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VECTOR BUNDLES, ISOPARAMETRIC FUNCTIONS AND RADON TRANSFORMS ON SYMMETRIC SPACES

Dedicated to the memory of Professor Tsunero Takahashi

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Abstract

We systematically construct isoparametric functions on compact symmetric spaces using vector bundles and sections of the bundles. We establish a relation between invariants of vector bundles and invariants of hypersurfaces which are the level sets of the isoparametric functions induced by sections of the bundles. We hope that this approach provides a new method for computing invariants of hypersurfaces. The Radon transform is performed to derive isoparametric functions on spheres from our functions.

1. Introduction

One of the main purposes in the present paper is to construct isoparametric functions on symmetric spaces of compact type systematically. The research of an isoparametric hypersurface, which is the regular level set of an isoparametric function, has a long history, going back to Levi-Civita and É.Cartan. We have a lot of literatures about isoparametric hypersurfaces of spaces of constant curvatures, which have constant principal curvatures. We denote by g the number of distinct principal curvatures. Amongst all, the research of an isoparametric hypersurface of a sphere is extensive and well-known. Substantial results are exhibited in [1], [3], [9], [13] and [14], etc. In [9], Münzner shows that $g = 1, 2, 3, 4, 6$ and in [3], a lot of isoparamertric functions on a sphere are systematically constructed by an algebraic method, which are called isoparametric functions of OT-FKM type. By contrast, we have few explicit examples of isoparametric functions on general Riemannian manifolds.

We utilise a homogeneous vector bundle and a section to construct an isoparametric function on an irreducible symmetric space, say G/K. To choose a vector bundle and a section, we consider an irreducible G-module W of spherical type. This means that the principal orbits are hyperspheres of W and so, we obtain a subgroup $H \subset G$ as a stabilizer. Though this hypothesis adds restriction to a number of considerable pairs $(G/K, W)$ (Table 3.2), the reason of the choice will be clear in the last section. If the representation is restricted to a subgroup K, then K-submodules of W induce homogeneous vector bundles over G/K.

Needless to say, a relation between the zero locus of a section of a vector bundle and the vector bundle itself is deeply understood in topology and algebraic geometry. In this paper, the zero locus of a section also has prominent features, see §3. In particular, the zero locus turns out to be a totally geodesic submanifold of G/K (Theorem 3.6). However, another

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interplay of vector bundles and submanifolds will be established in the next section, which is one of our purposes in the present paper.

In §4, we develop geometry of submanifolds of symmetric spaces involving vector bundles and sections. Frobenius reciprocity makes it possible that W is regarded as a space of sections and the chosen section is really an eigensection of the Laplace operator acting on sections. Using an invariant metric on the bundle, we define a function $f : G/K \to \mathbf{R}$ as the square of the norm of the section. The function has a symmetry, in other words, the function has an invariance under the action of an isometry group $H \subset G$ of G/K, because the section has the same symmetry. In addition, we show that the function satisfies the condition (2) (Theorem 4.3) in the definition of an isoparametric function (Definition 2.1).

If the action of H on G/K is of cohomogeneity one, then it is clear that f is an isoparametric function by a symmetry. In this case, we compute $|df|^2$ explicitly (Theorem 4.13) to check that the function satisfies the condition (1) in Definition 2.1. Though we already see that f satisfies the condition (2) in Definition 2.1, we subtract an appropriate constant from f to obtain an eigenfunction denoted by $\tilde{f}$ (see the Remark after Theorem 4.13).

Next, the mean curvature of the level hypersurface is also computed (Theorem 4.14). We have common description of $|df|^2$ and the mean curvature on any pairs $(G/K, W)$. As a by-product, we can specify the precise value whose inverse image of f is a minimal hypersurface in a family given by the isoparametric function. On the contrary, when we compute the principal curvature, we have distinct difference between pairs and no unified way (Theorems 4.17, 4.18, 4.19, 4.20 and 4.21). These invariants of submanifolds are related to invariants of vector bundles and sections involving the eigenvalues and the dimension of the eigenspaces. In those computations, the second fundamental forms of vector bundles [8] play essential roles and the theory developed by the first author in [11] provides us with a unified method. From this viewpoint, the present paper can be considered as a sequel of [11], where we focus attention on all sections in W, but in this paper, we pay our attention on a section in W.

If the cohomogeneity of the action of H on G/K is greater than one, then the function $f : G/K \to \mathbf{R}$ is not an isoparametric function. However, we can construct a new isoparametric function $F : G/K \to \mathbf{R}^k$ in the sense of Wang [18] (see also [2, p.55]), where k denotes the cohomogeneity of H-action. One component of F consists of the function f. In the case that a chosen pair is $(\mathrm{Sp}(n)/\mathrm{U}(n), \mathbf{C}^{2n})$, F coincides with a moment map for an Sp(1)-action on $\mathrm{Sp}(n)/\mathrm{U}(n)$.

Moreover, we can find a new isoparametric function $\tilde{f} : G/K \to \mathbf{R}$. The function $\tilde{f}$ has a larger symmetry than the original f. In short, a subgroup $\tilde{H} \subset G$ such that $H \subset \tilde{H}$ enters into our theory and $\tilde{f}$ is invariant under the action of $\tilde{H}$. The appearance of $\tilde{f}$ and $\tilde{H}$ is not accidental. We use other vector bundles and spaces of sections to explain in an algebraic and geometric way that the chosen section in §3 has really a hidden symmetry $\tilde{H} \subset G$. The $\tilde{H}$-action on G/K turns out to be of cohomogeneity one. The relation between $\tilde{f}$ and F makes some properties of H-action and level sets of F transparent. In particular, any submanifold in our family induced by F is *not* an equifocal submanifold. Equifocal submanifolds are one of generalizations of isoparametric hypersurfaces, see Terng-Thorbergsson [16]. In case of codimension one, isoparametric hypersurfaces are all equifocal and the converse is also true. Since we adopt Wang's definition of an isoparametric function, we obtain submanifolds of higher codimension, which are not equifocal. Terng's definition of an isoparametric function [15] gives us a deep structural theory and close relations to equifocal submanifolds. Since our $F : G/K \to \mathbf{R}^k$ does not satisfy Terng's conditions, we mainly focus our attention on cases of hypersurfaces.

In the final section, we interpret the reason that representations of spherical type are chosen. One of our aims in the present paper is to provide a geometric mean of constructing an isoparametric function on a sphere. By our assumption, the quotient space of G by H is a sphere $S^{N-1} \subset W$. Hence, we have a double fibration $\pi : G \to G/K$ and $\psi : G \to S^{N-1}$. Then we can define a Radon transform $R : C^\infty(G/K) \to C^\infty(S^{N-1})$ using the normalized Haar measure on H:

$$R(f)(x) = \int_{\psi^{-1}(x)} \pi^* f \, d\mu, \quad x \in S^{N-1}.$$

Then the Radon transform of $\tilde{f}$ turns out to be an isoparametric function on a sphere. More precisely, in the case that the H-action is of cohomogeneity one, $R(\tilde{f})$ is an isoparametric function corresponding to an isoparametric hypersurface with $g = 2$. If the cohomogeneity of H-action is greater than one, then $R(\tilde{f})$ is an isoparametric function whose regular hypersurface is an isoparametric hypersurface with $g = 4$.

Finally note that we have another fibrations $\tilde{\psi} : S^{N-1} \to G/\tilde{H}$, which are all (generalized) Hopf fibrations. In a similar way, we can define a Radon transform $\tilde{R} : C^\infty(G/K) \to C^\infty(G/\tilde{H})$. Then we can show that $\tilde{R}(\tilde{f})$ is also an isoparametric function on $G/\tilde{H}$ (Theorem 5.6), because the fibration $\tilde{\psi} : S^{N-1} \to G/\tilde{H}$ has totally geodesic fibres and $\tilde{\psi}^* \tilde{R}(\tilde{f}) = R(\tilde{f})$.

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2. Preliminaries

2.1. Isoparametric functions. First of all, we give a definition of an isoparametic function on a Riemannian manifold in this paper.

Definition 2.1. Let $f : M \to \mathbf{R}$ be a function on a Riemannian manifold (M, g_M). The function f is called an *isoparametric function* if there exist functions $F, G : \mathbf{R} \to \mathbf{R}$ such that

$$(1)\ g_M(df, df) = F(f), \quad (2)\ \Delta f = G(f).$$

The regular level set of an isoparametric function is called an *isoparametric hypersurface*. We recommend [17] for a review of isoparametric hypersurfaces.

Amongst isoparametric hypersurfaces, an isoparametric hypersurface of a sphere is well-known and has been researched for a long time. An isoparametric hypersurface of a sphere has g distinct constant principal curvatures, where $g = 1, 2, 3, 4, 6$ [9]. We give examples of isoparametric functions on a sphere.

Example. ($g = 2$) Let $S^{N-1} \subset \mathbf{R}^N$ be a unit sphere. If we denote a standard coordinate functions on $\mathbf{R}^N$ by $(x_1, \cdots, x_N)$, then

$$\frac{1}{N}\left\{q\sum_{i=1}^{p}x_i^2 - p\sum_{\alpha=1}^{q}x_\alpha^2\right\},$$

where $2 \leqq p \leqq N-2$ and $p+q=N$, is an isoparametric function. The regular level set is identified with $S^{p-1}\times S^{q-1}$.

Each isoparametic hypersurface with $g=1,2,3$ is homogeneous in the sense that it is one of orbits of an isometry group of a sphere. Such homogeneous isoparametric hypersurfaces of a sphere are completely classified in Takagi-Takahashi [14] using a result in Hsiang-Lawson [7]. However, there exist a lot of examples of non-homogeneous isoparametric hypersurfaces of a sphere with $g=4$.

First of all, Nomizu [12] found an isoparametric function with $g=4$.

Example. ($g=4$) Let $S^{2N-1}\subset \mathbf{C}^N$ ($N \geqq 3$) be a unit sphere. If a standard coordinate functions on $\mathbf{C}^N$ are denoted by $(x_1+iy_1,\cdots,x_N+iy_N)$, then

$$\left(\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}y_i^2\right)^2+4\left(\sum_{i=1}^{n}x_iy_i\right)^2$$

is an isoparametric function.

The regular level set is homogeneous in this example.

Ozeki and Takeuchi [13] gave first examples of non-homogeneous isoparametric hypersurfaces with $g=4$ and Ferus, Karcher and Münzner systematically constructed such hypersurfaces [3], which are nowadays called of OT-FKM type.

2.2. Geometry of Grassmannian. Next, we review geometry of Grassmannian manifolds, in order to fix notation and our convention in this paper. For proofs, see [11].

Let W be an N-dimensional vector space. In the case that W is a real vector space, we also consider the orientation of W.

Let $Gr_p(W)$ be a Grassmannian manifold of (oriented) p-planes in W and $S\to Gr_p(W)$ a tautological vector bundle. Since $S\to Gr_p(W)$ is regarded as a subbundle of a trivial vector bundle $\underline{W}\to Gr_p(W)$ of fibre W, we have an exact sequence of vector bundles:

$$0\to S\xrightarrow{i_S}\underline{W}\xrightarrow{\pi_Q}Q\to 0.$$

The quotient bundle $Q\to Gr_p(W)$ is called the *universal quotient bundle*. The tangent bundle is identified with $S^*\otimes Q$. (More precisely, the holomorphic tangent bundle is identified with $S^*\otimes Q$ in case of complex Grassmannian.)

We fix a scalar product $(\cdot,\cdot)$ on W. On the one hand, the orthogonal projection gives a bundle surjection $\pi_S:\underline{W}\to S$. On the other hand, $Q\to Gr_p(W)$ is regarded as the orthogonal complementary bundle $S^\perp\to Gr_p(W)$ to $S\to Gr_p(W)$, and so we obtain a bundle injection $i_Q:Q\to\underline{W}$. The vector bundles $S\to Gr_p(W)$ and $Q\to Gr_p(W)$ are equipped with metrics g_S and g_Q, respectively.

We can define a connection ∇^Q on $Q\to Gr_p(W)$ using a trivialization of $\underline{W}\to Gr_p(W)$ with an orthonormal basis. If t is a section of $Q\to Gr_p(W)$, then $i_Q(t)$ is considered as a W-valued function. Then we have

$$d\left(i_Q(t)\right) = \pi_S\left(d\left(i_Q(t)\right)\right) + \pi_Q\left(d\left(i_Q(t)\right)\right).$$

The connection $\nabla^Q t = \pi_Q\left(d\left(i_Q(t)\right)\right)$ is nothing but the canonical connection. The other term in right hand side $\pi_S\left(d\left(i_Q(t)\right)\right)$ is a 1-form with values in $\mathrm{Hom}(Q, S) \cong Q^* \otimes S$ which is called *the second fundamental form* in the sense of Kobayashi [8] and denoted by J.

In a similar way, if s is a section of $S \to Gr_p(W)$, then we have

$$d\left(i_S(s)\right) = \pi_S\left(d\left(i_S(s)\right)\right) + \pi_Q\left(d\left(i_S(s)\right)\right).$$

The canonical connection is expressed as $\nabla^S s = \pi_S\left(d\left(i_S(s)\right)\right)$ and we define the second fundamental form $I = \pi_Q d i_S$, which is a 1-form with values in $\mathrm{Hom}(S, Q) \cong S^* \otimes Q$.

In the case of a complex Grassmannian, we can also consider complex analytical structures. Canonical connections give holomorphic structures to $S \to Gr_p(W)$ and $Q \to Gr_p(W)$. In particular, W can be regarded as the space of holomorphic sections of $Q \to Gr_p(W)$ by a theorem of Borel-Weil. The second fundamental form $I \in \Omega^1(\mathrm{Hom}(S, Q))$ is of type $(1, 0)$ and The second fundamental form $J \in \Omega^1(\mathrm{Hom}(Q, S))$ is of type $(0, 1)$.

Since the (holomorphic) tangent bundle is identified with $S^* \otimes Q$, we can induce a Riemannian metric g_{Gr} on a Grassmannian.

• **Real case.** We have

$$g_{Gr}(X, Y) = -\mathrm{trace}\, J_Y I_X = -\mathrm{trace}\, I_Y J_X,$$

where X and Y are tangent vectors.

• **Complex case.** Let h_{Gr} be the Hermitian metric on the holomorphic tangent bundle $T_{1,0}$ induced by Hermitian metrics g_S and g_Q. The definition yields that

$$h_{Gr}(Z, W) = -\mathrm{trace}\, J_{\overline{W}} I_Z,$$

where Z and W are $(1, 0)$-vectors. Consequently we have

$$\begin{aligned} g_{Gr}(X, Y) &= -\,\mathrm{trace}\, J_Y I_X - \mathrm{trace}\, J_X I_Y \\ &= -\,\mathrm{trace}\, I_Y J_X - \mathrm{trace}\, I_X J_Y, \end{aligned}$$

where X and Y are (real) tangent vectors.

The Levi-Civita connection D is nothing but a connection induced by ∇^S and ∇^Q.

Proposition 2.2. *The second fundamental forms I and J are parallel.*

For a vector $w \in W$, we have two sections $s = \pi_S(w)$ and $t = \pi_Q(w)$, each of which is sometimes called *the section corresponding to w*. Obviously, we have

Proposition 2.3. *If s and t are the sections corresponding to $w \in W$, then*

$$\nabla^S s = -Jt, \quad \nabla^Q t = -Is.$$

Lemma 2.4. *The second fundamental forms I and J satisfy*

$$g_Q(Is, t) = -g_S(s, Jt).$$

We now can easily compute $(\nabla^S)^2$ and $(\nabla^Q)^2$. If s and t are the corresponding sections to $w \in W$, then we have

$$(\nabla^S)^2 s = \nabla^S(-Jt) = -(\nabla J)(t) - J(\nabla^Q t) = JIs,$$

$$(\nabla^Q)^2 t = \nabla^S(-Is) = -(\nabla I)(s) - I(\nabla^S s) = IJt.$$

More precisely, we have

$$\nabla^S_X(\nabla^S s)(Y) = J_Y I_X s, \quad \nabla^Q_X(\nabla^Q t)(Y) = I_Y J_X t.$$

For instance, we take the trace of $(\nabla^S)^2$ to define the Laplace operator: $\Delta s = -\sum_{i=1}^n \nabla^S_{e_i}(\nabla^S s)(e_i)$. We see that sections s and t are eigensections of the Laplacian ($\Delta s = qs$, $\Delta t = pt$, where $q = N - p$).

2.3. Totally geodesic immersions into Grassmannians. Let (G, K) be an irreducible symmetric pair of compact type, where G is a simply-connected compact Lie group and K is a closed subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebras. The standard decomposition is expressed as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.

