whether n is odd or even.)

Chapter 4

Binary dihedral quotient families

4.1 Binary dihedral quotient families

The dihedral group $D_n = \langle a, b : a^n = b^2 = 1,\ bab^{-1} = a^{-1}\rangle$ acts on the regular n-gon Δ_n as a is a $1/n$-rotation and b is a reflection. Thickening the edges of Δ_n yields a cable surface (torus) Σ with D_n-action. As before we give a complex structure to Σ such that the D_n-action is holomorphic. Consider next the *binary* dihedral group $\widetilde{D}_n = \langle \widetilde{a}, \widetilde{b} : \widetilde{a}^{2n} = 1, \widetilde{a}^n = \widetilde{b}^2, \widetilde{b}\widetilde{a}\widetilde{b}^{-1} = \widetilde{a}^{-1}\rangle$ and the double covering $q : \widetilde{D}_n \to D_n$ given by $\widetilde{a} \mapsto a$ and $\widetilde{b} \mapsto b$. Let $\widetilde{D}_n$ act on Σ via q, i.e. $g \in \widetilde{D}_n$ acts as $q(g)$ (this action is *not* effective). To each representation $\widetilde{D}_n \to GL_m(\mathbb{C})$, the associated quotient family $(\Sigma \times \mathbb{C}^m)/\widetilde{D}_n \to \mathbb{C}^m/\widetilde{D}_n$ of Σ is called a *binary dihedral* quotient family or a quotient family of *type* $\widetilde{D}_n$. We describe such quotient families for all irreducible representations of $\widetilde{D}_n$. For representations of $\widetilde{D}_n$, there is no suitable reference for our purpose, so we describe them herein. We determine all irreducible representations of $\widetilde{D}_n$. Note next that:

Lemma 4.1.1. *For a finite group G, let $p : G \to A := G/[G,G]$ be its abelianization. Then there is a one-to-one correspondence between the* 1-

dimensional representations of G and those of A. In fact, $\boldsymbol{\chi} \in \mathrm{Hom}(A, GL_1(\mathbb{C})) \mapsto \boldsymbol{\chi} \circ p \in \mathrm{Hom}(G, GL_1(\mathbb{C}))$ is bijective.

Proof. Surjective: For any $\tau \in \mathrm{Hom}(G, GL_1(\mathbb{C}))$, $\tau|_{[G,G]} \equiv 1$, so τ factorize through A, that is, $\tau = \boldsymbol{\chi} \circ p$ for some $\boldsymbol{\chi} \in \mathrm{Hom}(A, GL_1(\mathbb{C}))$. Injective: Suppose $\boldsymbol{\chi}_1 \circ p = \boldsymbol{\chi}_2 \circ p$. We then show $\boldsymbol{\chi}_1 = \boldsymbol{\chi}_2$, i.e. $\boldsymbol{\chi}_1(a) = \boldsymbol{\chi}_2(a)$ for any $a \in A$. Since $p : G \to A$ is surjective, we may take $\widetilde{a} \in G$ such that $p(\widetilde{a}) = a$. Then from $\boldsymbol{\chi}_1 \circ p = \boldsymbol{\chi}_2 \circ p$ we have $\boldsymbol{\chi}_1 \circ p(\widetilde{a}) = \boldsymbol{\chi}_2 \circ p(\widetilde{a})$, i.e. $\boldsymbol{\chi}_1(a) = \boldsymbol{\chi}_2(a)$. $\square$

We return to $\widetilde{D}_n = \langle \widetilde{a}, \widetilde{b} : \widetilde{a}^{2n} = 1, \widetilde{a}^n = \widetilde{b}^2, \widetilde{b}\widetilde{a}\widetilde{b}^{-1} = \widetilde{a}^{-1} \rangle$. Its abelianization $A_n := \widetilde{D}_n / [\widetilde{D}_n, \widetilde{D}_n]$ amounts to adding a relation $\widetilde{a}\widetilde{b} = \widetilde{b}\widetilde{a}$; then the last relation $\widetilde{b}\widetilde{a}\widetilde{b}^{-1} = \widetilde{a}^{-1}$ becomes $\widetilde{a} = \widetilde{a}^{-1}$, i.e. $\widetilde{a}^2 = 1$. Thus $A_n = \langle \widetilde{a}, \widetilde{b} : \widetilde{a}^2 = 1, \widetilde{a}^n = \widetilde{b}^2, \widetilde{a}\widetilde{b} = \widetilde{b}\widetilde{a} \rangle$. Note that:

(i) For even n, from $\widetilde{a}^2 = 1$ we have $\widetilde{a}^n = 1$, so $\widetilde{b}^2 = \widetilde{a}^n = 1$. Thus $A_n = \langle \widetilde{a}, \widetilde{b} : \widetilde{a}^2 = \widetilde{b}^2 = 1, \widetilde{a}\widetilde{b} = \widetilde{b}\widetilde{a} \rangle$.

(ii) For odd n, from $\widetilde{a}^2 = 1$ we have $\widetilde{a}^n = \widetilde{a}$, so $\widetilde{b}^2 = \widetilde{a}^n = \widetilde{a}$, accordingly $\widetilde{b}^4 = \widetilde{a}^2 = 1$. Thus $A_n = \langle \widetilde{a}, \widetilde{b} : \widetilde{b}^4 = 1, \widetilde{a} = \widetilde{b}^2, \widetilde{a}\widetilde{b} = \widetilde{b}\widetilde{a} \rangle$.

Hence:

$$A_n \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \widetilde{a} \rangle \times \langle \widetilde{b} \rangle & \text{if } n \text{ is even,} \\ \mathbb{Z}_4 = \langle \widetilde{b} \rangle & \text{if } n \text{ is odd.} \end{cases}$$

Depending on whether n is even or odd, the representations of A_n are $\boldsymbol{\chi}_i^{\mathrm{even}} : \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \widetilde{a} \rangle \times \langle \widetilde{b} \rangle \to GL_1(\mathbb{C})$ or $\boldsymbol{\chi}_i^{\mathrm{odd}} : \mathbb{Z}_4 = \langle \widetilde{b} \rangle \to GL_1(\mathbb{C})$ $(i = 1, 2, 3, 4)$ given by

$$(\boldsymbol{\chi}_i^{\mathrm{even}}(\widetilde{a}),\ \boldsymbol{\chi}_i^{\mathrm{even}}(\widetilde{b})) = \begin{cases} (1, 1) \\ (1, -1) \\ (-1, 1) \\ (-1, -1), \end{cases} \qquad \boldsymbol{\chi}_i^{\mathrm{odd}}(\widetilde{b}) = \begin{cases} 1 & i = 1 \\ -1 & i = 2 \\ \mathrm{i} & i = 3 \\ -\mathrm{i} & i = 4. \end{cases} \tag{4.1.1}$$

By Lemma 4.1.1, the following holds:

Proposition 4.1.2. *Let $p : \widetilde{D}_n \to A_n$ be the abelianization. Then the 1-dimensional representations of $\widetilde{D}_n$ are $\boldsymbol{\chi}_i^{\text{even}} \circ p$ for even n and $\boldsymbol{\chi}_i^{\text{odd}} \circ p$ for odd n.*

Definition 4.1.3. Let $q : \widetilde{D}_n \to D_n$ be the double covering given by $\widetilde{a} \mapsto a, \widetilde{b} \mapsto b$. For a representation $\rho : D_n \to GL_m(\mathbb{C})$, the composition $\widetilde{\rho} := \rho \circ q : \widetilde{D}_n \to GL_m(\mathbb{C})$ is the *lift* of ρ. A representation of $\widetilde{D}_n$ is *lifted* if it is the lift of some representation of D_n, otherwise *unlifted*.

$$\begin{array}{ccccc} & & \widetilde{D}_n & & \\ \text{abelianization } p & \swarrow & & \searrow & q \text{ double covering} \\ A_n & & & & D_n. \end{array}$$

Lemma 4.1.4. *The 1-dimensional representations $\boldsymbol{\chi}_i^{\text{even}} \circ p$ and $\boldsymbol{\chi}_i^{\text{odd}} \circ p$ of $\widetilde{D}_n$ are lifts of 1-dimensional representations of D_n. In fact the following hold:*

- Even n: $\boldsymbol{\chi}_i^{\text{even}} \circ p$ $(i = 1, 2, 3, 4)$ *is the lift of* χ_i *defined by* (3.2.1)*, that is,* $\boldsymbol{\chi}_i^{\text{even}} \circ p = \widetilde{\chi}_i \,(:= \chi_i \circ q)$.
- Odd n: $\boldsymbol{\chi}_i^{\text{odd}} \circ p$ $(i = 1, 2)$ *is the lift of* χ_i*, that is,* $\boldsymbol{\chi}_i^{\text{odd}} \circ p = \widetilde{\chi}_i \,(:= \chi_i \circ q)$*, whereas* $\boldsymbol{\chi}_i^{\text{odd}} \circ p$ $(i = 3, 4)$ *is not the lift of a representation of* D_n*.* Notation: Set $\sigma_1 := \boldsymbol{\chi}_3^{\text{odd}} \circ p$ and $\sigma_2 := \boldsymbol{\chi}_4^{\text{odd}} \circ p$.

Proof. We only show that σ_i $(i = 1, 2)$ are not the lifts of representations of D_n (the other statements are shown by easy computation). If $\sigma_i = \rho \circ q$ for some representation ρ of D_n, then $\sigma_i(\widetilde{a}^n) = \rho \circ q(\widetilde{a}^n)$. This however does not hold, as $\sigma_i(\widetilde{a}^n) = -1$ while $\rho \circ q(\widetilde{a}^n) = 1$. Indeed for $i = 1$: $\sigma_1(\widetilde{a}) = \boldsymbol{\chi}_3^{\text{odd}} \circ p(\widetilde{a}) = \boldsymbol{\chi}_3^{\text{odd}}(\widetilde{a}) = \boldsymbol{\chi}_3^{\text{odd}}(\widetilde{b}^2) = -1$ (as $\widetilde{a} = \widetilde{b}^2$ in A_n) while $\rho \circ q(\widetilde{a}^n) = \rho(1) = 1$ (as $q(\widetilde{a}^n) = 1$). Similarly for $i = 2$, this is confirmed. □

We next consider two kinds of 2-dimensional representations of $\widetilde{D}_n$: the lift $\widetilde{\rho}_j := \rho_j \circ q$ of $\rho_j : D_n \to GL_2(\mathbb{C})$ given by (3.3.1) and $\tau_k : \widetilde{D}_n \to GL_2(\mathbb{C})$

(k: odd, $1 \leq k < n$) defined by

$$\tau_k(\widetilde{a}) = \begin{pmatrix} e^{\pi \mathrm{i} k/n} & 0 \\ 0 & e^{-\pi \mathrm{i} k/n} \end{pmatrix}, \qquad \tau_k(\widetilde{b}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.1.2}$$

Lemma 4.1.5. (1) *$\widetilde{\rho}_j$ is irreducible.* (2) *τ_k is irreducible and unlifted.*

Proof. We show (1). Otherwise $\widetilde{\rho}_j$ is written as the sum of two 1-dimensional representations of $\widetilde{D}_n$, say $\widetilde{\rho}_j = f_1 \oplus f_2$. Here all 1-dimensional representations of $\widetilde{D}_n$ are exhausted by those in Proposition 4.1.2, and f_i is one of them. Now $\mathrm{tr}(\widetilde{\rho}_j) = \mathrm{tr}(f_1) + \mathrm{tr}(f_2)$, but there is no combination of f_1 and f_2 that satisfy this. We next show (2). The same argument used in (1) shows the irreducibility of τ_k. We show that τ_k is unlifted. If $\tau_k = \rho \circ q$ for some representation ρ of D_n, then $\tau_k(\widetilde{a}^n) = \rho \circ q(\widetilde{a}^n)$. This however does not hold, as $\tau_k(\widetilde{a}^n) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ while $\rho \circ q(\widetilde{a}^n) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. □

Even n:

	lifted	unlifted
1-dim	$\widetilde{\chi}_i$ ($i = 1, 2, 3, 4$)	none
2-dim	$\widetilde{\rho}_j$ ($j = 1, 2, \dots \frac{n}{2} - 1$)	τ_k ($k = 1, 3, \dots, n-1$)

Odd n:

	lifted	unlifted
1-dim	$\widetilde{\chi}_i$ ($i = 1, 2$)	σ_i ($i = 1, 2$)
2-dim	$\widetilde{\rho}_j$ ($j = 1, 2, \dots \frac{n-1}{2}$)	τ_k ($k = 1, 3, \dots, n-2$)

Table 4.1.1: Irreducible representations of $\widetilde{D}_n$

Proposition 4.1.6. *The representations in Table 4.1.1 exhaust all irreducible representations of $\widetilde{D}_n$.*

Proof. This is checked by the sum of squares formula ([Ser] Corollary 2 (a) p.18). For even n,

$$\sum_i (\dim \widetilde{\chi}_i)^2 + \sum_j (\dim \widetilde{\rho}_j)^2 + \sum_k (\dim \tau_k)^2 = |\widetilde{D}_n|,$$

indeed $4 \times 1^2 + \frac{n-2}{2} \times 2^2 + \frac{n}{2} \times 2^2 = 4n$. For odd n,

$$\sum_i (\dim \widetilde{\chi}_i)^2 + \sum_i (\dim \sigma_i)^2 + \sum_j (\dim \widetilde{\rho}_j)^2 + \sum_k (\dim \tau_k)^2 = |\widetilde{D}_n|,$$

indeed $2 \times 1^2 + 2 \times 1^2 + \frac{n-1}{2} \times 2^2 + \frac{n-1}{2} \times 2^2 = 4n$. $\square$

We give the explicit forms of the representations $\widetilde{\chi}_i, \sigma_i, \widetilde{\rho}_j$ (for τ_k see (4.1.2)):

$$(\widetilde{\chi}_i(\widetilde{a}),\, \widetilde{\chi}_i(\widetilde{b})) = \begin{cases} (1,1) & i = 1, \\ (1,-1) & i = 2, \\ (-1,1) & i = 3, \\ (-1,-1) & i = 4, \end{cases} \quad (\sigma_i(\widetilde{a}),\, \sigma_i(\widetilde{b})) = \begin{cases} (-1,\mathrm{i}) & i = 1, \\ (-1,-\mathrm{i}) & i = 2, \end{cases} \tag{4.1.3}$$

$$\widetilde{\rho}_j(\widetilde{a}) = \begin{pmatrix} e^{2\pi \mathrm{i} j/n} & 0 \\ 0 & e^{-2\pi \mathrm{i} j/n} \end{pmatrix}, \quad \widetilde{\rho}_j(\widetilde{b}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.1.4}$$

4.1.1 Lifted case

We describe the quotient families associated with the *lifted* irreducible representations of $\widetilde{D}_n$. We begin with preparation.

