

Title	THE FINITE GROUP ACTION AND THE EQUIVARIANT DETERMINANT OF ELLIPTIC OPERATORS III
Author(s)	Tsuboi, Kenji
Citation	Osaka Journal of Mathematics. 2019, 56(4), p. 759-785
Version Type	VoR
URL	https://doi.org/10.18910/73627
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

THE FINITE GROUP ACTION AND THE EQUIVARIANT DETERMINANT OF ELLIPTIC OPERATORS III

KENJI TSUBOI

(Received July 11, 2017, revised April 17, 2018)

Abstract

Let M be an even-dimensional closed oriented manifold and g a periodic automorphism of M of order p . In this paper, under the assumption that the fixed points of g^k ($1 \leq k \leq p-1$) are isolated, a calculation formula is provided for the homomorphism $I_D : \mathbb{Z}_p \rightarrow \mathbb{R}/\mathbb{Z}$ defined in [6] for equivariant twisted signature operators D over M . The formula gives a new method to study the periodic automorphisms of oriented manifolds. As examples of the application of the formula, results about the existence of the cyclic group action for 2,4,6-dimensional closed oriented manifolds are obtained.

1. Introduction

Let M be a $2m$ -dimensional closed oriented manifold and G a finite group acting on M . We assume that the action of G is effective and orientation-preserving. Let g be an element of G of order $p \geq 2$ and $\mathbb{Z}_p = \langle g \rangle$ the cyclic group generated by g . In this paper, we set the following assumption.

Assumption 1.1. *Some g^k ($1 \leq k \leq p-1$) has a fixed point, and any fixed point of g^k is isolated for $1 \leq k \leq p-1$ if g^k has a fixed point.*

In [6] we introduce a group homomorphism I_D by using an elliptic operator D adapted to a geometric structure of a manifold, and in [7] we apply this homomorphism to the existence problem of finite group actions on almost complex manifolds under the assumption above.

Let D be a G -equivariant elliptic operator. Then a homomorphism I_D from G to \mathbb{R}/\mathbb{Z} is defined by

$$I_D(g) = \frac{1}{2\pi\sqrt{-1}} \log \det(D, g) \in \mathbb{R}/\mathbb{Z}$$

for $g \in G$, where $\det(D, g)$ is defined by

$$\det(D, g) = \det(g|_{\ker D}) / \det(g|_{\text{coker } D}) \in S^1 \subset \mathbb{C}^*$$

(see [6] Definition 2.1). Then as we see in [6] (3) the next equality holds

$$(1) \quad I_D(g) \equiv \frac{p-1}{2p} \text{Ind}(D) - \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - \xi_p^{-k}} \text{Ind}(D, g^k) \pmod{\mathbb{Z}}$$

where Ind is the Atiyah-Singer index (see [3]) and ξ_p is the primitive p -th root of unity defined by $\xi_p = e^{2\pi\sqrt{-1}/p}$.

We can express the value $I_D(g)$ by the fixed point data of the g^k -action ($1 \leq k \leq p - 1$) by using the equality (1) and the fixed point formula of Atiyah-Segal-Singer [2], [3]. Since I_D is a homomorphism, the equalities $I_D(g^z) = zI_D(g)$, $I_D(gh) = I_D(g) + I_D(h)$ hold for any $g, h \in G$ and any integer z because \mathbb{R}/\mathbb{Z} is an abelian group. These properties of I_D impose conditions on the fixed point data and a calculation formula to obtain the precise value of I_D can be used to examine the existence of a finite group action on M .

When M is a $2m$ -dimensional closed almost complex manifold and the g -action preserves the almost complex structure of M , we give a formula to calculate $I_D(g)$ precisely for the $\otimes_{j=1}^m (\otimes_{\mathbb{C}}^j (\wedge_{\mathbb{C}}^j TM))$ -valued Dolbeault operator D over M in [7] Theorem 2.2. Though the formula is useful for closed almost complex manifolds, we need a formula to calculate the precise value of $I_D(g)$ for equivariant elliptic operators D which is adapted for closed oriented manifolds, which do not necessarily have almost complex structure.

In [9] Zagier gives a formula which enables us to calculate the precise value of $I_D(g)$ for the equivariant non-twisted signature operator D under Assumption 1.1.

DEFINITION 1.2. For a real number x , a real number $((x))$ is defined by

$$((x)) := \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer} \\ 0 & \text{if } x \text{ is an integer} \end{cases}$$

where $[x]$ is the Gauss' symbol. Note that $((x + a)) = ((x))$ for an integer a and that $((-x)) = -((x))$ because $[-x] = -[x] - 1$ if x is not an integer.

Theorem 1.3 ([9] p.103). *Let p be a natural number which is greater than 1 and a_1, \dots, a_n natural numbers which are prime to p . Then the following equality holds:*

$$(-\sqrt{-1})^n \sum_{k=1}^{p-1} \cot \frac{\pi a_1 k}{p} \cdots \cot \frac{\pi a_n k}{p} = 2^n p \sum_{\substack{1 \leq k_1, \dots, k_n < p \\ p | (a_1 k_1 + \dots + a_n k_n)}} \left(\left(\frac{k_1}{p} \right) \right) \cdots \left(\left(\frac{k_n}{p} \right) \right).$$

As we see later (see (4)) the left hand side of the equality above is equal to the sum of indices for the equivariant non-twisted signature operator, and the equality implies that the sum is a rational number.

In this paper, generalizing the Zagier's formula, we give a formula to calculate the precise value of $I_D(g)$ for equivariant twisted signature operators D over $2m$ -dimensional closed oriented manifolds to examine the existence of a finite group action on the manifolds.

2. Rotation angles of periodic automorphisms

Let M be a $2m$ -dimensional closed oriented manifold and p a natural number which is greater than 1. Assume that the cyclic group $\mathbb{Z}_p = \langle g \rangle$ acts on M . Under Assumption 1.1, let Ω be the union of the fixed points of g^k for $1 \leq k \leq p - 1$ and suppose that the image $\pi(\Omega)$ consists of b points $y_1, \dots, y_b \in M/\mathbb{Z}_p$ where $\pi : M \rightarrow M/\mathbb{Z}_p$ is the projection. In this paper, the \mathbb{Z}_p -action is called the \mathbb{Z}_p -action of isotropy orders (p_1, \dots, p_b) if the isotropy group at a point $q_i \in \pi^{-1}(y_i)$ ($1 \leq i \leq b$) is the cyclic group of order p_i . Then for $1 \leq i \leq b$ the isotropy group at any points in $\pi^{-1}(y_i)$ is the cyclic group of order p_i generated by g^{r_i}

where $r_i = p/p_i$ and $\pi^{-1}(y_i)$ consists of r_i points $q_i, g \cdot q_i, \dots, g^{r_i-1} \cdot q_i$.

REMARK 2.1. Let $e(M)$ be the Euler number of M . Then since \mathbb{Z}_p acts freely on the punctured manifold $M_0 = M \setminus \{\cup_{i=1}^b \pi^{-1}(y_i)\}$, the next equality holds:

$$(2) \quad e(M) \equiv \sum_{i=1}^b r_i \pmod{p}.$$

Let $\xi_{p_i}^{\tau_{ij}}, \xi_{p_i}^{-\tau_{ij}}$ ($1 \leq j \leq m$) be the eigenvalues of $g^{r_i}|_{T_{q_i}M}$, and

$$\vec{u}_{ij} - \sqrt{-1}\vec{v}_{ij}, \vec{u}_{ij} + \sqrt{-1}\vec{v}_{ij} \quad (\vec{u}_{ij}, \vec{v}_{ij} \in T_{q_i}M \simeq \mathbb{R}^{2m})$$

the corresponding eigenvectors such that

$$\omega_{q_i}(\vec{u}_{i1} \wedge \vec{v}_{i1} \wedge \dots \wedge \vec{u}_{im} \wedge \vec{v}_{im})$$

is positive with respect to a positive $2m$ -form ω where τ_{ij} ($1 \leq j \leq m$) are natural numbers such that $1 \leq \tau_{ij} \leq p_i - 1$ (see ([1] p.473). Since the eigenvalues of $g^{r_i \kappa}|_{T_{q_i}M}$ for $1 \leq \kappa \leq p_i - 1$ are $\xi_{p_i}^{\kappa \tau_{ij}}, \xi_{p_i}^{-\kappa \tau_{ij}}$ ($1 \leq j \leq m$), it follows from Assumption 1.1 that τ_{ij} is prime to p_i for $1 \leq i \leq b, 1 \leq j \leq m$. In this paper the set $\{\tau_{ij}\}$ ($1 \leq j \leq m$) is called the rotation angle of g^{r_i} around the points in $\pi^{-1}(y_i)$ and the set $\{\tau_{ij}\}$ ($1 \leq i \leq b, 1 \leq j \leq m$) is called the rotation angle of g . The rotation angle of g is also expressed as

$$((\tau_{11}, \dots, \tau_{1m}), \dots, (\tau_{b1}, \dots, \tau_{bm})) .$$

Note that the equality

$$g^{r_i} \cdot (\vec{u}_{ij} - \sqrt{-1}\vec{v}_{ij}) = \xi_{p_i}^{\tau_{ij}} (\vec{u}_{ij} - \sqrt{-1}\vec{v}_{ij})$$

implies the equality

$$(g^{r_i} \cdot \vec{u}_{ij}, g^{r_i} \cdot \vec{v}_{ij}) = (\vec{u}_{ij}, \vec{v}_{ij}) \begin{pmatrix} \cos \frac{2\pi\tau_{ij}}{p_i} & -\sin \frac{2\pi\tau_{ij}}{p_i} \\ \sin \frac{2\pi\tau_{ij}}{p_i} & \cos \frac{2\pi\tau_{ij}}{p_i} \end{pmatrix}$$

and that $\{\tau_{i1}, \dots, p_i - \tau_{ij}, \dots, p_i - \tau_{ik}, \dots, \tau_{im}\}$ ($j \neq k$) is also the rotation angle of g^{r_i} when $m \geq 2$ because

$$\omega_{q_i}(\vec{u}_{i1} \wedge \vec{v}_{i1} \wedge \dots \wedge \vec{v}_{ij} \wedge \vec{u}_{ij} \wedge \dots \wedge \vec{v}_{ik} \wedge \vec{u}_{ik} \wedge \dots \wedge \vec{u}_{im} \wedge \vec{v}_{im})$$

is also positive. Hence if p_i is even and some τ_{ij} equals $p_i/2$, $\{\tau_{i1}, \dots, p_i - \tau_{ik}, \dots, \tau_{im}\}$ is also the rotation angle of g^{r_i} for $k \neq j$ and therefore we can assume that $1 \leq \tau_{ij} \leq p_i/2$.

Corresponding to the irreducible representations of the rotation group, complex \mathbb{Z}_p -vector bundles $E^{\ell_1, \dots, \ell_m}$ are defined for non-negative integers ℓ_1, \dots, ℓ_m as follows:

$$E^{\ell_1, \dots, \ell_m} = \otimes_{j=1}^m \left(\otimes_{\mathbb{C}}^{\ell_j} \left(\wedge_{\mathbb{C}}^j (TM \otimes \mathbb{C}) \right) \right),$$

and the $E^{\ell_1, \dots, \ell_m}$ -valued signature operator $D_{\ell_1, \dots, \ell_m}$ is defined. Note that $E^{0, \dots, 0}$ is a trivial line bundle and $D_{0, \dots, 0}$ is the equivariant non-twisted signature operator. Suppose that the rotation angle of g^{r_i} around the points q_i in $\pi^{-1}(y_i)$ ($1 \leq i \leq b$) is $\{\tau_{ij}\}$. Then the fixed point set of g^k ($1 \leq k \leq p - 1$) exists if and only if k equals $r_i \kappa$ for $1 \leq i \leq b, 1 \leq \kappa \leq p_i - 1$ and it follows from the signature theorem and the G -signature theorem that

$$(3) \quad \text{Ind}(D_{\ell_1, \dots, \ell_m}) = 2^m \text{Ch}(E^{\ell_1, \dots, \ell_m}) \widehat{\mathbf{L}}(M)[M] \left(\widehat{\mathbf{L}}(M) = \prod_{j=1}^m \frac{x_j/2}{\tanh(x_j/2)} \right),$$

$$(4) \quad \text{Ind}(D_{\ell_1, \dots, \ell_m}, g^{r_i \kappa}) = \sum_{i=1}^b r_i \text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, g^{r_i \kappa}) \prod_{j=1}^m \left(-\sqrt{-1} \cot \frac{\pi \kappa \tau_{ij}}{p_i} \right)$$

where $\sigma_j(x_1^2, \dots, x_m^2)$ for the j -th elementary symmetric function σ_j is the j -th Pontrjagin class of M (see [3] (6.5), (6.6), [5] Theorem 13.9), $E_{q_i}^{\ell_1, \dots, \ell_m}$ is the fiber of $E^{\ell_1, \dots, \ell_m}$ at q_i and $\text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, g^{r_i \kappa})$ is the trace of the linear transformation $g^{r_i \kappa}$ of the fiber $E_{q_i}^{\ell_1, \dots, \ell_m}$ (see [3] (6.12), (6.19), [4] Theorem 2.5), and hence we have

$$(5) \quad \begin{aligned} & I_{D_{\ell_1, \dots, \ell_m}}(g) \\ & \equiv \frac{p-1}{p} 2^{m-1} \text{Ch}(E^{\ell_1, \dots, \ell_m}) \widehat{\mathbf{L}}(M)[M] \\ & \quad - \frac{1}{p} \sum_{i=1}^b r_i \sum_{\kappa=1}^{p_i-1} \frac{1}{1 - \xi^{p_i \kappa}} \text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, g^{r_i \kappa}) \prod_{j=1}^m \left(-\sqrt{-1} \cot \frac{\pi \kappa \tau_{ij}}{p_i} \right) \pmod{\mathbb{Z}}. \end{aligned}$$

REMARK 2.2. Since the Chern classes of $TM \otimes \mathbb{C}$ is expressed by the Pontrjagin classes of TM , the Chern character $\text{Ch}(E^{\ell_1, \dots, \ell_m})$ is expressed by the Pontrjagin classes of TM (see Proposition 2.6). Hence the equality (3) above implies that $\text{Ind}(D_{\ell_1, \dots, \ell_m}) = 0$ if m is odd.

