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LARGE TIME BEHAVIOR OF GLOBAL SOLUTIONS TO NONLINEAR WAVE EQUATIONS WITH FRICTIONAL AND VISCOELASTIC DAMPING TERMS

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Abstract
In this paper, we study the Cauchy problem for a nonlinear wave equation with frictional and viscoelastic damping terms in $\mathbb{R}^n$. As is pointed out by [10], in this combination, the frictional damping term is dominant for the viscoelastic one for the global dynamics of the linear equation. In this note we observe that if the initial data is small, the frictional damping term is again dominant even in the nonlinear equation case. In other words, our main result is diffusion phenomena: the solution is approximated by the heat kernel with a suitable constant. Especially, the result obtained for the $n = 3$ case is essentially new. Our proof is based on several estimates for the corresponding linear equations.

1. Introduction

In this paper we are concerned with the following Cauchy problem for the wave equation with two types of damping terms:

$$(1.1)\qquad \begin{cases} \partial_t^2 u - \Delta u + \partial_t u - \Delta\partial_t u = f(u), \quad t > 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases}$$

where $u_0(x)$ and $u_1(x)$ are given initial data, and about the nonlinearity $f(u)$ we shall consider only the typical case such as

$$f(r) := |r|^p, \quad (p > 1),$$

without loss of generality (see Remark 1.3 below).

Concerning the following equation with frictional damping:

$$(1.2)\qquad \begin{cases} \partial_t^2 u - \Delta u + \partial_t u = f(u), \quad t > 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases}$$

nowadays one can find an important result called as the critical exponent problem such as following: there exists an exponent $p_* > 1$ such that if the power p of nonlinearity $f(u)$ satisfies $p_* < p$, then the corresponding problem (1.2) has a small data global in time solution, while in the case when $1 < p \leq p_*$ the problem (1.2) does not admit any nontrivial global solutions for some initial data. We call p_* as the critical exponent. In the frictional

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damping case, we have $p_* = p_F := 1 + \frac{2}{n}$, which is called as the Fujita exponent in the semi-linear heat equation case. For those results, we refer to [4], [5], [8], [12], [15], [16], [18], [19], [20], [24], [25], [26] and the references therein.

Quite recently, Ikehata-Takeda [11] has treated the original problem (1.1) motivated by a previous result concerning the linear equation due to Ikehata-Sawada [10], and solved the Fujita critical exponent one. They have discovered the value $p_* = 1 + \frac{2}{n}$ again only in the low dimensional case (i.e., $n = 1, 2$). Then, the problem of the critical exponent to (1.1) is still open for all $n \geq 3$. This result due to [11] implies an important recognition that the dominant term is still the frictional damping $\partial_t u$, although the equation (1.1) has two types of damping terms. Note that in the viscoelastic damping case,

$$(1.3) \qquad \begin{cases} \partial_t^2 u - \Delta u - \Delta \partial_t u = f(u), & t > 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

we still do not know the "exact" critical exponent p_*. Several interesting results about this critical exponent problem including optimal linear estimates for (1.3) can be observed in the literature due to D'Abbicco-Reissig [2, see Theorem 2, and Section 4]. But, it seems to be a little far from complete results on the critical exponent problem of (1.3). In fact, in [2] they studied a more general form of equations such that

$$\partial_t^2 u - \Delta u + (-\Delta)^\sigma \partial_t u = \mu f(u)$$

with $\sigma \in [0, 1]$ and $\mu \geq 0$. Pioneering and/or important contributions for the case $\sigma = 1$ (i.e., strong damping one) can be found in several papers due to [7], [13] (both in abstract theory), [21], [23] and the references therein.

We should also mention some results for the asymptotic behavior of solutions to the linearized compressible Navier-Stokes systems, since the main results of this paper are largely overlapping with the results in that fields. We also note that it is well known that the solution of (1.3) is corresponding to the density of the Navier-Stokes system. In this case, Hoff-Zumbrun [6] firstly pointed out that the asymptotic behavior of the solution in terms of L^p-norms with $H^s \cap L^1$ data for some $s \geq 0$, has two possibilities as $t \to \infty$: When $p > 2$, the dominant term is given by the pure diffusive part. On the other hand, if $p < 2$, the solution asymptotically behaves like the diffusion wave. Kagei-Kobayashi [14] extended the results of [6] to the half space case. In a more simple setting, Kobayashi- Shibata [17] proved sharp decay estimates of the solutions and recently, Ikehata-Onodera [9] obtained the lower bound of solutions in terms of L^2.

From observations above one naturally encounters an important problem such that even in the higher dimensional case for $n \geq 3$, can one also solve the critical exponent problem of (1.1)?