Let $\rho : G \to GL(W)$ be an irreducible representation with an G-invariant scalar product. For simplicity, we do not distinguish a representation $\rho : G \to GL(W)$ from the representation space W. We assume that W has a non-trivial K-invariant orthogonal decomposition $W = U \oplus V$ such that $\mathfrak{m}U \subset V$ and $\mathfrak{m}V \subset U$. (Non-trivial decomposition means that neither U nor V is zero-dimensional.) Such a decomposition is called a *generalised Cartan decomposition* of W. More generally, we define

Definition 2.5. Let $\varrho : G \to GL(W)$ be an orthogonal or unitary representation of G. The (ϱ, W) has a generalised Cartan decomposition (for the symmetric pair (G, K)) if W is decomposed into two non-zero K-modules $W = U_0 \oplus V_0$ over the same coefficient field as that of W under the restriction of the homomorphism ϱ to a subgroup K, in such a way that

$$\varrho(\mathfrak{m})U_0 \subset V_0, \quad \varrho(\mathfrak{m})V_0 \subset U_0, \quad U_0 \perp V_0,$$

and neither U_0 or V_0 is a G-module (in other words, $\varrho(\mathfrak{m})U_0 \neq \{0\}$ and $\varrho(\mathfrak{m})V_0 \neq \{0\}$). The decomposition $W = U_0 \oplus V_0$ is called a generalised Cartan decomposition, more accurately, a real generalised Cartan decomposition or a complex generalised Cartan decomposition according to the coefficient field of W.

Assume that W has a generalised Cartan decomposition : $W = U \oplus V$. Let $\dim U = p$ and $\dim V = q$. We define an immersion $i : G/K \to Gr_p(W)$ by

$$i(gK) = \varrho(g)U, \quad g \in G.$$

We assume throughout this paper that a Riemannian metric on G/K is provided in such a way that the immersion $i : G/K \to Gr_p(W)$ is an *isometric* immersion. Then $i : G/K \to Gr_p(W)$ is indeed a totally geodesic immersion.

We can define two homogeneous vector bundles $G \times_K U$ and $G \times_K V$ with canonical connections, which are denoted by $\mathbf{U} \to G/K$ and $\mathbf{V} \to G/K$. Frobenius reciprocity yields that W can be regarded as a finite dimensional space of sections of $\mathbf{U} \to G/K$ and $\mathbf{V} \to G/K$. More precisely, $\pi_U : W \to U$ and $\pi_V : W \to V$ denote the orthogonal projections. For $w \in W$, we put

$$s([g]) := \left[g, \pi_U(g^{-1}w)\right], \quad t([g]) := \left[g, \pi_V(g^{-1}w)\right],$$

where $g \in G$ and $[g] \in G/K$. The sections $s \in \Gamma(U)$ and $t \in \Gamma(V)$ are also called the *corresponding sections to $w \in W$.*

From the construction, $\mathbf{U} \to G/K$ and $\mathbf{V} \to G/K$ are pull-back bundles of the tautological bundle and the universal quotient bundle over $Gr_p(W)$, respectively. Then the pull-back connections are the same as the canonical connections. We can also pull-back the second fundamental forms I and J which are sections of $i^*T^* \otimes \mathrm{Hom}(\mathbf{U}, \mathbf{V})$ and $i^*T^* \otimes \mathrm{Hom}(\mathbf{V}, \mathbf{U})$, respectively, where T^* is the cotangent bundle of Grassmannian. Using the projection $i^*T^* \to T^*G/K$, the pull-backs of I and J are the second fundamental forms of vector bundles, and so we denote by the same symbol the pull-backs of the second fundamental forms.

Theorem 2.6 ([11, Lemma 4.1])**.** *A map $f : G/K \to Gr_p(W)$ is totally geodesic (i.e.$\nabla df = 0$) if and only if the second fundamental form I of vector bundles is parallel.*

Proof. Since we have a fundamental relation $\nabla I = I_{\nabla df}$, the result follows. □

We define an endomorphism $A \in \Gamma(\mathrm{End}(\mathbf{V}))$ by

$$A = \sum_{i=1}^{n} I_{e_i} J_{e_i}, \quad n = \dim G/K,$$

where $e_1, \cdots, e_n$ is an orthonormal basis of the tangent space of G/K. We call A the *mean curvature operator*. Notice that A can be defined in a similar way, even if the domain is a Riemannian manifold [11]. Then we have

Theorem 2.7 ([11, Theorem 3.5])**.** *Let (M, g) be an n-dimensional Riemannian manifold and $F : M \to Gr_p(W)$ a smooth map. We fix an inner product or a Hermitian inner product $(\cdot, \cdot)$ on W.*

Then, the following two conditions are equivalent.

(1) *$F : M \to Gr_p(W)$ is a harmonic map.*
(2) *$\Delta t + At = 0$ for an arbitrary $t \in W$, where the vector space W is regarded as a space of sections of the pull-back bundle $F^*Q \to M$.*

Under these conditions, we have

$$|df|^2 = -\mathrm{trace}\, A.$$

The role of the universal quotient bundle in Theorem 2.7 can be replaced by the tautological bundle. To do so, we define an endomorphism B of $\mathbf{U} \to G/K$ by

$$B = \sum_{i=1}^{n} J_{e_i} I_{e_i},$$

which is also called the mean curvature operator.

3. Critical Submanifolds

Let (G, K) be an irreducible symmetric pair of compact type, where G is a simply-connected compact Lie group and K is a closed subgroup of G. The standard involution gives a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the corresponding Lie algebras of G and K, respectively.

We denote by W an irreducible G-module with a G-invariant scalar product, which has a hypersphere as a principal orbit. Such a representation W is called a representation of *spherical type*. Those are classified in Hsiang-Hsiang [6].

Table 1.

G	SU(n)	Spin(n)	Spin(7)	Spin(9)	Sp(n)
W	$\mathbb{C}^n, \mathbb{C}^{n^*}$	$\mathbb{R}^n$	S_7	S_9	$\mathbb{C}^{2n} \cong \mathbb{C}^{2n^*}$

G	Spin(8)	G_2
W	S_8^+, S_8^-	$\mathbf{R}^7$

In this table, S_n denotes a spin representation of Spin(n) and $S_n^{\pm}$ denote half-spin representations of Spin(n).

Then, it is easily checked that the following happens: either W is decomposed into two irreducible components as K-module $W = U \oplus V$, or W itself is an irreducible K-module. We consider only the former cases. Then on a case-by-case basis, we can show

Lemma 3.1. *The decomposition $W = U \oplus V$ is a generalised Cartan decomposition.*

We define two irreducible vector bundles $G \times_K U$ and $G \times_K V$, which are denoted by the same symbols $\mathbf{U} \to G/K$ and $\mathbf{V} \to G/K$, with canonical connections ∇^U and ∇^V, respectively.

Fix an element $w \in W$ such that $|w| = 1$ and consider the corresponding section $s \in \Gamma(U)$. Denote by H the isotropy subgroup of G at $w \in W$. Our assumption yields that the homogeneous space G/H is a unit sphere in W.

The square of a pointwise norm $f([g]) = |s|^2([g])$ $(g \in G)$ of the section s is a function on G/K. Here, we can take $w \in U \subset W$ without loss of generality, since W is of spherical type.

First of all, we can show

Lemma 3.2. *Only the zero set S_0 and the set S_M where the function f attains the maximum value (, which is called the maximum set) are critical submanifolds of $f : G/K \to \mathbf{R}$.*

Lemma 3.3. *If neither U nor V is a trivial representation of K, then both sets S_0 and S_M are connected and H-orbits.*

Lemma 3.4. *The function is a Morse-Bott function.*

For proofs, see [10] Lemmas 7.3, 7.8 and 7.10. The assumption that W is a G-representation of spherical type is exploited in proofs and we have that K-modules U and V are K-representations of spherical type, if they are not trivial representations of dimension 1. Indeed, we obtain

$$S_0 = \left\{[g] \in G/K \,|\, \pi_U([g^{-1}w]) = 0\right\}, \tag{3.1}$$

$$S_M = \left\{[g] \in G/K \,|\, \pi_V([g^{-1}w]) = 0\right\}. \tag{3.2}$$

If we denote by T_0 the zero set and by T_M the maximum set of $|t|^2$, then $T_0 = S_M$ and $T_M = S_0$. For this duality, we do not distinguish module U from V. In the case that neither U nor V is a trivial module of K, S_0 and S_M are assumed to be expressed as H/H_0 and H/H_M, respectively, as homogeneous spaces.

Lemma 3.5. *If U is not a trivial module of H_0, then S_0 is a singular H-orbit.*

Proof. Since W globally generates a bundle $\mathbf{U} \to G/K$, generic sections in W are transverse to the zero section. The hypothesis that W is a representation of spherical type implies that every section in W except zero is transverse to the zero section. From the transversality of the section, the normal spaces of S_0 can be identified with U. Then the assumption yields the result by so-called slice theorem. □

If we replace U, H_0 and S_0 by V, H_M and S_M, respectively, then the same conclusion holds. In this case, by Hsiang-Lawson [7], S_0 and S_M are minimal submanifolds. However, we can say more.

Theorem 3.6. *The critical submanifolds S_0 and S_M are totally geodesic submanifolds of G/K.*

Proof. First of all, we can consider a map into a Grassmannian $i : G/K \to Gr_p(W)$ as the induced map by $(\mathbf{V} \to G/K, W)$ [11, Definition 3.2] (and so, p denotes the dimension of U). Then i is a totally geodesic immersion from Lemma 3.1.

On a Grassmannian $Gr_p(W)$, the module U gives the tautological vector bundle $S \to Gr_p(W)$ in a similar fashion, whose pull-back bundle by i is naturally identified with $U \to G/K$. Then the element $w \in W$ also gives a section $\tilde{s}$ of $S \to Gr_p(W)$ and the pull-back of $\tilde{s}$ is nothing but the section s. Let $\tilde{S}_0$ and $\tilde{S}_M$ be the zero set and the maximum set of $|\tilde{s}|^2$. We take the orthogonal complement space $W^\perp$ of w in W. Then (3.1) and (3.2) imply that

$$\tilde{S}_0 = Gr_p(W^\perp), \quad \tilde{S}_M = Gr_{p-1}(W^\perp),$$

which are totally geodesic submanifolds of $Gr_p(W)$.

Then S_0 and S_M are the intersections of two totally geodesic submanifolds of $Gr_p(W)$ respectively ($S_0 = G/K \cap \tilde{S}_0$ and $S_M = G/K \cap \tilde{S}_M$), which yields the desired result. □

We give a table which includes symmetric spaces G/K, representation spaces W, stabilizers H, decompositions as K-modules $W = U \oplus V$ and pairs S_0 and S_M. We give a complete list in the table. To do so, we use the coincidences that happen in low dimensions between the various classical Lie groups, which are listed in the Remark after the Table 2.

Table 2.

G/K	W	H	$U \oplus V$	S_0, S_M
SU(n)/SO(n)	$\mathbf{C}^n$	SU($n-1$)	$\mathbf{R}^n \oplus \mathbf{R}^n$	SU($n-1$)/SO($n-1$)
$Gr_p(\mathbf{C}^n)$	$\mathbf{C}^n$	SU($n-1$)	$\mathbf{C}^p \oplus \mathbf{C}^q$	$Gr_p(\mathbf{C}^{n-1})$, $Gr_{p-1}(\mathbf{C}^{n-1})$
$Gr_p(\mathbf{R}^n)$	$\mathbf{R}^n$	Spin($n-1$)	$\mathbf{R}^p \oplus \mathbf{R}^q$	$Gr_p(\mathbf{R}^{n-1})$, $Gr_{p-1}(\mathbf{R}^{n-1})$
S^{n-1}	$\mathbf{R}^n$	Spin($n-1$)	$\mathbf{R} \oplus \mathbf{R}^{n-1}$	S^{n-1}, 2points
$Gr_4(\mathbf{R}^7)$	S_7	G_2	$\mathbf{R}^4 \oplus \mathbf{R}^4$	G_2/SO(4), G_2/SO(4)
$Gr_4(\mathbf{R}^8)$	$S_8^{\pm}$	Spin(7)	$\mathbf{R}^4 \oplus \mathbf{R}^4$	$Gr_4(\mathbf{R}^7)$, $Gr_3(\mathbf{R}^7)$
$Gr_4(\mathbf{R}^9)$	S_9	Spin(7)	$\mathbf{R}^8 \oplus \mathbf{R}^8$	$Gr_4(\mathbf{R}^7)$, $Gr_3(\mathbf{R}^7)$
Sp(n)/U(n)	$\mathbf{C}^{2n}$	Sp($n-1$)	$\mathbf{C}^n \oplus \mathbf{C}^{n^*}$	Sp($n-1$)/U($n-1$)
$Gr_p(\mathbf{H}^n)$	$\mathbf{H}^n$	Sp($n-1$)	$\mathbf{H}^p \oplus \mathbf{H}^q$	$Gr_p(\mathbf{H}^{n-1})$, $Gr_{p-1}(\mathbf{H}^{n-1})$
G_2/SO(4)	$\mathbf{R}^7$	SU(3)	$\mathbf{R}^4 \oplus \mathbf{R}^3$	SU(3)/SO(3), $\mathbf{C}P^2$

Remark. We now list the coincidences of a pair of symmetric spaces and representations W omitted in the table.

$$
\begin{aligned}
&(\mathrm{SU}(2)/\mathrm{SO}(2), \mathfrak{su}(2)) = (S^2, \mathbf{R}^3),\\
&\left(\mathrm{SU}(4)/\mathrm{SO}(4), \mathbf{R}^6 = \wedge^2 \mathbf{C}^{4^{\mathbf{R}}}\right) = \left(Gr_3(\mathbf{R}^6), \mathbf{R}^6\right),\\
&\left(\mathrm{SU}(4)/\mathrm{Sp}(2), \mathbf{R}^6\right) = \left(S^5, \mathbf{R}^6\right),\\
&\left(Gr_4(\mathbf{R}^6), \mathbf{C}^4\right) = \left(Gr_2(\mathbf{C}^4), \mathbf{C}^4\right),\\
&\left(Gr_2(\mathbf{R}^5), \mathbf{C}^4\right) = \left(\mathrm{Sp}(2)/\mathrm{U}(2), \mathbf{C}^4\right),\\
&\left(\mathrm{SO}(6)/\mathrm{U}(3), \mathbf{C}^4\right) = (\mathbf{C}P^3, \mathbf{C}^4),\\
&(\mathrm{Sp}(1)/\mathrm{U}(1), \mathfrak{sp}(1)) = (S^2, \mathbf{R}^3).
\end{aligned}
$$

4. Isoparametric functions

Let G/K, W, H and f be as in the previous section. In this section, the level set of the function $f : G/K \to \mathbb{R}$ is our main concern. Since $H \subset G$ is an isotropy subgroup at $w \in W$, f is invariant under the action of H. Hence, H acts on the level set of f.

We can easily show

Lemma 4.1. *If the action of H on G/K is of cohomogeneity one, then f is an isoparametric function.*

Because $|\mathrm{grad}\, f|^2$ and Δf are also invariant under the action of H, and so they are constant functions on the level set of f.

The actions of H are of cohomogeneity one except the following cases:

$$
(\mathrm{SU}(n)/\mathrm{SO}(n), \mathbf{C}^n), \quad \left(\mathrm{Sp}(n)/\mathrm{U}(n), \mathbf{C}^{2n}\right), \quad \left(Gr_4(\mathbf{R}^9), S_9\right).
$$

In the above cases, the cohomogeneity of the actions are 2, 3 and 2, respectively.

In the case of cohomogeneity one, we can easily describe the level set of f as a unit sphere bundle of S_0 or S_M, and show that all level sets are H-orbits, which are left to the reader.

From now on, we would like to compute geometric invariants of submanifolds, more precisely, mean curvatures and principal curvatures. These invariants are related to invariants of vector bundles.

Theorem 4.2. *We have*

$$
\Delta s = \frac{n}{p} s, \quad \Delta t = \frac{n}{q} t, \quad n := \dim G/K,
$$

for arbitrary $s \in W \subset \Gamma(\mathbf{U})$ and $t \in W \subset \Gamma(\mathbf{V})$, when W is an orthogonal representation.

We also have

$$
\Delta s = \frac{n}{2p} s, \quad \Delta t = \frac{n}{2q} t, \quad n := \dim G/K,
$$

for arbitrary $s \in W \subset \Gamma(\mathbf{U})$ and $t \in W \subset \Gamma(\mathbf{V})$, when W is a unitary representation.