REMARK 2.3. Assume that $p_i = 2$ for some i . Then since $\kappa = \tau_{ij} = 1$, we have

$$\prod_{j=1}^m \left(-\sqrt{-1} \cot \frac{\pi \kappa \tau_{ij}}{p_i} \right) = \prod_{j=1}^m \left(-\sqrt{-1} \cot \frac{\pi}{2} \right) = 0$$

in the equalities (4), (5).

DEFINITION 2.4. Let p be a natural number which is greater than 1 and a, b integers where b is prime to p . Then using the Euler’s function φ , we define integers $\langle a \rangle, \bar{b}$ as follows:

$$\begin{aligned} \langle a \rangle &= a - p \left\lfloor \frac{a}{p} \right\rfloor \iff 0 \leq \langle a \rangle \leq p - 1, \langle a \rangle \equiv a \pmod{p}, \\ \bar{b} &= \langle b^{\varphi(p)-1} \rangle \iff b \bar{b} \equiv 1 \pmod{p}, \quad 1 \leq \bar{b} \leq p - 1. \end{aligned}$$

In this paper, we call \bar{b} the mod. p -inverse of b . Note that $\langle a + p \rangle = \langle a \rangle, \overline{b + p} = \bar{b}, \bar{\bar{b}} \equiv b \pmod{p}$ because $(\bar{b} - b) \bar{b} \equiv 0 \pmod{p}$ and that $\langle a_1 \rangle \cdot \langle a_2 \rangle \equiv \langle a_1 a_2 \rangle, \bar{b}_1 \cdot \bar{b}_2 \equiv \overline{b_1 b_2} \pmod{p}$ because $(\bar{b}_1 \cdot \bar{b}_2 - \overline{b_1 b_2}) b_1 b_2 \equiv 0 \pmod{p}$ where b_1, b_2 are prime to p . Note moreover that the equality $\langle a_1 \rangle + \langle a_2 \rangle \equiv \langle a_1 + a_2 \rangle \pmod{p}$ holds but the equality $\bar{b}_1 + \bar{b}_2 \equiv \overline{b_1 + b_2} \pmod{p}$ does not hold.

REMARK 2.5. Assume that a natural number z is prime to p . Then we have $(g^{kz})^{\bar{z}} = g^k$ for $1 \leq k \leq p - 1$, and hence the fixed point set of g^{kz} coincides with that of g^k and the rotation angle of $g^{r_i z}$ is equivalent to $\{z \tau_{ij}\} \pmod{p_i}$ if the rotation angle of g^{r_i} is $\{\tau_{ij}\}$.

Proposition 2.6. *Suppose that the total Chern class of $TM \otimes \mathbb{C}$ equals $\prod_{j=1}^m (1 - x_j^2)$ and hence the j -th Pontrjagin class $p_j(TM)$ of M equals $\sigma_j(x_1^2, \dots, x_m^2)$ where σ_j is the j -th*

elementary symmetric function (see [5] p.228). Then we have

$$(6) \quad \text{Ch}(E^{\ell_1, \dots, \ell_m}) = \prod_{j=1}^m \sigma_j(e^{x_1}, e^{-x_1}, \dots, e^{x_m}, e^{-x_m})^{\ell_j}.$$

Proof. Since the total Chern class of $TM \otimes \mathbb{C}$ equals $\prod_{j=1}^m (1 + x_j)(1 - x_j)$, we have

$$\text{Ch}(\wedge_{\mathbb{C}}^j(TM \otimes \mathbb{C})) = \sigma_j(e^{x_1}, e^{-x_1}, \dots, e^{x_m}, e^{-x_m}).$$

Hence it follows that

$$\text{Ch}(E^{\ell_1, \dots, \ell_m}) = \prod_{j=1}^m \sigma_j(e^{x_1}, e^{-x_1}, \dots, e^{x_m}, e^{-x_m})^{\ell_j}. \quad \square$$

EXAMPLE 2.7. Assume that $m = 2$ and let p_1 be the first Pontrjagin number of M . Then we have

$$2^2 \widehat{\mathbf{L}}(M) = 2^2 \prod_{j=1}^2 \frac{x_j/2}{\tanh(x_j/2)} = 4 + \frac{1}{3} p_1(TM),$$

$$\text{Ch}(E^j) = \sigma_j(e^{x_1}, e^{-x_1}, e^{x_2}, e^{-x_2}) = \begin{cases} 4 + p_1(TM) & (j = 1, 3), \\ 6 + 2p_1(TM) & (j = 2) \\ 1 & (j = 0, 4) \end{cases},$$

where $E^j = \wedge_{\mathbb{C}}^j(TM \otimes \mathbb{C})$, and hence it follows from Proposition 2.6 that

$$\begin{aligned} \text{Ch}(\otimes^{\ell_j} E^j) &= (\text{Ch}(E^j))^{\ell_j} = \begin{cases} 4^{\ell_j} + \ell_j 4^{\ell_j-1} p_1(TM) & (j = 1, 3), \\ 6^{\ell_j} + 2\ell_j 6^{\ell_j-1} p_1(TM) & (j = 2) \\ 1 & (j = 0, 4) \end{cases} \\ \implies \text{Ch}(E^{\ell_1, \ell_2}) &= (\text{Ch}(E^1))^{\ell_1} (\text{Ch}(E^2))^{\ell_2} \\ &= 4^{\ell_1} 6^{\ell_2} + 2 \cdot 4^{\ell_1-1} 6^{\ell_2-1} (3\ell_1 + 4\ell_2) p_1(M) \\ \implies \text{Ind}D_{\ell_1, \ell_2} &= 2^2 \text{Ch}(E^{\ell_1, \ell_2}) \widehat{\mathbf{L}}(M)[M] \\ &= 2 \cdot 4^{\ell_1} 6^{\ell_2-1} (3\ell_1 + 4\ell_2 + 1) p_1 = 2^{2\ell_1 + \ell_2} 3^{\ell_2-1} (3\ell_1 + 4\ell_2 + 1) p_1. \end{aligned}$$

DEFINITION 2.8. Let p be a natural number which is greater than 1. Then for any integer λ , a real number $\rho_p(\lambda)$ is defined by

$$\rho_p(\lambda) = \xi_p^\lambda + \xi_p^{-\lambda}.$$

Clearly ρ_p has the following properties:

$$\rho_p(-\lambda) = \rho_p(\lambda), \quad \rho_p(0) = 2, \quad \rho_p(\lambda)\rho_p(\mu) = \rho_p(\lambda + \mu) + \rho_p(\lambda - \mu),$$

which implies that

$$(7) \quad \rho_p(\lambda_1)\rho_p(\lambda_2) \cdots \rho_p(\lambda_j) = \rho_p(\lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_j)$$

for integers $\mu, \lambda_1, \lambda_2, \dots, \lambda_j$ where

$$\rho_p(\lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_j) = \sum_{\varepsilon_i = \pm 1 (2 \leq i \leq j)} \rho_p(\lambda_1 + \varepsilon_2 \lambda_2 + \cdots + \varepsilon_j \lambda_j).$$

We set that the binomial coefficient $\binom{0}{0}$ and 0^0 are equal to 1 hereafter.

Proposition 2.9. For $1 \leq i \leq b$, $1 \leq j \leq m$, $1 \leq k \leq p_i - 1$ set

$$C_{ij}^k = \sum_{r=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{m-j+2r}{r} \sigma_{j-2r}(\rho_{p_i}(k\tau_{i1}), \dots, \rho_{p_i}(k\tau_{im})) + \varepsilon_j \binom{m}{\frac{j}{2}}$$

where σ_{j-2s} is the $j - 2s$ -th elementary symmetric function and

$$\varepsilon_j = \frac{1 + (-1)^j}{2} = \begin{cases} 1 & (j : \text{even}) \\ 0 & (j : \text{odd}) \end{cases} .$$

Then we have

$$\text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, g^{r;k}) = \prod_{j=1}^m (C_{ij}^k)^{\ell_j} .$$

Proof. Since the eigenvalues of $g^{r;k}(T_{q_i}M \otimes \mathbb{C})$ are $\lambda_1, \dots, \lambda_{2m}$ where $\lambda_{2j-1} = \xi_{p_i}^{k\tau_{ij}}$, $\lambda_{2j} = \xi_{p_i}^{-k\tau_{ij}}$ ($1 \leq j \leq m$), we have

$$\begin{aligned} \text{Ch}(\wedge_{\mathbb{C}}^j(T_{q_i}M \otimes \mathbb{C}), g^{r;k}) &= \text{Tr} \left(g^{r;k} \Big| \wedge_{\mathbb{C}}^j(T_{q_i}M \otimes \mathbb{C}) \right) \\ &= \sum_{1 \leq i_1 < \dots < i_j \leq 2m} \lambda_{i_1} \cdots \lambda_{i_j} = \sigma_j(\xi_{p_i}^{k\tau_{i1}}, \xi_{p_i}^{-k\tau_{i1}}, \dots, \xi_{p_i}^{k\tau_{im}}, \xi_{p_i}^{-k\tau_{im}}) \\ &= \frac{1}{j!} \lim_{t \rightarrow 0} \left(\frac{d}{dt} \right)^j (1 + t\xi_{p_i}^{k\tau_{i1}})(1 + t\xi_{p_i}^{-k\tau_{i1}}) \cdots (1 + t\xi_{p_i}^{k\tau_{im}})(1 + t\xi_{p_i}^{-k\tau_{im}}) \\ &= \frac{1}{j!} \lim_{t \rightarrow 0} \left(\frac{d}{dt} \right)^j \prod_{k=1}^m (t^2 + \rho_{p_i}(k\tau_{ik})t + 1) \\ &= \sum_{n_1 + \dots + n_m = j} \prod_{k=1}^m \lim_{t \rightarrow 0} \frac{1}{n_k!} \left(\frac{d}{dt} \right)^{n_k} (t^2 + \rho_{p_i}(k\tau_{ik})t + 1) \\ (8) \quad &= \sum_{n_1 + \dots + n_m = j} \prod_{k=1}^m \chi(n_k) \end{aligned}$$

where n_1, \dots, n_m are integers such that $0 \leq n_1, \dots, n_m \leq 2$ and

$$\chi(n_k) = \begin{cases} \rho_{p_i}(k\tau_{ik}) & \text{if } n_k = 1 \\ 1 & \text{if } n_k = 0, 2 \end{cases} .$$

Let $r, j - 2r$ be the numbers of 2's and 1's in n_1, \dots, n_m . If $2r < j$ and $n_{s_1} = \dots = n_{s_{j-2r}} = 1$, we have

$$(8) = \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{1 \leq s_1 < \dots < s_{j-2r} \leq m} \binom{m-j+2r}{r} \rho_{p_i}(k\tau_{is_1}) \cdots \rho_{p_i}(k\tau_{is_{j-2r}})$$

because the number of (t_1, \dots, t_r) such that $1 \leq t_1 < \dots < t_r \leq m$ and that $n_{t_1} = \dots = n_{t_r} = 2$ is $\binom{m-j+2r}{r}$. Hence we have

$$(8) = \sum_{r=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{m-j+2r}{r} \sigma_{j-2r}(\rho_{p_i}(k\tau_{i1}), \dots, \rho_{p_i}(k\tau_{im})) + \varepsilon_j \binom{m}{\frac{j}{2}} = C_{ij}^k,$$

because $\prod_{k=1}^m \chi(n_k) = 1$ if $2r = j$, $\lfloor \frac{j-1}{2} \rfloor = \lfloor \frac{j}{2} \rfloor - 1$ if j is even and $\lfloor \frac{j-1}{2} \rfloor = \lfloor \frac{j}{2} \rfloor$ if j is odd. Therefore we have

$$\text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, g^{r_i k}) = \prod_{j=1}^m (C_{ij}^k)^{\ell_j}. \quad \square$$

EXAMPLE 2.10. When $m = 1$, it follows from Proposition 2.9 that

$$\text{Ch}(E_{q_i}^{\ell_1}, g^{r_i}) = (C_{i1}^1)^{\ell_1} = (\sigma_1(\rho_{p_i}(\tau_{i1})))^{\ell_1} = (\rho_{p_i}(\tau_{i1}))^{\ell_1}.$$

Next assume that $m = 2$ and that $(\ell_1, \ell_2) \neq (0, 0)$. Let σ_k the k -th elementary symmetric polynomial in $\rho_{p_i}(\tau_{i1}), \rho_{p_i}(\tau_{i2})$. Then it follows from Proposition 2.9 that

$$\begin{aligned} \text{Ch}(E_{q_i}^{\ell_1, \ell_2}, g^{r_i}) &= (C_{i1}^1)^{\ell_1} (C_{i2}^1)^{\ell_2} = \sigma_1^{\ell_1} (\sigma_2 + 2)^{\ell_2} = \sum_{\kappa_2=0}^{\ell_2} \binom{\ell_2}{\kappa_2} 2^{\ell_2-\kappa_2} \sigma_1^{\ell_1} \sigma_2^{\kappa_2} \\ &= \sum_{\kappa_2=0}^{\ell_2} \binom{\ell_2}{\kappa_2} 2^{\ell_2-\kappa_2} (\rho_{p_i}(\tau_{i1}) + \rho_{p_i}(\tau_{i2}))^{\ell_1} (\rho_{p_i}(\tau_{i1}) \rho_{p_i}(\tau_{i2}))^{\kappa_2} \\ &= \sum_{\mu_1=0}^{\ell_1} \sum_{\kappa_2=0}^{\ell_2} \binom{\ell_1}{\mu_1} \binom{\ell_2}{\kappa_2} 2^{\ell_2-\kappa_2} \rho_{p_i}(\tau_{i1})^{\mu_1+\kappa_2} \rho_{p_i}(\tau_{i2})^{\ell_1-\mu_1+\kappa_2}, \end{aligned}$$

and using the property (7) we can express the value above as a summation of ρ_{p_i} 's. When $(\ell_1, \ell_2) = (0, 0)$, we have

$$\text{Ch}(E_{q_i}^{0,0}, g^{r_i}) = \sigma_1^0 (\sigma_2 + 2)^0 = 1 \implies 2\text{Ch}(E_{q_i}^{0,0}, g^{r_i}) = \rho(0).$$

REMARK 2.11. Let $h(\ell_1, \dots, \ell_m)$ be a non-negative integer and $\theta(\ell_1, \dots, \ell_m)$ a natural number defined by

$$h(\ell_1, \dots, \ell_m) = \prod_{j=1}^m \left(\varepsilon_j \binom{m}{\frac{j}{2}} \right)^{\ell_j}, \quad \theta(\ell_1, \dots, \ell_m) = \begin{cases} 2 & (h(\ell_1, \dots, \ell_m) \text{ is odd}) \\ 1 & (h(\ell_1, \dots, \ell_m) \text{ is even}) \end{cases}.$$

It follows from Proposition 2.9 that $\text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, g^{r_i})$ is expressed as a summation of products of $\rho_{p_i}(\mu_i)$'s with integer coefficients and $h(\ell_1, \dots, \ell_m)$ where μ_i 's are non-negative integers. Hence $\theta(\ell_1, \dots, \ell_m) \text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, g^{r_i})$ is expressed as a summation of $\rho_{p_i}(\mu)$'s with integer coefficients because $2 = \rho(0)$.