Our first purpose is to prove the following global existence result of the solution together with suitable decay properties to problem (1.1).

Theorem 1.1. *Let $n = 1, 2, 3$, $\varepsilon > 0$ and $p > 1 + \frac{2}{n}$. Assume that $(u_0, u_1) \in (W^{\frac{n}{2}+\varepsilon,1} \cap W^{\frac{n}{2}+\varepsilon,\infty}) \times (L^1 \cap L^\infty)$ with sufficiently small norms. Then, there exists a unique global solution*

$u \in C([0,\infty); L^1 \cap L^\infty)$ to problem (1.1) *satisfying*

$$\|u(t,\cdot)\|_{L^q(\mathbb{R}^n)} \le C(\|u_0\|_{W^{\frac{n}{2}+\varepsilon,1}\cap W^{\frac{n}{2}+\varepsilon,\infty}(\mathbb{R}^n)} + \|u_1\|_{L^1\cap L^\infty(\mathbb{R}^n)})(1+t)^{-\frac{n}{2}(1-\frac{1}{q})} \tag{1.4}$$

for $q \in [1,\infty]$.

Our second aim is to study the large time behavior of the global solution given in Theorem 1.1. For this we define the Gauss kernel by

$$G_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.$$

Theorem 1.2. *Under the same assumptions as in* Theorem 1.1*, the corresponding global solution $u(t,x)$ satisfies*

$$\lim_{t\to\infty} t^{\frac{n}{2}(1-\frac{1}{q})}\|u(t,\cdot) - MG_t\|_{L^q(\mathbb{R}^n)} = 0, \tag{1.5}$$

for $1 \le q \le \infty$, where $M := \int_{\mathbb{R}^n}(u_0(y)+u_1(y))dy + \int_0^\infty \int_{\mathbb{R}^n} f(u(s,y))dyds.$

Remark 1.3. We should remark that our results are easily extended to the nonlinear term $f(u)$ satisfying the locally Lipschitz growth condition

$$|f(u)| \le C|u|^p,$$
$$|f(u) - f(v)| \le C(|u|^{p-1} + |v|^{p-1})|u - v|$$

for some constant $C > 0$, with minor modification of the proofs.

Remark 1.4. By combining the blowup result given in [11, Theorem 1.3] and Theorems 1.1 and 1.2 with $n = 3$, one can make sure that even in the $n = 3$ case the critical exponent p_* to (1.1) is given by the Fujita exponent $p_* = p_F$. Such sharpness has already been announced in the low dimensional cases (i.e., $n = 1, 2$) by [11, Theorems 1.1 and 1.3]. So, the result for $n = 3$ is essentially new. It is also worth mentioning that we need not to assume the upper bound of the growth order p in Theorems 1.1 and 1.2, since we construct the global solution of (1.1) in the class $C([0,\infty); L^1 \cap L^\infty)$ under the restriction $n \le 3$. These are our main contributions to problem (1.1) in this paper. It is still open to show the global existence part for all $n \ge 4$, however, this part will be studied in our forthcoming project.

Before closing this section, we summarize notation, which will be used throughout this paper.

Let $\hat{f}$ denote the Fourier transform of f defined by

$$\hat{f}(\xi) := c_n \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$$

with $c_n = (2\pi)^{-\frac{n}{2}}$. Also, let $\mathcal{F}^{-1}[f]$ or $\check{f}$ denote the inverse Fourier transform.

We introduce smooth, radial cut-off functions to localize the frequency region as follows: χ_L, χ_M and $\chi_H \in C^\infty(\mathbb{R}^n)$ are defined by

$$\chi_L(\xi) = \begin{cases} 1, & |\xi| \le \frac{1}{2}, \\ 0, & |\xi| \ge \frac{3}{4}, \end{cases} \qquad \chi_H(\xi) = \begin{cases} 1, & |\xi| \ge 3, \\ 0, & |\xi| \le 2, \end{cases}$$

$$\chi_M(\xi) = 1 - \chi_L(\xi) - \chi_H(\xi).$$

For $k \geq 0$ and $1 \leq p \leq \infty$, let $W^{k,p}(\mathbb{R}^n)$ be the usual Sobolev spaces

$$W^{k,p}(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{R}; \|f\|_{W^{k,p}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \||\nabla_x|^k f\|_{L^p(\mathbb{R}^n)} < \infty \right\},$$

where $L^p(\mathbb{R}^n)$ is the Lebesgue space for $1 \leq p \leq \infty$ as usual. When $p = 2$, we denote $W^{k,2}(\mathbb{R}^n) = H^k(\mathbb{R}^n)$. For the notation of the function spaces, the domain $\mathbb{R}^n$ is often abbreviated. We frequently use the notation $\|f\|_p = \|f\|_{L^p(\mathbb{R}^n)}$ without confusion. Furthermore, in the following C denotes a positive constant, which may change from line to line.