Lemma 4.1.7 ([Ta,VI])**.** *Let $q : \widetilde{G} \to G$ be a surjective homomorphism between finite groups. Suppose that G acts on a complex analytic variety Y holomorphically and let $\widetilde{G}$ act on Y via q. Then for any representation $\rho : G \to GL_m(\mathbb{C})$ and its lift $\widetilde{\rho} := \rho \circ q : \widetilde{G} \to GL_m(\mathbb{C})$, the quotient family $\widetilde{\eta} : (Y \times \mathbb{C}^m)/\widetilde{G} \to \mathbb{C}^m/\widetilde{G}$ associated with $\widetilde{\rho}$ is isomorphic to the quotient family $\eta : (Y \times \mathbb{C}^m)/G \to \mathbb{C}^m/G$ associated with ρ.*

Proof. Since $\widetilde{G}$ acts on Y via q and on $\mathbb{C}^m$ via $\widetilde{\rho}_l := \rho_l \circ q$, the kernel $K := \mathrm{Ker}(q)$ acts trivially on both Y and $\mathbb{C}^m$, so $(Y \times \mathbb{C}^m)/K \cong Y \times \mathbb{C}^m$, and then

$$\begin{aligned}(Y \times \mathbb{C}^m)/\widetilde{G} &\cong (Y \times \mathbb{C}^m)/K \big/ \widetilde{G}/K \\ &\cong (Y \times \mathbb{C}^m)/G.\end{aligned}$$

Similarly we can confirm that $\mathbb{C}^m/\widetilde{G} \cong \mathbb{C}^m/G$. Moreover the following diagram commutes:

$$\begin{array}{ccc} (Y \times \mathbb{C}^m)/\widetilde{G} & \xrightarrow{\cong} & (Y \times \mathbb{C}^m)/G \\ \Big\downarrow{\scriptstyle \widetilde{\eta}} & & \Big\downarrow{\scriptstyle \eta} \\ \mathbb{C}^m/\widetilde{G} & \xrightarrow{\cong} & \mathbb{C}^m/G. \end{array}$$

□

Lemma 4.1.8. *In Lemma 4.1.7, the covering multiplicity of $\widetilde{\eta}^{-1}(s)$ is equal to $|H_t|/|K|$ (not $|H_t|$). In particular, the covering multiplicity of $\widetilde{\eta}^{-1}(s)$ is equal to that of $\eta^{-1}(s)$.*

Proof. Since $K \subset H_t$ acts on Σ trivially and H_t/K acts on Σ effectively, the covering degree of $\Sigma \to \Sigma/H_t$ (the covering multiplicity of $\widetilde{\eta}^{-1}(s)$) is equal to $|H_t|/|K|$. □

A binary dihedral quotient family associated with a lifted representation is isomorphic to a dihedral quotient family. In fact, let $\rho : D_n \to GL_m(\mathbb{C})$ be a representation and $\widetilde{\rho} := \rho \circ q : \widetilde{D}_n \to GL_m(\mathbb{C})$ be its lift, then application of Lemma 4.1.7 to $G = D_n$ and $\widetilde{G} = \widetilde{D}_n$ yields the following:

Corollary 4.1.9. *The quotient family $(\Sigma \times \mathbb{C}^m)/\widetilde{D}_n \to \mathbb{C}^m/\widetilde{D}_n$ associated with $\widetilde{\rho}$ is isomorphic to the quotient family $(\Sigma \times \mathbb{C}^m)/D_n \to \mathbb{C}^m/D_n$ associated with ρ.*

The lifted representations of $\widetilde{D}_n$ are $\widetilde{\chi}_i : \widetilde{D}_n \to GL_1(\mathbb{C})$ $(i = 1, 2, 3, 4)$ given by (4.1.3) and $\widetilde{\rho}_l : \widetilde{D}_n \to GL_2(\mathbb{C})$ (where $1 \leq l < \frac{n}{2}$) given by (4.1.4). Let $\widetilde{\xi}_i : (\Sigma \times \mathbb{C})/\widetilde{D}_n \to \mathbb{C}/\widetilde{D}_n$ and $\widetilde{\eta}_{n,l} : (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n \to \mathbb{C}^2/\widetilde{D}_n$ be their associated quotient families. By Corollary 4.1.9,

- $\widetilde{\xi}_i$ is isomorphic to the quotient family ξ_i associated with χ_i; so $\widetilde{\xi}_i^{-1}(s) \cong \xi_i^{-1}(s)$.
- $\widetilde{\eta}_{n,l}$ is isomorphic to the quotient family $\eta_{n,l}$ associated with ρ_l; so $\widetilde{\eta}_{n,l}^{-1}(s) \cong \eta_{n,l}^{-1}(s)$.

Here by Lemma 4.1.8,

$$\text{c.m. of } \widetilde{\xi}_i^{-1}(s) = \text{c.m. of } \xi_i^{-1}(s), \qquad \text{c.m. of } \widetilde{\eta}_{n,l}^{-1}(s) = \text{c.m. of } \eta_{n,l}^{-1}(s).$$

The kaleido fibers of ξ_i and $\eta_{n,l}$ and their covering multiplicities are determined in Theorems 3.2.6 and 3.4.11. Note next that

$$\mathrm{SL}_{\widetilde{\eta}_{n,l}} = \mathrm{KL}_{\eta_{n,l}} = \begin{cases} \overline{L}_0 & \text{if } n \text{ is odd,} \\ \overline{L}_0 \cup \overline{L}_1 & \text{if } n \text{ is even,} \end{cases} \tag{4.1.5}$$

where for the first equality, see Theorem 2.1.9 (1) and for the second equality, see (3.4.1). We formalize the results so far obtained as follows:

Proposition 4.1.10. *Let* $\widetilde{\xi}_i : (\Sigma \times \mathbb{C})/\widetilde{D}_n \to \mathbb{C}/\widetilde{D}_n$ $(i = 1, 2, 3, 4)$ *be the quotient family associated with* $\widetilde{\chi}_i$ *and* $\widetilde{\eta}_{n,l} : (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n \to \mathbb{C}^2/\widetilde{D}_n$ *(where* $1 \le l < \frac{n}{2}$*) be the quotient family associated with* $\widetilde{\rho}_l$*. Then the kaleido fibers and the covering multiplicities are as in* Table 4.1.2 (1) *and* (2) *respectively.*

(1)

	$\widetilde{\xi}_i^{-1}(0)$	$\widetilde{\xi}_1^{-1}(s)$ $(s \in (\mathbb{C}/\widetilde{D}_n) \setminus \{0\})$	$\widetilde{\xi}_2^{-1}(s)$ $(s \in (\mathbb{C}/\widetilde{D}_n) \setminus \{0\})$	$\widetilde{\xi}_3^{-1}(s)$ $(s \in (\mathbb{C}/\widetilde{D}_n) \setminus \{0\})$	$\widetilde{\xi}_4^{-1}(s)$ $(s \in (\mathbb{C}/\widetilde{D}_n) \setminus \{0\})$
fiber	Σ/D_n $(\mathbb{P}^1)$	Σ/D_n (elliptic curve)	$\Sigma/\langle a \rangle$ (elliptic curve)	$\Sigma/\langle a^2, b \rangle$ $(\mathbb{P}^1)$	$\Sigma/\langle a^2, ab \rangle$ $(\mathbb{P}^1)$
c.m.	$2n$	$2n$	n	n	n

(2)

	$\widetilde{\eta}_{n,l}^{-1}(0)$	$\widetilde{\eta}_{n,l}^{-1}(s)$ $(s \in \mathrm{SL}_{\widetilde{\eta}_{n,l}} \setminus \{0\})$	$\widetilde{\eta}_{n,l}^{-1}(s)$ $(s \in (\mathbb{C}^2/\widetilde{D}_n) \setminus \mathrm{SL}_{\widetilde{\eta}_{n,l}})$
fiber	Σ/D_n $(\mathbb{P}^1)$	$\Sigma/\langle a^{n'}, b \rangle$ $(\mathbb{P}^1)$	$\Sigma/\langle a^{n'} \rangle$ (elliptic curve)
c.m.	$2n$	$2d$	d

Table 4.1.2: $d := \gcd(n, l)$. In (2), $\mathrm{SL}_{\widetilde{\eta}_{n,l}}$ is the special locus of $\widetilde{\eta}_{n,l}$.

4.1.2 Unlifted case

Before proceeding, note that using the relations $\widetilde{a}^{2n} = 1, \widetilde{a}^n = \widetilde{b}^2$ and $\widetilde{b}\widetilde{a}\widetilde{b}^{-1} = \widetilde{a}^{-1}$, any element of $\widetilde{D}_n$ is written as either $\widetilde{a}^k$ or $\widetilde{a}^k\widetilde{b}$:

$$\widetilde{D}_n = \{\widetilde{a}^k, \widetilde{a}^k\widetilde{b} \,:\, k = 0, 1, \ldots, 2n-1\} \quad \text{(no overlap)}. \tag{4.1.6}$$

We shall describe the quotient families associated with the *unlifted* irreducible representations of $\widetilde{D}_n$.

1-dim case Recall that the unlifted representations exist only for odd n: they are $\sigma_1, \sigma_2 : \widetilde{D}_n \to GL_1(\mathbb{C})$ given by $(\sigma_1(\widetilde{a}), \sigma_1(\widetilde{b})) = (-1, \mathrm{i})$ and $(\sigma_2(\widetilde{a}), \sigma_2(\widetilde{b})) = (-1, -\mathrm{i})$.

Lemma 4.1.11. *For odd n,* $\mathrm{Ker}(\sigma_1) = \mathrm{Ker}(\sigma_2) = \langle \widetilde{a}^2 \rangle$.

Proof. Any element of $\widetilde{D}_n$ is expressed as either $\widetilde{a}^k$ or $\widetilde{a}^k\widetilde{b}$ (see (4.1.6)). Here $\sigma_1(\widetilde{a}^k) = (-1)^k$ and $\sigma_1(\widetilde{a}^k\widetilde{b}) = (-1)^k\mathrm{i}$. So $\sigma_1(g) = 1$ if and only if $g = \widetilde{a}^k$ for some *even* k. Thus $\mathrm{Ker}(\sigma_1) = \{\widetilde{a}^k \,:\, k \text{ is even}\} = \langle \widetilde{a}^2 \rangle$. Similarly $\mathrm{Ker}(\sigma_2) = \langle \widetilde{a}^2 \rangle$. □

Before proceeding, note the following:

Lemma 4.1.12. *For odd n,* $\Sigma/\langle \widetilde{a}^2 \rangle \cong \Sigma/\langle a \rangle$.

Proof. The action of $\langle \widetilde{a}^2 \rangle$ on Σ is, by definition, given by $\langle a^2 \rangle$. Here $\langle a^2 \rangle = \langle a \rangle$ (as n is odd), so $\Sigma/\langle \widetilde{a}^2 \rangle \cong \Sigma/\langle a \rangle$. □

Proposition 4.1.13. *Let* $\varpi_i : (\Sigma \times \mathbb{C})/\widetilde{D}_n \to \mathbb{C}/\widetilde{D}_n$ $(i = 1, 2)$ *be the quotient family of* Σ *associated with* $\sigma_i : \widetilde{D}_n \to GL_1(\mathbb{C})$*. Then the following holds* (c.m. means covering multiplicity):

$$\varpi_i^{-1}(s) = \begin{cases} \text{projective line } \Sigma/D_n \quad (\text{c.m. } 2n) & \text{if } s = 0, \\ \text{elliptic curve } \Sigma/\langle a \rangle \quad (\text{c.m. } n) & \text{if } s \neq 0. \end{cases}$$

Proof. By Lemma 2.1.3, $\varpi_i^{-1}(0) = \Sigma/\widetilde{D}_n$, which is identical to Σ/D_n, and whose covering multiplicity is $|D_n| = 2n$.