If $\theta \text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, g^{r_i}) = \sum_s n_s \rho_{p_i}(\mu_{is})$ and hence $\theta \text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, (g^{r_i})^{kz}) = \sum_s n_s \rho_{p_i}(kz\mu_{is})$ where $\theta = 1$ or 2 , n_s 's are natural numbers, z is a natural number which is prime to p and μ_{is} 's are non-negative integers, it follows from (3), (4) and (5) that

$$(9) \quad \begin{aligned} \theta I_{D_{\ell_1, \dots, \ell_m}}(g^z) &\equiv \frac{p-1}{p} 2^{m-1} \theta \text{Ch}(E^{\ell_1, \dots, \ell_m}) \widehat{\mathbf{L}}(M)[M] \\ &\quad - \frac{1}{p} \sum_{i=1}^b r_i \sum_s n_s \sum_{\kappa=1}^{p_i-1} \frac{\rho_{p_i}(kz\mu_{is})}{1 - \xi_{p_i}^{-\kappa}} \prod_{j=1}^m \left(-\sqrt{-1} \cot \frac{\pi k z \tau_{ij}}{p_i} \right) \pmod{\mathbb{Z}}. \end{aligned}$$

So we can calculate $\theta I_{D_{\ell_1, \dots, \ell_m}}(g^z)$ precisely if we can express the value

$$\sum_{k=1}^{p_i-1} \frac{\rho_{p_i}(kz\mu_{is})}{1 - \xi_{p_i}^{-k}} \prod_{j=1}^m \left(-\sqrt{-1} \cot \frac{\pi k z \tau_{ij}}{p_i} \right)$$

as a rational number (see Theorem 3.6).

Proposition 2.12. *Let p be a natural number which is greater than 1 and $R(x)$ a rational function with real coefficients. Then $\sum_{k=1}^{p-1} R(\xi_p^k)$ is a real number.*

Proof. Since R is a rational function with real coefficients, we have

$$\overline{\sum_{k=1}^{p-1} R(\xi_p^k)} = \sum_{k=1}^{p-1} R(\overline{\xi_p^k}) = \sum_{k=1}^{p-1} R(\xi_p^{p-k}) = \sum_{p-k=1}^{p-1} R(\xi_p^{p-k}) = \sum_{k=1}^{p-1} R(\xi_p^k). \quad \square$$

Proposition 2.13. *Let p be a natural number which is greater than 1, a an integer and b_j ($1 \leq j \leq 2\gamma$) natural numbers which are prime to p . Then we have*

$$\begin{aligned} (-1)^{\gamma+1} \sum_{k=1}^{p-1} \xi_p^{-ka} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma} \frac{\xi_p^{b_j k} + 1}{\xi_p^{b_j k} - 1} &= \sum_{k=1}^{p-1} \sin \frac{2\pi ak}{p} \cot \frac{\pi k}{p} \prod_{j=1}^{2\gamma} \cot \frac{\pi b_j k}{p}, \\ (-1)^\gamma \sum_{k=1}^{p-1} \xi_p^{-ka} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma-1} \frac{\xi_p^{b_j \overline{b_{2\gamma} k}} + 1}{\xi_p^{b_j \overline{b_{2\gamma} k}} - 1} &= \sum_{k=1}^{p-1} \cos \frac{2\pi ak}{p} \cot \frac{\pi k}{p} \prod_{j=1}^{2\gamma-1} \cot \frac{\pi b_j \overline{b_{2\gamma} k}}{p}. \end{aligned}$$

Proof. It follows from Proposition 2.12 that the left-hand sides of the equalities above are real numbers. Hence it follows from the equalities below

$$\begin{aligned} (10) \quad \frac{\xi_p^{k\tau} + 1}{\xi_p^{k\tau} - 1} &= \frac{e^{2\pi i k \tau / p} + 1}{e^{2\pi i k \tau / p} - 1} = \frac{i^{-1} \frac{e^{\pi i k \tau / p} + e^{-\pi i k \tau / p}}{2}}{\frac{e^{\pi i k \tau / p} - e^{-\pi i k \tau / p}}{2i}} = -\sqrt{-1} \cot \frac{\pi k \tau}{p} \\ \frac{2}{1 - \xi_p^{-k}} &= \frac{2\xi_p^k}{\xi_p^k - 1} = 1 + \frac{\xi_p^k + 1}{\xi_p^k - 1} = 1 - \sqrt{-1} \cot \frac{\pi k}{p}, \end{aligned}$$

that

$$\begin{aligned} &(-1)^{\gamma+1} \sum_{k=1}^{p-1} \xi_p^{-ka} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma} \frac{\xi_p^{b_j k} + 1}{\xi_p^{b_j k} - 1} \\ &= (-1)^{\gamma+1} \sum_{k=1}^{p-1} \left(\cos \frac{2\pi ak}{p} - \sqrt{-1} \sin \frac{2\pi ak}{p} \right) \left(-\sqrt{-1} \cot \frac{\pi k}{p} \right) \prod_{j=1}^{2\gamma} \left(-\sqrt{-1} \cot \frac{\pi b_j k}{p} \right) \\ &= \sum_{k=1}^{p-1} \sin \frac{2\pi ak}{p} \cot \frac{\pi k}{p} \prod_{j=1}^{2\gamma} \cot \frac{\pi b_j k}{p}, \\ &(-1)^\gamma \sum_{k=1}^{p-1} \xi_p^{-ka} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma-1} \frac{\xi_p^{b_j \overline{b_{2\gamma} k}} + 1}{\xi_p^{b_j \overline{b_{2\gamma} k}} - 1} \\ &= (-1)^\gamma \sum_{k=1}^{p-1} \left(\cos \frac{2\pi ak}{p} - \sqrt{-1} \sin \frac{2\pi ak}{p} \right) \left(-\sqrt{-1} \cot \frac{\pi k}{p} \right) \prod_{j=1}^{2\gamma-1} \left(-\sqrt{-1} \cot \frac{\pi b_j \overline{b_{2\gamma} k}}{p} \right) \end{aligned}$$

$$= \sum_{k=1}^{p-1} \cos \frac{2\pi ak}{p} \cot \frac{\pi k}{p} \prod_{j=1}^{2\gamma-1} \cot \frac{\pi b_j \overline{b_{2\gamma} k}}{p}. \quad \square$$

DEFINITION 2.14. Let p be a natural number which is greater than 1 and b_1, \dots, b_{2n} natural numbers which are prime to p . Then a rational number $\zeta_j(p; b_1, \dots, b_{2j})$ ($0 \leq j \leq n$) is defined by

$$\begin{aligned} \zeta_j(p; b_1, \dots, b_{2j}) &= (-1)^j 2^{2j} p \sum_{\substack{1 \leq k_1, \dots, k_{2j} < p \\ p|(b_1 k_1 + \dots + b_{2j} k_{2j})}} \left(\left(\frac{k_1}{p} \right) \right) \dots \left(\left(\frac{k_{2j}}{p} \right) \right) \\ &= \frac{(-1)^j}{p^{2j-1}} \sum_{\substack{1 \leq k_1, \dots, k_{2j} < p \\ p|(b_1 k_1 + \dots + b_{2j} k_{2j})}} (2k_1 - p) \dots (2k_{2j} - p) \end{aligned}$$

for $1 \leq j \leq n$ and $\zeta_0 = -1$. Set

$$c_k = \langle \overline{b_{2k-1} b_{2k}} \rangle \ (1 \leq k \leq n), \quad d_k = \langle \overline{b_{2k} b_{2k+1}} \rangle \ (1 \leq k \leq n-1)$$

and let a be a non-negative integer. Then an integer $\eta_j(p; a, b_{2n-2j+1}, \dots, b_{2n})$ ($0 \leq j \leq n$) is defined by $\eta_0 = 1$, $\eta_j(p; 0, b_{2n-2j+1}, \dots, b_{2n}) = 0$ ($1 \leq j \leq n$) and if $a > 0$

$$\begin{aligned} &\eta_j(p; a, b_{2n-2j+1}, \dots, b_{2n}) \\ &= \sum_{t_{n-1}=0}^{ac_n} \psi(a, c_n; t_{n-1}) \sum_{t_{n-2}=0}^{c_{n-1} d_{n-1} t_{n-1}} \psi(d_{n-1} t_{n-1}, c_{n-1}; t_{n-2}) \\ &\quad \dots \sum_{t_{n-j}=0}^{c_{n-j+1} d_{n-j+1} t_{n-j+1}} \psi(d_{n-j+1} t_{n-j+1}, c_{n-j+1}; t_{n-j}) \end{aligned}$$

for $2 \leq j \leq n$ and

$$\eta_1(p; a, b_{2n-1}, b_{2n}) = \sum_{t_{n-1}=0}^{ac_n} \psi(a, c_n; t_{n-1})$$

where ψ is an integer defined by

$$\psi(a, c; t) = \begin{cases} -1 & (t = ac) \\ 4 \left(\frac{t}{c} - a \right) & (0 < t < ac \text{ and } t \text{ is a multiple of } c) \\ 4 \left(\left[\frac{t}{c} \right] - a \right) + 2 & (0 < t < ac \text{ and } t \text{ is not a multiple of } c) \\ -2a + 1 & (t = 0) \end{cases} .$$

REMARK 2.15. For $j > 0$ it follows from Theorem 1.3 that

$$\zeta_j(p; b_1, \dots, b_{2j}) = \sum_{k=1}^{p-1} \cot \frac{\pi b_1 k}{p} \dots \cot \frac{\pi b_{2j} k}{p} .$$

DEFINITION 2.16. For a non-negative integer n and narural numbers a, c, p, q , integers $\phi_n(a, c)$ and $\beta(n, p, q)$ are defined by

$$\phi_n(a, c) = \sum_{t=0}^{ac} \psi(a, c; t)t^n, \quad \beta(n, p, q) = \sum_{k=p+1}^q k^n.$$

Proposition 2.17. For natural numbers a, c , we have

$$(11) \quad \phi_0(a, c) = -2a^2c,$$

$$(12) \quad \phi_n(a, c) = 4 \sum_{j=1}^{a-1} j\beta(n, jc-1, (j+1)c-1) - 2(2a-1)\beta(n, 0, ac-1) \\ - c^n \{2\beta(n, 0, a-1) + a^n\} \quad (n \geq 1).$$

Proof. Since there are $a-1$ t 's $\{c, 2c, \dots, (a-1)c\}$ which satisfy the conditions that $0 < t < ac$ and that t is a multiple of c , we have

$$\begin{aligned} \phi_0(a, c) &= \sum_{t=0}^{ac} \psi(a, c; t) = -2a + 1 - 1 + \sum_{t=1}^{ac-1} \left\{ 4 \left(\left\lfloor \frac{t}{c} \right\rfloor - a \right) + 2 \right\} - 2(a-1) \\ &= 2ac(1-2a) + 4 \sum_{t=1}^{ac-1} \left\lfloor \frac{t}{c} \right\rfloor = 2ac(1-2a) + 4 \left(\sum_{j=1}^{a-1} \sum_{t=jc}^{(j+1)c-1} j \right) \\ &= 2ac(1-2a) + 4 \left(\sum_{j=1}^{a-1} cj \right) = 2ac(1-2a) + 4c \frac{a(a-1)}{2} = -2a^2c. \end{aligned}$$

For $n \geq 1$ we have

$$\begin{aligned} \phi_n(a, c) &= \sum_{t=0}^{ac} \psi(a, c; t)t^n \\ &= (-2a+1) \cdot 0^n + (-1)(ac)^n + \sum_{t=1}^{ac-1} \left\{ 4 \left(\left\lfloor \frac{t}{c} \right\rfloor - a \right) + 2 \right\} t^n - 2 \sum_{j=1}^{a-1} (jc)^n \\ &= -a^n c^n + 4 \sum_{t=1}^{ac-1} t^n \left\lfloor \frac{t}{c} \right\rfloor + (2-4a)\beta(n, 0, ac-1) - 2c^n \beta(n, 0, a-1) \\ &= 4 \left(\sum_{j=1}^{a-1} \sum_{t=jc}^{(j+1)c-1} jt^n \right) - 2(2a-1)\beta(n, 0, ac-1) - c^n \{2\beta(n, 0, a-1) + a^n\} \\ &= 4 \sum_{j=1}^{a-1} j\beta(n, jc-1, (j+1)c-1) - 2(2a-1)\beta(n, 0, ac-1) \\ &\quad - c^n \{2\beta(n, 0, a-1) + a^n\}. \quad \square \end{aligned}$$