The paper is organized as follows. Section 2 presents some preliminaries. In Section 3, we show several point-wise estimates of the propagators for the corresponding linear equation in the Fourier space. Section 4 is devoted to the proof of linear estimates, which play crucial roles to get main results. In sections 5 and 6, we give the proof of our main results.

2. Preliminaries

In this section, we collect several basic facts on the Fourier multiplier theory, the decay estimates of the solution for the heat equation and elementary inequalities to obtain the decay property of the solutions.

2.1. Fourier multiplier. For $f \in L^2 \cap L^p$, $1 \leq p \leq \infty$, let $m(\xi)$ be the Fourier multiplier defined by

$$\mathcal{F}^{-1}[m\hat{f}](x) = c_n \int_{\mathbb{R}^n} e^{ix\cdot\xi} m(\xi) \hat{f}(\xi) d\xi.$$

We define M_p as the class of the Fourier multiplier with $1 \leq p \leq \infty$:

$$M_p := \Big\{ m : \mathbb{R}^n \to \mathbb{R}; \text{measurable} |$$
$$\text{There exists a constant } A_p > 0 \text{ such that } \|\mathcal{F}^{-1}[m\hat{f}]\|_p \leq A_p \|f\|_p \Big\}.$$

For $m \in M_p$, we let

$$M_p(m) := \sup_{f \neq 0} \frac{\|\mathcal{F}^{-1}[m\hat{f}]\|_p}{\|f\|_p}.$$

The following lemma describes the inclusion among the class of multipliers.

Lemma 2.1. *Let* $\frac{1}{p} + \frac{1}{p'} = 1$ *with* $1 \leq p \leq p' \leq \infty$. *Then* $M_p = M_{p'}$ *and for* $m \in C^\infty(\mathbb{R}^n)$, *it holds that*

$$M_p(m) = M_{p'}(m).$$

Moreover, if $m \in M_p$, *then* $m \in M_q$ *for all* $q \in [p, p']$ *and*

$$M_q(m) \leq M_p(m) = M_{p'}(m). \tag{2.1}$$

We recall the Carleson-Beurling inequality, which is used to show the L^p boundedness of the Fourier multipliers.

Lemma 2.2 (Carleson-Beurling's inequality)**.** *If* $m \in H^s$ *with* $s > \frac{n}{2}$*, then* $m \in M_r$ *for all* $1 \le r \le \infty$*. Moreover, there exists a constant* $C > 0$ *such that*

$$M_\infty(m) \le C\|m\|_2^{1-\frac{n}{2s}}\|m\|_{\dot{H}^s}^{\frac{n}{2s}}. \tag{2.2}$$

For the proof of Lemmas 2.1 and 2.2, see [1].

2.2. Decay property of the solution of heat equations. The following lemma is also well-known as the decay property and approximation formula of the solution of the heat equation. For the proof, see e.g., [3].

Lemma 2.3. *Let* $n \ge 1$*,* $\ell \ge 0$*,* $k \ge \tilde{k} \ge 0$ *and* $1 \le r \le q \le \infty$*. Then there exists a constant* $C > 0$ *such that*

$$\|\partial_t^\ell \nabla_x^k e^{t\Delta} g\|_q \le C t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})-\ell-\frac{k-\tilde{k}}{2}}\|\nabla_x^{\tilde{k}} g\|_r. \tag{2.3}$$

Moreover, if $g \in L^1 \cap L^q$*, then it holds that*

$$\lim_{t\to\infty} t^{\frac{n}{2}(1-\frac{1}{q})+\frac{k}{2}}\|\nabla_x^k(e^{t\Delta}g - mG_t)\|_q = 0, \tag{2.4}$$

where $m = \int_{\mathbb{R}^n} g(y)dy$.

2.3. Useful formula. In this subsection, we recall useful estimates to show several results in this paper. The following well-known estimate will be frequently used to obtain time decay estimates.

Lemma 2.4. *Let* $n \ge 1$*,* $k \ge 0$ *and* $1 \le r \le 2$*. Then there exists a constant* $C > 0$ *such that*

$$\|\,|\xi|^k e^{-(1+t)|\xi|^2}\|_r \le C(1+t)^{-\frac{n}{2r}-\frac{k}{2}}. \tag{2.5}$$

The next lemma is also useful to compute the decay order of the nonlinear term in the integral equation.