By Lemma 2.1.3, $\varpi_i^{-1}(s) = \Sigma/\mathrm{Ker}(\sigma_i)$, which is identical to $\Sigma/\langle\widetilde{a}^2\rangle$ (Lemma 4.1.11), that is equal to $\Sigma/\langle a\rangle$ (Lemma 4.1.12), whose covering multiplicity is $|\langle a\rangle| = n$. □

2-dim case Recall that the unlifted irreducible representations are $\tau_m : \widetilde{D}_n \to GL_2(\mathbb{C})$ (where m is odd and $1 \le m < n$) given by (see (4.1.2)):

$$\tau_m(\widetilde{a}) = \begin{pmatrix} e^{\pi i m/n} & 0 \\ 0 & e^{-\pi i m/n} \end{pmatrix}, \qquad \tau_m(\widetilde{b}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 4.1.14. $\mathrm{Ker}(\tau_m) = \langle\widetilde{a}^{2n'}\rangle$, *where we set* $n' := n/\gcd(m,n)$. (Thus the order of $\mathrm{Ker}(\tau_m)$ is $\gcd(m,n)$. In particular τ_m is injective if and only if $\gcd(m,n) = 1$.)

Proof. Note first that any element of $\widetilde{D}_n$ is expressed as either $\widetilde{a}^k$ or $\widetilde{a}^k\widetilde{b}$ ($k = 0, 1, \ldots, 2n-1$); see (4.1.6). Note next that $\mathrm{Ker}(\tau_m) \subset \langle\widetilde{a}\rangle$ as $\tau_m(\widetilde{a}^k\widetilde{b}) = \begin{pmatrix} 0 & -e^{\pi i m k/n} \\ e^{-\pi i m k/n} & 0 \end{pmatrix} \ne I$ for any k. The assertion is then immediate from the following equivalence:

$$\tau_m(\widetilde{a}^k) = \begin{pmatrix} e^{\pi i m k/n} & 0 \\ 0 & e^{-\pi i m k/n} \end{pmatrix} = I \iff k \text{ is a multiple of } 2n' \text{ (in other words } 0, 2n', 4n', \ldots, 2n-2n').$$

□

We shall describe the quotient family associated with τ_m. We separate into two cases depending on whether τ_m is injective or not (equivalently m is coprime to n or not).

Case: τ_m is injective Let $\widetilde{D}_n$ act on $\mathbb{C}^2$ via τ_m.

Lemma 4.1.15. *For* $g \in \widetilde{D}_n$, *set* $\mathrm{Fix}(g) := \{t \in \mathbb{C}^2 : \tau_m(g)t = t\}$. *If* $\tau_m : \widetilde{D}_n \to GL_2(\mathbb{C})$ *is injective, then* $\mathrm{Fix}(\widetilde{a}^k) = \{0\}$ *for* $k \ne 0$ (*while* $\mathrm{Fix}(\widetilde{a}^0) = \mathbb{C}^2$) *and* $\mathrm{Fix}(\widetilde{a}^k\widetilde{b}) = \{0\}$ *for any* k.

Proof. This is immediate from

$$\tau_m(\widetilde{a}^k) = \begin{pmatrix} e^{\pi \mathrm{i} mk/n} & 0 \\ 0 & e^{-\pi \mathrm{i} mk/n} \end{pmatrix}, \qquad \tau_m(\widetilde{a}^k \widetilde{b}) = \begin{pmatrix} 0 & -e^{\pi \mathrm{i} mk/n} \\ e^{-\pi \mathrm{i} mk/n} & 0 \end{pmatrix}.$$

□

Let $\xi_{n,m} : (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n \to \mathbb{C}^2/\widetilde{D}_n$ be the quotient family associated with τ_m. We now determine its kaleido locus $\mathrm{KL}_{\xi_{n,m}}$. The preimage $\widetilde{\mathrm{KL}}_{\xi_{n,m}}$ of $\mathrm{KL}_{\xi_{n,m}}$ under the quotient map $\mathbb{C}^2 \to \mathbb{C}^2/\widetilde{D}_n$ is given by $\bigcup_{g \in \widetilde{D}_n \setminus \{1\}} \mathrm{Fix}(g)$ (see the proof of Proposition 2.1.5). By Lemma 4.1.15, $\widetilde{\mathrm{KL}}_{\xi_{n,m}} = \{0\}$, so $\mathrm{KL}_{\xi_{n,m}} = \{0\}$. Thus $\xi_{n,m}^{-1}(s)$ $(s \in \mathbb{C}^2/\widetilde{D}_n)$ is kaleido if and only if $s = 0$. The only one kaleido fiber $\xi_{n,m}^{-1}(0)$ is $\Sigma/\widetilde{D}_n$. Here note that the action of $\widetilde{D}_n$ on Σ is equivalent to that of D_n on Σ, so $\Sigma/\widetilde{D}_n \cong \Sigma/D_n$, and thus $\xi_{n,m}^{-1}(0) = \Sigma/D_n$.

Theorem 4.1.16. *Let $\xi_{n,m} : (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n \to \mathbb{C}^2/\widetilde{D}_n$ be the quotient family of Σ associated with $\tau_m : \widetilde{D}_n \to GL_2(\mathbb{C})$. If τ_m is injective, then the following holds* (c.m. means covering multiplicity):

$$\xi_{n,m}^{-1}(s) = \begin{cases} \text{projective line } \Sigma/D_n \quad (\text{c.m. } 2n) & \text{if } s = 0, \\ \text{elliptic curve } \Sigma \quad (\text{c.m. } 1) & \text{if } s \neq 0. \end{cases}$$

In particular if $s \neq 0$, then all fibers of $\xi_{n,m}$ are Σ, so $\xi_{n,m}$ has a single kaleido fiber. We thus obtain the following:

Corollary 4.1.17. *The quotient family $\xi_{n,m} : (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n \to \mathbb{C}^2/\widetilde{D}_n$ of Σ associated with any unlifted irreducible representation $\tau_m : \widetilde{D}_n \to GL_2(\mathbb{C})$ has a single singular fiber — a kaleido fiber of covering multiplicity $2n$ (see Figure 4.1.1).*

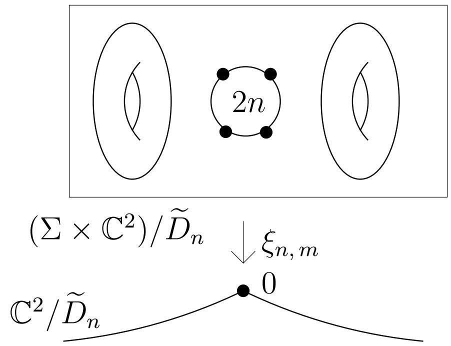


Figure 4.1.1:

Remark 4.1.18. (i) While the base space $\mathbb{C}^2/D_n$ of $\eta_{n,l}$ is smooth (Lemma 3.4.6), the base space $\mathbb{C}^2/\widetilde{D}_n$ of $\xi_{n,m}$ has a D-singularity. (ii) The singular locus of $(\Sigma \times \mathbb{C}^2)/\widetilde{D}_n$ consists of the four points on $\xi_{n,m}^{-1}(0)$ (see Proposition 4.1.29 below).

Case: τ_m is non-injective Let $\xi_{n,m} : (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n \to \mathbb{C}^2/\widetilde{D}_n$ be the quotient family associated with the irreducible representation $\tau_m : \widetilde{D}_n \to GL_2(\mathbb{C})$ given by

$$\tau_m(\widetilde{a}) = \begin{pmatrix} e^{\pi i m/n} & 0 \\ 0 & e^{-\pi i m/n} \end{pmatrix}, \qquad \tau_m(\widetilde{b}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In what follows, consider the case that τ_m is *not* injective. The kaleido locus $\mathrm{KL}_{\xi_{n,m}}$ of $\xi_{n,m}$ is then the whole of $\mathbb{C}^2/D_n$, i.e. every fiber of $\xi_{n,m}$ is kaleido (Proposition 2.1.5 (1)). By Theorem 2.1.9 (1), $\xi_{n,m} : (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n \to \mathbb{C}^2/\widetilde{D}_n$ is canonically isomorphic to the quotient family $\overline{\xi}_{n,m} : \big(\Sigma/\mathrm{Ker}(\tau_m) \times \mathbb{C}^2\big)/\big(\widetilde{D}_n/\mathrm{Ker}(\tau_m)\big) \to \mathbb{C}^2/\big(\widetilde{D}_n/\mathrm{Ker}(\tau_m)\big)$ associated with $\overline{\tau}_m : \widetilde{D}_n/\mathrm{Ker}(\tau_m) \to GL_2(\mathbb{C})$. Now set $d := \gcd(n,m)$, $n' := n/d$, and $m' := m/d$. By Lemma 4.1.14, $\gcd(n,m) \geq 2$ and $\mathrm{Ker}(\tau_m) = \langle \widetilde{a}^{n'} \rangle$, so $\widetilde{D}_n/\mathrm{Ker}(\tau_m) \cong \widetilde{D}_{n'}$. Set $\Sigma' := \Sigma/\langle \widetilde{a}^{n'} \rangle$; this coincides with $\Sigma/\langle a^{n'} \rangle$ (as $\widetilde{a}$ acts on Σ as a), so this is an elliptic curve (Lemma 3.2.4 (1)).

Lemma 4.1.19. *Two representations $\overline{\tau}_m : \widetilde{D}_{n'} \to GL_2(\mathbb{C})$ and $\tau_{m'} : \widetilde{D}_{n'} \to GL_2(\mathbb{C})$ coincide.*

Proof. The images α, β of $\widetilde{a}, \widetilde{b}$ under the quotient map $\widetilde{D}_n \to \widetilde{D}_{n'}$ $(\cong \widetilde{D}_n/\langle \widetilde{a}^{n'} \rangle)$ generate $\widetilde{D}_{n'}$, and

$$\overline{\tau}_m(\alpha) = \tau_{m'}(\alpha) = \begin{pmatrix} e^{\pi i m'/n'} & 0 \\ 0 & e^{-\pi i m'/n'} \end{pmatrix}, \qquad \overline{\tau}_m(\beta) = \tau_{m'}(\beta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

□

The above results combined with Theorem 2.1.9 yield the following:

Lemma 4.1.20. (1) *The quotient family* $\xi_{n,m} : (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n \to \mathbb{C}^2/\widetilde{D}_n$ *associated with* $\tau_m : \widetilde{D}_{n'} \to GL_2(\mathbb{C})$ *is canonically isomorphic to the quotient family* $\xi_{n',m'} : (\Sigma' \times \mathbb{C}^2)/\widetilde{D}_{n'} \to \mathbb{C}^2/\widetilde{D}_{n'}$ *associated with the injective representation* $\tau_{m'} : \widetilde{D}_{n'} \to GL_2(\mathbb{C})$*. Here* $\Sigma' := \Sigma/\mathrm{Ker}(\tau_m)$ *is an elliptic curve and* $|\mathrm{Ker}(\tau_m)| = d\,(= \gcd(n,m))$*, so*

$$(\text{covering multiplicity of } \xi_{n,m}^{-1}(s)) = d \times (\text{covering multiplicity of } \xi_{n',m'}^{-1}(s)).$$

(2) *Let* $\mathrm{SL}_{\xi_{n,m}}$ *be the special locus of* $\xi_{n,m}$ *(Remark 2.1.8) and* $\mathrm{KL}_{\xi_{n',m'}}$ *be the kaleido locus of* $\xi_{n',m'}$*. Under the isomorphism in* (1)*,* $\mathrm{SL}_{\xi_{n,m}} = \mathrm{KL}_{\xi_{n',m'}}$*.*

Here by Theorem 4.1.16,

$$\xi_{n',m'}^{-1}(s) = \begin{cases} \text{projective line } \Sigma'/D_{n'}\ (\cong \Sigma/D_n) & (\text{c.m. } 2n') \quad \text{if } s = 0, \\ \text{elliptic curve } \Sigma'\ (\cong \Sigma/\langle \widetilde{a}^{n'} \rangle) & (\text{c.m. } 1) \quad \text{if } s \neq 0. \end{cases}$$

Consequently the following holds:

Proposition 4.1.21. *Let* $\xi_{n,m} : (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n \to \mathbb{C}^2/\widetilde{D}_n$ *be the quotient family of* Σ *associated with* $\tau_m : \widetilde{D}_n \to GL_2(\mathbb{C})$*. Set* $d := \gcd(m,n)$ *and* $n' := n/d$*. Then the following holds* (c.m. means the covering multiplicity):

$$\xi_{n,m}^{-1}(s) = \begin{cases} \text{projective line } \Sigma/D_n & (\text{c.m. } 2n'd\,(= 2n)) \quad \text{if } s = 0, \\ \text{elliptic curve } \Sigma/\langle a^{n'} \rangle & (\text{c.m. } d) \quad \text{if } s \neq 0. \end{cases}$$

4.1.3 Singular loci of total spaces

Let $\tau_m : \widetilde{D}_n \to GL_2(\mathbb{C})$ be the representation given by (4.1.2) and $\xi_{n,m} : (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n \to \mathbb{C}^2/\widetilde{D}_n$ be its associated quotient family. We determine the type of the singular locus of $(\Sigma \times \mathbb{C}^2)/\widetilde{D}_n$. Note first the following (the proof is same as that of Lemma 3.5.1):

Lemma 4.1.22. *If $y \in (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n$ is a singularity, then $H_{\widetilde{y}} \neq \{1\}$, where $H_{\widetilde{y}}$ is the stabilizer of a lift $\widetilde{y} \in \Sigma \times \mathbb{C}^2$ of y.* (If moreover τ_m is injective, the converse holds (Lemma 4.1.26 below).)