EXAMPLE 2.18. Using the proposition above, we have

$$\begin{aligned} \eta_1(p; a, b_{2n-1}, b_{2n}) &= \phi_0(a, c_n) = -2a^2c_n, \\ \phi_2(a, c) &= 4 \sum_{j=1}^{a-1} j\beta(2, jc-1, (j+1)c-1) - 2(2a-1)\beta(2, 0, ac-1) - c^2 \{2\beta(2, 0, a-1) + a^2\} \end{aligned}$$

$$\begin{aligned}
 &= 4 \sum_{j=1}^{a-1} j \sum_{k=jc}^{(j+1)c-1} k^2 - 2(2a-1) \sum_{k=1}^{ac-1} k^2 - c^2 \left\{ 2 \sum_{k=1}^{a-1} k^2 + a^2 \right\} \\
 &= -\frac{1}{3} a^2 c_n (1 + c_n^2) - \frac{1}{3} a^4 c_n^3,
 \end{aligned}$$

and hence it follows that

$$\begin{aligned}
 \eta_2 &= \eta_2(p; a, b_{2n-3}, \dots, b_{2n}) \\
 &= \sum_{t_{n-1}=0}^{ac_n} \psi(a, c_n; t_{n-1}) \sum_{t_{n-2}=0}^{c_{n-1}d_{n-1}t_{n-1}} \psi(d_{n-1}t_{n-1}, c_{n-1}; t_{n-2}) \\
 &= \sum_{t_{n-1}=0}^{ac_n} \psi(a, c_n; t_{n-1}) \phi_0(d_{n-1}t_{n-1}, c_{n-1}) \\
 &= \sum_{t_{n-1}=0}^{ac_n} \psi(a, c_n; t_{n-1}) (-2c_{n-1}d_{n-1}^2t_{n-1}^2) = (-2c_{n-1}d_{n-1}^2) \phi_2(a, c_n) \\
 &= (-2c_{n-1}d_{n-1}^2) \left(-\frac{1}{3} a^2 c_n (1 + c_n^2) - \frac{1}{3} a^4 c_n^3 \right) = \frac{2}{3} a^2 c_{n-1} d_{n-1}^2 c_n (a^2 c_n^2 + c_n^2 + 1).
 \end{aligned}$$

3. Main results

Theorem 3.1. *Let p be a natural number which is greater than 1, n a natural number, μ a non-negative integer and b_1, \dots, b_{2n} natural numbers which are prime to p . Then we have*

$$\sum_{k=1}^{p-1} \cos \frac{2\pi\mu k}{p} \prod_{j=1}^{2n} \cot \frac{\pi b_j k}{p} \equiv \sum_{k=0}^n \zeta_k(p; b_1, \dots, b_{2k}) \eta_{n-k}(p; a, b_{2k+1}, \dots, b_{2n}) \pmod{p}$$

where $a = \overline{\mu b_{2n}}$.

Proof. When $\mu = 0$, as we see in Remark 2.15, the equality above holds. So we assume that $\mu > 0$ hereafter. Set

$$a_k = a_k(t_k) = \begin{cases} 0 & (k = 0) \\ d_k t_k (= \langle \overline{b_{2k} b_{2k+1}} \rangle t_k) & (1 \leq k \leq n-1) \\ a & (k = n) \end{cases} .$$

For $\gamma \geq 0$, set

$$\begin{aligned}
 F_\gamma(a, b_1, \dots, b_{2\gamma}) &= \begin{cases} (-1)^{\gamma+1} \sum_{k=1}^{p-1} \xi_p^{-ak} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma} \frac{\xi_p^{b_j k} + 1}{\xi_p^{b_j k} - 1} & (\gamma \geq 1) \\ - \sum_{k=1}^{p-1} \xi_p^{-ak} \frac{\xi_p^k + 1}{\xi_p^k - 1} & (\gamma = 0) \end{cases} , \\
 G_\gamma(a, b_1, \dots, b_{2\gamma}) &= \begin{cases} (-1)^\gamma \sum_{k=1}^{p-1} \xi_p^{-ak} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma-1} \frac{\xi_p^{b_j \overline{b_{2\gamma} k}} + 1}{\xi_p^{b_j \overline{b_{2\gamma} k}} - 1} & (\gamma \geq 1) \\ \sum_{k=1}^{p-1} \xi_p^{-ak} & (\gamma = 0) \end{cases} ,
 \end{aligned}$$

$$f_\gamma(a, b_1, \dots, b_{2\gamma}) = \begin{cases} \sum_{k=1}^{p-1} \sin \frac{2\pi ak}{p} \cot \frac{\pi k}{p} \prod_{j=1}^{2\gamma} \cot \frac{\pi b_j k}{p} & (\gamma \geq 1) \\ \sum_{k=1}^{p-1} \sin \frac{2\pi ak}{p} \cot \frac{\pi k}{p} & (\gamma = 0) \end{cases},$$

$$g_\gamma(a, b_1, \dots, b_{2\gamma}) = \begin{cases} \sum_{k=1}^{p-1} \cos \frac{2\pi ak}{p} \cot \frac{\pi k}{p} \prod_{j=1}^{2\gamma-1} \cot \frac{\pi b_j \overline{b_{2\gamma} k}}{p} & (\gamma \geq 1) \\ \sum_{k=1}^{p-1} \cos \frac{2\pi ak}{p} & (\gamma = 0) \end{cases}.$$

Note that the values $F_\gamma, G_\gamma, f_\gamma, g_\gamma$ are invariant under the substitution $a \rightarrow a+p, b_j \rightarrow b_j+p$, and that for any integer a we have

$$G_0(a) = g_0(a) = \begin{cases} p-1 & (\text{if } a \text{ is a multiple of } p) \\ -1 & (\text{if } a \text{ is not a multiple of } p) \end{cases} \equiv -1 \pmod{p}.$$

Note moreover that for a natural number λ which is prime to p , we have

$$(13) \quad g_\gamma(a, \lambda b_1, \dots, \lambda b_{2\gamma}) = g_\gamma(a, b_1, \dots, b_{2\gamma})$$

because $\lambda b_j \overline{\lambda b_{2\gamma}} \equiv \lambda \overline{\lambda} b_j \overline{b_{2\gamma}} \equiv b_j \overline{b_{2\gamma}} \pmod{p}$. As we see in Proposition 2.13, F_γ, G_γ are real numbers and the following equalities hold for $\gamma \geq 1$:

$$F_\gamma(a, b_1, \dots, b_{2\gamma}) = f_\gamma(a, b_1, \dots, b_{2\gamma}),$$

$$G_\gamma(a, b_1, \dots, b_{2\gamma}) = g_\gamma(a, b_1, \dots, b_{2\gamma}).$$

Moreover it follows from Proposition 2.12 and (10) that

$$F_0(a) = - \sum_{k=1}^{p-1} \xi_p^{-ak} \frac{\xi_p^k + 1}{\xi_p^k - 1} = - \sum_{k=1}^{p-1} \left(\cos \frac{2\pi ak}{p} - \sqrt{-1} \sin \frac{2\pi ak}{p} \right) \left(-\sqrt{-1} \cot \frac{\pi k}{p} \right)$$

$$= \sum_{k=1}^{p-1} \sin \frac{2\pi ak}{p} \cot \frac{\pi k}{p} = f_0(a),$$

and similarly that $G_0(a) = g_0(a)$.

Lemma 3.2. For $0 \leq k \leq n$ the following equality holds:

$$g_k(0, b_1, \dots, b_{2k}) \equiv \zeta_k(p; b_1, \dots, b_{2k}) \pmod{p}.$$

Proof. When $k = 0$, we have

$$g_0(0) = \sum_{k=1}^{p-1} \cos 0 = p-1 \equiv \zeta_0 \pmod{p}.$$

For $1 \leq k \leq n$ since the map $\overline{b_{2k} k} \rightarrow k$ gives a bijection of \mathbb{Z}_p , we have

$$g_k(0, b_1, \dots, b_{2k}) = \sum_{k=1}^{p-1} \cot \frac{\pi b_{2k} \overline{b_{2k} k}}{p} \prod_{j=1}^{2k-1} \cot \frac{\pi b_j \overline{b_{2k} k}}{p} = \sum_{k=1}^{p-1} \prod_{j=1}^{2k} \cot \frac{\pi b_j k}{p},$$

which is equal to $\zeta_k(p; b_1, \dots, b_{2k})$ as we see in Remark 2.15. □

Lemma 3.3. *We have*

$$f_0(a) = \sum_{k=1}^{p-1} \sin \frac{2\pi ak}{p} \cot \frac{\pi k}{p} \equiv -2a \pmod{p}.$$

Proof. It follows from the Eisenstein’s formula

$$(14) \quad \sum_{k=1}^{p-1} \sin \frac{2\pi ak}{p} \cot \frac{\pi k}{p} = -2p \left(\left(\frac{a}{p} \right) \right).$$

(see [9] p.103 (22)) that

$$f_0(a) = -2p \left(\left(\frac{a}{p} \right) \right) = -2p \left(\frac{a}{p} - \left[\frac{a}{p} \right] - \frac{1}{2} \right) \equiv -2a \pmod{p}. \quad \square$$

The next Lemma is the key lemma to prove the theorem.

Lemma 3.4. *For $\gamma \geq 1$ we have*

$$g_\gamma(a, b_1, \dots, b_{2\gamma}) = \sum_{t_{\gamma-1}=0}^{ac_\gamma} \psi(a, c_\gamma; t_{\gamma-1}) g_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1, \dots, b_{2\gamma-2}) + g_\gamma(0, b_1, \dots, b_{2\gamma}).$$

Proof. Since the map $b_{2\gamma}k \rightarrow k$ gives a bijection of \mathbb{Z}_p , for $\gamma \geq 1$ we have

$$\begin{aligned} & F_\gamma(a+1, b_1, \dots, b_{2\gamma}) - F_\gamma(a, b_1, \dots, b_{2\gamma}) \\ &= (-1)^{\gamma+1} \sum_{k=1}^{p-1} \xi_p^{-k(a+1)} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma} \frac{\xi_p^{b_j k} + 1}{\xi_p^{b_j k} - 1} (1 - \xi_p^k) \\ &= (-1)^\gamma \sum_{k=1}^{p-1} (\xi_p^{-ak} + \xi_p^{-k(a+1)}) \prod_{j=1}^{2\gamma} \frac{\xi_p^{b_j k} + 1}{\xi_p^{b_j k} - 1} \\ &= (-1)^\gamma \sum_{k=1}^{p-1} (\xi_p^{-kab_{2\gamma}} + \xi_p^{-k(a+1)b_{2\gamma}}) \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma-1} \frac{\xi_p^{b_j \overline{b_{2\gamma} k}} + 1}{\xi_p^{b_j \overline{b_{2\gamma} k}} - 1} \\ &= G_\gamma(\overline{ab_{2\gamma}}, b_1, \dots, b_{2\gamma}) + G_\gamma((a+1)\overline{b_{2\gamma}}, b_1, \dots, b_{2\gamma}), \end{aligned}$$

and

$$F_0(a+1) - F_0(a) = (-1)^0 \sum_{k=1}^{p-1} (\xi_p^{-kab_{2\gamma}} + \xi_p^{-k(a+1)b_{2\gamma}}) = G_0(\overline{ab_{2\gamma}}) + G_0((a+1)\overline{b_{2\gamma}}),$$

$$F_\gamma(0, b_1, \dots, b_{2\gamma}) = f_\gamma(0, b_1, \dots, b_{2\gamma}) = 0.$$

Hence it follows that

$$\begin{aligned} & F_\gamma(a, b_1, \dots, b_{2\gamma}) = F_\gamma(a, b_1, \dots, b_{2\gamma}) - F_\gamma(0, b_1, \dots, b_{2\gamma}) \\ &= \sum_{t=0}^{a-1} \{ F_\gamma(t+1, b_1, \dots, b_{2\gamma}) - F_\gamma(t, b_1, \dots, b_{2\gamma}) \} \\ &= \sum_{t=0}^{a-1} \{ G_\gamma(\overline{tb_{2\gamma}}, b_1, \dots, b_{2\gamma}) + G_\gamma((t+1)\overline{b_{2\gamma}}, b_1, \dots, b_{2\gamma}) \} \end{aligned}$$

$$(15) \quad = 2 \sum_{t=0}^a G_\gamma(t\overline{b_{2\gamma}}, b_1, \dots, b_{2\gamma}) - G_\gamma(a\overline{b_{2\gamma}}, b_1, \dots, b_{2\gamma}) - G_\gamma(0, b_1, \dots, b_{2\gamma}).$$