Proof. From Theorem 2.6, we see that the mean curvature operators A and B are parallel. Since $\mathbf{U} \to G/K$ and $\mathbf{V} \to G/K$ are irreducible, we have

$$B = -\mu Id_U, \quad A = -\nu Id_V$$

for some constant μ and ν. Since $i : G/K \to Gr_p(W)$ is totally geodesic (hence harmonic), Theorem 2.7 yields that

$$\Delta s = \mu s, \quad \Delta t = \nu t.$$

Since $i : G/K \to Gr_p(W)$ is an isometric immersion, the definition of the Riemannian metric g_{Gr} yields that

$$n = \sum g_{Gr}(e_i, e_i) = -\sum \operatorname{trace} J_{e_i} I_{e_i} = -\operatorname{trace} A = -\operatorname{trace} B,$$

when W is a real representation, and

$$n = -2 \operatorname{trace} A = -2 \operatorname{trace} B,$$

when W is a complex representation. Hence we have our desired results. □

We fix $w \in W$ ($|w| = 1$) again and consider the function $f = |s|^2$.

Theorem 4.3. *We have that*

$$\Delta f = \frac{2nN}{pq}\left(f - \frac{p}{N}\right),$$

when W is an orthogonal representation and

$$\Delta f = \frac{nN}{pq}\left(f - \frac{p}{N}\right),$$

when W is a unitary representation.

Proof. Notice that $w \in W$ also induces a section of $S \to Gr_p(W)$ denoted by $\tilde{s}$. It follows that the pull-back section of $\tilde{s}$ is nothing but $s \in \Gamma(\mathbf{U})$. From Proposition 2.3, we see that $\nabla^S \tilde{s} = -J\tilde{t}$ on Grassmannian, where $\tilde{t}$ is the corresponding section. Since $i : G/K \to Gr_p(W)$ is a totally geodesic immersion and ∇^U is regarded as the pull-back connection of ∇^S, we also have $\nabla^U s = -Jt$. Then we obtain

$$|Jt|^2 = \sum g_U\left(J_{e_i}t, J_{e_i}t\right) = -g_V(At, t) = g_V(\Delta t, t).$$

The well-known formula

$$\Delta |s|^2 = g_U(\Delta s, s) + g_U(s, \Delta s) - 2|\nabla^U s|^2$$

yields that

$$\Delta |s|^2 = 2 g_U(\Delta s, s) - 2 g_V(\Delta t, t).$$

Theorem 4.2 yields the result. □

Hence, the function f always satisfies the condition (2) of the definition of an isoparametric function.

However, $|\operatorname{grad} f|^2 = |df|^2$ does not satisfy the condition (1) in general. We distinguish the case that the action of H is of the cohomogeneity one from others.

4.1. The case of cohomogeneity one. In this subsection, we omit the case that G/K is a sphere. Hence, in the decomposition $W = U \oplus V$, U and V are K-representations of spherical type. Moreover, S_0 and S_M are singular H-orbits, which are expressed as H/H_0 and H/H_M, respectively. Since the action of H is of cohomogeneity one, U is a representation of H_0 of spherical type and V is a representation of H_M of spherical type.

Let $\mathbf{n}$ be a unit normal vector field defined by

$$\mathbf{n} = \frac{\operatorname{grad} f}{|\operatorname{grad} f|},$$

on the regular point of f. We denote by $A_{\mathbf{n}}$ the shape operator of $f^{-1}(c)$, where c is a regular value. By definition, we have that

$$A_{\mathbf{n}}X = -D_X\mathbf{n} = -X\left(\frac{1}{|df|}\right)\operatorname{grad} f - \frac{1}{|df|}D_X\operatorname{grad} f,$$

where X is a tangent vector to $f^{-1}(c)$ and D is the Levi-Civita connection on G/K. Since f is an isoparametric function, the first term of the right-hand-side vanishes. Consequently, we have that

$$g\left(A_{\mathbf{n}}X, Y\right) = -\frac{1}{|df|}\left(D_X df\right)(Y),$$

where X and Y are tangent vectors to $f^{-1}(c)$ and g is the Riemannian metric on G/K. The definition of f yields that

$$\begin{aligned}(D_X df)(Y) = {} & g_U\left(\nabla^U_X\left(\nabla^U s\right)(Y), s\right) + g_U\left(s, \nabla^U_X\left(\nabla^U s\right)(Y)\right) \\ & + g_U\left(\nabla^U_X s, \nabla^U_Y s\right) + g_U\left(\nabla^U_Y s, \nabla^U_X s\right).\end{aligned}$$

Since $W = U \oplus V$ is a generalised Cartan decomposition, Proposition 2.3 yields that

$$\nabla^U_X s = -Jt, \quad \nabla^U_X\left(\nabla^U s\right)(Y) = J_Y I_X s,$$

where t is the corresponding section. It follows that

$$\begin{aligned}(D_X df)(Y) = {} & -g_V\left(I_X s, I_Y s\right) - g_V\left(I_Y s, I_X s\right) \\ & + g_U\left(J_X t, J_Y t\right) + g_U\left(J_Y t, J_X t\right).\end{aligned}$$

We define endomorphisms $\tilde{I}$ and $\tilde{J}$ of the tangent bundle of G/K by

$$g(\tilde{I}X, Y) = \frac{1}{2}\left\{g_V\left(I_X s, I_Y s\right) + g_V\left(I_Y s, I_X s\right)\right\}$$

and

$$g(\tilde{J}X, Y) = \frac{1}{2}\left\{g_U\left(J_X t, J_Y t\right) + g_U\left(J_Y t, J_X t\right)\right\}.$$

By definition, we obtain

$$A_{\mathbf{n}} = \frac{2}{|df|}\left(\tilde{I} - \tilde{J}\right). \tag{4.1}$$

We can immediately see

Lemma 4.4. *The endomorphisms $\tilde{I}$ and $\tilde{J}$ are H-invariant symmetric operators.*

To see properties of $\tilde{I}$ and $\tilde{J}$, we give a key algebraic theorem.

We denote by $\mathfrak{h}$ the corresponding Lie subalgebra to H and a natural projection by $\pi : G \to G/K$.

Theorem 4.5. *In the case that the H-action on G/K is of cohomogeneity one, for an arbitrary $\xi \in \mathfrak{m}$ such that $\xi \perp \mathfrak{h}$ and $|\xi w| = 1$, we have that*

$$\xi^2 w = -w.$$

Proof. Let N be the normal space of S_M at $\pi(e)$, where e is a unit element of G. The subgroup $L \subset G$ defined as $L := K \cap H$ is isomorphic to H_M and acts on N as a representation of spherical type, since the action of H is of cohomogeneity one.

Since W globally generates $\mathbf{V} \to G/K$ and is a representation of spherical type, t is transverse to the zero section. Hence we have that

$$T_x S_M = \operatorname{Ker} \nabla^V t = \operatorname{Ker} Is = \{X \in T_x G/K \mid I_X s = 0\}.$$

It follows that $T_{\pi(e)} G/K = \operatorname{Ker} Is \oplus_\perp N$.

We may regard $Is : T_{\pi(e)} G/K \to V_{\pi(e)}$ as an homomorphism $Is : \mathfrak{m} \to V$ and consider $N \subset \mathfrak{m}$. Since $s(\pi(e)) = [e, w]$, Is is an L-equivariant homomorphism. Hence V is also an L-representation of spherical type which is isomorphic to N.

Let $\xi \in N$ such that $|I_\xi s| = |\xi w| = 1$. Then we obtain $L_\xi \subset L$ as an isotropy subgroup at ξ and $I_\xi : \mathbf{U} \to V$ is an L_ξ-equivariant homomorphism. The endomorphism $J_\xi I_\xi : U \to U$ can be now regarded as $\xi^2 : U \to U$, which is a restriction of $\xi^2 : W \to W$ to $U \subset W$. Note that the eigenvalues except zero of $\xi^2|_U$ are the same as ones of $\xi^2|_V$ with multiplicities, since $W = U \oplus V$ is a generalised Cartan decomposition. Then L_ξ irreducible decompositions of U and V, which are given after the proof, yield that $\xi^2 w = cw$ with some constant $c \in \mathbf{R}$ by Schur's lemma. It follows that $c = \langle \xi^2 w, w \rangle = -\langle \xi w, \xi w \rangle = -1$. □

We shall exploit L_ξ-decomposition in the sequel. We denote by $\mathfrak{l}$ and $\mathfrak{l}_\xi$ the corresponding Lie subalgebras to L and L_ξ, respectively.

• L_ξ-decomposition of $(G/K, W)$.

(1) $\left(Gr_p(\mathbf{R}^N), \mathbf{R}^N\right)$.

Let $e_1, \cdots, e_N$ be an orthonormal basis of $\mathbf{R}^N$ such that $e_1, \cdots, e_p$ spans $\mathbf{R}^p$. We take $w = e_1$ and so, $\mathfrak{l} = \mathfrak{so}(p-1) \oplus \mathfrak{so}(q)$, where $q := N - p$. Let ξ be a skew endomorphism of $\mathbf{R}^N$ such that

$$\xi e_1 = e_{p+1}, \;\; \xi e_{p+1} = -e_1, \text{and } \xi e_A = 0, \;\; A \neq 1, p+1.$$

Notice that $\xi \in \mathfrak{m} \cap \mathfrak{h}^\perp$ with $|\xi w| = 1$. It follows that $\mathfrak{l}_\xi$ is isomorphic to $\mathfrak{so}(p-1) \oplus \mathfrak{so}(q-1)$. Then we have

$$U = \mathbf{R}^p = \mathbf{R}w \oplus \mathbf{R}^{p-1}, \quad V = \mathbf{R}^q = \mathbf{R}\xi w \oplus \mathbf{R}^{q-1}.$$

(2) $\left(Gr_p(\mathbf{C}^N), \mathbf{C}^N\right)$.

Let $e_1, \cdots, e_N$ be a unitary basis of $\mathbf{C}^N$ such that $e_1, \cdots, e_p$ spans $\mathbf{C}^p$. We take $w = e_1$ and so, $\mathfrak{l} = \mathfrak{u}(1) \oplus \mathfrak{su}(p-1) \oplus \mathfrak{su}(q)$. Let ξ be a skew Hermitian endomorphism of $\mathbf{C}^N$ such that

$$\xi e_1 = e_{p+1}, \;\; \xi e_{p+1} = -e_1, \text{and } \xi e_A = 0, \;\; A \neq 1, p+1.$$

Notice that $\xi \in \mathfrak{m} \cap \mathfrak{h}^\perp$ with $|\xi w| = 1$. It follows that $\mathfrak{l}_\xi$ is isomorphic to $\mathfrak{u}(1) \oplus \mathfrak{su}(p-1) \oplus$

$\mathfrak{su}(q-1)$. Then we have

$$U = \mathbf{C}^p = \mathbf{C}w \oplus \mathbf{C}^{p-1}, \quad V = \mathbf{C}^q = \mathbf{C}\xi w \oplus \mathbf{C}^{q-1}.$$

(3) $\left(Gr_p(\mathbf{H}^N), \mathbf{H}^N\right)$.

Let $e_1, \cdots, e_N$ be a quaternion-unitary basis of $\mathbf{H}^N$ such that $e_1, \cdots, e_p$ spans $\mathbf{H}^p$. We take $w = e_1$ and so, $\mathfrak{l} = \mathfrak{sp}(p-1) \oplus \mathfrak{sp}(q)$. Let ξ be a quaternion-skew Hermitian endomorphism of $\mathbf{H}^N$ such that

$$\xi e_1 = e_{p+1},\ \xi e_{p+1} = -e_1, \text{and } \xi e_A = 0,\ A \neq 1, p+1.$$

Notice that $\xi \in \mathfrak{m} \cap \mathfrak{h}^\perp$ with $|\xi w| = 1$. It follows that $\mathfrak{l}_\xi$ is isomorphic to $\mathfrak{sp}(p-1) \oplus \mathfrak{sp}(q-1)$. Then we have

$$U = \mathbf{H}^p = \mathbf{H}w \oplus \mathbf{H}^{p-1}, \quad V = \mathbf{H}^q = \mathbf{H}\xi w \oplus \mathbf{H}^{q-1}.$$

(4) $\left(Gr_4(\mathbf{R}^7), S_7\right)$.

The isotropy subalgebra is isomorphic to $\mathfrak{so}(4) \oplus \mathfrak{sp}(1)$. The Lie algebra $\mathfrak{so}(4)$ is a direct sum of two copies of $\mathfrak{sp}(1)$. To distinguish these copies of $\mathfrak{sp}(1)$, the isotropy subalgebra is denoted by $\mathfrak{sp}_+(1) \oplus \mathfrak{sp}_-(1) \oplus \mathfrak{sp}(1)$.

Under the action of the isotropy subalgebra on S_7, we have an irreducible decomposition:

$$S_7 = \left(\mathbf{C}^2_+ \otimes \mathbf{C}^2\right)^{\mathbf{R}} \oplus \left(\mathbf{C}^2_- \otimes \mathbf{C}^2\right)^{\mathbf{R}},$$

where $\mathbf{C}^2_{(\pm)}$ denote the standard representations of $\mathfrak{sp}_{(\pm)}(1)$, respectively and $\left(\mathbf{C}^2_\pm \otimes \mathbf{C}^2\right)^{\mathbf{R}}$ denote real invariant spaces of $\mathbf{C}^2{}_\pm \otimes \mathbf{C}^2$, respectively.

We pick up a unit vector $w \in \left(\mathbf{C}^2_+ \otimes \mathbf{C}^2\right)^{\mathbf{R}}$ and so, $\mathfrak{l}$ is regarded as the diagonal subalgebra of $\mathfrak{sp}_+(1) \oplus \mathfrak{sp}(1)$. Let $v \in \left(\mathbf{C}^2_- \otimes \mathbf{C}^2\right)^{\mathbf{R}}$ be a unit vector. Since L_ξ ($\xi \in \mathfrak{m} \cap \mathfrak{h}^\perp$ with $|\xi w| = 1$) can be identified with an isotropy subgroup of the L-action on S_7 at v, it follows that $\mathfrak{l}_\xi$ is isomorphic to the subalgebra $\{(X, X, X)\}$ of $\mathfrak{sp}_+(1) \oplus \mathfrak{sp}_-(1) \oplus \mathfrak{sp}(1)$. Then we have

$$U = \left(\mathbf{C}^2_+ \otimes \mathbf{C}^2\right)^{\mathbf{R}} = \mathbf{R}w \oplus \mathbf{R}^3, \quad V = \left(\mathbf{C}^2_- \otimes \mathbf{C}^2\right)^{\mathbf{R}} = \mathbf{R}v \oplus \mathbf{R}^3,$$

where $\mathbf{R}^3$ denotes the adjoint representation of $\mathfrak{l}_\xi$.

(5) $\left(G_2/\mathrm{SO}(4), \mathbf{R}^7\right)$.

To distinguish two copies of $\mathfrak{sp}(1)$, the isotropy subalgebra is denoted by $\mathfrak{sp}_+(1) \oplus \mathfrak{sp}_-(1)$.

Under the action of the isotropy subalgebra on $\mathbf{R}^7$, we have an irreducible decomposition:

$$\mathbf{R}^7 = \left(\mathbf{C}^2_+ \otimes \mathbf{C}^2_-\right)^{\mathbf{R}} \oplus \mathfrak{sp}_-(1),$$

where $\mathbf{C}^2_\pm$ denote the standard representations of $\mathfrak{sp}_\pm(1)$, respectively and $\left(\mathbf{C}^2_+ \otimes \mathbf{C}^2_-\right)^{\mathbf{R}}$ denotes a real invariant space of $\mathbf{C}^2{}_+ \otimes \mathbf{C}^2_-$.

We pick up a unit vector $w \in \left(\mathbf{C}^2_+ \otimes \mathbf{C}^2_-\right)^{\mathbf{R}}$ and so, $\mathfrak{l}$ is regarded as the diagonal subalgebra Δ of $\mathfrak{sp}_+(1) \oplus \mathfrak{sp}_-(1)$. Let $v \in \mathfrak{sp}_-(1)$ be a unit vector. Since L_ξ ($\xi \in \mathfrak{m} \cap \mathfrak{h}^\perp$ with $|\xi w| = 1$) can be identified with an isotropy subgroup of L-action on $\mathbf{R}^7$ at v, we have that $\mathfrak{l}_\xi$ is isomorphic to $\mathfrak{u}(1)$ which is the standard subalgebra of Δ. Then we have

$$U = \left(\mathbf{C}^2_+ \otimes \mathbf{C}^2_-\right)^{\mathbf{R}} = \mathbf{R}w \oplus \mathbf{R} \oplus \mathbf{C}_2, \quad V = \mathfrak{sp}_-(1) = \mathbf{R}v \oplus \mathbf{C}_2,$$

where $\mathbf{C}_\alpha$ denotes an irreducible representation of $\mathfrak{u}(1)$ with weight α.

Remark. We should consider the case of $\left(Gr_4(\mathbf{R}^8), S_8^{\pm}\right)$. However, the triality gives the same picture as in the case of $\left(Gr_4(\mathbf{R}^8), \mathbf{R}^8\right)$, and so we omit it.