Since the $\widetilde{D}_n$-action on $\Sigma \times \mathbb{C}^2$ is diagonal $\big((z,t) \mapsto (gz, \tau_m(g)t)\big)$, the following holds:

Lemma 4.1.23. *For any $(z,t) \in \Sigma \times \mathbb{C}^2$, $H_{(z,t)} = H_z \cap H_t$.* (Thus $H_{(z,t)} \neq \{1\}$ is restated as $H_z \cap H_t \neq \{1\}$.)

To determine the singularities of $(\Sigma \times \mathbb{C}^2)/\widetilde{D}_n$, we shall determine (z,t) such that $H_{(z,t)} \neq \{1\}$. Thanks to Lemma 4.1.20 (1), *we may assume that the representation τ_m is injective.*

The action of $\widetilde{D}_n = \{\widetilde{a}^k, \widetilde{a}^k\widetilde{b} : k = 0, 1, \ldots, 2n-1\}$ on Σ is not effective; the kernel of $\widetilde{D}_n \to \mathrm{Aut}(\Sigma)$ is $\langle \widetilde{a}^n \rangle$. Since $\widetilde{a}, \widetilde{b} \in \widetilde{D}_n$ acts on Σ as $a, b \in D_n$, by (i) of Lemma 3.5.3, we have

$$H_z = \begin{cases} \langle \widetilde{a}^k\widetilde{b}, \widetilde{a}^n \rangle & \text{if } z \in \mathrm{Fix}_\Sigma(\widetilde{a}^k\widetilde{b}), \\ \langle \widetilde{a}^n \rangle & \text{otherwise.} \end{cases}$$

Here $(\widetilde{a}^k\widetilde{b})^2 = (\widetilde{a}^k\widetilde{b})(\widetilde{a}^k\widetilde{b}) = \widetilde{a}^k\widetilde{a}^{-k}\widetilde{b}\widetilde{b} = \widetilde{b}^2 = \widetilde{a}^n$, so $\langle \widetilde{a}^k\widetilde{b}, \widetilde{a}^n \rangle = \langle \widetilde{a}^k\widetilde{b} \rangle$. This confirms (i) of the following (while (ii) is nothing but Lemma 4.1.15; note τ_m is injective):

Lemma 4.1.24.

(i) $$H_z = \begin{cases} \langle \widetilde{a}^k\widetilde{b} \rangle \ (\cong \mathbb{Z}_4) & \textit{if } z \in \mathrm{Fix}_\Sigma(\widetilde{a}^k\widetilde{b}), \\ \langle \widetilde{a}^n \rangle \ (\cong \mathbb{Z}_2) & \textit{otherwise,} \end{cases}$$ (ii) $$H_t = \begin{cases} \widetilde{D}_n & \textit{if } t = 0, \\ \{1\} & \textit{otherwise.} \end{cases}$$

From $H_{(z,t)} = H_z \cap H_t$ and Lemma 4.1.24, we have:

Corollary 4.1.25. *Suppose that τ_m is injective. Then the following conditions are equivalent:*

(i) $H_{(z,t)} \neq \{1\}$.

(ii) $H_{(z,t)} = \langle \widetilde{a}^k \widetilde{b} \rangle$ *for some* k.

(iii) $z \in \mathrm{Fix}_\Sigma(\widetilde{a}^k \widetilde{b})$ *and* $t = 0$.

The image of $(z, t) \in \Sigma \times \mathbb{C}^2$ under the quotient map $\Sigma \times \mathbb{C}^2 \to (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n$ is denoted by $[z, t] \in (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n$. If $[z, t] \in (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n$ is a singularity, then $H_{(z,t)} \neq \{1\}$ (Lemma 4.1.22). Conversely the following holds:

Lemma 4.1.26. *Suppose that τ_m is injective. If $H_{(z,t)} \neq \{1\}$, then $[z, t] \in (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n$ is a singularity.*

Proof. If $H_{(z,t)} \neq \{1\}$, then $H_{(z,t)} = \langle a^k b \rangle$ for some k (Corollary 4.1.25). Noting that $\widetilde{a}^k \widetilde{b}$ is *not* a pseudo-reflection (Lemma 4.1.32 below), $H_{(z,t)} = \langle \widetilde{a}^k \widetilde{b} \rangle$ is a small group, so $[z, t] \in (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n$ is a singularity. □

Combining Lemma 4.1.22, Corollary 4.1.25, and Lemma 4.1.26 yield the following:

Proposition 4.1.27. *Suppose that τ_m is injective. Then the following are equivalent:*

(i) $[z, t] \in (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n$ *is a singularity.*

(ii) $H_{(z,t)} \neq \{1\}$.

(iii) $H_{(z,t)} = \langle \widetilde{a}^k \widetilde{b} \rangle$ *for some* k.

(iv) $z \in \mathrm{Fix}_\Sigma(\widetilde{a}^k \widetilde{b})$ *and* $t = 0$. (In case of D_n, this condition instead $(z, t) \in \mathrm{Fix}_\Sigma(a^k b) \times \mathrm{Fix}_{\mathbb{C}^2}(a^k b)$ (Proposition 3.5.6 (iv)), in the present case, $\mathrm{Fix}_{\mathbb{C}^2}(\widetilde{a}^k \widetilde{b}) = \{0\}$.)

Regard the quotient map $\Sigma \times \{0\} \to (\Sigma \times \{0\})/\widetilde{D}_n$ as $\Sigma \to \Sigma/\widetilde{D}_n\,(= \Sigma/D_n)$. This has four branch points (Lemma 3.2.5 (2)). Write them as $p_i \in (\Sigma \times \{0\})/\widetilde{D}_n$ $(i = 1, 2, 3, 4)$. In what follows, regard $(\Sigma \times \{0\})/\widetilde{D}_n$ as a subspace of $(\Sigma \times \mathbb{C}^2)/\widetilde{D}_n$.

Lemma 4.1.28. *$z \in \mathrm{Fix}_\Sigma(\widetilde{a}^k\widetilde{b})$ for some k $(k = 0, 1, \ldots, 2n-1)$ if and only if z is a ramification point of $\Sigma \to \Sigma/\widetilde{D}_n\,(= \Sigma/D_n)$.*

Proof. $\Longrightarrow$ is obvious. We show $\Longleftarrow$. If z is a ramification point of $\Sigma \to \Sigma/D_n$, then its stabilizer $H_z \subset D_n$ is nontrivial. By Lemma 3.5.3, $z \in \mathrm{Fix}_\Sigma(a^k b)$ for some k, so $z \in \mathrm{Fix}_\Sigma(\widetilde{a}^k\widetilde{b})$ for some k. □

Recall that the ramification points of $\Sigma \to \Sigma/D_n$ are $p_k, q_k, r_k, s_k \in \Sigma$ $(k = 0, 1, \ldots, n-1)$ (see Figure 3.4.3 for odd n, Figure 3.4.5 for even n), and the branch points are $[p_0], [q_0], [r_0], [s_0] \in \Sigma/D_n$ (Lemma 3.2.5 (2)). By Proposition 4.1.27 and Lemma 4.1.28, we obtain the following:

Proposition 4.1.29. *Suppose that τ_m is injective. Then the singular locus of $(\Sigma \times \mathbb{C}^2)/\widetilde{D}_n$ is isolated, indeed consists of the branch points $[p_0, 0]$, $[q_0, 0]$, $[r_0, 0]$, $[s_0, 0]$ of $\Sigma \times \{0\} \to (\Sigma \times \{0\})/\widetilde{D}_n$.*

Consider next the case that τ_m is *not* injective. We reduce this to the injective case: Set $n' := n/\gcd(m,n)$ and $m' := m/\gcd(m,n)$ and $\Sigma' := \Sigma/\langle \widetilde{a}^{n'} \rangle$. Then note that the quotient family $\xi_{n,m} : (\Sigma \times \mathbb{C}^2)/\widetilde{D}_n \to \mathbb{C}^2/\widetilde{D}_n$ associated with $\tau_m : \widetilde{D}_n \to GL_2(\mathbb{C})$ is naturally identified with the quotient family $\xi_{n',m'} : (\Sigma' \times \mathbb{C}^2)/\widetilde{D}_{n'} \to \mathbb{C}^2/\widetilde{D}_{n'}$ associated with the injective representation $\tau_{m'}\,(= \overline{\tau}_m) : \widetilde{D}_{n'} \to GL_2(\mathbb{C})$. In fact,

(i)
$$\begin{aligned}(\Sigma \times \mathbb{C}^2)/\widetilde{D}_n &= (\Sigma \times \mathbb{C}^2)/\langle \widetilde{a}^{n'} \rangle\big/(\widetilde{D}_n/\langle \widetilde{a}^{n'} \rangle) \\ &\cong \big(\Sigma/\langle \widetilde{a}^{n'} \rangle \times \mathbb{C}^2\big)\big/(\widetilde{D}_n/\langle \widetilde{a}^{n'} \rangle) \quad \text{as } \langle \widetilde{a}^{n'} \rangle\text{-action on } \mathbb{C}^2 \text{ is trivial} \\ &= (\Sigma' \times \mathbb{C}^2)/\widetilde{D}_{n'},\end{aligned}$$

(ii)
$$\begin{aligned}\mathbb{C}^2/\widetilde{D}_n &= (\mathbb{C}^2/\langle \widetilde{a}^{n'} \rangle)\big/(\widetilde{D}_n/\langle \widetilde{a}^{n'} \rangle) \\ &\cong \mathbb{C}^2\big/(\widetilde{D}_n/\langle \widetilde{a}^{n'} \rangle) \quad \text{as } \langle \widetilde{a}^{n'} \rangle\text{-action on } \mathbb{C}^2 \text{ is trivial} \\ &= \mathbb{C}^2/\widetilde{D}_{n'},\end{aligned}$$

and the following diagram commutes:

$$\begin{array}{ccc} (\Sigma\times\mathbb{C}^2)/\widetilde{D}_n & \xrightarrow{\cong} & (\Sigma'\times\mathbb{C}^2)/\widetilde{D}_{n'} \\ {\scriptstyle \xi_{n,m}}\downarrow & & \downarrow{\scriptstyle \xi_{n',m'}} \\ \mathbb{C}^2/\widetilde{D}_n & \xrightarrow{\cong} & \mathbb{C}^2/\widetilde{D}_{n'}. \end{array} \tag{4.1.7}$$

Consequently the assumption "τ_m is injective" in Proposition 4.1.29 may be omitted:

Proposition 4.1.30. *The singular locus of* $(\Sigma\times\mathbb{C}^2)/\widetilde{D}_n$ *is isolated, consisting of the four points on* $(\Sigma\times\{0\})/\widetilde{D}_n$ $(=\xi_{n,m}^{-1}(0))$ *that are the branch points of* $\Sigma\times\{0\}\to(\Sigma\times\{0\})/\widetilde{D}_n$*.*

Remark 4.1.31. The isomorphism $(\Sigma\times\mathbb{C}^2)/\widetilde{D}_n \xrightarrow{\cong} (\Sigma'\times\mathbb{C}^2)/\widetilde{D}_{n'}$ in (4.1.7) is explicitly given by $[y,t]\mapsto[y \bmod \langle\widetilde{a}^{n'}\rangle, t]$.

We next determine the type of each singularity of $(\Sigma\times\mathbb{C}^2)/\widetilde{D}_n$. The following is needed:

Lemma 4.1.32. *The action of* $\widetilde{a}^k\widetilde{b}$ *on* $\Sigma\times\mathbb{C}^2$ *is given by* $M := \begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\mathrm{i} \end{pmatrix}$ *up to conjugation.* (Note: M is *not* a pseudo-reflection.)

Proof. For $z\in \mathrm{Fix}_\Sigma(\widetilde{a}^k\widetilde{b})$, take an $\langle\widetilde{a}^k\widetilde{b}\rangle$-invariant small disk Δ in Σ centered at z. Then the action of $\widetilde{a}^k\widetilde{b}$ on Δ is given by $x\mapsto -x$ (Lemma 3.1.3 (2)) and that on $\mathbb{C}^2$ is given by $\begin{pmatrix} 0 & -e^{\pi \mathrm{i}mk/n} \\ e^{-\pi \mathrm{i}mk/n} & 0 \end{pmatrix}$ (see (4.1.2)). So the action of $\widetilde{a}^k\widetilde{b}$ on $\Delta\times\mathbb{C}^2$ is given by $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -e^{\pi \mathrm{i}mk/n} \\ 0 & e^{-\pi \mathrm{i}mk/n} & 0 \end{pmatrix}$, which is diagonalized to $M := \begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\mathrm{i} \end{pmatrix}$ in $GL_3(\mathbb{C})$. □

Let $\mathbb{Z}_4$ be the cyclic subgroup of order 4 in $GL_3(\mathbb{C})$ generated by M. We may express $M = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^3 \end{pmatrix}$, where $\zeta := e^{2\pi i/4}$. So $\mathbb{C}^3/\mathbb{Z}_4$ is a singularity of type $\dfrac{1}{4}(1,2,3)$ and *not* terminal (see Remark 4.1.34 below). By Proposition 4.1.30, the singular locus of $(\Sigma \times \mathbb{C}^2)/\widetilde{D}_n$ consists of four isolated singularities and by Lemma 4.1.32, each is isomorphic to $\mathbb{C}^3/\mathbb{Z}_4$.