Moreover since the map $b_{2\gamma-1}\overline{b_{2\gamma}k} \rightarrow k$ gives a bijection of \mathbb{Z}_p , for $\gamma \geq 1$ we have

$$\begin{aligned} & G_\gamma(a+1, b_1, \dots, b_{2\gamma}) - G_\gamma(a, b_1, \dots, b_{2\gamma}) \\ &= (-1)^\gamma \sum_{k=1}^{p-1} \xi_p^{-k(a+1)} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma-1} \frac{\xi_p^{b_j\overline{b_{2\gamma}k}} + 1}{\xi_p^{b_j\overline{b_{2\gamma}k}} - 1} (1 - \xi_p^k) \\ &= (-1)^{\gamma+1} \sum_{k=1}^{p-1} (\xi_p^{-ka} + \xi_p^{-k(a+1)}) \frac{\xi_p^{b_{2\gamma-1}\overline{b_{2\gamma}k}} + 1}{\xi_p^{b_{2\gamma-1}\overline{b_{2\gamma}k}} - 1} \prod_{j=1}^{2\gamma-2} \frac{\xi_p^{b_j\overline{b_{2\gamma}k}} + 1}{\xi_p^{b_j\overline{b_{2\gamma}k}} - 1} \\ &= (-1)^{\gamma+1} \sum_{k=1}^{p-1} \left(\xi_p^{-kab_{2\gamma-1}b_{2\gamma}} + \xi_p^{-k(a+1)b_{2\gamma-1}b_{2\gamma}} \right) \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma-2} \frac{\xi_p^{b_j\overline{b_{2\gamma-1}k}} + 1}{\xi_p^{b_j\overline{b_{2\gamma-1}k}} - 1} \\ &= -\left\{ F_{\gamma-1}(a\overline{b_{2\gamma-1}b_{2\gamma}}, b_1\overline{b_{2\gamma-1}}, \dots, b_{2\gamma-2}\overline{b_{2\gamma-1}}) \right. \\ &\quad \left. + F_{\gamma-1}((a+1)\overline{b_{2\gamma-1}b_{2\gamma}}, b_1\overline{b_{2\gamma-1}}, \dots, b_{2\gamma-2}\overline{b_{2\gamma-1}}) \right\} \\ &= -\left\{ F_{\gamma-1}(a\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^\gamma, \dots, b_{2\gamma-2}^\gamma) + F_{\gamma-1}((a+1)\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^\gamma, \dots, b_{2\gamma-2}^\gamma) \right\}, \end{aligned}$$

where $b_j^\gamma = b_j\overline{b_{2\gamma-1}}$ ($1 \leq j \leq 2\gamma - 2$) and hence it follows that

$$\begin{aligned} & G_\gamma(a, b_1, \dots, b_{2\gamma}) - G_\gamma(0, b_1, \dots, b_{2\gamma}) \\ &= \sum_{s=0}^{a-1} \left\{ G_\gamma(s+1, b_1, \dots, b_{2\gamma}) - G_\gamma(s, b_1, \dots, b_{2\gamma}) \right\} \\ &= -\sum_{s=0}^{a-1} \left\{ F_{\gamma-1}(s\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^\gamma, \dots, b_{2\gamma-2}^\gamma) + F_{\gamma-1}((s+1)\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^\gamma, \dots, b_{2\gamma-2}^\gamma) \right\} \\ &= -2 \sum_{s=0}^a F_{\gamma-1}(s\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^\gamma, \dots, b_{2\gamma-2}^\gamma) + F_{\gamma-1}(a\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^\gamma, \dots, b_{2\gamma-2}^\gamma) \end{aligned}$$

because $F_{\gamma-1}(0, b_1^\gamma, \dots, b_{2\gamma-2}^\gamma) = 0$. Therefore for $\gamma \geq 1$ it follows from (15) that

$$\begin{aligned} & G_\gamma(a, b_1, \dots, b_{2\gamma}) \\ &= -2 \sum_{s=0}^a F_{\gamma-1}(s\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^\gamma, \dots, b_{2\gamma-2}^\gamma) \\ &\quad + F_{\gamma-1}(a\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^\gamma, \dots, b_{2\gamma-2}^\gamma) + G_\gamma(0, b_1, \dots, b_{2\gamma}) \\ (16) \quad &= -2 \sum_{s_{\gamma-1}=0}^a F_{\gamma-1}(s_{\gamma-1}c_\gamma, b_1^\gamma, \dots, b_{2\gamma-2}^\gamma) + F_{\gamma-1}(ac_\gamma, b_1^\gamma, \dots, b_{2\gamma-2}^\gamma) \\ &\quad + G_\gamma(0, b_1, \dots, b_{2\gamma}) \\ &\quad \left(\text{Since } t_{\gamma-1}\overline{b_{2\gamma-2}^\gamma} \equiv a_{\gamma-1}(t_{\gamma-1}), \quad s_{\gamma-1}c_\gamma\overline{b_{2\gamma-2}^\gamma} \equiv a_{\gamma-1}(s_{\gamma-1}c_\gamma) \pmod{p}. \right) \end{aligned}$$

$$\begin{aligned}
 &= -2 \sum_{s_{\gamma-1}=0}^a \left\{ 2 \sum_{t_{\gamma-1}=0}^{s_{\gamma-1}c_{\gamma}} G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \right. \\
 &\quad \left. - G_{\gamma-1}(a_{\gamma-1}(s_{\gamma-1}c_{\gamma}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) - G_{\gamma-1}(0, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \right\} \\
 &\quad + 2 \sum_{t_{\gamma-1}=0}^{ac_{\gamma}} G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \\
 &\quad - G_{\gamma-1}(a_{\gamma-1}(ac_{\gamma}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) - G_{\gamma-1}(0, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \\
 &\quad + G_{\gamma}(0, b_1, \dots, b_{2\gamma}) \\
 (17) \quad &= -4 \sum_{s_{\gamma-1}=0}^a \sum_{t_{\gamma-1}=0}^{s_{\gamma-1}c_{\gamma}} G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \\
 &\quad + 2 \sum_{s_{\gamma-1}=0}^a G_{\gamma-1}(a_{\gamma-1}(s_{\gamma-1}c_{\gamma}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) + 2 \sum_{s_{\gamma-1}=0}^a G_{\gamma-1}(0, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \\
 &\quad + 2 \sum_{t_{\gamma-1}=0}^{ac_{\gamma}} G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) - G_{\gamma-1}(a_{\gamma-1}(ac_{\gamma}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \\
 &\quad - G_{\gamma-1}(0, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) + G_{\gamma}(0, b_1, \dots, b_{2\gamma}) \\
 &= \sum_{s_{\gamma-1}=0}^a \sum_{t_{\gamma-1}=0}^{s_{\gamma-1}c_{\gamma}} \begin{pmatrix} -4 + 2\delta(t_{\gamma-1}, s_{\gamma-1}c_{\gamma}) \\ +2\delta(t_{\gamma-1}, 0) + 2\delta(s_{\gamma-1}, a) \\ -\delta(s_{\gamma-1}, a)\delta(t_{\gamma-1}, s_{\gamma-1}c_{\gamma}) \\ -\delta(s_{\gamma-1}, a)\delta(t_{\gamma-1}, 0) \end{pmatrix} G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \\
 &\quad + g_{\gamma}(0, b_1, \dots, b_{2\gamma}) \\
 &= \sum_{t_{\gamma-1}=0}^{ac_{\gamma}} \Lambda(t_{\gamma-1}) G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) + g_{\gamma}(0, b_1, \dots, b_{2\gamma})
 \end{aligned}$$

where

$$\Lambda(t) = \sum_{s_{\gamma-1}=N(t)}^a \begin{pmatrix} -4 + 2\delta(t, s_{\gamma-1}c_{\gamma}) \\ +2\delta(t, 0) + 2\delta(s_{\gamma-1}, a) \\ -\delta(s_{\gamma-1}, a)\delta(t, s_{\gamma-1}c_{\gamma}) \\ -\delta(s_{\gamma-1}, a)\delta(t, 0) \end{pmatrix}, \quad N(t) = \begin{cases} \frac{t}{c_{\gamma}} & (\text{if } c_{\gamma}|t) \\ \left\lfloor \frac{t}{c_{\gamma}} \right\rfloor + 1 & (\text{otherwise}) \end{cases}.$$

Here we have

$$\begin{aligned}
 \Lambda(ac_{\gamma}) &= -4 + 2 + 0 + 2 - 1 - 0 = -1, \\
 \Lambda(t) &= -4 \left(a - \frac{t}{c_{\gamma}} + 1 \right) + 2 + 0 + 2 - 0 - 0 = 4 \left(\frac{t}{c_{\gamma}} - a \right) \\
 &\quad \text{if } 0 < t < ac_{\gamma} \text{ and } t \text{ is a multiple of } c_{\gamma}, \\
 \Lambda(t) &= -4 \left(a - \left\lfloor \frac{t}{c_{\gamma}} \right\rfloor \right) + 0 + 0 + 2 - 0 - 0 = 4 \left(\left\lfloor \frac{t}{c_{\gamma}} \right\rfloor - a \right) + 2 \\
 &\quad \text{if } 0 < t < ac_{\gamma} \text{ and } t \text{ is not a multiple of } c_{\gamma}, \\
 \Lambda(0) &= -4(a + 1) + 2 + 2(a + 1) + 2 - 0 - 1 = -2a + 1,
 \end{aligned}$$

which implies that $\Lambda(t_{\gamma-1})$ is equal to $\psi(a, c_{\gamma}; t_{\gamma-1})$. Therefore it follows from (13) that

$$\begin{aligned}
 g_\gamma(a, b_1, \dots, b_{2\gamma}) &= G_\gamma(a, b_1, \dots, b_{2\gamma}) \\
 &= \sum_{t_{\gamma-1}=0}^{ac_\gamma} \Lambda(t_{\gamma-1})G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^\gamma, \dots, b_{2\gamma-2}^\gamma) + g_\gamma(0, b_1, \dots, b_{2\gamma}) \\
 &= \sum_{t_{\gamma-1}=0}^{c_\gamma a} \psi(a, c_\gamma; t_{\gamma-1})g_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1, \dots, b_{2\gamma-2}) + g_\gamma(0, b_1, \dots, b_{2\gamma}). \quad \square
 \end{aligned}$$

For $2 \leq \nu \leq n - 1$ set $\psi_\nu(t_{\nu-1}) = \psi(a_\nu(t_\nu), c_\nu; t_{\nu-1})$.

It follows from the result of Lemma 3.4 that

$$\begin{aligned}
 g_n(a, b_1, \dots, b_{2n}) &= \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1})g_{n-1}(a_{n-1}(t_{n-1}), b_1, \dots, b_{2n-2}) + g_n(0, b_1, \dots, b_{2n}) \\
 &= \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) \left\{ \sum_{t_{n-2}=0}^{a_{n-1}c_{n-1}} \psi_{n-1}(t_{n-2})g_{n-2}(a_{n-2}(t_{n-2}), b_1, \dots, b_{2n-4}) \right. \\
 &\quad \left. + g_{n-1}(0, b_1, \dots, b_{2n-2}) \right\} \\
 &\quad + g_n(0, b_1, \dots, b_{2n}) \\
 &= \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) \sum_{t_{n-2}=0}^{a_{n-1}c_{n-1}} \psi_{n-1}(t_{n-2})g_{n-2}(a_{n-2}(t_{n-2}), b_1, \dots, b_{2n-4}) \\
 &\quad + g_{n-1}(0, b_1, \dots, b_{2n-2}) \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) + g_n(0, b_1, \dots, b_{2n}) \\
 (18) \quad &= \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) \cdots \sum_{t_2=0}^{a_3c_3} \psi_3(t_2) \sum_{t_1=0}^{a_2c_2} \psi_2(t_1)g_1(a_1(t_1), b_1, b_2) \\
 &\quad + \sum_{k=2}^{n-1} g_k(0, b_1, \dots, b_{2k}) \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) \cdots \sum_{t_k=0}^{a_{k+1}c_{k+1}} \psi_{k+1}(t_k) \\
 &\quad + g_n(0, b_1, \dots, b_{2n}).
 \end{aligned}$$

Here it follows from (16), Example 2.18 and Lemma 3.3 that

$$\begin{aligned}
 g_1(a_1, b_1, b_2) &= -2 \sum_{s=0}^{a_1} f_0(sc_1) + f_0(a_1c_1) + g_1(0, b_1, b_2) \\
 &\equiv -2 \sum_{s=0}^{a_1} (-2sc_1) - 2a_1c_1 + g_1(0, b_1, b_2) \\
 &= 2a_1^2c_1 + g_1(0, b_1, b_2) = -\phi_0(a_1, c_1) + g_1(0, b_1, b_2) \\
 &\equiv g_0(0) \sum_{t_0=0}^{a_1c_1} \psi(a_1, c_1; t_0) + g_1(0, b_1, b_2) \pmod{p},
 \end{aligned}$$

and hence we have

$$\begin{aligned}
 &\sum_{t_{n-1}=0}^{a_n c_n} \psi(a_n, c_n; t_{n-1}) \cdots \sum_{t_1=0}^{a_2 c_2} \psi(a_2, c_2; t_1)g_1(a_1, b_1, b_2) \\
 &\equiv g_0(0) \sum_{t_{n-1}=0}^{a_n c_n} \psi(a_n, c_n; t_{n-1}) \cdots \sum_{t_1=0}^{a_2 c_2} \psi(a_2, c_2; t_1) \sum_{t_0=0}^{a_1 c_1} \psi(a_1, c_1; t_0)
 \end{aligned}$$

$$\begin{aligned}
 &+ g_1(0, b_1, b_2) \sum_{t_{n-1}=0}^{a_n c_n} \psi(a_n, c_n; t_{n-1}) \cdots \sum_{t_1=0}^{a_2 c_2} \psi(a_2, c_2; t_1) \\
 &\equiv g_0(0) \eta_n(p; a, b_1, \dots, b_{2n}) + g_1(0, b_1, b_2) \eta_{n-1}(p; a, b_3, \dots, b_{2n}) \pmod{p}.
 \end{aligned}$$

Moreover we have

$$\begin{aligned}
 &\sum_{k=2}^{n-1} g_k(0, b_1, \dots, b_{2k}) \sum_{t_{n-1}=0}^{c_n a} \psi_n(t_{n-1}) \cdots \sum_{t_k=0}^{c_{k+1} a_{k+1}} \psi_{k+1}(t_k) \\
 &= \sum_{k=2}^{n-1} g_k(0, b_1, \dots, b_{2k}) \eta_{n-k}(p; a, b_{2k+1}, \dots, b_{2n}),
 \end{aligned}$$

and it follows from $\eta_0(a) = 1$ that

$$g_n(0, b_1, \dots, b_{2n}) = g_n(0, b_1, \dots, b_{2n}) \eta_0(a).$$

Here since $k \rightarrow \overline{b_{2n}}k$ gives a bijection of \mathbb{Z}_p , it follows from Lemma 3.2 that

$$\begin{aligned}
 &\sum_{k=1}^{p-1} \cos \frac{2\pi\mu k}{p} \prod_{j=1}^{2n} \cot \frac{\pi b_j k}{p} = \sum_{k=1}^{p-1} \cos \frac{2\pi a k}{p} \prod_{j=1}^{2n} \cot \frac{\pi b_j \overline{b_{2n}} k}{p} = g_n(a, b_1, \dots, b_{2n}) \\
 &\equiv \sum_{k=0}^n g_k(0, b_1, \dots, b_{2k}) \eta_{n-k}(p; a, b_{2k+1}, \dots, b_{2n}) \\
 &\equiv \sum_{k=0}^n \zeta_k(p; b_1, \dots, b_{2k}) \eta_{n-k}(p; a, b_{2k+1}, \dots, b_{2n}) \pmod{p}.
 \end{aligned}$$

This completes the proof of Theorem 3.1. □

Corollary 3.5. *Let p be a natural number which is greater than 1, μ a non-negative integer, q a natural number and b_1, \dots, b_{2q} natural numbers which are prime to p . Then*

$$p^{2q-1} \sum_{k=1}^{p-1} \cos \frac{2\pi\mu k}{p} \prod_{j=1}^{2q} \cot \frac{\pi b_j k}{p}$$

is an integer.