Corollary 4.6. *We can find a geodesic on G/K which intersects all H-orbits orthogonally.*

Proof. For any $\xi \in \mathfrak{m}$ such that $\xi \perp \mathfrak{h}$ and $|\xi w| = 1$, Theorem 4.5 yields that

$$e^{t\xi} w = \sum \frac{1}{n!} \xi^n w = \cos tw + \sin tv,$$

where we put $v := \xi w \in V$. Then, $\pi(e^{t\xi})$ is a geodesic through $\pi(e)$.

Moreover, we get

$$s\left(\pi(e^{t\xi})\right) = \left[e^{t\xi}, \pi_U\left(e^{-t\xi} w\right)\right] = \cos t \left[e^{t\xi}, w\right] = \cos t e^{t\xi} s\left(\pi(e)\right).$$

Hence,

$$f\left(\pi(e^{t\xi})\right) = \cos^2 t,$$

and so, the geodesic $\pi(e^{t\xi})$ meets all H-orbits.

Since $\xi \perp \mathfrak{h}$, ξ can be regarded as a normal vector of S_M in G/K. We can identify the normal bundle of an H-orbit with a neighbourhood of the H-orbit G-equivariantly via an exponential map restricted to the normal space. Hence the geodesic $\pi(e^{t\xi})$ intersects all H-orbits orthogonally by Gauss's lemma. □

Remark. The existence of a geodesic which intersects all orbits orthogonally is well-known in the case that the action is of cohomogeneity one. However, we exploit our geodesic $\pi(e^{t\xi})$ to compute submanifold-geometric invariants including principal curvatures of the regular level set explicitly. To do so, we fix the notation $\pi(e^{t\xi})$ to express the specified geodesic.

For simplicity, we put $o := \pi(e) \in G/K$.

Theorem 4.7. *The endomorphism $\tilde{I}$ has only two eigenspaces, which is expressed as $T_xG/K = E_1 \oplus E_2$, where $s(x) \neq 0$. The eigenspace E_1 with zero eigenvalue is indeed* Ker Is, *where we regard Is as a homomorphism $Is : TG/K \to \mathbf{V}$. The both E_1 and E_2 can be identified with T_oS_M and V, respectively, via a parallel transport along the geodesic $\pi\left(e^{t\xi}\right)$ and an action of H, where $\xi \in \mathfrak{m} \cap \mathfrak{h}^\perp$.*

Proof. As we already show,

$$T_oS_M = \operatorname{Ker} \nabla^V t = \left\{X \in T_oG/K \mid \nabla^V_X t = 0\right\}.$$

It follows from $\nabla^V t = -Is$ that T_oS_M is included in the eigenspace of $\tilde{I}$ with zero eigenvalue.

Let L be an isotropy subgroup of H at $o \in S_M$. Then we already see that $N(\cong V)$ is an irreducible representation of L. From Lemma 4.4, V must be an eigenspace of $\tilde{I}$, because L acts on each eigenspace.

Let $x \in G/K$ be a point outside S_M and suppose that $s(x) \neq 0$. It follows from H-invariance of $\tilde{I}$ that x can be assumed to be joined to o by $\pi(e^{t\xi})$ and $x = \pi(e^{t_0\xi})$ for some

$\xi \in \mathfrak{m} \cap \mathfrak{h}^{\perp}$. If we put $g(x) = e^{t_0 \xi}$, then $x = g(x)o$ and $s(x) = \cos t_0 g(x)s(o)$.

Since I is G-invariant, if X and $Y \in T_x G/K$, then we obtain

$$\begin{aligned}
g_x\left(\tilde{I}X, Y\right) &= \frac{1}{2}\{g_{V_x}(I_X s(x), I_Y s(x)) + g_{V_x}(I_Y s(x), I_X s(x))\} \\
&= \frac{1}{2}\Big\{g_{V_x}\left(I_{g(x)g(x)^{-1}X} \cos t_0 g(x)s(o), I_{g(x)g(x)^{-1}Y} \cos t_0 g(x)s(o)\right) \\
&\qquad + g_{V_x}\left(I_{g(x)g(x)^{-1}Y} \cos t_0 g(x)s(o), I_{g(x)g(x)^{-1}X} \cos t_0 g(x)s(o)\right)\Big\} \\
&= \frac{f(y)}{2}\Big\{g_{V_y}\left(g(x)I_{g(x)^{-1}X} s(o), g(x)I_{g(x)^{-1}Y} s(o)\right) \\
&\qquad + g_{V_y}\left(g(x)I_{g(x)^{-1}Y} s(o), g(x)I_{g(x)^{-1}X} s(o)\right)\Big\} \\
&= \frac{f(y)}{2}\Big\{g_{V_o}\left(I_{g(x)^{-1}X} s(o), I_{g(x)^{-1}Y} s(o)\right) + g_{V_o}\left(I_{g(x)^{-1}Y} s(o), I_{g(x)^{-1}X} s(o)\right)\Big\} \\
&= f(y)\, g_o\left(\tilde{I}g(x)^{-1}X, g(x)^{-1}Y\right).
\end{aligned}$$

It follows that $T_x G/K = g(x)T_o S_M \oplus g(x)V_o$ is the eigenspace decomposition of the endomorphism $\tilde{I}_x$. It also follows that $g(x)T_o S_M = \operatorname{Ker} Is$. □

Lemma 4.8. *The normal vector field* **n** *belongs to E_2, where $df \neq 0$.*

Proof. From Corollary 4.6, the velocity vector of the geodesic $\pi(e^{t\xi})$ is a constant multiple of the unit normal vector field **n**.

By Theorem 4.7, the eigenspace E_1 corresponding to zero eigenvalue is the image of a parallel transport of TS_M along $\pi(e^{t\xi})$. Then, we have that $\mathbf{n} \perp E_1$. The H-invariance gives our desired result. □

Remark. It is well-known (and easily shown) that the unit normal vector field **n** generates a geodesic if the function satisfies the condition (1) of Definition 2.1.

We denote by λ an eigenvalue of $\tilde{I}$ whose eigenspace is $E_2 \cong V$.

Theorem 4.9. *The eigenvalue λ is equal to $\frac{n}{pq}|s|^2$ when W is real, $\frac{n}{4pq}|s|^2$ when W is complex.*

Proof. In both cases, we have

$$\sum g(\tilde{I}e_i, e_i) = g_V(I_{e_i}s, I_{e_i}s) = -g_U(Bs, s) = g_U(\Delta s, s). \tag{4.2}$$

• W: **real**. By definition, we have $\sum g(\tilde{I}e_i, e_i) = q\lambda$. From (4.2) and Theorem 4.2, we get

$$q\lambda = \frac{n}{p}|s|^2.$$

• W: **complex**. By definition, we have $\sum g(\tilde{I}e_i, e_i) = 2q\lambda$. From (4.2) and Theorem 4.2, we get

$$2q\lambda = \frac{n}{2p}|s|^2.$$

□

In a similar way, we have

Theorem 4.10. *The eigenspaces of $\tilde{J}$ can be identified with U and TS_0 via a parallel transport along the geodesic $\pi(e^{t\xi})$ and an H-action. The eigenvalue corresponding to the eigenspace U is $\frac{n}{pq}|t|^2$ when W is real, $\frac{n}{4pq}|t|^2$ when W is complex. The eigenspace TS_0 is the kernel of $\tilde{J}$.*

For simplicity, it is said that the eigenspaces of $\tilde{I}$ are V and TS_M and the eigenspaces of $\tilde{J}$ are U and TS_0, when no confusion can arise.

Lemma 4.11. *The unit normal vector field* **n** *is the eigenvector of $\tilde{J}$ which belongs to U.*

We can compute the norm of the velocity vector of the geodesic $\pi(e^{t\xi})$.

Lemma 4.12. *Let $\xi \in \mathfrak{m} \cap \mathfrak{h}^{\perp}$ with $|\xi w| = 1$. The square of the norm $|\xi|^2$ is equal to $\frac{pq}{n}$, when W is real, $\frac{4pq}{n}$, when W is complex.*

Proof. On the one hand, since ξ is a constant multiple of **n**, we get $\tilde{I}\xi = \lambda\xi$, from Lemma 4.8, where λ is the eigenvalue different from zero. It follows that

$$g(\tilde{I}\xi, \xi) = \lambda|\xi|^2$$

On the other hand, the definition gives $g(\tilde{I}\xi, \xi) = g_V\left(I_\xi s, I_\xi s\right)$. Since G/K is a totally geodesic submanifold of $Gr_p(W)$, we can compute

$$I_\xi s\left(\pi(e^{t\xi})\right) = \left[e^{t\xi}, \xi \cos tw\right] = \cos t\left[e^{t\xi}, \xi w\right].$$

Since $|\xi w| = 1$, we obtain $|I_\xi s|^2 = \cos^2 t|\xi w|^2 = |s|^2$.

We immediately get $\lambda|\xi|^2 = |s|^2$, which provides us with the result by Theorem 4.9. □

Theorem 4.13. *The norm of the gradient vector grad f is given by*

$$|df| = \begin{cases} 2|s||t|\sqrt{\frac{n}{pq}}, & \text{when } W \text{ is real,} \\ |s||t|\sqrt{\frac{n}{pq}}, & \text{when } W \text{ is complex.} \end{cases}$$

Proof. Let $\xi \in \mathfrak{m} \cap \mathfrak{h}^{\perp}$ with $|\xi w| = 1$. It is enough to compute the norm on the geodesic $\pi(e^{t\xi})$ due to H-invariance. Note that the corresponding section t is expressed as

$$t\left(\pi(e^{t\xi})\right) = \left[e^{t\xi}, -\sin tv\right] = -\sin t\left[e^{t\xi}, v\right] = -\sin te^{t\xi}t(o).$$

Moreover, we get from $\xi v = -w$ that

$$J_\xi t\left(\pi(e^{t\xi})\right) = -\sin t\left[e^{t\xi}, \xi v\right] = \tan t\left[e^{t\xi}, w\right] = \frac{|t|}{|s|}s\left(\pi(e^{t\xi})\right).$$

Then, we have

$$\begin{aligned} |df|^2 &= \sum\left(g_U\left(\nabla^U_{e_i}s, s\right) + g_U\left(s, \nabla^U_{e_i}s\right)\right)^2 \\ &= \sum\left(g_U\left(J_{e_i}t, s\right) + g_U\left(s, J_{e_i}t\right)\right)^2 \\ &= \sum\frac{|s|^2}{|t|^2}\left(g_U\left(J_{e_i}t, J_\xi t\right) + g_U\left(J_\xi t, J_{e_i}t\right)\right)^2 = \sum 4\frac{|s|^2}{|t|^2}\left(g(\tilde{J}e_i, \xi)\right)^2. \end{aligned}$$

We can take $e_n = \mathbf{n}$ and already see that $\xi = |\xi|\mathbf{n}$ (up to a sign). Theorem 4.10 and Lemma 4.12 yield that

$$|df|^2 = 4\frac{|t|^2}{|s|^2}|\xi|^2\mu^2,$$

where μ is the eigenvalue of $\tilde{J}$ different from zero. □

REMARK. From Theorems 4.3 and 4.13, it follows that

$$\Delta f = \frac{2nN}{pq}\left(f - \frac{p}{N}\right), \quad |df|^2 = \frac{4n}{pq}f(1-f),$$

when W is real, and

$$\Delta f = \frac{nN}{pq}\left(f - \frac{p}{N}\right), \quad |df|^2 = \frac{n}{pq}f(1-f),$$

when W is complex. If we define a new function $\tilde{f}$ by

$$\tilde{f} = f - \frac{p}{N},$$

then we have

$$\Delta\tilde{f} = \frac{2nN}{pq}\tilde{f}, \quad |d\tilde{f}|^2 = \frac{4n}{pq}\left(\tilde{f} + \frac{p}{N}\right)\left(\frac{q}{N} - \tilde{f}\right),$$

when W is real, and

$$\Delta\tilde{f} = \frac{nN}{pq}\tilde{f}, \quad |d\tilde{f}|^2 = \frac{n}{pq}\left(\tilde{f} + \frac{p}{N}\right)\left(\frac{q}{N} - \tilde{f}\right),$$

when W is complex.

Let c be a regular value of the function $f : G/K \to \mathbf{R}$. We can compute the mean curvature m of the hypersurface $f^{-1}(c)$. Notice that $|s|^2 = c$ and $|t|^2 = \sqrt{1-c}$, by definition. Hence, instead of using c, we employ $|s|$ and $|t|$ to express invariants on $f^{-1}(c)$.

Theorem 4.14. *Let m be the mean curvature of the regular level set $f^{-1}(c)$. Then m is expressed:*

$$m = \begin{cases} \frac{1}{|s||t|}\sqrt{\frac{n}{pq}}\left\{|s|^2(q-1) - |t|^2(p-1)\right\}, & \text{when } W \text{ is real}, \\ \frac{1}{2|s||t|}\sqrt{\frac{n}{pq}}\left\{|s|^2(2q-1) - |t|^2(2p-1)\right\}, & \text{when } W \text{ is complex}. \end{cases}$$

Proof. From (4.1), Theorems 4.7, 4.9 and 4.10 and Lemmas 4.8 and 4.12, it follows that

$$m = \sum_{i=1}^{n-1} g(A_{\mathbf{n}}e_i, e_i) = \frac{2}{|df|}\left\{\operatorname{trace}\tilde{I} - g(\tilde{I}\mathbf{n}, \mathbf{n}) - \operatorname{trace}\tilde{J} + g(\tilde{J}\mathbf{n}, \mathbf{n})\right\},$$

where $e_1, \cdots, e_n = \mathbf{n}$ is an orthonormal basis of TG/K. Using again Lemmas 4.8 and 4.12, Theorem 4.13 yield the result. □

REMARK. Using only the function f, m is described as

$$m = \begin{cases} \sqrt{\frac{n}{pqf(1-f)}}\{(N-2)f - (p-1)\}, & \text{when } W \text{ is real}, \\ \sqrt{\frac{n}{4pqf(1-f)}}\{2(N-1)f - (2p-1)\}, & \text{when } W \text{ is complex}. \end{cases}$$

Corollary 4.15. *There exists one and only one minimal regular level set of the function f. More precisely, $f^{-1}(c)$ is a minimal hypersurface, where*

$$c = \begin{cases} \frac{p-1}{N-2}, & \text{when } W \text{ is real,} \\ \frac{2p-1}{2(N-1)}, & \text{when } W \text{ is complex.} \end{cases}$$

Next, we compute principal curvatures, in other words, the eigenvalues of $A_{\mathbf{n}}$. From (4.1), we should see how the eigenspaces of $\tilde{I}$ and $\tilde{J}$ intersect with each other.

As we have already seen, the eigendecomposition of $\tilde{I}$ is expressed as

$$T_oG/K \cong \mathfrak{m} = \{X \in \mathfrak{m} \mid Xw = 0\} \oplus_\perp V = TS_M \oplus_\perp V.$$

We put $g_0 := e^{\frac{\pi}{2}\xi}$. In a similar way, we have

$$g_0^{-1}\left(T_{\pi(g_0)}G/K\right) \cong \mathfrak{m} = \{X \in \mathfrak{m} \mid Xv = 0\} \oplus_\perp U = g_0^{-1}\left(T_{\pi(g_0)}S_0\right) \oplus_\perp U.$$

We use the same notation as in L_ξ-decomposition of $(G/K, W)$ after Theorem 4.5.

• Principal curvatures.

By (4.1), Theorems 4.7, 4.9, 4.10 and 4.13 imply

Lemma 4.16. *The shape operator $A_{\mathbf{n}}$ satisfies*

$$A_{\mathbf{n}} = \begin{cases} 0 & (\text{on } T_oS_M \cap g_0^{-1}(T_{\pi(g_0)}S_0)) \\ \frac{|s|}{a|t|}\sqrt{\frac{n}{pq}} & (\text{on } g_0^{-1}(T_{\pi(g_0)}S_0) \cap V) \\ -\frac{|t|}{a|s|}\sqrt{\frac{n}{pq}} & (\text{on } T_oS_M \cap U) \\ \frac{|s|^2 - |t|^2}{a|s||t|}\sqrt{\frac{n}{pq}} & (\text{on } U \cap V), \end{cases}$$

where $a = 1$ (resp. 2) if W is real (resp. complex).