We formalize the results so far obtained as follows:

Theorem 4.1.33. *The singular locus of* $(\Sigma \times \mathbb{C}^2)/\widetilde{D}_n$ *consists of four isolated singularities — any of which is of type* $\dfrac{1}{4}(1,2,3)$ (this is not a terminal singularity)*.*

Remark 4.1.34. For a cyclic subgroup $\mathbb{Z}_m$ of order m in $GL_3(\mathbb{C})$ generated by an element of the form $\begin{pmatrix} \zeta^{n_1} & 0 & 0 \\ 0 & \zeta^{n_2} & 0 \\ 0 & 0 & \zeta^{n_3} \end{pmatrix}$ where $\zeta := e^{2\pi i/m}$ and n_i $(i = 1,2,3)$ are integers such that $0 < n_i < m$, the quotient singularity $\mathbb{C}^3/\mathbb{Z}_m$ is called *of type* $\dfrac{1}{m}(n_1, n_2, n_3)$. This singularity is terminal if and only if $(n_1, n_2, n_3) = (1, \ell, -\ell)$ for some ℓ relatively prime to m (see [Ish] p.185 Theorem 8.3.17).

Summary The singular locus of the total space of a dihedral quotient family (associated with a representation of D_n) is not isolated, consisting of four ridges — their configuration depends on the parity of n; compare Figure 3.4.4 and Figure 3.4.6. On the other hand, the singular locus of the total space of a binary dihedral quotient family (associated with a representation of $\widetilde{D}_n$) is isolated, consisting of four points irrespective of the parity of n. These differences arise from the following differences between D_n and $\widetilde{D}_n$:

	$D_n = \{a^k, a^k b \,:\, k = 0, \ldots, n-1\}$	$\widetilde{D}_n = \{\widetilde{a}^k, \widetilde{a}^k\widetilde{b} \,:\, k = 0, \ldots, 2n-1\}$
(I)	$\mathrm{Fix}(a^k) = \{0\}$ and $\mathrm{Fix}(a^k b) = \mathbb{C}$ (see Lemma 3.3.1)	$\mathrm{Fix}(\widetilde{a}^k) = \mathrm{Fix}(\widetilde{a}^k\widetilde{b}) = \{0\}$ (see Lemma 4.1.15)
(II)	n even: $\{a^k b \,:\, k \text{ is odd}\}$, $\{a^k b \,:\, k \text{ is even}\}$ are distinct D_n-orbits n odd: $\{a^k b : k \text{ is any}\}$ is a single D_n-orbit (see Corollary 3.3.4)	$\{\widetilde{a}^k\widetilde{b} \,:\, k \text{ is odd}\}$, $\{\widetilde{a}^k\widetilde{b} \,:\, k \text{ is even}\}$ are distinct $\widetilde{D}_n$-orbits (shown by the argument used in the proof of Corollary 3.3.4 (1))

Table 4.1.3: Comparison of D_n and $\widetilde{D}_n$

Part II

The Family of Tetra Riemann Surfaces

Chapter 5

Aspects of Group Actions

5.1 Description of tetrahedral group action

Let Σ be the cable surface of the tetrahedron, on which the tetrahedral group $\mathfrak{T}$ naturally acts. Thanks to Kerckhoff's theorem [Ker], we may give a complex structure to it such that the $\mathfrak{T}$-action is holomorphic. We determine the branch data of the quotient map $\psi : \Sigma \to \Sigma/\mathfrak{T}$. We first review terminology with the intension of fixing notation.

Note first that ψ is a $|\mathfrak{T}|$-fold covering.

- For $y \in \Sigma/\mathfrak{T}$, if $\#\psi^{-1}(y) < |\mathfrak{T}|$, then y is a *branch point* (with branch index $|\mathfrak{T}|/\#\psi^{-1}(y)$).
- If $y \in \Sigma/\mathfrak{T}$ is a branch point, then $x \in \psi^{-1}(y)$ is a *ramification point* (with ramification index $|\mathfrak{T}|/\#\psi^{-1}(y)$).

A ramification point is alternatively characterized as a point with non-trivial stabilizer. For a point $x \in \Sigma$, its stabilizer $\mathfrak{T}_x$ is a subgroup of $\mathfrak{T}$ given by

$$\mathfrak{T}_x := \{g \in \mathfrak{T} : gx = x\}.$$

Now for $y \in \Sigma/\mathfrak{T}$, take $x \in \psi^{-1}(y)$. Then $\mathfrak{T}$ acts transitively on the points of $\psi^{-1}(y)$ while $\mathfrak{T}_x$ fixing x. Thus $\psi^{-1}(y) \cong \mathfrak{T}/\mathfrak{T}_x$ (as sets), and $\#\psi^{-1}(y) =$

$|\mathfrak{T}|/|\mathfrak{T}_x|$. Hence $|\mathfrak{T}|/\#\psi^{-1}(y) = |\mathfrak{T}|\Big/|\mathfrak{T}|/|\mathfrak{T}_x| = |\mathfrak{T}_x|$.

We thus obtain:

Lemma 5.1.1. *The ramification index of x is $|\mathfrak{T}_x|$. Thus:*

$$\#\psi^{-1}(y) < |\mathfrak{T}| \quad (\text{i.e. } y \text{ is a branch point}) \iff 1 < |\mathfrak{T}_x| \iff \mathfrak{T}_x \neq \{1\}.$$

Remark 5.1.2. $|\mathfrak{T}_x|$ is independent of the choice of $x \in \psi^{-1}(y)$. In fact for another $x' \in \psi^{-1}(y)$, $\mathfrak{T}_x$ and $\mathfrak{T}_{x'}$ are conjugate: There exists $g \in \mathfrak{T}$ such that $x' = gx$, for which $\mathfrak{T}_{x'} = g\mathfrak{T}_x g^{-1}$.

Take a ramification point $x \in \Sigma$. Then to each conjugate $g\mathfrak{T}_x g^{-1}$ $(g \in \mathfrak{T})$ of $\mathfrak{T}_x$, a ramification point $y = gx$ is associated; note that $\mathfrak{T}_y = g\mathfrak{T}_x g^{-1}$ and the ramification index $|\mathfrak{T}_y|$ of y is equal to $|\mathfrak{T}_x|$. Now denote by H the conjugacy class $\{g\mathfrak{T}_x g^{-1} : g \in \mathfrak{T}\}$ of $\mathfrak{T}_x$. The ramification points associated with the subgroups in this conjugacy class are called *H-ramification points*.

Definition 5.1.3. Let $y_1, y_2, \ldots, y_l$ be the branch points of ψ, and for each $y_i \in \Sigma/\mathfrak{T}$, let $e_i := |\mathfrak{T}_x|$ $(x \in \psi^{-1}(y_i))$ be the *branch index* of y_i. Then the tuple $\big(\text{genus}(\Sigma/\mathfrak{T}); e_1, e_2, \ldots, e_l\big)$ is called the *branch data* (*signature*) of ψ (or of the $\mathfrak{T}$-action on Σ).

The ramification points of $\Sigma \to \Sigma/\mathfrak{T}$ are the points of Σ with nontrivial stabilizers (Lemma 5.1.1). To determine such points, identify $\mathfrak{T}$ with the alternating group $\mathfrak{A}_4$ under the canonical isomorphism induced from the permutation of the vertices of the tetrahedron. Here the (proper) nontrivial subgroups of $\mathfrak{A}_4$ are $\mathbb{Z}_2, \mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ up to conjugation. Among them, $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are stabilizers of some points of Σ. In fact $\mathbb{Z}_2$ acts as a 1/2-rotation fixing four points as illustrated in Figure 5.1.1 (there are three conjugate $\mathbb{Z}_2$'s in $\mathfrak{A}_4$) and $\mathbb{Z}_3$ acts as a 1/3-rotation fixing two points as illustrated in Figure 5.1.2 (there are four conjugate $\mathbb{Z}_3$'s in $\mathfrak{A}_4$), while $\mathbb{Z}_2 \times \mathbb{Z}_2$ fixes *no* point (as a whole group) and is generated by a pair of 1/2-rotations (there are three conjugate $\mathbb{Z}_2 \times \mathbb{Z}_2$'s in $\mathfrak{A}_4$ respectively generated by (1) and (2), (2) and (3), or (3) and (1) in Figure 5.1.1). The total number of $\mathbb{Z}_2$-ramification points are $4 \times 3 = 12$ and the total number of $\mathbb{Z}_3$-ramification points are $2 \times 4 = 8$. The ramification index of each $\mathbb{Z}_2$-ramification point is $|\mathbb{Z}_2| = 2$ and the ramification index of each $\mathbb{Z}_3$-ramification point is $|\mathbb{Z}_3| = 3$.

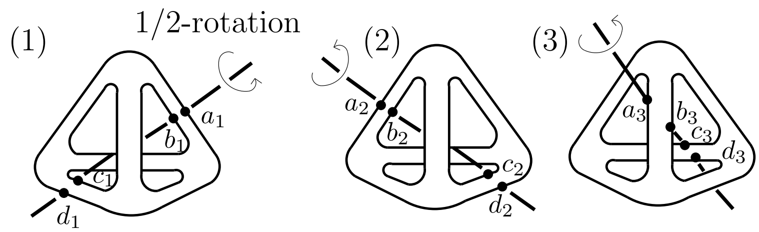


Figure 5.1.1:

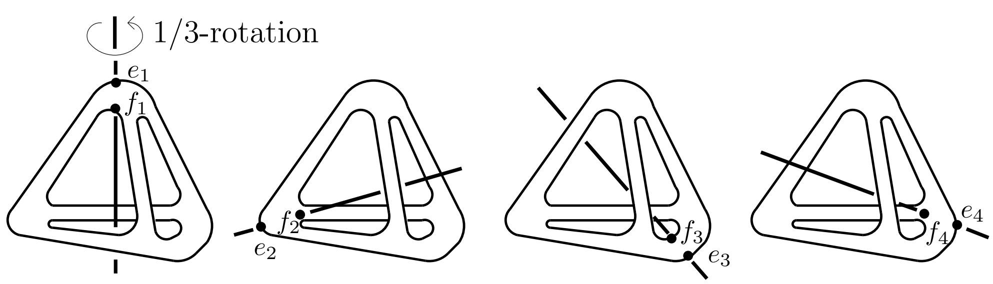


Figure 5.1.2:

The images of the ramification points under the quotient map $\psi : \Sigma \to \Sigma/\mathfrak{T}$ are the *branch points*. Note:

- a_i, b_i, c_i, d_i $(i = 2, 3)$ are identified with a_1, b_1, c_1, d_1 respectively.
- a_1 (resp. b_1) is identified with d_1 (resp. c_1) via a 1/2-rotation as illustrated in Figure 5.1.3.

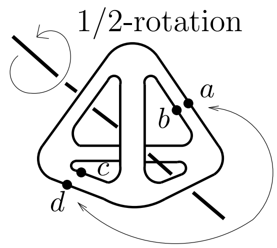


Figure 5.1.3:

Hence the images of the $\mathbb{Z}_2$-ramification points are two points $\overline{a}_1$ and $\overline{b}_1$ (with branch index 2).

Next under $\psi : \Sigma \to \Sigma/\mathfrak{T}$, e_i, f_i $(i = 2, 3, 4)$ are identified with e_1, f_1 respectively. Hence the images of the $\mathbb{Z}_3$-ramification points are two points $\overline{e}_1$ and $\overline{f}_1$ (with branch index 3).

We summarize the above as follows:

Lemma 5.1.4. *The quotient map* $\Sigma \to \Sigma/\mathfrak{T}$ *has four branch points with branch indices* $(2, 2, 3, 3)$.

We next show that $\Sigma/\mathfrak{T} \cong \mathbb{P}^1$ by applying the Riemann–Hurwitz formula:

$$\chi(\Sigma) = |\mathfrak{T}|\ \chi\big(\Sigma/\mathfrak{T}\big) - \sum_{p\in\mathcal{R}}(e_p - 1), \tag{5.1.1}$$

where $\mathcal{R}$ is the set of the ramification points and e_p is its ramification index of $p \in \mathcal{R}$. In the present case, $\chi(\Sigma) = -4$, $|\mathfrak{T}| = 12$ and $\sum_p(e_p - 1) = 12(2 - 1) + 8(3 - 1) = 28$. Thus from(5.1.1), $\chi(\Sigma/\mathfrak{T}) = -2$, implying that $\Sigma/\mathfrak{T} \cong \mathbb{P}^1$. This with Lemma 5.1.4 yields the following:

Proposition 5.1.5. *Let* Σ *be the cable surface of the tetrahedron, on which the tetrahedral group* $\mathfrak{T}$ *acts. Then* $\Sigma/\mathfrak{T} \cong \mathbb{P}^1$ *and the quotient map* $\Sigma \to \Sigma/\mathfrak{T}$ *has four branch points with branch indices* $(2, 2, 3, 3)$. (Thus the branch data of the $\mathfrak{T}$-action on Σ is $(0; 2, 2, 3, 3)$.)