Proof. It follows from the theorem above that

$$\sum_{k=1}^{p-1} \cos \frac{2\pi\mu k}{p} \prod_{j=1}^{2q} \cot \frac{\pi b_j k}{p} = \sum_{k=0}^q \zeta_k(p; b_1, \dots, b_{2k}) \eta_{q-k}(p; \mu \overline{b_{2q}}, b_{2k+1}, \dots, b_{2q}) + p\nu$$

where ν is an integer. Here since $\eta_{q-k}(p; \mu \overline{b_{2q}}, b_{2k+1}, \dots, b_{2q})$ and

$$p^{2q-1} \zeta_k(p; b_1, \dots, b_{2k}) = (-1)^k p^{2q-2k} \sum_{\substack{1 \leq k_1, \dots, k_{2k} < p \\ p | (b_1 k_1 + \dots + b_{2k} k_{2k})}} (2k_1 - p) \cdots (2k_{2k} - p)$$

are integers for $0 \leq k \leq q$, it follows that

$$p^{2q-1} \sum_{k=1}^{p-1} \cos \frac{2\pi\mu k}{p} \prod_{j=1}^{2q} \cot \frac{\pi b_j k}{p}$$

$$= p^{2q-1} \sum_{j=0}^q \eta_{q-j}(p; \mu \overline{b_{2q}}, b_{2j+1}, \dots, b_{2q}) \zeta_j(p; b_1, \dots, b_{2j}) + p^{2q} \nu$$

is an integer. □

Theorem 3.6. *Suppose that the total Chern class of $TM \otimes \mathbb{C}$ equals $\prod_{j=1}^m (1 - x_j^2)$ and that $\theta \text{Ch}(E_{q_i}^{\ell_1 \dots \ell_m}, g^{r_i}) = \sum_s n_s \rho_{p_i}(\mu_{is})$ where $\theta = \theta(\ell_1 \dots \ell_m) = 1$ or 2 (see Remark 2.11), n_s 's are natural numbers and μ_{is} 's are non-negative integers. Let a_{is}, b_{ij} ($1 \leq i \leq b, 1 \leq j \leq m$), h_m be integers defined by*

$$a_{is} = \begin{cases} \mu_{is} \overline{\tau_{i2n}} & (\text{if } m \text{ is even}) \\ z \mu_{is} & (\text{if } m \text{ is odd}) \end{cases}, \quad b_{ij} = \begin{cases} \overline{z} & (\text{if } m \text{ is odd and } j = 2n) \\ \tau_{ij} & (\text{otherwise}) \end{cases},$$

$$h_m = \begin{cases} 2^{m-1} & (\text{if } m \text{ is even}) \\ 0 & (\text{if } m \text{ is odd}) \end{cases}$$

and z a natural number which is prime to p . Then the next equality holds as an element of \mathbb{R}/\mathbb{Z} .

$$\theta I_{D_{\ell_1, \dots, \ell_m}}(g^z) = \frac{p-1}{p} \theta h_m \prod_{j=1}^m \sigma_j(e^{x_1}, e^{-x_1}, \dots, e^{x_m}, e^{-x_m})^{\ell_j} \prod_{j=1}^m \frac{x_j/2}{\tanh(x_j/2)} [M]$$

$$+ (-1)^{n+1} \frac{1}{p} \sum_{i=1}^b r_i \sum_s n_s \sum_{k=0}^n \zeta_k(p; b_{i1}, \dots, b_{i2k}) \eta_{n-k}(p; a_{is}, b_{i2k+1}, \dots, b_{i2n}).$$

Proof. It follows from (10) that

$$(19) \quad \sum_{k=1}^{p-1} \frac{\rho_{p_i}(kz\mu_{is})}{1 - \xi_p^{-k}} \prod_{j=1}^m \left(-\sqrt{-1} \cot \frac{\pi k z \tau_{ij}}{p} \right) = \sum_{k=1}^{p-1} R(\xi_p^k)$$

where

$$R(\xi_p^k) = \frac{\xi_p^{kz\mu_{is}} + \xi_p^{-kz\mu_{is}}}{1 - \xi_p^{-k}} \prod_{j=1}^m \frac{\xi_p^{kz\tau_{ij}} + 1}{\xi_p^{kz\tau_{ij}} - 1}.$$

Then Proposition 2.12 implies that the both sides of (19) are real numbers, and the map $kz \rightarrow k$ gives a bijection of \mathbb{Z}_p because z is prime to p . Hence we have

$$\sum_{k=1}^{p-1} \frac{\rho_{p_i}(kz\mu_{is})}{1 - \xi_p^{-k}} \prod_{j=1}^m \left(-\sqrt{-1} \cot \frac{\pi k z \tau_{ij}}{p} \right)$$

$$= \sum_{k=1}^{p-1} \frac{\xi_p^{kz\mu_{is}} + \xi_p^{-kz\mu_{is}}}{2} \frac{2}{1 - \xi_p^{-k}} \prod_{j=1}^m \left(-\sqrt{-1} \cot \frac{\pi k z \tau_{ij}}{p} \right)$$

$$= \sum_{k=1}^{p-1} \cos \frac{2\pi k z \mu_{is}}{p} \left(1 - \sqrt{-1} \cot \frac{\pi k}{p} \right) \prod_{j=1}^m \left(-\sqrt{-1} \cot \frac{\pi k z \tau_{ij}}{p} \right)$$

$$\begin{aligned}
 &= \left\{ \begin{aligned}
 &(-1)^n \sum_{k=1}^{p-1} \cos \frac{2\pi k z \mu_{is}}{p} \prod_{j=1}^{2n} \cot \frac{\pi k z \tau_{ij}}{p} \\
 &= (-1)^n \sum_{k=1}^{p-1} \cos \frac{2\pi k \mu_{is}}{p} \prod_{j=1}^{2n} \cot \frac{\pi k \tau_{ij}}{p} \quad (m = 2n) \\
 &(-1)^n \sum_{k=1}^{p-1} \cos \frac{2\pi k z \mu_{is}}{p} \cot \frac{\pi k}{p} \prod_{j=1}^{2n-1} \cot \frac{\pi k z \tau_{ij}}{p} \\
 &= (-1)^n \sum_{k=1}^{p-1} \cos \frac{2\pi k \mu_{is}}{p} \cot \frac{\pi k \bar{z}}{p} \prod_{j=1}^{2n-1} \cot \frac{\pi k \tau_{ij}}{p} \quad (m = 2n - 1)
 \end{aligned} \right. \\
 &= (-1)^n \sum_{k=1}^{p-1} \cos \frac{2\pi \mu_{is} k}{p} \prod_{j=1}^{2n} \cot \frac{\pi b_j k}{p}.
 \end{aligned}$$

Hence it follows from Theorem 3.1 that

$$\begin{aligned}
 (20) \quad &\sum_{k=1}^{p-1} \frac{\rho_{p_i}(kz\mu_{is})}{1 - \xi_p^{-k}} \prod_{j=1}^m \left(-\sqrt{-1} \cot \frac{\pi k z \tau_{ij}}{p} \right) \\
 &\equiv (-1)^n \sum_{j=0}^n \zeta_j(p; b_1, \dots, b_{2j}) \eta_{n-j}(p; a_{is}, b_{2j+1}, \dots, b_{2n}).
 \end{aligned}$$

Therefore it follows from (3), Proposition 2.6 and (9) that

$$\begin{aligned}
 \theta I_{D_{\ell_1, \dots, \ell_m}}(g^z) &= \frac{p-1}{p} \theta h_m \prod_{j=1}^m \sigma_j(e^{x_1}, e^{-x_1}, \dots, e^{x_m}, e^{-x_m})^{\ell_j} \prod_{j=1}^m \frac{x_j/2}{\tanh(x_j/2)} [M] \\
 &+ (-1)^{n+1} \frac{1}{p} \sum_{i=1}^b r_i \sum_s n_s \sum_{k=0}^n \zeta_k(p; b_{i1}, \dots, b_{i2k}) \eta_{n-k}(p; a_{is}, b_{i2k+1}, \dots, b_{i2n}).
 \end{aligned}$$

This completes the proof of Theorem 3.6. □

Corollary 3.7. *Assume that the cyclic group $\mathbb{Z}_p = \langle g \rangle$ acts on $2m$ -dimensional closed oriented manifold where m is even. Then for non-negative integers ℓ_1, \dots, ℓ_m we have $I_{D_{\ell_1, \dots, \ell_m}}(g) = 0$ if p is odd and $2\theta I_{D_{\ell_1, \dots, \ell_m}}(g) = 0$ if p is even.*

Proof. Since m is even, $\theta I_{D_{\ell_1, \dots, \ell_m}}(g^z)$ does not depend on the natural number z which is prime to p because a_{is}, b_{ij} in the theorem above does not depend on z . Hence setting $z = p - 1$ which is prime to p , we have

$$\begin{cases} p I_{D_{\ell_1, \dots, \ell_m}}(g) = I_{D_{\ell_1, \dots, \ell_m}}(g^p) = 0, \\ \theta I_{D_{\ell_1, \dots, \ell_m}}(g) = \theta I_{D_{\ell_1, \dots, \ell_m}}(g^{p-1}) = \theta(p-1) I_{D_{\ell_1, \dots, \ell_m}}(g) \end{cases} \\
 \implies \theta(2-p) I_{D_{\ell_1, \dots, \ell_m}}(g) = 2\theta I_{D_{\ell_1, \dots, \ell_m}}(g) = 0,$$

which imply that $I_{D_{\ell_1, \dots, \ell_m}}(g) = 0$ if p is odd because 2θ is prime to p . □

4. Examples

EXAMPLE 4.1. In this example we consider the case that $m = 1$, namely the case that M is a compact Riemann surface. Let p be an odd prime number and \mathbb{Z}_p a cyclic group generated by g . Then we have $p_i = p$, in other words, $r_i = 1$ for $1 \leq i \leq b$. Let t_1, \dots, t_b be natural numbers such that $1 \leq t_1, \dots, t_b \leq p - 1$, z a natural number which is not a multiple of p , and set

$$f(t_1, \dots, t_b, z) = \frac{1}{p} \sum_{i=1}^b (2zt_i + \zeta_1(p; t_i, \bar{z})) .$$

Then we have

$$\begin{aligned} f(t_1, \dots, t_b, z) &\equiv \frac{1}{p} \sum_{i=1}^b \left(-2(zt_i)^2 \langle \bar{t}_i \bar{z} \rangle \cdot (-1) + 1 \cdot \zeta_1(p; t_i, \bar{z}) \right) \pmod{\mathbb{Z}} \\ &= \frac{1}{p} \sum_{i=1}^b (\zeta_0 \eta_1(p; zt_i, t_i, \bar{z}) + \zeta_1(p; t_i, \bar{z}) \eta_0) . \end{aligned}$$

If $\{t_1, \dots, t_b\}$ is the rotation angle of a \mathbb{Z}_p -action, namely if there exists an action of $\mathbb{Z}_p = \langle g \rangle$ on a compact Riemann surface such that the rotation angle of g is $\{t_1, \dots, t_b\}$, it follows from Theorem 3.6 that $f(t_1, \dots, t_b, z) = I_{D_1}(g^z) \in \mathbb{R}/\mathbb{Z}$ because $\text{Ind}(D_1) = 0$ and $\text{Ch}(E_{q_i}^1, g) = \rho_p(t_i)$ (see Example 2.10), and hence the following equality holds:

$$(21) \quad f(t_1, \dots, t_b, z) - zf(t_1, \dots, t_b, 1) \equiv 0 \pmod{\mathbb{Z}} .$$

Here as we see in the proof of Proposition 4.6 in [7], $\{t_1, \dots, t_b\}$ is the rotation angle of a \mathbb{Z}_p -action if the equality

$$(22) \quad \sum_{i=1}^b \bar{t}_i \equiv 0 \pmod{p}$$

holds. In this example we show that the equality (22) follows from the equality (21), which implies that the equality (21), which corresponds to the equality $I_{D_1}(g^z) = zI_{D_1}(g)$, is the necessary and sufficient condition for $\{t_1, \dots, t_b\}$ to be the rotation angle of a \mathbb{Z}_p -action.