Lemma 2.5. (*i*) *Let* $a > 0$ *and* $b > 0$ *with* $\max\{a, b\} > 1$*. There exists a constant* $C > 0$ *depending only on a and b such that for* $t \ge 0$ *it is true that*

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}ds \le C(1+t)^{-\min\{a,b\}}. \tag{2.6}$$

(*ii*) *Let* $1 > a \ge 0$*,* $b > 0$ *and* $c > 0$*. There exists a constant* $C > 0$*, which is independent of t such that for* $t \ge 0$ *it holds that*

$$\int_0^t e^{-c(t-s)}(t-s)^{-a}(1+s)^{-b}ds \le C(1+t)^{-b}. \tag{2.7}$$

The proof of Lemma 2.5 is well-known (see e.g. [22]).

3. Point-wise estimates in the Fourier space

In this section, we show point-wise estimates of the Fourier multipliers, which are important to obtain linear estimates in the next section. Now, we recall the Fourier multiplier expression of the evolution operators to the linear problem. According to the notation of

[10] and [11] we define the Fourier multipliers $\mathcal{K}_0(t,\xi)$ and $\mathcal{K}_1(t,\xi)$ as

$$\mathcal{K}_0(t,\xi) := \frac{-\lambda_- e^{\lambda_+ t} + \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \frac{e^{-t|\xi|^2} - |\xi|^2 e^{-t}}{1-|\xi|^2},$$

$$\mathcal{K}_1(t,\xi) := \frac{-e^{\lambda_- t} + e^{\lambda_+ t}}{\lambda_+ - \lambda_-} = \frac{e^{-t|\xi|^2} - e^{-t}}{1-|\xi|^2},$$

and the evolution operators $K_0(t)g$ and $K_1(t)g$ to problem (1.1) by

$$K_j(t)g := \mathcal{F}^{-1}[\mathcal{K}_j(t,\xi)\hat{g}] \tag{3.1}$$

for $j = 0, 1$, where $\lambda_\pm$ are the characteristic roots computed through the corresponding algebraic equations (see Section 3 of [11])

$$\lambda^2 + (1+|\xi|^2)\lambda + |\xi|^2 = 0.$$

Moreover, using the cut-off functions χ_k ($k = L, M, H$), we introduce the "localized" evolution operators by

$$K_{jk}(t)g := \mathcal{F}^{-1}[\mathcal{K}_{jk}(t,\xi)\hat{g}], \tag{3.2}$$

where $\mathcal{K}_{jk}(t,\xi) := \mathcal{K}_j(t,\xi)\chi_k$, for $j = 0, 1$, $k = L, M, H$.

3.1. Estimates for the low frequency parts. We begin with the following point-wise estimates on small $|\xi|$ region in the Fourier space.

Lemma 3.1. *Let $n \geq 1$ be an integer and $|\xi| \leq 1/2$. Then there exists a constant $C > 0$ such that*

$$|e^{-t|\xi|^2} - e^{-t}|\xi|^2| \leq Ce^{-(1+t)|\xi|^2}, \tag{3.3}$$

$$|\nabla_\xi(e^{-t|\xi|^2} - e^{-t}|\xi|^2)| \leq Ce^{-(1+t)|\xi|^2}(1+t)|\xi|, \tag{3.4}$$

$$|\nabla_\xi^2(e^{-t|\xi|^2} - e^{-t}|\xi|^2)| \leq Ce^{-(1+t)|\xi|^2}(1+t+t^2|\xi|^2). \tag{3.5}$$

Proof. The proof is straightforward. Noting $|\xi| \leq \frac{1}{2}$, we easily see that

$$|e^{-t|\xi|^2} - e^{-t}|\xi|^2| \leq C(e^{-t|\xi|^2} + e^{-t}) \leq Ce^{-(1+t)|\xi|^2},$$

and

$$|\nabla_\xi(e^{-t|\xi|^2} - e^{-t}|\xi|^2)| \leq Ce^{-t|\xi|^2}t|\xi| + Ce^{-t}|\xi| \leq Ce^{-(1+t)|\xi|^2}(1+t)|\xi|,$$

which prove the estimates (3.3) and (3.4), respectively. Finally we show the estimate (3.5). Taking the second derivative and using $|\xi| \leq \frac{1}{2}$ again, we have

$$\begin{aligned} |\nabla_\xi^2(e^{-t|\xi|^2} - e^{-t}|\xi|^2)| &= 2|\nabla_\xi(e^{-t|\xi|^2}t\xi - e^{-t}\xi)| \\ &\leq C(e^{-t|\xi|^2}(|t\xi|^2 + t) + e^{-t}) \\ &\leq Ce^{-(1+t)|\xi|^2}(1+t+t^2|\xi|^2), \end{aligned}$$

which is the desired estimate (3.5), and the proof is complete. □

The following estimates are useful to obtain the decay property and the large time behavior of the evolution operator $K_1(t)g$.