We regard the branch points on $\Sigma/\mathfrak{T}$ as "marked points"; observe that the complex structure on $\Sigma/\mathfrak{T}\,(\cong \mathbb{P}^1)$ with four marked points admits a 1-parameter family of deformations (caused by moving one point among the four points — three points on $\mathbb{P}^1$ are normalized as $0, 1, \infty$ under some element of $PSL_2(\mathbb{C})$). Varying one branch point (in $\mathbb{P}^1 \setminus \{\text{other branch points}\}$) yields a family of *topologically equivalent* coverings. The complex structures on the covering spaces are given by the pull back of the complex structures on $\Sigma/\mathfrak{T}$ with four marked points via the quotient map $\Sigma \to \Sigma/\mathfrak{T}$. We thus obtain a 1-parameter family of complex structures on Σ with the same covering transformation group, that is, $\mathfrak{T}$ (their branch data remain $(0; 2, 2, 3, 3)$). We formalize this as follows:

Corollary 5.1.6. (Non-rigidity) *Let Σ be the cable surface of the tetrahedron, on which the tetrahedral group $\mathfrak{T}$ acts. Give a complex structure to Σ such that the $\mathfrak{T}$-action is holomorphic, and regard Σ as a Riemann surface with $\mathfrak{T}$-action. Then Σ admits a "$\mathfrak{T}$-action preserving" 1-parameter deformation — there exists a 1-parameter family of Riemann surfaces with $\mathfrak{T}$-actions starting from Σ such that their branch data remain $(0; 2, 2, 3, 3)$.*

Chapter 6

The Family of Tetra Curves

6.1 Defining equations of $\mathfrak{A}_4$-curves of genus 3

In what follows, unless otherwise mentioned, all curves are of genus 3*.* We identify the tetrahedral group $\mathfrak{T}$ with the alternating group $\mathfrak{A}_4$ under the canonical isomorphism (recall: $\mathfrak{T}$ permutes the four vertices of the tetrahedron, which induces $\mathfrak{T} \cong \mathfrak{A}_4$). A curve C with $\mathfrak{A}_4$-action (i.e. $\mathfrak{A}_4 \subset \mathrm{Aut}(C)$) is called an $\mathfrak{A}_4$*-curve*. The aim of this section is to show the following:

Theorem 6.1.1. *The* $\mathfrak{A}_4$*-curves of genus* 3 *are as follows:* (i) *There is a unique hyperelliptic one:* $B : y^2 = x^8 + 14x^4 + 1$*.* (ii) *Non-hyperelliptic ones form a* 1*-parameter family* $C_t : x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0$ $(t \in \mathbb{C} \setminus \{\pm 2, -1\})$.

Note first that: *If* C *is an* $\mathfrak{A}_4$*-curve, then* $\mathfrak{A}_4 \subset \mathrm{Aut}(C)$*, so* $|\mathrm{Aut}(C)|$ *is divisible by* $|\mathfrak{A}_4| = 12$*.* We thus consider curves C such that $|\mathrm{Aut}(C)|$ is divisible by 12. We separate into hyperelliptic case and non-hyperelliptic case:

(H) The list of hyperelliptic curves C such that $|\mathrm{Aut}(C)|$ is divisible by 12 is as follows ([GSS] p.118 Table 1):

Aut	$\lvert\mathrm{Aut}\rvert$	C
D_{12}	12	H1 : $y^2 = x(x^6 + tx^2 + 1)$
U_6	24	H2 : $y^2 = x(x^6 - 1)$
$\mathbb{Z}_2 \times \mathfrak{S}_4$	48	H3 : $y^2 = x^8 + 14x^4 + 1$

Table 6.1.1: $\mathfrak{S}_n$: symmetric group of degree n. $U_6 := \langle a, b : a^2 = b^{12} = abab^7 = 1\rangle$ (or $\langle a, b : a^2, b^{12}, abab^7\rangle$ in [BGG] p.272 Table 2, 3.e). $D_{2n} := \langle a, b : a^n = b^2 = abab = 1\rangle$: dihedral group of order $2n$. NOTE: $U_6 = \langle a, b : a^2, b^6, abab^4\rangle$ in [GSS] p.118 Table 1 seems a typo, because in which case $|U_6| \neq 24$ (but $|U_6| = 6$).

Here:

- H1 and H2 are *not* $\mathfrak{A}_4$-curves, as $\mathfrak{A}_4 \not\subset \mathrm{Aut}(\mathrm{H1})\,(= D_{12})$ and $\mathfrak{A}_4 \not\subset \mathrm{Aut}(\mathrm{H2})\,(= U_6)$ by Lemma 6.1.3 below.
- H3 is an $\mathfrak{A}_4$-curve, indeed $\mathfrak{A}_4 \subset \mathbb{Z}_2 \times \mathfrak{S}_4$. (Note: The Galois group of $x^8 + 14x^4 + 1$ is $\mathfrak{S}_4$, see [Kle] p.58.)

This confirms (i) of Theorem 6.1.1.

(NH) The list of non-hyperelliptic curves C such that $|\mathrm{Aut}(C)|$ is divisible by 12 is as follows ([Bars] Theorem 16 p.10) — note that any non-hyperelliptic curve of genus 3 is realized as a quadric in $\mathbb{P}^2$:

Aut	$\lvert\mathrm{Aut}\rvert$	C
$\mathfrak{S}_4$	24	NH1 : $x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0$, where $t \in \mathbb{C} \setminus \{0, \frac{-3\pm3\sqrt{-7}}{2}, \pm2, -1\}$; this is singular for $t = \pm2, -1$ (Lemma 7.1.6), Fermat for $t = 0$, Klein for $t = \frac{-3\pm3\sqrt{-7}}{2}$.
$\mathbb{Z}_4 \circledcirc \mathfrak{A}_4 \cong SL_2(\mathbb{F}_3) \rtimes \mathbb{Z}_2$	48	NH2 : $x^4 + y^4 + z^3x = 0$
$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathfrak{S}_3$	96	NH3 : $x^4 + y^4 + z^4 = 0$ (Fermat curve)
$PSL_2(\mathbb{F}_7)$	168	NH4 : $z^3y + y^3x + x^3z = 0$ (Klein curve)

Table 6.1.2: $A \rtimes B$ is the semidirect product of A and B. For $\mathbb{Z}_4 \circledcirc \mathfrak{A}_4$, see [Bars] p.10. NOTE: Both $\mathbb{Z}_4 \circledcirc \mathfrak{A}_4$ and $SL_2(\mathbb{F}_3) \rtimes \mathbb{Z}_2$ have the same identification number of finite group: "GAP Id. $[48, 33]$" ([Bars] p.10, [PaWi] p.9), so they are isomorphic.

Here:

- NH1 is an $\mathfrak{A}_4$-curve, as $\mathfrak{A}_4 \subset \mathrm{Aut}(\mathrm{NH1})\,(= \mathfrak{S}_4)$.
- NH2 is excluded, as $\mathfrak{A}_4 \not\subset \mathrm{Aut}(\mathrm{NH2})\,(= SL_2(\mathbb{F}_3) \rtimes \mathbb{Z}_2)$; see [PaWi] p.9.
- NH3 and NH4 are special cases of NH1 at the value of $t = 0$ and $t = \frac{-3\pm 3\sqrt{-7}}{2}$ respectively. In fact, NH1 for $t = 0$ is NH3, and NH1 for $t = \frac{-3\pm 3\sqrt{-7}}{2}$ is isomorphic to NH4 ([KuSe] p.121 Theorem 2).

This confirms (ii) of Theorem 6.1.1.

Supplement: Technical lemmas on groups

Lemma 6.1.2. *For $U_6 := \langle a, b \,:\, a^2 = b^{12} = abab^7 = 1\rangle$, the following hold:*

(i) $ba = ab^5$.

(ii) *Any element of U_6 is written as b^k or ab^k $(k = 0, 1, \ldots, 11)$. Consequently*
$U_6 = \{b^k, ab^k : k = 0, 1, \ldots, 11\}$.

(iii) *Any subgroup $H \subset U_6$ of order 12 is normal in U_6 and $U_6/H \cong \mathbb{Z}_2$. Moreover $b^2 \in H$.*

Proof. (i): The relation $abab^7 = 1$ is rewritten as $ba = a^{-1}b^{-7}$. Here $a^{-1} = a$ and $b^{-7} = b^5$ (as $a^2 = b^{12} = 1$), thus $ba = ab^5$.
(ii): Use (i).
(iii): Since $|U_6| = 24$ and $|H| = 12$, H is of index 2 in U_6, so normal. We show that $b^2 \in H$. If $b \in H$, this is trivial. If $b \notin H$, then b determines the generator $\overline{b}$ of $U_6/H \cong \mathbb{Z}_2$, so $\overline{b}^2 = 1$, thus $b^2 \in H$. □

Lemma 6.1.3. (1) *$\mathfrak{A}_4$ is "not" a subgroup of D_{12}.*

(2) *$\mathfrak{A}_4$ is "not" a subgroup of U_6.*

Proof. (1): Since $|\mathfrak{A}_4| = |D_{12}|\,(= 12)$, if $\mathfrak{A}_4 \subset D_{12}$ then $\mathfrak{A}_4 = D_{12}$, which contradicts the fact that D_{12} has an element of order 6 while $\mathfrak{A}_4$ does not (because the order of any element of $\mathfrak{A}_4$ is either 1, 2 or 3).

(2): Since $|\mathfrak{A}_4| = 12$, if $\mathfrak{A}_4 \subset U_6$ then $b^2 \in \mathfrak{A}_4$ (Lemma 6.1.2 (iii)). The order of this element is 6, which is a contradiction. □

6.2 Proof of main results in Part II

Unless otherwise mentioned, all curves are assumed to be of genus 3. The tetrahedral group $\mathfrak{T}$ naturally acts on the cable surface Σ of the tetrahedron. By Kerckhoff's theorem [Ker], we may give a complex structure on Σ such that this action is holomorphic. Recall that $\mathfrak{T} \cong \mathfrak{A}_4$, so Σ is an $\mathfrak{A}_4$-curve. Its branch data on Σ is $(0; 2, 2, 3, 3)$ (Proposition 5.1.5). An $\mathfrak{A}_4$-curve is said to be *of tetra type* if the $\mathfrak{A}_4$-action is topologically equivalent to the standard tetrahedral group action on Σ.

Definition 6.2.1. An $\mathfrak{A}_4$-curve of tetra type is called a *tetra curve.*

We determine all tetra curves, in fact we show that B and C_t ($t \in \mathbb{C} \setminus \{\pm 2, -1\}$) in Theorem 6.1.1 exhaust all tetra curves. This is a consequence of a chain of claims:

Claim I *C_t for some t is a tetra curve.*

Proof. If none of C_t is a tetra curve, then *only* B is a tetra curve, which however cannot occur due to the non-rigidity of a tetra surface (Corollary 5.1.6). □

From the non-rigidity of a tetra surface (Corollary 5.1.6), the following holds:

Claim II *Let C_{t_0} be a tetra curve, then there exists an open neighborhood U of t_0 in $\mathbb{C} \setminus \{\pm 2, -1\}$ such that for any $t \in U$, C_t is a tetra curve.*

In fact every C_t ($t \in \mathbb{C} \setminus \{\pm 2, -1\}$) is a tetra curve. To show this, we need preparation. Let S be the complex surface in $\mathbb{P}^2 \times \mathbb{C}$ defined by

$$S := \Big\{ \big([x : y : z],\, t\big) \in \mathbb{P}^2 \times \mathbb{C} : x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0 \Big\},$$

and let $p : S \to \mathbb{C}$ be the projection $\big([x : y : z],\, t\big) \mapsto t$; and $C_t = p^{-1}(t)$.