Here we assume that the equality (21) holds. Since

$$k_1 t_i + k_2 \bar{z} \equiv 0 \iff (k_1 t_i + k_2 \bar{z})z \equiv k_1 t_i z + k_2 \equiv 0 \iff k_2 \equiv -k_1 t_i z \pmod{p} ,$$

it follows that

$$\begin{aligned} \zeta_1(p; t_i, \bar{z}) &= -4p \sum_{\substack{1 \leq k_1, k_2 \leq p \\ p | (t_i k_1 + \bar{z} k_2)}} \left(\left(\frac{k_1}{p} \right) \right) \left(\left(\frac{k_2}{p} \right) \right) = -4p \sum_{k_1=1}^{p-1} \left(\left(\frac{k_1}{p} \right) \right) \left(\left(\frac{-k_1 t_i z}{p} \right) \right) \\ &= 4p \sum_{k=1}^{p-1} \left(\left(\frac{k}{p} \right) \right) \left(\left(\frac{kt_i z}{p} \right) \right) , \end{aligned}$$

and hence we have

$$f(t_1, \dots, t_b, z) = \frac{1}{p} \sum_{i=1}^b \left(2zt_i + 4p \sum_{k=1}^{p-1} \left(\left(\frac{k}{p} \right) \right) \left(\left(\frac{kt_i z}{p} \right) \right) \right) .$$

Then since

$$\begin{aligned} 2 \left\{ z \left(\binom{kt_i}{p} \right) - \left(\binom{kt_i z}{p} \right) \right\} &= 2z \left(\frac{kt_i}{p} - \left[\frac{kt_i}{p} \right] - \frac{1}{2} \right) - 2 \left(\frac{kt_i z}{p} - \left[\frac{kt_i z}{p} \right] - \frac{1}{2} \right) \\ &= 2 \left[\frac{kt_i z}{p} \right] - 2z \left[\frac{kt_i}{p} \right] + 1 - z \end{aligned}$$

is an integer and the map $kt_i \rightarrow k$ gives a bijection of \mathbb{Z}_p , it follows that

$$\begin{aligned} z f(t_1, \dots, t_b, 1) - f(t_1, \dots, t_b, z) &= 4 \sum_{i=1}^b \sum_{k=1}^{p-1} \left(\binom{k}{p} \right) \left\{ z \left(\binom{kt_i}{p} \right) - \left(\binom{kt_i z}{p} \right) \right\} \\ &= 2 \sum_{i=1}^b \sum_{k=1}^{p-1} \left(\frac{2k}{p} - 1 \right) \left\{ z \left(\binom{kt_i}{p} \right) - \left(\binom{kt_i z}{p} \right) \right\} \\ &= \frac{4}{p} \sum_{i=1}^b \sum_{k=1}^{p-1} k \left\{ z \left(\binom{kt_i}{p} \right) - \left(\binom{kt_i z}{p} \right) \right\} \\ &= \frac{4}{p} \sum_{i=1}^b \sum_{k=1}^{p-1} k \bar{t}_i \left\{ z \left(\binom{k}{p} \right) - \left(\binom{kz}{p} \right) \right\} = \frac{1}{p} T \times F(p, z) \end{aligned}$$

where

$$T = \sum_{i=1}^b \bar{t}_i, \quad F(p, z) = 4 \sum_{k=1}^{p-1} k \left\{ z \left(\binom{k}{p} \right) - \left(\binom{kz}{p} \right) \right\}.$$

Therefore it follows from the assumption that

$$(23) \quad 2p \{ 2f(t_1, \dots, t_b, 1) - f(t_1, \dots, t_b, 2) \} = T \times 2F(p, 2) \equiv 0 \pmod{p}.$$

Here we have

$$\begin{aligned} 2F(p, 2) &= 8 \sum_{k=1}^{p-1} k \left\{ 2 \left(\binom{k}{p} \right) - \left(\binom{2k}{p} \right) \right\} \\ &= 8 \sum_{k=1}^{p-1} k \left\{ \left(2 \frac{k}{p} - 0 - 1 \right) - \left(\frac{2k}{p} - \left[\frac{2k}{p} \right] - \frac{1}{2} \right) \right\} \\ &= 8 \sum_{k=1}^{p-1} k \left[\frac{2k}{p} \right] - 4 \sum_{k=1}^{p-1} k = 8 \sum_{k=1}^{p-1} k \left[\frac{2k}{p} \right] - 2p(p-1) \equiv 8 \sum_{k=1}^{p-1} k \left[\frac{2k}{p} \right] \pmod{p} \\ &= 8 \sum_{k=\frac{p+1}{2}}^{p-1} k = (3p-1)(p-1) \equiv 1 \pmod{p}, \end{aligned}$$

and hence it follows from (23) that $T \equiv 0 \pmod{p}$, namely the equality (22) holds.

EXAMPLE 4.2. In this example we consider the case that $m = 2$, namely the case that M is a 4-dimensional closed oriented manifold, which does not necessarily admit an almost complex structure (see Example 3.2 in [7]). In this example using Theorem 3.6, we show that a relationship between the rotation angle of a \mathbb{Z}_p -action on M and the signature of M exists (see (28),(29) below). Let g be an orientation-preserving periodic diffeomorphism of M of

order p . Let $\sigma(M)$ be the signature of M and suppose that $\theta\text{Ch}(E_{q_i}^{\ell_1, \ell_2}, g^{r_i}) = \sum_s n_s \rho_{p_i}(\mu_{is})$ ($\theta = 1$ or 2). Then since the first Pontrjagin number p_1 is equal to $3\sigma(M)$, it follows from Example 2.7, Example 2.18 and Theorem 3.6 that

$$\begin{aligned} \theta I_{D_{\ell_1, \ell_2}}(g) &= \frac{p-1}{2p} \theta 2^{2\ell_1 + \ell_2} 3^{\ell_2} (3\ell_1 + 4\ell_2 + 1) \sigma(M) \\ &\quad + \sum_{i=1}^b \frac{1}{p_i} \sum_s n_s \sum_{k=0}^1 \zeta_k(p_i; \tau_{i1}, \dots, \tau_{i,2k}) \eta_{1-k}(p; \mu_{is} \overline{\tau_{i2}}, \tau_{i,2k+1}, \tau_{i2}) \\ &= \frac{p-1}{2p} \theta 2^{2\ell_1 + \ell_2} 3^{\ell_2} (3\ell_1 + 4\ell_2 + 1) \sigma(M) \\ &\quad + \sum_{i=1}^b \frac{1}{p_i} \sum_s n_s \{ (-1) (-2(\mu_{is} \overline{\tau_{i2}})^2 \overline{\tau_{i1}} \tau_{i2}) + \zeta_1(p_i; \tau_{i1}, \tau_{i2}) \cdot 1 \} \\ &\iff \\ (24) \quad p \theta I_{D_{\ell_1, \ell_2}}(g) &= \frac{p-1}{2} \theta 2^{2\ell_1 + \ell_2} 3^{\ell_2} (3\ell_1 + 4\ell_2 + 1) \sigma(M) \\ &\quad + \sum_{i=1}^b r_i \left(\sum_s 2n_s \mu_{is}^2 \right) \overline{\tau_{i1}} \tau_{i2} + \sum_{i=1}^b r_i \left(\sum_s n_s \right) \zeta_1(p_i; \tau_{i1}, \tau_{i2}) \pmod{p}. \end{aligned}$$

Here as we see in Example 2.10 we have

$$\text{Ch}(E_{q_i}^{\ell_1, \ell_2}, g) = \sigma_1^{\ell_1} (\sigma_2 + 2)^{\ell_2}$$

where σ_k is the k -th elementary symmetric polynomial in $\rho_{p_i}(\tau_{i1}), \rho_{p_i}(\tau_{i2})$. Hence we have

$$\begin{aligned} \text{Ch}(E_{q_i}^{0,0}, g) = 1 &= \frac{1}{2} \rho_{p_i}(0) \iff 2\text{Ch}(E_{q_i}^{0,0}, g) = \rho_{p_i}(0), \\ \text{Ch}(E_{q_i}^{1,0}, g) &= \sigma_1 = \rho_{p_i}(\tau_{i1}) + \rho_{p_i}(\tau_{i2}), \\ \text{Ch}(E_{q_i}^{0,1}, g) &= \sigma_2 + 2 = \rho_{p_i}(\tau_{i1}) \rho_{p_i}(\tau_{i2}) + 2 = \rho_{p_i}(\tau_{i1} + \tau_{i2}) + \rho_{p_i}(\tau_{i1} - \tau_{i2}) + \rho_{p_i}(0). \end{aligned}$$

Set $\lambda = 1$ if p is odd and $\lambda = 2$ if p is even. Then it follows from Corollary 3.7, (24) and the equalities above that

$$\begin{aligned} (25) \quad 2\lambda p I_{D_{0,0}}(g) &= \lambda(p-1)\sigma(M) + \lambda \sum_{i=1}^b r_i \zeta_1(p_i; \tau_{i1}, \tau_{i2}) \equiv 0 \pmod{p}, \\ p\lambda I_{D_{1,0}}(g) &= 8\lambda(p-1)\sigma(M) + 2\lambda \sum_{i=1}^b r_i (\tau_{i1}^2 + \tau_{i2}^2) \overline{\tau_{i1}} \tau_{i2} + 2\lambda \sum_{i=1}^b r_i \zeta_1(p_i; \tau_{i1}, \tau_{i2}) \\ (26) \quad &\equiv -8\lambda\sigma(M) + 2\lambda \sum_{i=1}^b r_i (\tau_{i1} \overline{\tau_{i2}} + \overline{\tau_{i1}} \tau_{i2}) + 2\lambda \sum_{i=1}^b r_i \zeta_1(p_i; \tau_{i1}, \tau_{i2}) \equiv 0 \pmod{p}. \end{aligned}$$

Since $(\tau_{i1} + \tau_{i2})^2 + (\tau_{i1} - \tau_{i2})^2 + 0^2 = 2(\tau_{i1}^2 + \tau_{i2}^2)$, the next equality also holds.

$$\begin{aligned} (27) \quad p\lambda I_{D_{0,1}}(g) &\equiv -15\lambda\sigma(M) + 4\lambda \sum_{i=1}^b r_i (\tau_{i1} \overline{\tau_{i2}} + \overline{\tau_{i1}} \tau_{i2}) + 3\lambda \sum_{i=1}^b r_i \zeta_1(p_i; \tau_{i1}, \tau_{i2}) \equiv 0 \pmod{p}. \end{aligned}$$

Here we have

$$\begin{cases} 2 \times (25) - (26) \equiv 6\lambda\sigma(M) - 2\lambda \sum_{i=1}^b r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) \equiv 0 \pmod{p} \\ 3 \times (25) - (27) \equiv 12\lambda\sigma(M) - 4\lambda \sum_{i=1}^b r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) \equiv 0 \pmod{p} \end{cases}$$

$$\iff 2\lambda \sum_{i=1}^b r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) \equiv 6\lambda\sigma(M) \pmod{p},$$

and since 2 is prime to p if p is odd, it follows that

$$(28) \quad 4 \sum_{i=1}^b r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) \equiv 12\sigma(M) \pmod{p} \quad \text{if } p \text{ is even,}$$

$$(29) \quad \sum_{i=1}^b r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) \equiv 3\sigma(M) \pmod{p} \quad \text{if } p \text{ is odd.}$$

For example, we consider the case that $M = \mathbb{C}P^2$. Let p be a prime number which is greater than 3 and a, b natural numbers such that $1 < a < b < p$. Then

$$\mathbb{C}P^2 \ni q = [z_0 : z_1 : z_2] \rightarrow g \cdot q = [z_0 : \xi_p^a z_1 : \xi_p^b z_2] \in \mathbb{C}P^2$$

defines an action of the cyclic group $\mathbb{Z}_p = \langle g \rangle$. The fixed point set of g^k ($1 \leq k \leq p - 1$) consists of $q_1 = [1 : 0 : 0]$, $q_2 = [0 : 1 : 0]$, $q_3 = [0 : 0 : 1]$, and the rotation angle of g is

$$((\tau_{11}, \tau_{12}), (\tau_{21}, \tau_{22}), (\tau_{31}, \tau_{32})) = ((a, b), (p - a, b - a), (p - b, p + a - b)).$$

Then $r_i = 1$ for $1 \leq i \leq 3$ and we have

$$\begin{aligned} \sum_{i=1}^3 r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) &\equiv \overline{ab} + \overline{ab} - \overline{a(b-a)} + \overline{-a(b-a)} - \overline{b(a-b)} - \overline{b(a-b)} \\ &\equiv \overline{ab} + \overline{ab} - \overline{a(b-a)} - \overline{ab} + 1 + \overline{b(b-a)} - \overline{ba} + 1 \\ &= (b-a)\overline{(b-a)} + 1 + 1 \equiv 3 = 3\sigma(\mathbb{C}P^2) \pmod{p}. \end{aligned}$$

On the other hand since $3 \equiv 3\sigma(M) \iff 3(\sigma(M) - 1) \equiv 0 \pmod{p}$, M does not admit an \mathbb{Z}_p -action with the rotation angle above if $\sigma(M) - 1$ is not a multiple of p .

For example let p be any prime number such that $p + 3$ is a multiple of 4 and set $n = (p+3)/2$. Let M be the connected sum of n copies of $\mathbb{C}P^2$ and $n-2$ copies of $\overline{\mathbb{C}P^2}$ which is the underlying manifold of $\mathbb{C}P^2$ with the orientation reversed. Then the Euler number $e(M)$ of M is equal to $2n$ and the equality (2) is satisfied for $b = 3$. But since $\sigma(M) - 1 = n - (n - 2) - 1 = 1$ is not a multiple of p , M does not admit an \mathbb{Z}_p -action with the rotation angle above. Note that M does not admit any almost complex structure because $e(M) + \sigma(M) = 2n + 2$ is not a multiple of 4 (see [8] p.1625).

EXAMPLE 4.3. Let M be a 6-dimensional closed oriented manifold, which does not necessarily admit an almost complex structure (see Example 3.3 in [7]). In this example using Theorem 3.6, we show that M does not admit the \mathbb{Z}_6 -action of isotropy orders

$$(p_1, p_2, p_3, \dots, p_{v+2}) = (6, 3, 2, \dots, 2).$$

Here we assume that M admits a \mathbb{Z}_6 -action of isotropy orders above. Then it follows from the equality (2) that the Euler number $e(M)$ of M satisfies the equality:

$$\sum_{i=1}^{\nu+2} r_i - e(M) = 1 + 2 + 3\nu - e(M) = 3 \left(\nu + 1 - \frac{e(M)}{3} \right) \equiv 0 \pmod{6},$$

which implies that there does not exist such a \mathbb{Z}_6 -action above unless the following condition is satisfied:

(30) $e(M)$ is a multiple of 3 and $\nu + 1 - \frac{e(M)}{3}$ is an even number.