(1) $\left(Gr_p(\mathbf{R}^N), \mathbf{R}^N\right)$,

The tangent space $\mathfrak{m}$ is regarded as $\mathbf{R}^p \otimes \mathbf{R}^q$. We get the $\mathfrak{l}_\xi$-decomposition of $\mathfrak{m}$:

$$\begin{aligned} \mathfrak{m} &= \mathbf{R}w \otimes \mathbf{R}\xi w \oplus_\perp \mathbf{R}^{p-1} \otimes \mathbf{R}\xi w \oplus_\perp \mathbf{R}w \otimes \mathbf{R}^{q-1} \oplus_\perp \mathbf{R}^{p-1} \otimes \mathbf{R}^{q-1} \\ &= \mathbf{R} \oplus_\perp \mathbf{R}^{p-1} \oplus_\perp \mathbf{R}^{q-1} \oplus_\perp \mathbf{R}^{p-1} \otimes \mathbf{R}^{q-1}. \end{aligned}$$

In this decomposition, we can identify:

$$\begin{aligned} TS_0 &= \mathbf{R}^{q-1} \oplus_\perp \mathbf{R}^{p-1} \otimes \mathbf{R}^{q-1}, \quad TS_M = \mathbf{R}^{p-1} \oplus_\perp \mathbf{R}^{p-1} \otimes \mathbf{R}^{q-1}, \\ U &= \mathbf{R} \oplus_\perp \mathbf{R}^{p-1}, \quad V = \mathbf{R} \oplus_\perp \mathbf{R}^{q-1}. \end{aligned}$$

Since the both $\tilde{I}$ and $\tilde{J}$ are $\mathfrak{l}_\xi$-invariant, Schur's lemma yields the eigendecomposition of $A_{\mathbf{n}}$. Then Lemma 4.16 implies

Theorem 4.17. *The principal curvatures of the regular level set $f^{-1}(c)$ of the function f are*

$$\frac{|s|}{|t|}, \quad -\frac{|t|}{|s|}, \quad 0,$$

with multiplicities $q-1$, $p-1$, $(p-1)(q-1)$, respectively.

(2) $\left(Gr_p(\mathbf{C}^N), \mathbf{C}^N\right)$,

The holomorphic tangent space at o is regarded as $\mathbf{C}^{p^*} \otimes \mathbf{C}^q$. We identify $\mathfrak{m}$ with the holomorphic tangent space at o. We get the $\mathfrak{l}_\xi$-decomposition of $\mathfrak{m}$:

$$\begin{aligned}\mathfrak{m} =& \mathbf{C}w^* \otimes \mathbf{C}\xi w \oplus_\perp \mathbf{C}^{p-1^*} \otimes \mathbf{C}\xi w \oplus_\perp \mathbf{C}w^* \otimes \mathbf{C}^{q-1} \oplus_\perp \mathbf{C}^{p-1^*} \otimes \mathbf{C}^{q-1}\\ =& \mathbf{C} \oplus_\perp \mathbf{C}^{p-1^*} \oplus_\perp \mathbf{C}^{q-1} \oplus_\perp \mathbf{C}^{p-1^*} \otimes \mathbf{C}^{q-1}.\end{aligned}$$

In this decomposition, we can identify:

$$\begin{aligned}TS_0 =& \mathbf{C}^{q-1} \oplus_\perp \mathbf{C}^{p-1^*} \otimes \mathbf{C}^{q-1}, \quad TS_M = \mathbf{C}^{p-1^*} \oplus_\perp \mathbf{C}^{p-1^*} \otimes \mathbf{C}^{q-1},\\ U^* =& \mathbf{C} \oplus_\perp \mathbf{C}^{p-1^*}, \quad V = \mathbf{C} \oplus_\perp \mathbf{C}^{q-1}.\end{aligned}$$

Since the both $\tilde{I}$ and $\tilde{J}$ are $\mathfrak{l}_\xi$-invariant, Schur's lemma yields the eigendecomposition of $A_{\mathbf{n}}$.

Then Lemma 4.16 implies

Theorem 4.18. *The principal curvatures of the regular level set $f^{-1}(c)$ of the function f are*

$$\frac{1}{\sqrt{2}|s||t|}(|s|^2 - |t|^2), \quad \frac{|s|}{\sqrt{2}|t|}, \quad -\frac{|t|}{\sqrt{2}|s|}, \quad 0,$$

with multiplicities 1, $2(q-1)$, $2(p-1)$, $2(p-1)(q-1)$, *respectively.*

(3) $\left(Gr_p(\mathbf{H}^N), \mathbf{H}^N\right)$,

The tangent space $\mathfrak{m}$ is regarded as $\mathbf{H}^p \otimes \mathbf{H}^q$, in an appropriate sense. We get the $\mathfrak{l}_\xi$-decomposition of $\mathfrak{m}$:

$$\begin{aligned}\mathfrak{m} =& \mathbf{H}w \otimes \mathbf{H}\xi w \oplus_\perp \mathbf{H}^{p-1} \otimes \mathbf{H}\xi w \oplus_\perp \mathbf{H}w \otimes \mathbf{H}^{q-1} \oplus_\perp \mathbf{H}^{p-1} \otimes \mathbf{H}^{q-1}\\ =& \mathbf{H} \oplus_\perp \mathbf{H}^{p-1} \oplus_\perp \mathbf{H}^{q-1} \oplus_\perp \mathbf{H}^{p-1} \otimes \mathbf{H}^{q-1}.\end{aligned}$$

In this decomposition, we can identify:

$$\begin{aligned}TS_0 =& \mathbf{H}^{q-1} \oplus_\perp \mathbf{H}^{p-1} \otimes \mathbf{H}^{q-1}, \quad TS_M = \mathbf{H}^{p-1} \oplus_\perp \mathbf{H}^{p-1} \otimes \mathbf{H}^{q-1},\\ U =& \mathbf{H} \oplus_\perp \mathbf{H}^{p-1}, \quad V = \mathbf{H} \oplus_\perp \mathbf{H}^{q-1}.\end{aligned}$$

Since the both $\tilde{I}$ and $\tilde{J}$ are $\mathfrak{l}_\xi$-invariant, Schur's lemma yields the eigendecomposition of $A_{\mathbf{n}}$.

Then Lemma 4.16 implies

Theorem 4.19. *The principal curvatures of the regular level set $f^{-1}(c)$ of the function f are*

$$\frac{1}{2|s||t|}(|s|^2 - |t|^2), \quad \frac{|s|}{2|t|}, \quad -\frac{|t|}{2|s|}, \quad 0,$$

with multiplicities 3, $4(q-1)$, $4(p-1)$, $4(p-1)(q-1)$, *respectively.*

(4) $\left(Gr_4(\mathbf{R}^7), S_7\right)$,

The tangent space $\mathfrak{m}$ is isomorphic to $\mathbf{R}^4 \otimes \mathbf{R}^3$ as $\mathfrak{so}(4) \oplus \mathfrak{sp}(1)$-module. Note that $\mathfrak{l}_\xi$ is isomorphic to the subalgebra $\{(X, X, X)\}$ of $\mathfrak{sp}_+(1) \oplus \mathfrak{sp}_-(1) \oplus \mathfrak{sp}(1)$, and so we have a decomposition of $\mathfrak{m}$ as $\mathfrak{l}_\xi$-module:

$$\mathfrak{m} = \mathbf{R}^5 \oplus_\perp 2\mathbf{R}^3 \oplus_\perp \mathbf{R}.$$

Since

$$TS_0 = \mathbf{R}^5 \oplus_\perp \mathbf{R}^3, \quad TS_M = \mathbf{R}^5 \oplus_\perp \mathbf{R}^3, \quad U = \mathbf{R} \oplus \mathbf{R}^3, \quad V = \mathbf{R} \oplus \mathbf{R}^3,$$

we can not obtain the same conclusion as before.

We consider a $\mathfrak{l}_\xi$-irreducible decomposition of S_7:

$$S_7 = \mathbf{R}w \otimes \mathbf{R}^3 \oplus \mathbf{R}v \otimes \mathbf{R}^3,$$

We already see that $\xi w = v$ and $\xi v = -w$. Let u_1, u_2, u_3 be an orthonormal basis of $\mathbf{R}^3 \subset U$. We can put $\xi^2 u_i = x u_i$ for $i = 1, 2, 3$. Using the relation that $|\xi|^2 = g_V(I_\xi w, I_\xi w) + \sum_i g_V(I_\xi u_i, I_\xi u_i)$, Lemma 4.12 yields that

$$\frac{pq}{n} = 1 - 3x.$$

If we substitute p, q and n by 4, 4 and 12, then we obtain

$$x = -\frac{1}{9}.$$

Hence we can take an orthonormal basis v_1, v_2 and v_3 of $\mathbf{R}^3 \subset V$ such that

$$\xi u_i = \frac{1}{3} v_i, \quad \xi v_i = \frac{-1}{3} u_i, \quad i = 1, 2, 3.$$

Let η be a normal vector of $T_o S_M$ which is orthogonal to ξ, (which yields that $\eta \in \mathbf{R}^3 \subset V$), and satisfies that $|\eta w| = 1$. Theorem 4.12 gives

$$|\eta|^2 = \frac{pq}{n} = \frac{4}{3}.$$

The relation $\xi \perp \eta$ yields that $\xi w \perp \eta w$. Hence we may suppose that

$$\eta w = v_1, \quad \eta v = \frac{-1}{3} u_1.$$

We put $\eta = \eta_0 + \eta_1$ according to the decomposition $\mathfrak{m} = TS_0 \oplus U$. Note that $\eta_0 \in \mathbf{R}^3 \subset TS_0$ and $\eta_1 \in \mathbf{R}^3 \subset U$. Then we have

$$\eta v = \eta_0 v + \eta_1 v = \eta_1 v,$$

and so, $|\eta_1 v|^2 = \frac{1}{9}$. Since $\eta_1 \in U$, we get

$$g(\tilde{J}\eta_1, \eta_1) = \frac{n}{pq}|t|^2|\eta_1|^2 = \frac{3}{4}|\eta_1|^2.$$

On the other hand, we have

$$g_o(\tilde{J}\eta_1, \eta_1) = g_V(J_{\eta_1} t, J_{\eta_1} t) = |t|^2|\eta_1 v|^2 = \frac{1}{9}.$$

Consequently, we have $|\eta_1|^2 = \frac{4}{27}, \quad |\eta_0|^2 = \frac{32}{27}$, and so,

$$|\eta_0| : |\eta_1| = 2\sqrt{2} : 1.$$

Hence, if X is a vector of $\mathbf{R}^3 \subset T_o S_M$ and $X = X_0 + X_1$, where $X_0 \in \mathbf{R}^3 \subset T_o S_0$ and $X_1 \in \mathbf{R}^3 \subset U$, then we have

$$|X_0| : |X_1| = 1 : 2\sqrt{2}.$$

Let X_1, X_2 and X_3 be an orthonormal basis of $\mathbf{R}^3 \subset T_oS_M$ and v_1, v_2 and v_3 an orthonormal basis of $\mathbf{R}^3 \subset V$. Then we can take an orthonormal basis Y_1, Y_2 and Y_3 of $\left(\mathbf{R}^3 \oplus \mathbf{R}^3\right) \cap T_oS_0$ and an orthonormal basis u_1, u_2 and u_3 of $\left(\mathbf{R}^3 \oplus \mathbf{R}^3\right) \cap U$ such that

$$X_i = \frac{1}{3}\left(Y_i - 2\sqrt{2}u_i\right), \quad v_i = \frac{1}{3}\left(2\sqrt{2}Y_i + u_i\right), \quad i = 1, 2, 3.$$

It follows from Theorem 4.10 that

$$\tilde{J}X_i = \frac{2\sqrt{2}}{9}\frac{n}{pq}|t|^2\left(2\sqrt{2}X_i - v_i\right), \quad \tilde{J}v_i = \frac{1}{9}\frac{n}{pq}|t|^2\left(-2\sqrt{2}X_i + v_i\right)$$

From (4.1) and Theorems 4.7, 4.9, 4.10 and 4.13, we need to compute the eigenvalues of

$$\frac{1}{9}\sqrt{\frac{n}{pq}}\frac{1}{|s||t|}\begin{pmatrix} -8|t|^2 & 2\sqrt{2}|t|^2 \\ 2\sqrt{2}|t|^2 & 9|s|^2 - |t|^2 \end{pmatrix}$$

to obtain the principal curvatures. Then we have

Theorem 4.20. *The principal curvatures of the regular level set $f^{-1}(c)$ of the function f are*

$$\frac{\sqrt{3}}{12}\frac{1}{|s||t|}\left\{3(|s|^2 - |t|^2) \pm \sqrt{9 - 4|s|^2|t|^2}\right\}, \quad 0,$$

with multiplicities 3, 3, 5, *respectively.*

(5) $\left(G_2/\mathrm{SO}(4), \mathbf{R}^7\right)$,

We can proceed in the almost same way as in case of $\left(Gr_4(\mathbf{R}^7), S_7\right)$. So we shall sketch a proof.

The tangent space $\mathfrak{m}$ is regarded as $\left(\mathbf{C}^2_+ \otimes S^3\mathbf{C}^2_-\right)^{\mathbf{R}}$. Since $\mathfrak{l}_\xi$ is isomorphic to $\mathfrak{u}(1) \subset \Delta \subset \mathfrak{sp}_+(1) \oplus \mathfrak{sp}_-(1)$, we get a decomposition of $\mathfrak{m}$ as $\mathfrak{l}_\xi$-module:

$$\mathfrak{m}^{\mathbf{C}} = \mathbf{C}_4 \oplus_\perp 2\mathbf{C}_2 \oplus_\perp 2\mathbf{C} \oplus_\perp 2\mathbf{C}_{-2} \oplus_\perp \mathbf{C}_{-4}.$$

Considering real representations, we can take

$$\mathfrak{m} = \mathbf{C}_4 \oplus_\perp 2\mathbf{C}_2 \oplus_\perp 2\mathbf{R}.$$

Since

$$TS_0 = \mathbf{C}_4 \oplus \mathbf{C}_2, \quad TS_M = \mathbf{C}_4 \oplus \mathbf{C}_2 \oplus \mathbf{R}, \quad U = \mathbf{R} \oplus \mathbf{C}_2 \oplus \mathbf{R}, \quad V = \mathbf{R} \oplus \mathbf{C}_2,$$

we can not obtain the same conclusion as before.

We consider a $\mathfrak{u}(1)$-irreducible decomposition of $\mathbf{R}^7$:

$$\mathbf{R}^7 = \mathbf{R}w \oplus \mathbf{R} \oplus \mathbf{C}_2 \oplus \mathbf{R}v \oplus \mathbf{C}_2.$$

Let u_1, u_2 be an orthonormal basis of $\mathbf{C}_2 \subset U$ and $u_3 \in \mathbf{R} \subset U$ be an unit vector. We can put $\xi^2 u_i = xu_i$ for $i = 1, 2$ and $\xi u_3 = 0$. If follows from Lemma 4.12 that

$$\frac{pq}{n} = 1 - 2x,$$

and so

$$x = -\frac{1}{4}.$$

Let η be a normal vector of $T_o S_M$ which is orthogonal to ξ, (which yields that $\eta \in \mathbf{C}_2 \subset V$), and satisfies that $|\eta w| = 1$. We put $\eta = \eta_0 + \eta_1$ according to the decomposition $\mathfrak{m} = TS_0 \oplus U$. Note that $\eta_0 \in \mathbf{C}_2 \subset TS_0$ and $\eta_1 \in \mathbf{C}_2 \subset U$. Then we have $|\eta_1|^2 = \frac{3}{8}, \quad |\eta_0|^2 = \frac{9}{8}$, and so,

$$|\eta_0| : |\eta_1| = 3 : \sqrt{3}.$$

Hence, if X is a vector of $\mathbf{C}_2 \subset T_o S_M$ and $X = X_0 + X_1$, where $X_0 \in \mathbf{C}_2 \subset T_o S_0$ and $X_1 \in \mathbf{C}_2 \subset U$, then we obtain

$$|X_0| : |X_1| = 1 : \sqrt{3}.$$

From (4.1) and Theorems 4.7, 4.9, 4.10 and 4.13, we need to compute the eigenvalues of

$$\frac{1}{4}\sqrt{\frac{n}{pq}}\frac{1}{|s||t|}\begin{pmatrix} -3|t|^2 & \sqrt{3}|t|^2 \\ \sqrt{3}|t|^2 & 4|s|^2 - |t|^2 \end{pmatrix}$$

to obtain the principal curvatures. Then we have

Theorem 4.21. *The principal curvatures of the regular level set $f^{-1}(c)$ of the function f are*

$$\frac{1}{\sqrt{6}}\frac{1}{|s||t|}\left\{(|s|^2 - |t|^2) \pm \sqrt{1 - |s|^2|t|^2}\right\}, \quad -\sqrt{\frac{2}{3}}\frac{|t|}{|s|}, \quad 0,$$

with multiplicities 2, 2, 1 *and* 2, *respectively.*

4.2. The case of cohomogeneity greater than one. In this subsection, we see that $f = |s|^2$ is not an isoparametic function in each case. However, if we adopt Wang's definition of isoparametric functions ([18] or see also [2, p.55]), it will be shown that we can find a vector valued isoparametirc function $F : G/K \to \mathbf{R}^k$ which has f as a component, where k is the cohomogeneity of the H-action. Every H-orbit is included in a level set of F.