Lemma 3.2. *Let $n \geq 1$ be an integer and $|\xi| \leq 1/2$. Then there exists a constant $C > 0$ such that*

$$|e^{-t|\xi|^2} - e^{-t}| \leq Ce^{-(1+t)|\xi|^2}, \tag{3.6}$$

$$|\nabla_\xi(e^{-t|\xi|^2} - e^{-t})| \leq Ce^{-(1+t)|\xi|^2} t|\xi|, \tag{3.7}$$

$$|\nabla_\xi^2(e^{-t|\xi|^2} - e^{-t})| \leq Ce^{-(1+t)|\xi|^2}(t + t^2|\xi|^2). \tag{3.8}$$

Proof. The proof is standard. We have (3.6) by similar arguments to (3.3). When $k > 0$, by applying $\nabla_\xi^k(e^{-t|\xi|^2} - e^{-t}) = \nabla_\xi^k e^{-t|\xi|^2}$, (3.7) and (3.8) can be derived. □

As an easy consequence of Lemmas 3.1 and 3.2, we arrive at the point-wise estimates for the Fourier multipliers with small $|\xi|$.

Corollary 3.3. *Under the assumptions as in* Lemmas 3.1 *and* Lemma 3.2*, it holds that*

$$|\mathcal{K}_{jL}(t,\xi)| \leq Ce^{-(1+t)|\xi|^2}\chi_L, \tag{3.9}$$

$$|\nabla_\xi \mathcal{K}_{jL}(t,\xi)| \leq Ce^{-(1+t)|\xi|^2}(1+t)|\xi|\chi_L + Ce^{-\frac{t}{4}}|\chi_L'|, \tag{3.10}$$

$$|\nabla_\xi^2 \mathcal{K}_{jL}(t,\xi)| \leq Ce^{-(1+t)|\xi|^2}(1 + t + t^2|\xi|^2)\chi_L + Ce^{-\frac{t}{4}}(|\chi_L'| + |\chi_L''|) \tag{3.11}$$

for $j = 0, 1$.

Proof. The estimates (3.9), (3.10) and (3.11) for $j = 1$ are shown by the same argument. Here we only show (3.11) with $j = 0$. We first note that

$$|\nabla_\xi^k(1 - |\xi|^2)^{-1}| \leq \begin{cases} C|\xi|, & \text{for } k = 1, \\ C, & \text{for integers } k \geq 0. \end{cases} \tag{3.12}$$

In addition, it is easy to see that

$$|\nabla_\xi^k \mathcal{K}_{0L}(t,\xi)| \leq Ce^{-\frac{t}{4}} \tag{3.13}$$

on $\operatorname{supp}\chi_L' \cup \operatorname{supp}\chi_L''$ by (3.3) - (3.5) and (3.12) with $k = 0, 1$. Thus, a direct calculation, (3.12), (3.13) and Lemma 3.1 show that

$$\begin{aligned}
|\nabla_\xi^2 \mathcal{K}_{0L}(t,\xi)| &\leq C\left|\nabla_\xi^2\left(\frac{e^{-t|\xi|^2} - e^{-t}|\xi|^2}{1 - |\xi|^2}\chi_L\right)\right| \\
&\leq C\chi_L|\nabla_\xi^2(e^{-t|\xi|^2} - e^{-t}|\xi|^2)| + C\chi_L|\xi||\nabla_\xi(e^{-t|\xi|^2} - e^{-t}|\xi|^2)| \\
&\quad + C\chi_L|e^{-t|\xi|^2} - e^{-t}|\xi|^2| + Ce^{-\frac{t}{4}}(|\chi_L'| + |\chi_L''|) \\
&\leq C\chi_L(1 + t + t^2|\xi|^2)e^{-(1+t)|\xi|^2} + C\chi_L|\xi|^2 e^{-(1+t)|\xi|^2} \\
&\quad + C\chi_L e^{-(1+t)|\xi|^2} + Ce^{-\frac{t}{4}}(|\chi_L'| + |\chi_L''|) \\
&\leq Ce^{-(1+t)|\xi|^2}(1 + t + t^2|\xi|^2)\chi_L + Ce^{-\frac{t}{4}}(|\chi_L'| + |\chi_L''|),
\end{aligned}$$

which is the desired estimate (3.11) with $j = 0$. The proof of Corollary 3.3 is now complete. □

The following result plays an important role to obtain asymptotic profiles of the evolution operators $K_0(t)g$ and $K_1(t)g$.