Lemma 6.2.2. (1) *$M := S \setminus (C_{\pm 2} \cup C_{-1})$ is non-singular.* (2) *Set $\Omega := \mathbb{C} \setminus \{\pm 2, -1\}$, then the restriction $p : M \to \Omega$ is a fibration.*

Proof. (1) follows from the fact that all of singular points of S lie on C_2 (Lemma 7.1.1 below). (2) is clear, because no degeneration occurs in $p : M \to \Omega$ (Lemma 7.1.6 below). □

Lemma 6.2.3. *For each point $b \in \Omega$, there exists an open neighborhood V of b in Ω such that the $\mathfrak{A}_4$-actions on all C_t ($t \in V$) are topologically the same.*

Proof. Since $p : M \to \Omega$ is a fibration, by the Ehresmann fibration theorem there exists a sufficiently small open neighborhood V of b in Ω such that the restriction $p : p^{-1}(V) \to V$ is diffeomorphically isomorphic to a projection $C \times V \to V$ (where $C = C_b$). The fiberwise $\mathfrak{A}_4$-action on $p^{-1}(V)$ corresponds to a fiberwise $\mathfrak{A}_4$-action on $C \times V$. Identifying the fiber $C \times \{t\}$ ($t \in V$) with C in an obvious way, we regard the $\mathfrak{A}_4$-action on $C \times \{t\}$ ($t \in V$) as a family of $\mathfrak{A}_4$-actions on a single C. This amounts to a family of injective homomorphisms $\iota_t : \mathfrak{A}_4 \to \mathrm{MCG}(C)$, where $\mathrm{MCG}(C)$ denotes the mapping class group of C. Since $\mathrm{MCG}(C)$ is discrete, ι_t must be constant. Therefore the $\mathfrak{A}_4$-actions on all C_t ($t \in V$) are topologically the same. □

Now we can show:

Claim III *Every C_t ($t \in \Omega$) is a tetra curve.*

Proof. It suffices to show that the $\mathfrak{A}_4$-actions on all C_t ($t \in \Omega$) are topologically the same. Take the open neighborhood U in Claim II as a *maximal* one. We claim that U is the whole of Ω. Otherwise there is a boundary point (say b) of U in Ω. By Lemma 6.2.3, there exists an open neighborhood V of

b in Ω such that the $\mathfrak{A}_4$-actions on all C_t $(t \in V)$ are topologically the same. So $V \subset U$, which contradicts the fact that $b \notin U$. $\square$

Our next task is to show that B is also a tetra curve. Let Δ be a sufficiently small disk centered at $t = 2$ in $\mathbb{C}$ and set $W := p^{-1}(\Delta)$. Consider the restriction $p : W \to \Delta$ of $p : S \to \mathbb{C}$ around the singular fiber $C_2 = p^{-1}(2)$ $(= 2\mathbb{P}^1$; see Lemma 7.1.4). After showing that B arises as the central fiber of a stable reduction of $p : W \to \Delta$, we will show that B is a tetra curve. Note first that W has eight isolated singularities, which lie on C_2 and exhaust all singularities of S (see Lemma 7.1.1 below). These eight singularities are A_1-singularities (see Lemma 7.1.3 below).

Now let $p' : W' \to \Delta$ be the family obtained from $p : W \to \Delta$ by the base change $t - 2 = s^2$, where explicitly

$$W' := \{([x : y : z], s) \in \mathbb{P}^2 \times \Delta \,:\, x^4 + y^4 + z^4 + (s^2 + 2)(x^2y^2 + y^2z^2 + z^2x^2) = 0\}.$$

The central fiber $p'^{-1}(0)$ of $p' : W' \to \Delta$ is identical to $p^{-1}(2)\,(= C_2)$, so $p'^{-1}(0) \cong \mathbb{P}^1$. Here W' is singular in codimension 1 (W' is 'bent' along $p'^{-1}(0)$), and so non-normal. Let $\nu : N \to W'$ be the normalization of W'. Then $p'' := p' \circ \nu : N \to \Delta$ is a (non-degenerating) family of smooth curves, which is the *stable reduction* of $p : W \to \Delta$.

$$\begin{array}{ccccc} N & \xrightarrow{\nu} & W' & & W \\ & \overset{p''}{\searrow} & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ & & \Delta & \xrightarrow[\text{change}]{\text{base}} & \Delta. \end{array} \tag{6.2.1}$$

Lemma 6.2.4. *Let $r_1, r_2, \ldots, r_8 \in p^{-1}(0)$ be the eight singularities of W and $r'_1, r'_2, \ldots, r'_8 \in p'^{-1}(0)$ be the corresponding points of W' under the identification of $p^{-1}(0)$ with $p'^{-1}(0)$. Then the restriction $\nu : p''^{-1}(0) \to p'^{-1}(0)\,(\cong \mathbb{P}^1)$ of $\nu : N \to W'$ is a double covering with eight branch points $r'_1, r'_2, \ldots, r'_8$.*

Proof. For each point $q \in p^{-1}(0)$, let $q' \in p'^{-1}(0)$ denote the corresponding point under the identification of $p^{-1}(0)$ with $p'^{-1}(0)$. To show the assertion, we describe the normalization $\nu : N \to W'$ around each point $q' \in p'^{-1}(0)$. We separate into two cases depending on the position of q' (below, we take a coordinate of the disk Δ so that the center is the origin: $\Delta = \{T \in \mathbb{C} : |T| < 1\}$):

Case 1. $q' \in \{r'_1, r'_2, \ldots, r'_8\}$: In this case W is defined by $TX = Y^2$ around q (and $p : W \to \Delta$ is given by $(X, Y, T) \mapsto T$ around q). Here q corresponds to the origin $(X, Y, T) = (0, 0, 0)$. The base change $T \mapsto T^2$ turns $TX = Y^2$ to $T^2X = Y^2$, which is the defining equation of W' around q' (and $p' : W' \to \Delta$ is given by $(X, Y, T) \mapsto T$ around q'). Here q' corresponds to the origin $(X, Y, T) = (0, 0, 0)$. Let $W_{q'}$ be a sufficiently small neighborhood of q'. Then $W_{q'} \cap p'^{-1}(0)$, given by $T = Y = 0$ (the X-axis), is the non-normal locus of $W_{q'}$. The normalization of $W_{q'}$ is given by $(u, v) \in \mathbb{C}^2 \mapsto (v^2, uv, u) \in W'_{q'}$; note that $(X, Y, T) = (v^2, uv, u)$ satisfies $T^2X = Y^2$, as $u^2v^2 = (uv)^2$. (Precisely speaking, we need to shrink $\mathbb{C}^2$ around the origin.) On the u-axis in $\mathbb{C}^2$, this normalization is given by $v \mapsto v^2$, which is a double covering over the origin q'. See Figure 6.2.1.

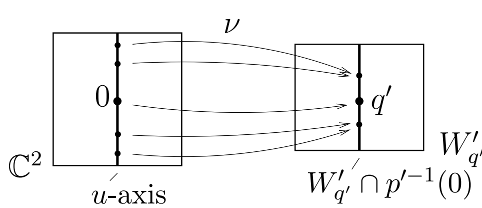


Figure 6.2.1: The restriction of $\nu : \mathbb{C}^2 \to W'_{q'}$ to the u-axis is two-to-one outside $0 \in \mathbb{C}^2$ while ramified at 0.

Case 2. $q' \in p'^{-1}(0) \setminus \{r'_1, r'_2, \ldots, r'_8\}$: In this case W is defined by $T = Y^2$ around q (and $p : W \to \Delta$ is given by $(X, Y, T) \mapsto T$ around q). Accordingly W' is defined by $T^2 = Y^2$ around q' (and $p' : W' \to \Delta$ is given by $(X, Y, T) \mapsto T$ around q'). Let $W_{q'}$ be a sufficiently small neighborhood of q'. Then $W_{q'} \cap p'^{-1}(0)$, given by $Y = T = 0$ (the X-axis), is the non-normal locus of $W_{q'}$. The normalization of $W_{q'} : T^2 = Y^2$ is explicitly given by $V_+ \amalg V_- \to W_{q'}$, where $V_+ := \{(u, v, w) \in \mathbb{C}^3 : w = v\}$ and $V_- := \{(u, v, w) \in \mathbb{C}^3 : w = -v\}$, and $\nu|_{V_+}$ and $\nu|_{V_-}$ are the 'identity' maps. (Precisely speaking, we need to shrink V_+ and V_- around the origins.) This normalization isomorphically maps the u-axis in V_+ and the u-axis in V_- to the X-axis in $W_{q'}$, which is an unramified double covering.

The descriptions in Case 1 and Case 2 together imply the assertion.

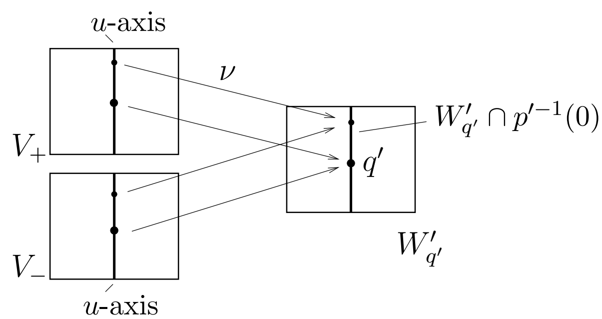


Figure 6.2.2: The restriction of $\nu : V_+ \amalg V_- \to W_{q'}$ to the u-axis in V_+ and the u-axis in V_- is two-to-one and unramified.

□

A genus 3 curve branched over $\mathbb{P}^1$ at eight points is necessarily hyperelliptic, and the double covering is the quotient under the hyperelliptic involution — the ramification points are the fixed points of hyperelliptic involution. Thus the following holds:

Corollary 6.2.5. *In Lemma 6.2.4, $p''^{-1}(0)$ is a hyperelliptic curve, and the eight ramification points of ν are the fixed points of its hyperelliptic involution ι, so that $p''^{-1}(0)/\iota = p'^{-1}(0)\,(= p^{-1}(0))$.*

We next show that the hyperelliptic curve $A := p''^{-1}(0)$ admits an $\mathfrak{A}_4$-action. Consider the commutative diagram:

$$\begin{array}{ccc} N & \longrightarrow & W \\ {\scriptstyle p''}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta & \longrightarrow & \Delta. \end{array} \tag{6.2.2}$$

Then the $\mathfrak{A}_4$-action on $W \setminus p^{-1}(0)$ lifts to an $\mathfrak{A}_4$-action on $N \setminus A$ such that it maps each fiber $p''^{-1}(u)$ $(u \neq 0)$ to itself (Remark 6.2.7 below). The commutativity of (6.2.2) implies that each fiber $p''^{-1}(u)$ $(u \neq 0)$ is isomorphic to C_s $(s = u^2)$, so from Claim III it is a tetra curve (equipped with the $\mathfrak{A}_4$-action). We thus obtain the following:

Lemma 6.2.6. (1) $A := p''^{-1}(0)$ *is a hyperelliptic curve.* (2) $p'' : N \setminus A \to \Delta \setminus \{0\}$ *is a family of smooth* $\mathfrak{A}_4$*-curves — the* $\mathfrak{A}_4$*-action on* $N \setminus A$ *maps each fiber* $p''^{-1}(u)$ $(u \neq 0)$ *to itself, and* $p''^{-1}(u)$ $(u \neq 0)$ *is a tetra curve.*

Remark 6.2.7. Any finite group action on a plane curve in $\mathbb{P}^2$ is the restriction of a projective linear action on $\mathbb{P}^2$, that is, the finite group acts as a subgroup of $PGL_3(\mathbb{C})$ ([Nam] Corollary 5.3.19 p.382). So in our context, the $\mathfrak{A}_4$-action on W is of the form $([x : y : z], s) \mapsto (g([x : y : z]), s)$, where $g \in PGL_3(\mathbb{C})$ (and $s := t - 2$). This action naturally defines an $\mathfrak{A}_4$-action on $N \setminus A$. Indeed as W is defined by $f([x : y : z], s) := x^4 + y^4 + z^4 + (s + 2)(x^2y^2 + y^2z^2 + z^2x^2) = 0$ in $\mathbb{P}^2 \times \Delta$, $N \setminus A$ is defined by $f([x : y : z], s^2) = 0$ in $\mathbb{P}^2 \times (\Delta \setminus \{0\})$, thus the $\mathfrak{A}_4$-action on W defines an $\mathfrak{A}_4$-action on $N \setminus A$. (Caution: N itself is not simply defined by $f([x : y : z], s^2) = 0$.)

The $\mathfrak{A}_4$-action on $N \setminus A$ uniquely extends to an $\mathfrak{A}_4$-action on N that maps $A = p''^{-1}(0)$ to itself (see Remark 6.2.8 below). In particular A is an $\mathfrak{A}_4$-curve. With Lemma 6.2.6 (1), *A is a hyperelliptic $\mathfrak{A}_4$-curve.* Such a curve is unique — it is B (see Theorem 6.1.1), thus $A = B$.

Remark 6.2.8. Let $\pi : M \to \Delta$ be a family of smooth curves and set $X := \pi^{-1}(0)$. Suppose that the restriction $\pi : M \setminus X \to \Delta \setminus \{0\}$ is a family of smooth G-curves (G: a finite group). Then the G-action on $M \setminus X$ uniquely extends to a G-action on M that maps X to itself (see [ACG] p.115).

We show that the $\mathfrak{A}_4$-actions on all fibers of $p'' : N \to \Delta$ are topologically equivalent. First by the Ehresmann fibration theorem, $p'' : N \to \Delta$ may be topologically considered as the projection $A \times \Delta \to \Delta$ (recall that Δ is a sufficiently small disk). Then applying the argument in the proof of Lemma

6.2.3 shows that the $\mathfrak{A}_4$-actions on all fibers of $p'' : N \to \Delta$ are topologically the same. Since the $\mathfrak{A}_4$-actions on all fibers of $p'' : N \setminus A \to \Delta \setminus \{0\}$ are of tetra type (Lemma 6.2.6 (2)), the $\mathfrak{A}_4$-action on A is also of tetra type. Thus $A\,(= B)$ is also a tetra curve.