Moreover, we shall show that there exists a hidden symmetry in each case, in other words, $w \in W$ determines another subgroup of G. We obtain a subgroup $\tilde{H} \subset G$ such that $H \subset \tilde{H}$. The action of $\tilde{H}$ on G/K is of cohomogeneity one. Finally, the corresponding isoparametric functions are specified and we shall detect the relation between $w \in W$ and the new function.

Remark. For completeness, we give a definition of an isoparametric function by Wang. Let $f = (f_1, \cdots, f_k) : M \to \mathbf{R}^k$ be a function on a Riemannian manifold (M, g_M) with values in $\mathbf{R}^k$. The function f is called an *isoparametric function* if there exist functions $F_{ij}, G_i : \mathbf{R}^k \to \mathbf{R}$ $(1 \leqq i, j \leqq k)$ such that

$$(1)\ g_M(df_i, df_j) = F_{ij}(f_1, \cdots, f_k) \quad (2)\ \Delta f_i = G_i(f_1, \cdots, f_k).$$

This definition is different from a definition of Terng [15]. Though Terng's definition is stronger than one of Wang, Terng get a deep and beautiful structural theory. See also Heintze, Liu and Olmos [4] for isoparametric submanifolds. In both, the principal orbit of an hyperpolar action is a typical example.

• $(\mathrm{SU}(n)/\mathrm{SO}(n), \mathbf{C}^n)$

The tangent space $\mathfrak{m}$ is identified with a representation $S^2_0\mathbf{R}^n$ of $\mathrm{SO}(n)$, where $S^2_0\mathbf{R}^n$ denotes the set of tracefree symmetric transformations on $\mathbf{R}^n$. We denote by $\pi_0 : \mathbf{R}^n \otimes \mathbf{R}^n \to S^2_0\mathbf{R}^n$ the indicated orthogonal projection.

According to a generalised Cartan decomposition of $\mathbf{C}^n$, we obtain $z = x+iy \in \mathbf{R}^n \oplus i\mathbf{R}^n \cong \mathbf{R}^n \oplus \mathbf{R}^n$. Hence the vector bundle $\mathbf{V} \to G/K$ is naturally identified with $U \to G/K$, and we do not distinguish one from the other.

Let Y be an element of $S^2_0\mathbf{R}^n$. Since $iY \in \mathfrak{m} \subset \mathfrak{su}(n)$, we have

$$(iY)(x+iy) = -Yy + iYx, \quad \text{and so,} \quad Y(x,y) = (-Yy, Yx).$$

When $\mathbf{C}^n$ is regarded as a *real* representation of $\mathrm{SU}(n)$ and the orthogonal projections are defined as $\pi_U(x+iy) = x$ and $\pi_V(x+iy) = y$, we have

$$\nabla_{\pi(L_g(iY))}s = \left[g, -(iY)\pi_V(g^{-1}w)\right] = \left[g, Y\pi_V(g^{-1}w)\right],$$
$$\nabla_{\pi(L_g(iY))}t = \left[g, -(iY)\pi_U(g^{-1}w)\right] = \left[g, -Y\pi_U(g^{-1}w)\right],$$

where $g \in G$. For simplicity, we identify $Y \in \mathfrak{m}$ with the tangent vector $\pi(L_gY)$ to G/K and $\nabla_Y s$ and $\nabla_Y t$ are abbreviated to Yt and $-Ys$, respectively.

Then we get

$$df = 2g_U\,(\nabla s, s) = 2g_U\,(Yt, s) = 2g_{Gr}(Y, s \otimes t),$$

where g_{Gr} is the Riemannian metric on $Gr_n(\mathbf{R}^{2n})$, which is the target of the totally geodesic immersion of $G/K \to Gr_n(\mathbf{R}^{2n})$. Hence we obtain

$$df = 2\pi_0(s \otimes t) = 2\left(s \cdot t - \frac{g_U(s,t)}{n}I_n\right),$$

where

$$s \cdot t = \frac{1}{2}\,(s \otimes t + t \otimes s)\,,$$

and I_n denotes the identity transformation of $U \to G/K$. Consequently, we have

$$|df|^2 = 2\left(|s|^2|t|^2 + \frac{n-2}{n}g_U(s,t)^2\right),$$

which shows that f is *not* an isoparametric function. Note that f is an isoparametric function in the case that $n = 2$, since we have

$$\left(\mathrm{SU}(2)/\mathrm{SO}(2), \mathbf{C}^2\right) = \left(\mathbf{C}P^1, \mathbf{C}^2\right),$$

which was already seen in the previous subsection.

We compute

$$dg_U(s,t)(Y) = g_U(Yt,t) - g_U(s,Ys) = g_{Gr}(Y, t \otimes t - s \otimes s)$$

and so, we get

$$dg_U(s,t) = \pi_0(t \cdot t - s \cdot s) = t \cdot t - s \cdot s + \frac{|s|^2 - |t|^2}{n}I_n.$$

It follows that

$$|dg_U(s,t)|^2 = |s|^4 - 2g_U(s,t)^2 + |t|^4 - \frac{1}{n}\left(|s|^2 - |t|^2\right)^2,$$
$$g\left(df, dg_U(s,t)\right) = -\frac{2(n-1)}{n} g_U(s,t)\left(|s|^2 - |t|^2\right),$$

Moreover, we have

$$\sum g_U(\nabla_{e_i} s, \nabla_{e_i} t) = -\sum g_U(e_i t, e_i s) = \sum g_U(e_i e_i t, s) = -g_U(\Delta t, s).$$

It follows from Theorem 4.2 that

$$\begin{aligned}\Delta g_U(s,t) =& g_U(\Delta s, t) - 2\sum g_U(\nabla s, \nabla t) + g_U(s, \Delta t)\\ =& \frac{2(n-1)(n+2)}{n} g_U(s,t).\end{aligned}$$

Consequently, we obtain an isoparametric function F with values in $\mathbf{R}^2$:

$$F := \left(|s|^2 - |t|^2, 2g_U(s,t)\right).$$

Since $g_U(s,t)$ is also H-invariant, the level sets of F consist of H-orbits.

We put $\tilde{f} = |F|^2 = \left(|s|^2 - |t|^2\right)^2 + 4g_U(s,t)^2$.

Theorem 4.22. *The function $\tilde{f}$ is an isoparametric function on the symmetric space $SU(n)/SO(n)$.*

Proof. Combined our direct computations with a well-known formula $\Delta f^2 = 2\left\{f\Delta f - |df|^2\right\}$, we have

$$\Delta\tilde{f} = 4(n+1)\tilde{f} - 8.$$

Moreover, it follows that

$$|d\tilde{f}|^2 = 8\tilde{f}(1 - \tilde{f}).$$

□

We explain how $w \in W$ relates to $\tilde{f}$. Let h be an invariant Hermitian product on $W \cong \mathbf{C}^n$. Then $iw \otimes h(\cdot, w) - \frac{i}{n}I_n \in W \otimes W^*$ can be considered as an element of $\mathfrak{su}(n)$. We have a generalised Cartan decomposition of $\mathfrak{su}(n)$, which is a standard decomposition $\mathfrak{su}(n) = \mathfrak{so}(n) \oplus \mathfrak{m}$. Hence $iw \otimes h(\cdot, w) - \frac{i}{n}I_n$ determines a section $\tilde{s}$ of the holonomy bundle $\mathrm{SU}(n) \times_{\mathrm{SO}(n)} \mathfrak{so}(n)$. Since

$$\tilde{s} = \left[g, pr\left(g^{-1}\left(iw \otimes h(\cdot, w) - \frac{i}{n}I_n\right)\right)\right], \quad g \in \mathrm{SU}(n),$$

where $pr : \mathfrak{su}(n) \to \mathfrak{so}(n)$ is the orthogonal projection and $w = s + it$, we have

$$\tilde{s} = s \otimes g_U(\cdot, t) - t \otimes g_U(\cdot, s).$$

Consequently, we obtain

$$2|\tilde{s}|^2 = 4\left(|s|^2|t|^2 - g_U(s,t)^2\right) = 1 - \left\{\left(|s|^2 - |t|^2\right)^2 + 4g_U(s,t)^2\right\} = 1 - \tilde{f}. \tag{4.3}$$

Since $iw \otimes h(\cdot, w) - \frac{i}{n}I_n$ is invariant under the action of $S\left(U(1) \times U(n-1)\right)$ which is denoted by $\tilde{H}$, we have

Lemma 4.23. *The function $\tilde{f}$ is invariant under the action of $\tilde{H}$.*

If we check the action of $\tilde{H}$ on $SU(n)/SO(n)$ at o infinitesimally, it follows that the action of $\tilde{H}$ on $SU(n)/SO(n)$ is of cohomogeneity one.

Next, we determine critical points of $\tilde{f}$. We begin with a simple algebraic lemma, whose proof is left to the reader.

Lemma 4.24. *Let u and v be vectors in $\mathbf{R}^n$. Then $\pi_0(u \cdot v)$ and $\pi_0(u^2 - v^2)$ are linearly independent if and only if u and v are linearly independent.*

We have

$$d\tilde{f} = 8\left(|s|^2 - |t|^2\right)\pi_0(s \cdot t) + 8g_U(s,t)\pi_0(t^2 - s^2). \tag{4.4}$$

Lemma 4.25. *The set of critical points of $\tilde{f}$ consists of those points in $\tilde{f}^{-1}(0)$ and $\tilde{f}^{-1}(1)$.*

Proof. From (4.4), $x \in SU(n)/SO(n)$ is a critical point of $\tilde{f}$ if and only if $F(x) = 0$ or $\pi_0(s \cdot t)(x)$ and $\pi_0(t^2 - s^2)(x)$ are linearly dependent and

$$\left(|s|^2 - |t|^2\right)\pi_0(s \cdot t) + g_U(s,t)\pi_0(t^2 - s^2) = 0 \tag{4.5}$$

at x.

Of course, $F(x) = 0$ is equivalent to $\tilde{f}(x) = 0$.

The latter condition yields that $s(x)$ and $t(x)$ are linearly dependent by lemma 4.24. Then the equation (4.5) is automatically satisfied. The Cauchy-Schwarz inequality implies that

$$\tilde{f} \leqq (|s|^2 - |t|^2)^2 + 4|s|^2|t|^2 = \left(|s|^2 + |t|^2\right)^2 = 1,$$

where the equality holds if and only if $s(x)$ and $t(x)$ are linearly dependent. □

We can describe $\tilde{f}^{-1}(0)$ and $\tilde{f}^{-1}(1)$ as $\tilde{H} \cong U(n-1)$-orbits, respectively. In fact, we have

$$\tilde{f}^{-1}(0) = U(n-1)/U(1) \times SO(n-2) \cong SU(n-1)/SO(n-2),$$
$$\tilde{f}^{-1}(1) = U(n-1)/SO(n-1) \supset S_0,\ S_M.$$

We already see that

$$dF(x) = \left(4\pi_0(s \cdot t), 2\pi_0(t^2 - s^2)\right).$$

If $F(x) = 0$, then $\tilde{f}(x) = 0$ and so, $s(x)$ and $t(x)$ are linearly independent. Lemma 4.24 yields that $\pi_0(s \cdot t)$ and $\pi_0(t^2 - s^2)$ is also linearly independent. Hence x is a regular point of F. Indeed, from the above description, though $\tilde{f}^{-1}(0)$ is a singular orbit of $\tilde{H}$, $\tilde{F}^{-1}(0)$ is not a singular orbit of H.

Lemma 4.26. *One orbit $F^{-1}(0)$ of the action of H on $SU(n)/SO(n)$, which is not a singular orbit, is a minimal submanifold of $SU(n)/SO(n)$.*

Proof. The orbit $F^{-1}(0)$ is equal to $\tilde{f}^{-1}(0)$ and $\tilde{f}^{-1}(0)$ is a singular orbit of $\tilde{H}$. The theorem of Hsiang-Lawson [7] yields the result. □

Lemma 4.27. *The action of H on $SU(n)/SO(n)$ is not a hyperpolar action.*

Proof. We can apply [5, Theorem 3.13, p.231] to get the result. (We also refer to [5] to see the definition of the hyperpolar action.) □

Corollary 4.28. *The submanifold $F^{-1}(c)$, where c is a regular value of F, is not an equifocal submanifold of $SU(n)/SO(n)$.*

See [16] for the definition of the equifocal submanifold, which is considered as a generalization of isoparametric hypersurfaces.

Next we focus our attention on $\tilde{f}^{-1}(1)$. From (4.3), $\tilde{f}^{-1}(1)$ is nothing but a zero locus of the section $\tilde{s}$. In addition, since $\mathfrak{su}(n) = \mathfrak{so}(n) \oplus \mathfrak{m}$ is a generalised Cartan decomposition, we have a totally geodesic immersion from $\mathrm{SU}(n)/\mathrm{SO}(n)$ to a real Grassmannian $Gr_p(\mathfrak{su}(n))$, where $p = \dim \mathrm{SO}(n)$. Then the same method as in the proof of Theorem 3.6 yields

Theorem 4.29. *The level set $\tilde{f}^{-1}(1)$ is a totally geodesic submanifold of $SU(n)/SO(n)$.*

- $(\mathrm{Sp}(n)/\mathrm{U}(n), \mathbf{C}^n)$

The holomorphic tangent space is identified with $S^2\mathbf{C}^{n^*}$ as complex $\mathrm{U}(n)$-module, where $S^2\mathbf{C}^{n^*}$ denotes a symmetric power of $\mathbf{C}^{n^*}$. Hence, $S^2\mathbf{C}^{n^*} \oplus S^2\mathbf{C}^n$ is regarded as the complexification of $\mathfrak{m}$, which is denoted by $\mathfrak{m}^{\mathbf{C}}$. Let $\sigma : \mathfrak{m}^{\mathbf{C}} \to \mathfrak{m}^{\mathbf{C}}$ be the real structure. If $Y \in \mathfrak{m}$ is a real vector, then there exists a unique $Z \in S^2\mathbf{C}^{n^*}$ such that $Y = (Z, \sigma(Z)) \in \mathfrak{m}^{\mathbf{C}}$.

Let $j : \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ be an invariant quaternion structure. We regard $\mathbf{C}^{2n}$ as a left $\mathbf{H}$-module with j. As $\mathrm{U}(n)$-module, we have $\mathbf{C}^{2n} = \mathbf{C}^n \oplus \mathbf{C}^{n^*}$. If $Z \in S^2\mathbf{C}^{n^*}$ is regarded as a homomorphism $Z : \mathbf{C}^n \to \mathbf{C}^{n^*}$, then we have $\sigma(Z) = jZj : \mathbf{C}^{n^*} \to \mathbf{C}^n$, where the quaternion structure j is restricted to $\mathbf{C}^{n^*}$. Consequently, $Y \in \mathfrak{m}$ acts on $(u, v) \in \mathbf{C}^n \oplus \mathbf{C}^{n^*}$ in the following way:

$$Y(u, v) = (\sigma(Z)v, Zu)\,,$$

where $Y = (Z, \sigma(Z)) \in \mathfrak{m}^{\mathbf{C}}$.

We put $U = G \times_K \mathbf{C}^n$ and $V = G \times_K \mathbf{C}^{n^*} \cong U^*$. With our convention, we have

$$\nabla_{\pi(L_g(Y))} s = \left[g, -\sigma(Z)\pi_V(g^{-1}w)\right], \quad \nabla_{\pi(L_g(Y))} t = \left[g, -Z\pi_U(g^{-1}w)\right],$$

where $g \in G$. For simplicity, we identify $Y \in \mathfrak{m}$ with the tangent vector $\pi(L_g Y)$ to G/K and $\nabla_Y s$ and $\nabla_Y t$ are abbreviated to $-\sigma(Z)t$ and $-Zs$, respectively.

Then we get

$$\begin{aligned} df(Y) &= g_U(\nabla_Y s, s) + g_U(s, \nabla_Y s) = -g_U(\sigma(Z)t, s) - g_U(s, \sigma(Z)t) \\ &= -h_{Gr}(\sigma(Z), g_V(\cdot, t) \otimes s) - h_{Gr}(g_V(\cdot, t) \otimes s, \sigma(Z))\,, \end{aligned}$$

where h_{Gr} is the Hermitian metric on $Gr_n(\mathbf{C}^{2n})$, which is the target of the totally geodesic immersion of $G/K \to Gr_n(\mathbf{C}^{2n})$.

Hence we obtain

$$df^{1,0} = s \cdot g_V(\cdot, t) = \frac{1}{2}(s \otimes g_V(\cdot, t) + g_V(\cdot, t) \otimes s)\,.$$

Consequently, we have

$$|df|^2 = \left(|s|^2|t|^2 + |(s, t)|^2\right),$$

where $(\cdot, \cdot)$ denotes the pairing between $\mathbf{U} \to G/K$ and $\mathbf{V} \to G/K$. This shows that f is *not* an isoparametric function.