So we assume that the condition above is satisfied.

Let g be a generator of \mathbb{Z}_6 and $\{\tau_{i1}, \tau_{i2}, \tau_{i3}\}$ the rotation angle of g^i . Note that $\tau_{1j} = 1$ or 5 and $\tau_{2j} = 1$ or 2 for $1 \leq j \leq 3$ and hence the mod. p_i -inverse of τ_{ij} is equal to τ_{ij} itself for $i = 1, 2$. Here it follows from Proposition 2.9 that

$$\text{Ch}(E_{q_i}^{0,0,1}, g) = C_{i3}^1 = \rho_{p_i}(\tau_{i1} \pm \tau_{i2} \pm \tau_{i3}) + 2 \sum_{1 \leq j \leq 3} \rho_{p_i}(\tau_{ij}) = \sum_s n_s \rho_{p_i}(\mu_{is})$$

where

$$\begin{aligned} (n_1, \mu_{i1}) &= (1, \tau_{i1} + \tau_{i2} + \tau_{i3}), & (n_2, \mu_{i2}) &= (1, \tau_{i1} + \tau_{i2} - \tau_{i3}), \\ (n_3, \mu_{i3}) &= (1, \tau_{i1} - \tau_{i2} + \tau_{i3}), & (n_4, \mu_{i4}) &= (1, \tau_{i1} - \tau_{i2} - \tau_{i3}), \\ (n_5, \mu_{i5}) &= (2, \tau_{i1}), & (n_6, \mu_{i6}) &= (2, \tau_{i2}), & (n_7, \mu_{i7}) &= (2, \tau_{i3}). \end{aligned}$$

Then since $\text{Ind}(D_{0,0,1})$ vanishes as we see in Remark 2.2 and $\zeta_0 = -1$, $\eta_0 = 1$, it follows from Theorem 3.6 that

(31)
$$I_{D_{0,0,1}}(g^z) = -\frac{1}{p} \sum_{i=1}^{\nu+2} r_i \sum_s n_s F_s(i, z)$$

where

$$F_s(i, z) = -\eta_2(p_i; z\mu_{is}, \tau_{i1}, \tau_{i2}, \tau_{i3}, \bar{z}) + \zeta_1(p_i; \tau_{i1}, \tau_{i2})\eta_1(p_i; z\mu_{is}, \tau_{i3}, \bar{z}) + \zeta_2(p_i; \tau_{i1}, \tau_{i2}, \tau_{i3}, \bar{z}).$$

Here it follows from Example 2.18 for $n = 2$ that

$$\begin{aligned} \zeta_1 &= \zeta_1(p_i; \tau_{i1}, \tau_{i2}) = -\frac{1}{p_i} \sum_{\substack{1 \leq k_1, k_2 < p_i \\ p_i | (\tau_{i1}k_1 + \tau_{i2}k_2)}} (2k_1 - p_i)(2k_2 - p_i), \\ \zeta_2 &= \zeta_2(p_i; \tau_{i1}, \tau_{i2}, \tau_{i3}, z) = \frac{1}{p_i^3} \sum_{\substack{1 \leq k_1, k_2, k_3, k_4 < p_i \\ p_i | (\tau_{i1}k_1 + \tau_{i2}k_2 + \tau_{i3}k_3 + \bar{z}k_4)}} (2k_1 - p_i)(2k_2 - p_i)(2k_3 - p_i)(2k_4 - p_i), \\ \eta_1 &= \eta_1(p_i; z\mu_{is}, \tau_{i3}, \bar{z}) = -2a_{is}^2 c_2 = -2(z\mu_{is})^2 \langle \bar{\tau}_{i3} \bar{z} \rangle, \\ \eta_2 &= \eta_2(p_i; z\mu_{is}, \tau_{i1}, \tau_{i2}, \tau_{i3}, \bar{z}) = \frac{2}{3} a_{is}^2 c_1 d_1^2 c_2 (a_{is}^2 c_2^2 + c_2^2 + 1) \\ &= \frac{2}{3} (z\mu_{is})^2 \langle \bar{\tau}_{i1} \tau_{i2} \rangle \langle \bar{\tau}_{i2} \tau_{i3} \rangle \langle \bar{\tau}_{i3} \bar{z} \rangle \left\{ (z\mu_{is})^2 \langle \bar{\tau}_{i3} \bar{z} \rangle^2 + \langle \bar{\tau}_{i3} \bar{z} \rangle^2 + 1 \right\}. \end{aligned}$$

When $i = 1$, $p_i = 6$, $z = 1$, $(\tau_{11}, \tau_{12}, \tau_{13}) = (1, 1, 1)$, we have

$$\mu_{11} = 1 + 1 + 1 = 3, \mu_{12} = 1 + 1 - 1 = 1, \mu_{13} = 1 - 1 + 1 = 1, \mu_{14} = 1 - 1 - 1 = -1,$$

$$\mu_{15} = \mu_{16} = \mu_{17} = 1, \quad \bar{z} = 1,$$

and direct computation shows that

$$\begin{aligned} \zeta_1(6; \tau_{11}, \tau_{12}) &= \zeta_1(6; 1, 1) = \frac{20}{3}, \\ \zeta_2(6; \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \zeta_2(6; 1, 1, 1, 1) = \frac{164}{9}, \\ \eta_1(6; z\mu_{11}, \tau_{13}, \bar{z}) &= \eta_1(6; 3, 1, 1) = -18, \\ \eta_1(6; z\mu_{14}, \tau_{13}, \bar{z}) &= \eta_1(6; -1, 1, 1) = -2, \\ \eta_1(6; z\mu_{1s}, \tau_{13}, \bar{z}) &= \eta_1(6; 1, 1, 1) = -2 \quad (s = 2, 3, 5, 6, 7), \\ \eta_2(6; z\mu_{11}, \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \eta_2(6; 3, 1, 1, 1, 1) = 66, \\ \eta_2(6; z\mu_{14}, \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \eta_2(6; -1, 1, 1, 1, 1) = 2, \\ \eta_2(6; z\mu_{1s}, \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \eta_2(6; 1, 1, 1, 1, 1) = 2 \quad (s = 2, 3, 5, 6, 7), \end{aligned}$$

and hence we have

$$\sum_s n_s F_s(1, 1) = \sum_{s=1}^4 \left\{ (-1)u_{1s} + \frac{20}{3}v_{1s} + \frac{164}{9} \right\} + 2 \sum_{s=5}^7 \left\{ (-1)u_s + \frac{20}{3}v_s + \frac{164}{9} \right\} = -\frac{1276}{9}$$

where $(u_{11}, v_{11}) = (66, -18)$, $(u_{1s}, v_{1s}) = (2, -2)$ ($s = 2, 3, 4, 5, 6, 7$).

When $i = 2$, $p_i = 3$, $z = 1$, $(\tau_{21}, \tau_{22}, \tau_{23}) = (1, 1, 2)$, we have

$$\begin{aligned} \mu_{21} &= 1 + 1 + 2 = 4, \quad \mu_{22} = 1 + 1 - 2 = 0, \quad \mu_{23} = 1 - 1 + 2 = 2, \quad \mu_{24} = 1 - 1 - 2 = -2, \\ \mu_{25} &= \mu_{26} = 1, \quad \mu_{27} = 2, \end{aligned}$$

and direct computation shows that

$$\begin{aligned} \zeta_1(3; \tau_{21}, \tau_{22}) &= \zeta_1(3; 1, 1) = \frac{2}{3}, \\ \zeta_2(3; \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \zeta_2(3; 1, 1, 2, 1) = -\frac{2}{9}, \\ \eta_1(3; z\mu_{21}, \tau_{23}, \bar{z}) &= \eta_1(3; 4, 2, 1) = -64, \\ \eta_1(3; z\mu_{22}, \tau_{23}, \bar{z}) &= \eta_1(3; 0, 2, 1) = 0, \\ \eta_1(3; z\mu_{2s}, \tau_{13}, \bar{z}) &= \eta_1(3; 2, 2, 1) = -16 \quad (s = 3, 7), \\ \eta_1(3; z\mu_{24}, \tau_{13}, \bar{z}) &= \eta_1(3; -2, 2, 1) = -16, \\ \eta_1(3; z\mu_{2s}, \tau_{13}, \bar{z}) &= \eta_1(3; 1, 2, 1) = -4 \quad (s = 5, 6), \\ \eta_2(3; z\mu_{21}, \tau_{21}, \tau_{22}, \tau_{23}, \bar{z}) &= \eta_2(3; 4, 1, 1, 2, 1) = 5888, \\ \eta_2(3; z\mu_{22}, \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \eta_2(3; 0, 1, 1, 2, 1) = 0, \\ \eta_2(3; z\mu_{2s}, \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \eta_2(3; 2, 1, 1, 2, 1) = 448 \quad (s = 3, 7), \\ \eta_2(3; z\mu_{24}, \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \eta_2(3; -2, 1, 1, 2, 1) = 448, \\ \eta_2(3; z\mu_{2s}, \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \eta_2(3; 1, 1, 1, 2, 1) = 48 \quad (s = 5, 6), \end{aligned}$$

and hence we have

$$\sum_s n_s F_s(2, 1) = \sum_{s=1}^4 \left\{ (-1)u_{2s} + \frac{2}{3}v_{2s} - \frac{2}{9} \right\} + 2 \sum_{s=5}^7 \left\{ (-1)u_{2s} + \frac{2}{3}v_{2s} - \frac{2}{9} \right\} = -\frac{71732}{9}$$

where $(u_{21}, v_{21}) = (5888, -64)$, $(u_{22}, v_{22}) = (0, 0)$, $(u_{2s}, v_{2s}) = (448, -16)$ ($s = 3, 4, 7$), $(u_{2s}, v_{2s}) = (48, -4)$ ($s = 5, 6$).

When $3 \leq i \leq \nu + 2$, $p_i = 2$ and it follows from Remark 2.3 and (20) that

$$\sum_s n_s F_s(i, 1) = 0.$$

Hence it follows from (5) and (31) that

$$\begin{aligned} I_{D_{0,0,1}}(g) &= -\frac{1}{6} \sum_{i=1}^{\nu+2} r_i \sum_s n_s F_s(i, z) \\ &= \frac{1}{6} \left(\frac{1276}{9} + 2 \cdot \frac{71732}{9} + 0 \right) = \frac{72370}{27} = \frac{10}{27} \in \mathbb{R}/\mathbb{Z}. \end{aligned}$$

Using the same method, we have

$$I_{D_{0,0,1}}(g^5) = \frac{17}{27} \in \mathbb{R}/\mathbb{Z},$$

and therefore it follows that

$$I_{D_{0,0,1}}(g^5) - 5I_{D_{0,0,1}}(g) = \frac{7}{9} \in \mathbb{R}/\mathbb{Z}.$$

Namely the equality

$$I_{D_{0,0,1}}(g^5) = 5I_{D_{0,0,1}}(g)$$

does not hold for $(\tau_{11}, \tau_{12}, \tau_{13}; \tau_{21}, \tau_{22}, \tau_{23}) = (1, 1, 1; 1, 1, 2)$.

The direct computation using the same argument shows that the equality

$$I_{D_{0,0,1}}(g^5) = 5I_{D_{0,0,1}}(g)$$

does not hold for $\tau_{1j} = 1, 5$ ($1 \leq j \leq 3$) and $\tau_{2j} = 1, 2$ ($1 \leq j \leq 3$). This is a contradiction and therefore M does not admit the \mathbb{Z}_6 -action of isotropy orders above.

For example let N be a 4-dimensional closed oriented manifold, R a compact Riemann surface of genus r such that $r \equiv 1 \pmod{3}$ and set $M = N \times R$. Then the Euler number of M is a multiple of 6 because the Euler number of R is $2(1 - r)$, and the condition (30) is satisfied for any odd natural number ν . Then it follows from the result above that M does not admit the \mathbb{Z}_6 -action of isotropy orders $(p_1, p_2, p_3, \dots, p_{\nu+2}) = (6, 3, 2, \dots, 2)$.

REMARK 4.4. In the example above, M is an almost complex manifold if N is an almost complex manifold. But the \mathbb{Z}_6 -action does not necessarily preserve the almost complex structure and the result above can not be obtained from the methods in [7], where the action is assumed to preserve the almost complex structure.

References

- [1] M.F. Atiyah and R. Bott: *A Lefschetz fixed point formula for elliptic complexes II*, Ann. of Math. **87** (1968), 451–491.

- [2] M.F. Atiyah and G.B. Segal: *The index of elliptic operators II*, Ann. of Math. **87** (1968), 531–545.
- [3] M.F. Atiyah and I.M. Singer: *The index of elliptic operators III*, Ann. of Math. **87** (1968), 546–604.
- [4] H. Donnelly: *Eta invariants for G-spaces*, Indiana Univ. Math. J. **27** (1978), 889–918.
- [5] H.B. Lawson and M.L. Michelsohn: *Spin Geometry*, Princeton Mathematical Series **38**, Princeton University Press, 1989.
- [6] K. Tsuboi: *The finite group action and the equivariant determinant of elliptic operators*, J. Math. Soc. Japan **57** (2005), 95–113.
- [7] K. Tsuboi: *The finite group action and the equivariant determinant of elliptic operators II*, J. Math. Soc. Japan **65** (2013), 797–827.
- [8] A. Van de Ven: *On the Chern numbers of certain complex and almost complex manifolds*, Proc. Nat. Acad. Sci. U.S.A. **55** (1966), 1624–1627.
- [9] D.B. Zagier: *Equivariant Pontriagin Classes and Applications to Orbit Spaces*, Lecture Notes in Mathematics **290**, Springer-Verlag Berlin-Heidelberg-New York, 1972.

Tokyo University of Marine science and technology
4–5–7 Kounan, Minato-ku
Tokyo 108–8477
Japan
e-mail: tsubois@kaiyodai.ac.jp