Corollary 3.4. *Under the same assumption as in* Lemmas 3.1 *and* Lemma 3.2*, it holds that*

$$|\mathcal{K}_{jL}(t,\xi) - e^{-t|\xi|^2}\chi_L| \le C|\xi|^2 e^{-(1+t)|\xi|^2}\chi_L, \tag{3.14}$$

$$|\nabla_\xi(\mathcal{K}_{jL}(t,\xi) - e^{-t|\xi|^2}\chi_L)| \le Ce^{-(1+t)|\xi|^2}|\xi|(1+t|\xi|^2)\chi_L + Ce^{-\frac{t}{4}}|\chi_L'|, \tag{3.15}$$

$$\begin{aligned}&|\nabla_\xi^2(\mathcal{K}_{jL}(t,\xi) - e^{-t|\xi|^2}\chi_L)| \\ &\quad\le Ce^{-(1+t)|\xi|^2}(1+t|\xi|^2+t^2|\xi|^4)\chi_L + Ce^{-\frac{t}{4}}(|\chi_L'|+|\chi_L''|)\end{aligned} \tag{3.16}$$

for $j = 0, 1$.

Proof. We first consider the case $j = 0$. Combining the estimate (3.9) with $j = 1$ and the fact that

$$\mathcal{K}_{0L}(t,\xi) - e^{-t|\xi|^2}\chi_L = |\xi|^2\mathcal{K}_{1L}(t,\xi), \tag{3.17}$$

one can get (3.14) with $j = 0$. In order to show (3.15) and (3.16), by using (3.17) again we see that

$$\begin{aligned}&|\nabla_\xi^k(\mathcal{K}_{0L}(t,\xi) - e^{-t|\xi|^2}\chi_L)| \\ &\le \begin{cases} C(|\xi||\mathcal{K}_{1L}(t,\xi)| + |\xi|^2|\nabla_\xi\mathcal{K}_{1L}(t,\xi)|) \text{ for } k=1, \\ C(|\mathcal{K}_{1L}(t,\xi)| + |\xi||\nabla_\xi\mathcal{K}_{1L}(t,\xi)| + |\xi|^2|\nabla_\xi^2\mathcal{K}_{1L}(t,\xi)|) \text{ for } k=2.\end{cases}\end{aligned} \tag{3.18}$$

Combining (3.18) and (3.10) with $j = 1$ yields the estimate (3.15) with $j = 0$. We now apply this argument again to (3.10) with $j = 1$ replaced by (3.11) with $j = 1$, to obtain the estimate (3.16) with $j = 0$. Finally we prove (3.14) - (3.16) with $j = 1$. Noting that

$$\mathcal{K}_{1L}(t,\xi) - e^{-t|\xi|^2}\chi_L = \frac{e^{-t|\xi|^2}|\xi|^2 - e^{-t}}{1-|\xi|^2}\chi_L, \tag{3.19}$$

and applying a similar argument to (3.6), one gets (3.14) with $j = 1$. Moreover, using $\nabla_\xi^k(e^{-t|\xi|^2}|\xi|^2 - e^{-t}) = \nabla_\xi^k(e^{-t|\xi|^2}|\xi|^2)$ for $k > 0$, we can deduce that

$$|\nabla_\xi(e^{-t|\xi|^2}|\xi|^2 - e^{-t})| \le C|\xi|(1+t|\xi|^2)e^{-t|\xi|^2}, \tag{3.20}$$

$$|\nabla_\xi^2(e^{-t|\xi|^2}|\xi|^2 - e^{-t})| \le C(1+t|\xi|^2+t^2|\xi|^4)e^{-t|\xi|^2}. \tag{3.21}$$

Therefore, by (3.14) with $j = 1$ and (3.20), we obtain (3.15) with $j = 1$. Likewise, we use (3.14) and (3.15) with $j = 1$ and (3.21) to meet (3.16) with $j = 1$, and the corollary follows. □

3.2. Estimates for the middle and high frequency parts. The following lemma states that the middle part for $|\xi|$ has a sufficient regularity and decays fast.

Lemma 3.5. *Let* $n \ge 1$ *and* $k \ge 0$*. Then there exists a constant* $C > 0$ *such that*

$$|\nabla_\xi^k\mathcal{K}_{jM}(t,\xi)\chi_M| \le Ce^{-\frac{t}{4}}\chi_M \tag{3.22}$$

for $j = 0, 1$.