We compute

$$d(s,t)(Y) = -(\sigma(Z)t, t) - (s, Zs) = -(\sigma(Z), t\otimes t) - (s\otimes s, Z),$$

where $(\cdot,\cdot)$ in the right-hand-side denotes the obvious pairing. It follows that

$$d(s,t) = -t\cdot t - s\cdot s.$$

As a result, we have

$$|d(s,t)|^2 = |s|^4 + |t|^4,$$
$$h\left(d(s,t), df\right) = -\left(|s|^2 + |t|^2\right)(s,t) = -(s,t),$$

Moreover, we have

$$\sum(\nabla_{e_i} s, \nabla_{e_i} t) = -(\Delta t, s).$$

It follows from Theorem 4.2 that

$$\Delta(s,t) = (\Delta s, t) - 2\sum(\nabla s, \nabla t) + (s, \Delta t) = 2(n+1)(s,t).$$

Consequently, we obtain an isoparametric function F with values in $\mathbf{R}^3$:

$$F := \left(|s|^2 - |t|^2, 2(s,t)\right).$$

Since (s,t) is also H-invariant, the level sets of F consists of H-orbits.

We put $\tilde{f} = |F|^2 = \left(|s|^2 - |t|^2\right)^2 + 4|(s,t)|^2$.

Theorem 4.30. *The function $\tilde{f}$ is an isoparametric function on the symmetric space $Sp(n)/U(n)$.*

Proof. In a similar computation to one in a proof of Theorem 4.22, we have

$$|d\tilde{f}|^2 = 4\tilde{f}(1-\tilde{f}),$$

and

$$\Delta\tilde{f} = 2(2n+1)\tilde{f} - 6.$$

□

We discuss a relation between $w \in W$ and $\tilde{f}$. Let ω be an invariant symplectic form on $W \cong \mathbf{C}^{2n}$ and we do not distinguish between $\mathbf{C}^{2n}$ and $\mathbf{C}^{2n^*}$. We can consider $w \wedge jw \in \wedge^2\mathbf{C}^{2n}$. We have an irreducible decomposition $\wedge^2\mathbf{C}^{2n} = \wedge^2_0\mathbf{C}^{2n} \oplus \mathbf{C}\omega$ as $\mathrm{Sp}(n)$-module, and so we define the orthogonal projection $\pi_0 : \wedge^2\mathbf{C}^{2n} \to \wedge^2_0\mathbf{C}^{2n}$. As a $\mathrm{U}(n)$-module, we have $\wedge^2_0\mathbf{C}^{2n} = \wedge^2\mathbf{C}^n \oplus \wedge^2\mathbf{C}^{n^*} \oplus \mathfrak{su}(n)^{\mathbf{C}}$. Taking a real part, we get the orthogonal projection $pr : \left(\wedge^2_0\mathbf{C}^{2n}\right)^{\mathbf{R}} \to \left(\wedge^2\mathbf{C}^n \oplus \wedge^2\mathbf{C}^{n^*}\right)^{\mathbf{R}}$. Hence $w \wedge jw$ determines a section $\tilde{s}$ of the bundle $\mathrm{Sp}(n) \times_{\mathrm{U}(n)} \left(\wedge^2\mathbf{C}^n \oplus \wedge^2\mathbf{C}^{n^*}\right)^{\mathbf{R}}$. Since

$$\tilde{s} = \left[g, pr\left(g^{-1}\pi_0\left(w \wedge jw\right)\right)\right], \quad g \in \mathrm{Sp}(n),$$

we have

$$\tilde{s} = -s \otimes g_V(\cdot, t) - t \otimes g_U(\cdot, s).$$

Consequently, we obtain

$$2|\tilde{s}|^2 = 4\left(|s|^2|t|^2 - |(s,t)|^2\right) = 1 - \left\{\left(|s|^2 - |t|^2\right)^2 + 4|(s,t)|^2\right\} = 1 - \tilde{f}.$$

Since $w \wedge jw$ is invariant under the action of $\mathrm{Sp}(1) \times \mathrm{Sp}(n-1)$ which is denoted by $\tilde{H}$, we have

Lemma 4.31. *The function $\tilde{f}$ is invariant under the action of $\tilde{H}$.*

From the infinitesimal action of $\tilde{H}$ on $\mathrm{Sp}(n)/\mathrm{U}(n)$ at o, it follows that the action of $\tilde{H}$ on $\mathrm{Sp}(n)/\mathrm{U}(n)$ is of cohomogeneity one.

Remark. From the viewpoint of Sp(1), the function F is a moment map for the action of Sp(1) on $\mathrm{Sp}(n)/\mathrm{U}(n)$. Hence $\mathrm{Sp}(n-1)$ acts on the Kähler quotient. Indeed, the Kähler quotient is identified with a flag manifold $\mathrm{Sp}(n-1)/\mathrm{S}\left(\mathrm{U}(n-2) \times \mathrm{U}(1) \times \mathrm{U}(2)\right)$.

Next, we determine critical points of $\tilde{f}$. We have

$$d\tilde{f}^{1,0} = 4\left(|s|^2 - |t|^2\right) s \cdot g_V(\cdot, t) - 4\overline{(s,t)} s^2 + 4(s,t) g_V(\cdot, t)^2. \tag{4.6}$$

Lemma 4.32. *The set of critical points of $\tilde{f}$ consists of those points in $\tilde{f}^{-1}(0)$ and $\tilde{f}^{-1}(1)$.*

Proof. If s and $g_V(\cdot, t)$ are linearly dependent, then we have $d\tilde{f}^{1,0} = 0$ by (4.6).

Suppose that s and $g_V(\cdot, t)$ are linearly independent. Then, $s \cdot g_V(\cdot, t)$, s^2 and $g_V(\cdot, t)^2$ are linearly independent. It follows from (4.6) that $d\tilde{f}^{1,0} = 0$ if and only if $\tilde{f} = 0$.

Since $(s,t) = g_U\left(s, g_V(\cdot, t)\right)$, the Cauchy-Schwarz inequality implies that

$$\tilde{f} \leqq (|s|^2 - |t|^2)^2 + 4|s|^2|t|^2 = \left(|s|^2 + |t|^2\right)^2 = 1,$$

where the equality holds if and only if s and $g_V(\cdot, t)$ are linearly dependent. □

We can describe $\tilde{f}^{-1}(0)$ and $\tilde{f}^{-1}(1)$ as $\tilde{H}$-orbits, respectively. In fact, we have

$$\tilde{f}^{-1}(0) = \mathrm{Sp}(1) \times \mathrm{Sp}(n-1)/\mathrm{Sp}(1) \times \mathrm{U}(n-2) \cong \mathrm{Sp}(n-1)/\mathrm{U}(n-2),$$
$$\tilde{f}^{-1}(1) = S^2 \times \mathrm{Sp}(n-1)/\mathrm{U}(n-1) \supset S_0,\ S_M.$$

In similar ways in the case of $(\mathrm{SU}(n)/\mathrm{SO}(n), \mathbf{C}^n)$, we have

Lemma 4.33. *One orbit $F^{-1}(0)$ of the action of H on $Sp(n)/U(n)$, which is not a singular orbit, is a minimal submanifold of $Sp(n)/U(n)$.*

Lemma 4.34. *The action of H on $Sp(n)/U(n)$ is not a hyperpolar action.*

Corollary 4.35. *The submanifold $F^{-1}(c)$, where c is a regular value of F, is not an equifocal submanifold of $Sp(n)/U(n)$.*

Theorem 4.36. *The level set $\tilde{f}^{-1}(1)$ is a totally geodesic submanifold of $Sp(n)/U(n)$.*

• $\left(Gr_4(\mathbf{R}^9), S_9\right)$

Since $S_9 = S_4^+ \otimes S_5 \oplus S_4^- \otimes S_5$ as $\mathrm{Spin}(4) \times \mathrm{Spin}(5)$-module, we put $U = S_4^+ \otimes S_5$ and $V = S_4^- \otimes S_5$. More precisely, though we need to take a real part of each space, we omit the notation to indicate it. According to the decomposition

$$U \otimes V = \mathbf{R}^4 \otimes \left(\mathbf{R} \oplus \mathbf{R}^5 \oplus \mathfrak{so}(5)\right),$$

we define two orthogonal projections $\pi_0 : U \otimes V \to \mathbf{R}^4$ and $\pi_T : U \otimes V \to \mathbf{R}^4 \otimes \mathbf{R}^5$. Note that $\mathbf{R}^4$ and $\mathbf{R}^4 \otimes \mathbf{R}^5$ can also be considered as the tautological bundle and the cotangent bundle on $Gr_4(\mathbf{R}^9)$ with our convention.

We have

$$d|s|^2 = 2s \otimes t$$

on $Gr_8(S_9)$, where s and t are regarded as sections of the tautological bundle and the universal quotient bundle on $Gr_8(S_9)$, respectively. Since $S \otimes Q$ can be regarded as the cotangent bundle on $Gr_8(S_9)$, using a totally geodesic immersion $i : Gr_4(\mathbf{R}^9) \to Gr_8(S_9)$, we obtain

$$df = 2\pi_T(s \otimes t).$$

Lemma 4.37. *We have*

$$|df|^2 = 2\left(|s|^2|t|^2 - 6\,|\pi_0(s \otimes t)|^2\right).$$

Proof. First of all, we pay attention on Spin(5)-modules. We identify Spin(5) with Sp(2). Then S_5 is recognized with the standard representation $\mathbf{C}^4$ with an invariant symplectic form ω of Sp(2) and we have $\mathbf{C}^4 \otimes \mathbf{C}^4 = \mathbf{C}\omega \oplus \wedge_0^2\mathbf{C}^4 \oplus \mathfrak{so}(5)^{\mathbf{C}}$. If $u, v \in \mathbf{C}^4$, then, under the decomposition

$$u \otimes v = u \wedge v + u \cdot v, \quad u \wedge v = \frac{1}{2}\left(u \otimes v - v \otimes u\right), \quad u \cdot v = \frac{1}{2}\left(u \otimes v + v \otimes u\right),$$

we have

$$u \wedge v \in \mathbf{C}\omega \oplus \wedge_0^2\mathbf{C}^4, \quad u \cdot v \in \mathfrak{so}(5)^{\mathbf{C}}.$$

It follows that

$$|u \wedge v|^2 = \frac{1}{2}\left(|u|^2|v|^2 - |h(u,v)|^2\right),$$

where $h(\cdot,\cdot)$ is an invariant Hermitian product on $\mathbf{C}^4$.

We denote two orthogonal projections by $p_0 : \wedge^2\mathbf{C}^4 \to \mathbf{C}\omega$ and $p_T : \wedge^2\mathbf{C}^4 \to \wedge_0^2\mathbf{C}^4$, respectively. It follows from $|u \wedge v|^2 = |p_0(u \wedge v)|^2 + |p_T(u \wedge v)|^2$ that

$$|p_0(u \wedge v)|^2 + |p_T(u \wedge v)|^2 = \frac{1}{2}\left(|u|^2|v|^2 - |h(u,v)|^2\right). \tag{4.7}$$

Since $|\omega|^2 = 4$, we get

$$p_0(u \wedge v) = \frac{1}{4}\omega(u,v)\omega, \quad |p_0(u \wedge v)|^2 = \frac{1}{4}\,|\omega(u,v)|^2.$$

It follows that

$$|p_T(u \wedge v)|^2 = \frac{1}{2}\left(|u|^2|v|^2 - |h(u,v)|^2\right) - \frac{1}{4}|\omega(u,v)|^2. \tag{4.8}$$

The subgroup Spin(4) is now identified with $\mathrm{Sp}_+(1) \times \mathrm{Sp}_-(1)$. Let $\mathbf{C}_\pm^2$ be standard representations with invariant quaternion structures $j_\pm$ of $\mathrm{Sp}_\pm(1)$, respectively. Note that $\mathbf{C}_\pm^2$ are equivalent to $S_4^\pm$, respectively. We denote by e_1, e_2 the standard basis of $\mathbf{C}^2_+$. This means that e_1, e_2 is a unitary basis with $e_2 = j_+ e_1$. The standard basis of $\mathbf{C}^2_-$ is denoted by f_1, f_2. Let $a = e_1 \otimes u_1 + e_2 \otimes u_2$ be a real vector in $\mathbf{C}^2_+ \otimes \mathbf{C}^4$. This yields that

$$ju_1 = u_2,$$

where j is an invariant quaternion structure on $\mathbf{C}^4$. We denote a real vector in $\mathbf{C}^2_- \otimes \mathbf{C}^4$ by $b = f_1 \otimes v_1 + f_2 \otimes v_2$ with $jv_1 = v_2$. We have

$$a \otimes b = \sum \left(e_i \otimes f_j\right) \otimes \left(u_i \otimes v_j\right).$$

By definition, we get

$$\pi_T(a \otimes b) = \sum \left(e_i \otimes f_j\right) \otimes p_T\left(u_i \wedge v_j\right),$$

and so,

$$|\pi_T(a \otimes b)|^2 = \sum \left|p_T\left(u_i \wedge v_j\right)\right|^2. \tag{4.9}$$

Since a and b are real vectors, we have, for instance,

$$h(u_1, v_1) = -h(u_1, jv_2) = \omega(u_1, v_2).$$

Consequently, it follows from (4.8) that

$$|p_T\,(u_1 \wedge v_1)|^2 = \frac{1}{2}\left(|u_1|^2|v_1|^2 - |\omega(u_1, v_2)|^2\right) - \frac{1}{4}|\omega(u_1, v_1)|^2.$$

and so, (4.9) yields that

$$|\pi_T(a \otimes b)|^2 = \frac{1}{2}\left(|u_1|^2 + |u_2|^2\right)\left(|v_1|^2 + |v_2|^2\right) - \frac{3}{4}\sum |\omega(u_i, v_j)|^2. \tag{4.10}$$

The definition yields that

$$\pi_0(a \otimes b) = \sum \left(e_i \otimes f_j\right) \otimes p_0\left(u_i \wedge v_j\right), \tag{4.11}$$

and so,

$$|\pi_0(a \otimes b)|^2 = \sum \left|p_0\left(u_i \wedge v_j\right)\right|^2 = \frac{1}{4}\sum \left|\omega(u_i, v_j)\right|^2. \tag{4.12}$$

It follows from (4.10) and (4.12) that

$$|\pi_T(a \otimes b)|^2 = \frac{1}{2}|a|^2|b|^2 - 3\,|\pi_0(a \otimes b)|^2,$$

which yields the result. □

If $\pi_0(s \otimes t) \neq 0$, then it follows that f is *not* an isoparametric function on $Gr_4(\mathbf{R}^9)$. Since $\pi_0(s \otimes t)$ is a section of $\mathbf{R}^4$ determined by $w \in S^9$, we need to see how $\pi_0(s \otimes t)$ corresponds to w. Note that $w \otimes w$ is an element of $S^2 S_9$ the symmetric power of S_9. As Spin(9)-module, we have a decomposition $S^2 S_9 = \mathbf{R} \oplus \mathbf{R}^9 \oplus \wedge^4 \mathbf{R}^9$. Let $\Pi : S^2 S_9 \to \mathbf{R}^9$ be the orthogonal projection. We define a Spin(9)-equivariant map $\alpha : S_9 \to \mathbf{R}^9$ as

$$\alpha(w) = \Pi(w \otimes w).$$

To describe $\alpha : S_9 \to \mathbf{R}^9$ explicitly, we use a diagonal subgroup $\Delta \subset \mathrm{Sp}_+(1) \times \mathrm{Sp}_-(1)$ and regard S_9 and $\mathbf{R}^9$ as $\Delta \times \mathrm{Sp}(2)$-modules:

$$S_9 = \left(\mathbf{C}^2 \otimes \mathbf{C}^4\right)^{\mathbf{R}} \oplus \left(\mathbf{C}^2 \otimes \mathbf{C}^4\right)^{\mathbf{R}}, \quad \mathbf{R}^9 = \mathbf{R} \oplus \left(S^2\mathbf{C}^2\right)^{\mathbf{R}} \oplus \left(\wedge_0^2 \mathbf{C}^4\right)^{\mathbf{R}},$$

where $\mathbf{C}^2$ denotes the standard representation of Δ. We use Δ to define a quaternion structure on $\left(\mathbf{C}^2 \otimes \mathbf{C}^4\right)^{\mathbf{R}}$ and so, $\mathbf{R} \oplus \left(S^2\mathbf{C}^2\right)^{\mathbf{R}} \subset \mathbf{R}^9$ is identified with a scalar field $\mathbf{H}$. Then we have

$$S_9 = \mathbf{H}^2 \oplus \mathbf{H}^2, \quad \mathbf{R}^9 = \mathbf{H} \oplus \left(\wedge_0^2 \mathbf{C}^4\right)^{\mathbf{R}}.$$

Using a quaternion structure, we can also show

Lemma 4.38. *For an arbitrary* $(u, v) \in S_9 = \mathbf{H}^2 \oplus \mathbf{H}^2$, $\alpha : S_9 \to \mathbf{R}^9$ *can be expressed as:*

$$\alpha(u, v) = c\left(h_{\mathbf{H}}(u, v), p_T(u \wedge ju) - p_T(v \wedge jv)\right),$$

where c *is a real non-zero constant and* $h_{\mathbf{H}}$ *denotes a quaternion hermitian inner product on* $\mathbf{H}^2$.