We summarize the results so far obtained as follows:

Theorem 6.2.9. (1) *The tetra curves are exhausted by B and C_t ($t \in \mathbb{C} \setminus \{\pm 2, -1\}$).*

(2) *W (and S) has eight singularities and they lie on C_2 and all are A_1-singularities.*

(3) *Let $p'' : N \to \Delta$ be the stable reduction of $p : W \to \Delta$ via a base change $\Delta \to \Delta$, $t-2 \mapsto (t-2)^2$. Then the central fiber of p'' is B.*

Remark 6.2.10. The $\mathfrak{A}_4$-action on B (and C_t) corresponds to an embedding of $\mathfrak{A}_4$ (as a subgroup) into the mapping class group MCG_3 of a genus 3 curve. Then in MCG_3, $\mathfrak{A}_4$ and the hyperelliptic involution ι commute, which follows from the commutativity of the $\mathfrak{A}_4$-action and the $\mathbb{Z}_2$-action ($\mathbb{Z}_2 = \langle \iota \rangle$) on $N \setminus A$; see the paragraph above Lemma 6.2.6.

Now let $\mathfrak{r} : M \to W$ be the minimal resolution of the eight A_1-singularities $r_1, r_2, \ldots, r_8$ (in Lemma 6.2.4), where each $E_i := \mathfrak{r}^{-1}(r_i)$ ($i = 1, 2, \ldots, 8$) is $\mathbb{P}^1$ with self-intersection number -2, that is, E_i is a (-2)-curve. The composition of $\mathfrak{r}$ with $p : W \to \Delta$ is a degeneration $\pi := p \circ \mathfrak{r} : M \to \Delta$ of smooth curves of genus 3, whose singular fiber is $2\mathbb{P}^1 + \sum_{i=1}^{8} E_i$ (Figure 6.2.3), where each E_i intersects $2\mathbb{P}^1$ transversally. The monodromy of $\pi : M \to \Delta$ is the hyperelliptic involution in Corollary 6.2.5.

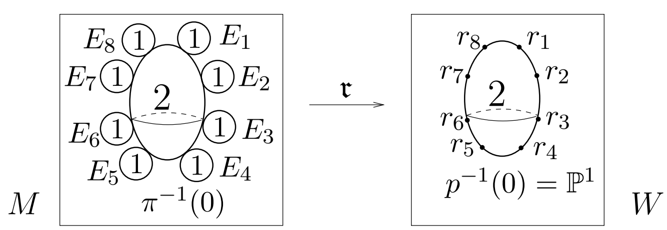


Figure 6.2.3: The minimal resolution $\mathfrak{r} : M \to W$.

We formalize the above as follows:

Proposition 6.2.11. *Let* $\mathfrak{r} : M \to W$ *be the minimal resolution of the eight* A_1*-singularities* $r_1, r_2, \ldots, r_8$ *of* W*; each* $E_i := \mathfrak{r}^{-1}(r_i)$ *is a* (-2)*-curve. Then the composition of* $\mathfrak{r}$ *with* $p : W \to \Delta$ *is a degeneration* $\pi := p \circ \mathfrak{r} : M \to \Delta$ *of smooth curves of genus* 3*, whose singular fiber is* $2\mathbb{P}^1 + \sum_{i=1}^{8} E_i$*, and the monodromy of* $\pi : M \to \Delta$ *is the hyperelliptic involution in Corollary 6.2.5.*

Chapter 7

Description of Singularities

7.1 The singularities and singular fibers of the $\mathfrak{A}_4$-family

Let S be the complex surface in $\mathbb{P}^2 \times \mathbb{C}$ defined by

$$S := \Big\{ \big([x:y:z],\, t\big) \in \mathbb{P}^2 \times \mathbb{C} : x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0 \Big\},$$

and let $p : S \to \mathbb{C}$ be the projection $\big([x:y:z],\, t\big) \mapsto t$ and $C_t := p^{-1}(t)$. The restriction of $p : S \to \mathbb{C}$ to $\mathbb{C} \setminus \{\pm 2, -1\}$ is the family of $\mathfrak{A}_4$-curves appearing in Theorem 6.1.1 (ii).

Lemma 7.1.1. *S has eight isolated singularities $\big([\pm\omega : \pm\omega^2 : 1], 2\big)$ and $\big([\pm\omega^2 : \pm\omega : 1], 2\big)$, where $\omega := e^{2\pi i/3}$, which lie on the fiber $C_2 = p^{-1}(2)$.*

Proof. Take an open covering $\mathbb{P}^2 = U \cup V \cup W$, where $U = \{z = 1\}$, $V = \{x = 1\}$, and $W = \{y = 1\}$. We show that the singularities of S lie on $(U \cap V \cap W) \times \mathbb{C}$ and they are $\big([\pm\omega : \pm\omega^2 : 1], 2\big)$ and $\big([\pm\omega^2 : \pm\omega : 1], 2\big)$.

We first determine the singularities of S on $U \times \mathbb{C}$. The defining equation of S on $U \times \mathbb{C}$ is given by $f(x, y, t) = x^4 + y^4 + 1 + t(x^2y^2 + x^2 + y^2)$. Let $q = (x, y, t) \in S|_{U \times \mathbb{C}}$, then

$$\text{(a)} \quad x^4 + y^4 + 1 + t(x^2y^2 + x^2 + y^2) = 0.$$

Suppose that q is a singularity, equivalently

$$\partial_x f(q) = \partial_y f(q) = \partial_t f(q) = 0 \quad \text{(Jacobi criterion)},$$

or explicitly

$$\begin{cases} \text{(b)} & x(4x^2 + 2t(y^2+1)) = 0, \\ \text{(c)} & y(4y^2 + 2t(x^2+1)) = 0, \\ \text{(d)} & x^2y^2 + x^2 + y^2 = 0. \end{cases}$$

We claim that $x \neq 0$ and $y \neq 0$. Indeed if $x = 0$ then (a) and (d) become (a)$'$ $y^4 + ty^2 + 1 = 0$ and (d)$'$ $y^2 = 0$, so $1 = 0$ (absurd!). Similarly if $y = 0$ then (a) and (d) yield a contradiction, so $y \neq 0$. Dividing now (b) by x and (c) by y yields

$$\begin{cases} \text{(b)}' & 4x^2 + 2t(y^2+1) = 0, \\ \text{(c)}' & 4y^2 + 2t(x^2+1) = 0. \end{cases}$$

We next claim that $x^2 \neq -1$ and $y^2 \neq -1$. If $x^2 = -1$ then (c)$'$ implies $y = 0$ (contradiction). Similarly if $y^2 = -1$ then (b)$'$ implies $x = 0$ (contradiction). Now eliminating t from (b)$'$ and (c)$'$ yields (e) $\dfrac{2x^2}{y^2+1} = \dfrac{2y^2}{x^2+1}$. From (d), $x^2 = \dfrac{-y^2}{y^2+1}$. Substituting this into (e) yields $y^2 = -2, \omega, \omega^2$, so $(x^2, y^2) = (-2,-2), (\omega, \omega^2), (\omega^2, \omega)$. Here the first one is excluded, as it does not satisfy (a). The others indeed satisfy all of (a), (b), (c), (d) for $t = 2$. Therefore the singularities of S on $U \times \mathbb{C}$ are $\big([\pm\omega : \pm\omega^2 : 1], 2\big)$ and $\big([\pm\omega^2 : \pm\omega : 1], 2\big)$.

Similarly the singularities of S on $V \times \mathbb{C}$ are $\big([1 : \pm\omega : \pm\omega^2], 2\big)$ and $\big([1 : \pm\omega^2 : \pm\omega], 2\big)$. They are 'equal' to $\big([\pm\omega : \pm\omega^2 : 1], 2\big)$ and $\big([\pm\omega^2 : \pm\omega : 1], 2\big)$ (projective coordinates!). Similarly the singularities of S on $W \times \mathbb{C}$ are $\big([\pm\omega^2 : 1 : \pm\omega], 2\big)$ and $\big([\pm\omega : 1 : \pm\omega^2], 2\big)$, and they are also 'equal' to $\big([\pm\omega : \pm\omega^2 : 1], 2\big)$ and $\big([\pm\omega^2 : \pm\omega : 1], 2\big)$. □

Remark 7.1.2. The eight points $[\pm\omega : \pm\omega^2 : 1], [\pm\omega^2 : \pm\omega : 1]$ on $\mathbb{P}^2$ are the base points of the pencil $\{C_t\}_{t\in\mathbb{P}^1}$.

Lemma 7.1.3. *All eight singularities* $\big([\pm\omega : \pm\omega^2 : 1], 2\big)$ *and* $\big([\pm\omega^2 : \pm\omega : 1], 2\big)$ *of* S *are an* A_1*-singularities.*

Proof. It suffices to check that the Hessian of $f(x, y, t) = x^4 + y^4 + 1 + t(x^2y^2 + x^2 + y^2)$ at each singularity is nonzero, that is, nondegenerate (as this is equivalent to the singularity being A_1; see, e.g. [Oka1]). The Hessian matrix of f is

$$H = \begin{pmatrix} 12x^2 + 2t(y^2+1) & 4txy & 2x(y^2+1) \\ 4txy & 12y^2 + 2t(x^2+1) & 2y(x^2+1) \\ 2x(y^2+1) & 2y(x^2+1) & 0 \end{pmatrix}.$$

At a singularity $(x, y, t) = (\omega, \omega^2, 2)$, H is given by

$$\begin{pmatrix} 12\omega^2 & 8 & -2 \\ 8 & 12\omega & -2 \\ -2 & -2 & 0 \end{pmatrix},$$

whose determinant is nonzero (indeed 16). Similarly for the other singularities, the Hessian is nonzero. □

We next determine the singular fibers of $p : S \to \mathbb{C}$.

Lemma 7.1.4. C_t *for* $t = \pm 2, -1$ *are reducible. In fact:*

(i) C_2 *is* $\mathbb{P}^1$ *of multiplicity* 2.

(ii) C_{-2} *consists of four* $\mathbb{P}^1$*'s and any two of them intersect at one point.*

(iii) C_{-1} *consists of two* $\mathbb{P}^1$*'s intersecting at four points.*

Proof. The defining equations of C_t for $t = \pm 2, -1$ factorize as follows (so C_t for $t = \pm 2, -1$ are reducible):
DE for $t = 2$:

$$x^4 + y^4 + z^4 + 2(x^2y^2 + y^2z^2 + z^2x^2) = (x^2 + y^2 + z^2)^2.$$

DE for $t = -2$:

$$x^4+y^4+z^4-2(x^2y^2+y^2z^2+z^2x^2) = (x+y+z)(x+y-z)(x-y+z)(x-y-z).$$

DE for $t = -1$: Where $\omega := e^{2\pi \mathrm{i}/3}$,

$$x^4 + y^4 + z^4 - (x^2y^2 + y^2z^2 + z^2x^2) = (x^2 + \omega y^2 + \omega^2 z^2)(x^2 + \omega^2 y^2 + \omega z^2).$$

Note that any factor of the above factorizations is linear or quadratic, so it defines $\mathbb{P}^1$. Thus each irreducible component of C_t for $t = \pm 2, -1$ is $\mathbb{P}^1$. The other assertions are immediate from the above factorizations. □

Lemma 7.1.5. *C_t for $t \neq \pm 2, -1$ is smooth.*

Proof. Set $F(x, y, z) := x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2)$. Then $[x : y : z] \in C_t$ is a singularity if and only if $\partial_x F = \partial_y F = \partial_z F = 0$, or explicitly

$$x\big(2x^2 + t(y^2 + z^2)\big) = y\big(2y^2 + t(z^2 + x^2)\big) = z\big(2z^2 + t(x^2 + y^2)\big) = 0. \quad (7.1.1)$$

We separate into two cases:
Case 1 $xyz \neq 0$: Then (7.1.1) is simplified into

$$2x^2 + t(y^2 + z^2) = 2y^2 + t(z^2 + x^2) = 2z^2 + t(x^2 + y^2) = 0.$$

Thus $\begin{pmatrix} 2 & t & t \\ t & 2 & t \\ t & t & 2 \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} = 0$. This has a nontrivial solution precisely when $\begin{vmatrix} 2 & t & t \\ t & 2 & t \\ t & t & 2 \end{vmatrix} = 0$, that is, $2t^3 - 6t^2 + 8 = 0$, so $t = 2, -1$.
Case 2 $xyz = 0$: Then *no* two of x, y, z can be 0 (for instance if $x = y = 0$, then from (7.1.1), $z = 0$, so $x = y = z = 0$, which contradicts $[x : y : z] \in \mathbb{P}^2$). We may thus assume that $x = 0$ and $yz \neq 0$. Then $2y^2 + tz^2 = 2z^2 + ty^2 = 0$, so $\begin{pmatrix} 2 & t \\ t & 2 \end{pmatrix} \begin{pmatrix} y^2 \\ z^2 \end{pmatrix} = 0$. This has a nontrivial solution precisely when $\begin{vmatrix} 2 & t \\ t & 2 \end{vmatrix} = 0$, that is, $-t^2 + 4 = 0$, so $t = \pm 2$.

We thus conclude that C_t is singular if and only if $t = \pm 2, -1$. □

By Lemmas 7.1.4 and 7.1.5, the following is obtained:

Lemma 7.1.6. C_t $(t \in \mathbb{C})$ *is singular precisely when* $t = \pm 2, -1$: *the singular curves* C_2, C_{-2}, C_{-1} *are explicitly described in Lemma 7.1.4.*

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