Proof. The support of the middle part $\nabla_\xi^k \mathcal{K}_{jM}(t,\xi)\chi_M$ is compact and does not contain a neighborhood of the origin $\xi = 0$. Therefore, we can estimate the polynomial of $|\xi|$ by a constant. This implies the desired estimate (3.22), and the proof is now complete. □

The rest part of this subsection is devoted to the point-wise estimates for the high frequency parts $K_{jH}(t)g$ for $j = 0, 1$.

Lemma 3.6. *Let* $n = 1, 2, 3$, $\varepsilon > 0$ *and* $\alpha \in \{2, \frac{n}{2} + \varepsilon\}$. *Then it holds that*

$$\nabla_\xi^k \left(\frac{|\xi|^{2-\alpha}}{1 - |\xi|^2} \chi_H \right) \in L^2(\mathbb{R}^n) \tag{3.23}$$

for $k = 0, 1, 2$.

Proof. It is easy to see that

$$\nabla_\xi^k \left(\frac{|\xi|^{2-\alpha}}{1 - |\xi|^2} \right) = O(|\xi|^{-\alpha-k}) \tag{3.24}$$

as $|\xi| \to \infty$ and $2(-\alpha - k) < -n$. Moreover the support of $\frac{|\xi|^{2-\alpha}}{1-|\xi|^2}\chi_H$ does not have a neighborhood of $|\xi| = 1$. Summing up these facts, we can assert (3.23), and the proof is complete. □

Lemma 3.7. *Let* $n \geq 1$ *and* $|\xi| \geq 3$. *Then there exists a constant* $C > 0$ *such that*

$$|\nabla_\xi (e^{-t|\xi|^2} - e^{-t})| \leq C e^{-t|\xi|^2} t|\xi|, \tag{3.25}$$

$$|\nabla_\xi^2 (e^{-t|\xi|^2} - e^{-t})| \leq C e^{-t|\xi|^2} (t + t^2|\xi|^2). \tag{3.26}$$

Proof. Applying $\nabla_\xi^k (e^{-t|\xi|^2} - e^{-t}) = \nabla_\xi^k e^{-t|\xi|^2}$ for $k > 0$ again, we easily have Lemma 3.7. □

Corollary 3.8. *Under the same assumptions as in* Lemma 3.7, *there exists a constant* $C > 0$ *such that*

$$|\mathcal{K}_{1H}(t,\xi)| \leq C e^{-t} |\xi|^{-2} \chi_H, \tag{3.27}$$

$$\left|\nabla_\xi \mathcal{K}_{1H}(t,\xi)\right| \leq C e^{-\frac{t}{2}} |\xi|^{-2} (\chi_H + |\chi'_H|), \tag{3.28}$$

$$\left|\nabla_\xi^2 \mathcal{K}_{1H}(t,\xi)\right| \leq C e^{-\frac{t}{2}} |\xi|^{-2} (\chi_H + |\chi'_H| + |\chi''_H|). \tag{3.29}$$

Proof. Since (3.27) - (3.29) are shown by the similar way, we only check the validity of (3.29). We first note that

$$\left|\mathcal{K}_j(t,\xi)\chi'_H\right| + \left|\nabla_\xi \mathcal{K}_j(t,\xi)\chi'_H\right| + \left|\mathcal{K}_j(t,\xi)\chi''_H\right| \leq C e^{-\frac{t}{2}} (|\chi'_H| + |\chi''_H|) \tag{3.30}$$

for $j = 0, 1$. Indeed, the support of χ'_H and χ''_H is compact and does not include a neighborhood of $\xi = 0$. So, the direct calculation and (3.24) - (3.26) show

$$
\begin{aligned}
(3.31)\qquad & |\nabla_\xi^2 \mathcal{K}_1(t,\xi)| \\
& \le C|\nabla_\xi^2 (e^{-t|\xi|^2})||\xi|^{-2} + C|\nabla_\xi e^{-t|\xi|^2}||\nabla_\xi (1-|\xi|^2)^{-1}| + Ce^{-t|\xi|^2}|\nabla_\xi^2 (1-|\xi|^2)^{-1}| \\
& \le Ce^{-t}e^{-ct|\xi|^2}\left\{(t+t^2|\xi|^2)|\xi|^{-2} + |\xi|^{-3}t|\xi| + |\xi|^{-4})\right\} \\
& \le Ce^{-\frac{t}{2}}|\xi|^{-2}
\end{aligned}
$$

for $|\xi| \ge 3$. Thus combining (3.30) and (3.31), we see

$$
\begin{aligned}
|\nabla_\xi^2 \mathcal{K}_{1H}(t,\xi)| &\le C\chi_H |\nabla_\xi^2 \mathcal{K}_1(t,\xi)| + Ce^{-\frac{t}{2}}(|\chi_H'| + |\chi_H''|) \\
&\le Ce^{-\frac{t}{2}}|\xi|^{-2}(\chi_H + |\chi_H'| + |\chi_H''|),
\end{aligned}
$$

which is the desired conclusion. □

The following estimates are useful for the estimates for $K_{0H}(t)g$.