The sections s and t are locally expressed as

$$s = e_1 \otimes s_1 + e_2 \otimes s_2, \quad t = f_1 \otimes t_1 + f_2 \otimes t_2,$$

where $\{e_1, e_2\}$ and $\{f_1, f_2\}$ are now regarded as local standard frames. Since s and t are real sections, we have

$$js_1 = s_2, \quad jt_1 = t_2.$$

Under the identification $S_9 = \mathbf{H}^2 \oplus \mathbf{H}^2$, this yields that

$$g^{-1}w = \sqrt{2}(s_1, t_1) \in \mathbf{H}^2 \oplus \mathbf{H}^2, \quad g \in \mathrm{Spin}(9).$$

It follows from (4.11) and our identification $\mathbf{R}^4 \cong \mathbf{H}$ that

$$\pi_0(s \otimes t) = \sqrt{2} h_{\mathbf{H}}(s_1, t_1), \tag{4.13}$$

which is nothing but the section of the tautological bundle corresponding to $\alpha(w)$ (up to constant) by Lemma 4.38. Consequently, f is *not* an isoparametric function on $Gr_4(\mathbf{R}^9)$, but a new function $\tilde{f} := |\pi_0(s \otimes t)|^2$ is an isoparametric function considered in the previous subsection. We have a subgroup $\mathrm{Spin}(8) \subset \mathrm{Spin}(9)$ as an isotropy subgroup at $\alpha(w)$, which is denoted by $\tilde{H}$. Of course, $\tilde{f}$ is invariant under the action of Spin(8). Since $|s|^2 = |s_1|^2 + |s_2|^2 = 2|s_1|^2$ and $|t|^2 = 2|t_1|^2$, the Cauchy-Schwarz inequality implies that

$$|\pi_0(s \otimes t)|^2 \leqq \frac{1}{2}|s|^2|t|^2 = \frac{1}{8}\left\{1 - \left(|s|^2 - |t|^2\right)^2\right\},$$

where the equality holds if and only if $|s|^2 = |t|^2 = \frac{1}{2}$. In particular, the maximum value of $\tilde{f}$ is $\frac{1}{8}$. This yields that $|\alpha(w)|^2 = \frac{1}{8}$. Hence we have

$$\alpha(u, v) = \sqrt{2}\left(h_{\mathbf{H}}(u, v), p_T(u \wedge ju) - p_T(v \wedge jv)\right). \tag{4.14}$$

It follows that

$$\tilde{f}^{-1}(0) = Gr_4(\mathbf{R}^8) \supset S_0, S_M, \quad \tilde{f}^{-1}\left(\frac{1}{8}\right) = Gr_3(\mathbf{R}^8).$$

We define a function $F : Gr_4(\mathbf{R}^9) \to \mathbf{R}^2$:

$$F := \left(|s|^2 - |t|^2, \tilde{f}\right).$$

Lemma 4.39. *The function F is an isoparametric function.*

Proof. From Lemma 4.37, we get

$$\left|d\left(|s|^2 - |t|^2\right)\right|^2 = \frac{1}{2}\left\{1 - \left(|s|^2 - |t|^2\right)^2 - 6\tilde{f}\right\}.$$

We need to compute $g\left(d\left(|s|^2 - |t|^2\right), d\tilde{f}\right)$. Since $\pi_0(s \otimes t)$ is a section of the tautological bundle corresponding to $\alpha(w)$, it follows from (4.14) that

$$d\tilde{f} = 4h_{\mathbf{H}}(s_1, t_1) \otimes \{p_T(s_1 \wedge s_2) - p_T(t_1 \wedge t_2)\}.$$

On the other hand, we see that

$$d\left(|s|^2 - |t|^2\right) = 2df = 4\pi_T(s \otimes t) = 4\sum \left(e_i \otimes f_j\right) \otimes p_T\left(s_i \wedge t_j\right).$$

It follows from $\mathbf{R}^4 \cong \mathbf{H}$ that

$$\begin{aligned}
&\frac{1}{4}g\left(d\tilde{f}, \pi_T(s \otimes t)\right)\\
=&\omega(s_1, t_1)g\left(p_T(s_1 \wedge s_2) - p_T(t_1 \wedge t_2), p_T\left(s_1 \wedge t_1\right)\right)\\
+&\overline{\omega(s_1, t_1)}g\left(p_T(s_1 \wedge s_2) - p_T(t_1 \wedge t_2), p_T\left(s_2 \wedge t_2\right)\right)\\
+&h(s_1, t_1)g\left(p_T(s_1 \wedge s_2) - p_T(t_1 \wedge t_2), p_T\left(s_1 \wedge t_2\right)\right)\\
-&\overline{h(s_1, t_1)}g\left(p_T(s_1 \wedge s_2) - p_T(t_1 \wedge t_2), p_T\left(s_2 \wedge t_1\right)\right)\\
=&\frac{1}{4}\left(|s|^2 - |t|^2\right)^2\tilde{f}.
\end{aligned}$$

□

Since $F^{-1}\left(0, \frac{1}{8}\right) = \tilde{f}^{-1}\left(\frac{1}{8}\right)$, we obtain

Lemma 4.40. *One orbit $F^{-1}\left(0, \frac{1}{8}\right)$ of the action of H on $Gr_4(\mathbf{R}^9)$ is a totally geodesic submanifold of $Gr_4(\mathbf{R}^9)$.*

Remark. From $F^{-1}\left(0, \frac{1}{8}\right) = \tilde{f}^{-1}\left(\frac{1}{8}\right)$, we can get the well-known fact that Spin(7)/Sp(1)× Sp(1) ≅ $Gr_3(\mathbf{R}^8)$.

Lemma 4.41. *The action of H on $Gr_4(\mathbf{R}^9)$ is not a hyperpolar action.*

Corollary 4.42. *The submanifold $F^{-1}(c)$, where c is a regular value of F, is not an equifocal submanifold of $Gr_4(\mathbf{R}^9)$.*

5. Radon transforms

We obtained isoparametric functions $\tilde{f}$ in the previous section. In the case that the action of H is of cohomogeneity one, $\tilde{f}$ is invariant under the action of H. Otherwise, $\tilde{f}$ is invariant under the action of $\tilde{H}$. In both cases, if we pull back $\tilde{f}$ to G under the natural fibration $\pi : G \to G/K$, then the pull-back function $\pi^*\tilde{f}$ is invariant under the action of $H \times K$ on G, where H acts on G on the left and K on the right. Hence, we can push down $\pi^*\tilde{f}$ to get a function on $H\backslash G$.

To be more precise, we introduce the Radon transform. Let $\psi : G \to H\backslash G$ be a natural fibration and $d\mu$ is the normalized Haar measure on H. We use the same notation to denote the measure on the fibre of $\psi : G \to H\backslash G$ induced by $d\mu$. We define a Radon transform $R : C^\infty(G/K) \to C^\infty(H\backslash G)$ for an arbitrary function f on G/K as

$$R(f)(x) = \int_{\psi^{-1}(x)} \pi^* f d\mu, \quad x \in H\backslash G.$$

By definition, the Radon transform is a G-equivariant linear map.

5.1. The case of cohomogeneity one. Let $\tilde{f} = |s|^2 - \frac{p}{N}$ be an isoparametric function defined in the Remark after Theorem 4.13. Let $\{e_1, \cdots, e_N\}$ be an orthogonal basis of a real representation W such that $\{w = e_1, \cdots, e_p\}$ is a basis of U and $\{e_{p+1}, \cdots, e_N\}$ is a basis of V. By definition, we have

$$\tilde{f}(\pi(g)) = \left|\pi_U\left(g^{-1}w\right)\right|^2 - \frac{p}{N}, \quad g \in G.$$

Let $\{x_1, \cdots, x_N\}$ be the standard coordinate functions with respect to $e_1, \cdots, e_N$ on W. We get

$$\begin{aligned}\left|\pi_U\left(g^{-1}w\right)\right|^2 - \frac{p}{N} &= \sum_{i=1}^{p} x_i(g^{-1}w)^2 - \frac{p}{N}\sum_{A=1}^{N} x_A(g^{-1}w)^2 \\ &= \frac{1}{N}\left\{q\sum_{i=1}^{p} x_i(g^{-1}w)^2 - p\sum_{\alpha=p+1}^{N} x_\alpha(g^{-1}w)^2\right\},\end{aligned}$$

and so,

$$R(\tilde{f}) = \frac{1}{N}\left\{q\sum_{i=1}^{p} x_i(g^{-1}w)^2 - p\sum_{\alpha=p+1}^{N} x_\alpha(g^{-1}w)^2\right\}.$$

If a real representation is replaced by a complex representation, then we have a similar result.

Theorem 5.1. *The Radon transform of $\tilde{f}$ in the case of cohomogeneity one is an isoparametric function on a unit sphere of W which induces an isoparametric hypersurface of a sphere with two distinct principal curvatures.*

5.2. The case of cohomogeneity greater than one. We obtain Radon transforms of $\tilde{f}$ on case-by-case computations.

• $(\mathrm{SU}(n)/\mathrm{SO}(n), \mathbf{C}^n)$ $n \geqq 3$.

Let $\{e_1, Je_1, \cdots, e_n, Je_n\}$ be an orthogonal basis of a real representation $\mathbf{C}^n = \left(\mathbf{R}^{2n}, J\right)$ such that $\{w = e_1, \cdots, e_n\}$ is a basis of U and $\{Je_1, \cdots, Je_n\}$ is a basis of V. Since $\tilde{f} = \left(|s|^2 - |t|^2\right)^2 + 4g(s,t)^2$, by definition, we have

$$\tilde{f}(\pi(g)) = \left(\left|\pi_U\left(g^{-1}w\right)\right|^2 - \left|\pi_V\left(g^{-1}w\right)\right|^2\right)^2 + 4g\left(\pi_U\left(g^{-1}w\right), \pi_V\left(g^{-1}w\right)\right)^2,$$

where we identify U with V in a standard way and $g \in \mathrm{SU}(n)$. Let $\{x_1, y_1, \cdots, x_n, y_n\}$ be the standard coordinate functions with respect to $e_1, Je_1, \cdots, e_n, Je_n$ on W. It follows that

$$R(\tilde{f})(x,y) = \left(\sum_{i=1}^{n} x_i(g^{-1}w)^2 - \sum_{i=1}^{n} y_i(g^{-1}w)^2\right)^2 + 4\left(\sum_{i=1}^{n} x_i(g^{-1}w)y_i(g^{-1}w)\right)^2.$$

Theorem 5.2. *In the case of* $(SU(n)/SO(n), \mathbf{C}^n)$ $(n \geqq 3)$*, the Radon transform of* $\tilde{f}$ *is an isoparametric function defined by Nomizu* [12] *on a unit sphere of* $\mathbf{C}^n$ *which induces an isoparametric hypersurface of a sphere with four distinct principal curvatures.*

• $\left(\mathrm{Sp}(n)/\mathrm{U}(n), \mathbf{C}^{2n}\right)$ $n \geqq 2$.

Let $\{e_1, je_1, \cdots, e_n, je_n\}$ be a unitary basis of a complex representation $\mathbf{C}^{2n}$ such that $\{w = e_1, \cdots, e_n\}$ is a basis of $U \cong \mathbf{C}^n$ and $\{je_1, \cdots, je_n\}$ is a basis of $V \cong \mathbf{C}^{n^*}$. Since $\tilde{f} = \left(|s|^2 - |t|^2\right)^2 + 4|(s,t)|^2$ by definition, we have

$$\tilde{f}(\pi(g)) = \left(\left|\pi_U\left(g^{-1}w\right)\right|^2 - \left|\pi_V\left(g^{-1}w\right)\right|^2\right)^2 + 4\left|\left(\pi_U\left(g^{-1}w\right), \pi_V\left(g^{-1}w\right)\right)\right|^2,$$

where $g \in \mathrm{Sp}(n)$. Let $\{z_1, w_1, \cdots, z_n, w_n\}$ be the standard coordinate functions with respect to $e_1, Je_1, \cdots, e_n, Je_n$ on W. It follows that

$$R(\tilde{f})(z,w) = \left(\sum_{i=1}^{n} |z_i(g^{-1}w)|^2 - \sum_{i=1}^{n} |w_i(g^{-1}w)|^2\right)^2 + 4\left|\sum_{i=1}^{n} z_i(g^{-1}w)w_i(g^{-1}w)\right|^2.$$

Theorem 5.3. *In the case of* $\left(Sp(n)/U(n), \mathbf{C}^{2n}\right)$ $(n \geqq 2)$*, the Radon transform of* $\tilde{f}$ *is an isoparametric function on a unit sphere of* $\mathbf{C}^{2n}$ *which induces an isoparametric hypersurface of a sphere with four distinct principal curvatures.*

From [3, Satz in §6.1], we have

Theorem 5.4. *In each case, every isoparametric hypersurface of a sphere in a family defined by* $R(\tilde{f})$ *is homogeneous, in the sense that it is an orbit of the action of isometry group.*

• $\left(Gr_4(\mathbf{R}^9), S_9\right)$

We use an identification between S_9 and $\mathbf{H}^2 \oplus \mathbf{H}^2$ in the previous section. It follows from (4.13) that

$$R(\tilde{f})(u,v) = 2\,|h_{\mathbf{H}}(u,v)|^2.$$

Theorem 5.5. *The Radon transform of* $\tilde{f}$ *is an isoparametric function on a unit sphere of* S_9 *which induces a family of isoparametric hypersurfaces of a sphere with four distinct principal curvatures. Every isoparametric hypersurface in our family is homogeneous.*

We will postpone a proof until the last paragraph.

Since $\tilde{f}$ is also invariant under $\tilde{H}$, we can easily obtain a Radon transform of $\tilde{f}$ on $\tilde{H}\backslash G$, which is denoted by $\tilde{R}(\tilde{f})$. In each case, we also have a fibration $\tilde{\psi} : H\backslash G \to \tilde{H}\backslash G$ with totally geodesic fibres. More concretely, we have

$$S^{2n-1} \to \mathbf{C}P^{n-1}, \quad S^{4n-1} \to \mathbf{H}P^{n-1}, \quad \text{and} \quad S^{15} \to S^8.$$

Using the normalized Haar measure on $\tilde{H}$, we have

$$\tilde{\psi}^*\tilde{R}(\tilde{f}) = R(\tilde{f}).$$

Since $R(\tilde{f})$ is constant on the fibre of $\tilde{\psi} : H\backslash G \to \tilde{H}\backslash G$, it follows from Theorems 5.2, 5.3 and 5.5 that

Theorem 5.6. *The Radon transform $\tilde{R}(\tilde{f})$ is an isoparametric function on $\tilde{H}\backslash G$.*

We describe $\tilde{R}(\tilde{f})$ in the last case. To do so, we "normalize" $\tilde{f}$ to get an eigenfunction. Since $\pi_0(s \otimes t)$ is the corresponding section to $\alpha(w) \in \mathbf{R}^9$ with $|\alpha(w)|^2 = \frac{1}{8}$, it follows from the Remark after Theorem 4.13 that $\hat{f} := \tilde{f} - \frac{1}{18}$ is an eigenfunction. According to the $SO(4) \times SO(5)$ decomposition of $\mathbf{R}^9$, we put $(\tilde{u}, \tilde{v}) \in \mathbf{R}^4 \oplus \mathbf{R}^5 = \mathbf{R}^9$. Then Theorem 5.1 yields that $5|\tilde{u}|^2 - 4|\tilde{v}|^2$ is an isoparametric function. If α is restricted to the unit sphere of S_9, we have that $\tilde{\psi} = \alpha$, It follows from (4.14) that

$$\begin{aligned}\tilde{R}(\hat{f})(\alpha(u,v)) &= \frac{2}{72}\left[5\,|h_{\mathbf{H}}(u,v)|^2 - 4\left\{\frac{1}{4}(|u|^2+|v|^2)^2 - |h_{\mathbf{H}}(u,v)|^2\right\}\right] \\ &= \frac{1}{36}\left\{9\,|h_{\mathbf{H}}(u,v)|^2 - (|u|^2+|v|^2)^2\right\}.\end{aligned}$$

From [3, Satz in §6.4], Theorem 5.5 holds. We can directly check that $(|u|^2 + |v|^2)^2 - 9\,|h_{\mathbf{H}}(u,v)|^2$ is a harmonic function on S_9, but in [3], a polynomial $(|u|^2 + |v|^2)^2 - 8\,|h_{\mathbf{H}}(u,v)|^2$ is introduced as an isoparametric function, which is called a Cartan-Münzner polynomial.

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