Corollary 3.9. *Under the same assumptions as in* Lemma 3.7*, there exists a constant* $C > 0$ *such that*

$$
\left|\mathcal{K}_0(t,\xi)|\xi|^{-(\frac{n}{2}+\varepsilon)}\chi_H\right| \le Ce^{-t}|\xi|^{-(\frac{n}{2}+\varepsilon)}\chi_H, \tag{3.32}
$$

$$
\left|\nabla_\xi\left(\mathcal{K}_0(t,\xi)|\xi|^{-(\frac{n}{2}+\varepsilon)}\chi_H\right)\right| \le Ce^{-\frac{t}{2}}|\xi|^{-(\frac{n}{2}+\varepsilon)}(\chi_H + |\chi_H'|), \tag{3.33}
$$

$$
\left|\nabla_\xi^2\left(\mathcal{K}_0(t,\xi)|\xi|^{-(\frac{n}{2}+\varepsilon)}\chi_H\right)\right| \le Ce^{-\frac{t}{2}}|\xi|^{-(\frac{n}{2}+\varepsilon)}(\chi_H + |\chi_H'| + |\chi_H''|). \tag{3.34}
$$

Proof. Let $k = 0, 1, 2$. Observing the fact that

$$
\begin{aligned}
(3.35)\qquad & \left|\nabla_\xi^k\left(\frac{e^{-t|\xi|^2}|\xi|^{-(\frac{n}{2}+\varepsilon)} - e^{-t}|\xi|^{2-(\frac{n}{2}+\varepsilon)}}{1-|\xi|^2}\chi_H\right)\right| \\
& \le \left|\nabla_\xi^k\left(\frac{e^{-t|\xi|^2}|\xi|^{-(\frac{n}{2}+\varepsilon)}}{1-|\xi|^2}\chi_H\right)\right| + e^{-t}\left|\nabla_\xi^k\left(\frac{|\xi|^{2-(\frac{n}{2}+\varepsilon)}}{1-|\xi|^2}\chi_H\right)\right|,
\end{aligned}
$$

we see that the first factor in the right hand side of (3.35) satisfy the following estimates

$$
\begin{aligned}
(3.36)\qquad & \left|\frac{e^{-t|\xi|^2}|\xi|^{-(\frac{n}{2}+\varepsilon)}}{1-|\xi|^2}\chi_H\right| \le Ce^{-t}|\xi|^{-(\frac{n}{2}+\varepsilon)-2}\chi_H, \\
& \left|\nabla_\xi\left(\frac{e^{-t|\xi|^2}|\xi|^{-(\frac{n}{2}+\varepsilon)}}{1-|\xi|^2}\chi_H\right)\right| \le Ce^{-\frac{t}{2}}|\xi|^{-(\frac{n}{2}+\varepsilon)-2}(\chi_H + |\chi_H'|), \\
& \left|\nabla_\xi^2\left(\frac{e^{-t|\xi|^2}|\xi|^{-(\frac{n}{2}+\varepsilon)}}{1-|\xi|^2}\chi_H\right)\right| \le Ce^{-\frac{t}{2}}|\xi|^{-(\frac{n}{2}+\varepsilon)-2}(\chi_H + |\chi_H'| + |\chi_H''|),
\end{aligned}
$$

as in Corollary 3.8. Furthermore, by using (3.24) with $\alpha = \dfrac{n}{2} + \varepsilon$, and (3.31) with $j = 1$, the second factor in the right hand side of (3.35) is estimated as follows

$$
e^{-t}\left|\nabla_\xi^k\left(\frac{|\xi|^{2-(\frac{n}{2}+\varepsilon)}}{1-|\xi|^2}\chi_H\right)\right| \le Ce^{-\frac{t}{2}}|\xi|^{-\frac{n}{2}-\varepsilon-k}\chi_H. \tag{3.37}
$$

Summing up these estimates (3.35) - (3.37), one can conclude (3.32) - (3.34). □

4. Linear estimates

In this section, we shall study an important decay property of the solution $u(t, x)$ to the corresponding linear equation:

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u - \Delta\partial_t u = 0, \quad t > 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), \quad x \in \mathbb{R}^n \end{cases} \tag{4.1}$$

in order to handle with the original semi-linear problem (1.1). Our purpose is to show the following proposition, which suggests large time behaviors of the solution to the linear problem above in $L^1 \cap L^\infty$ framework.

Proposition 4.1. *Let $n = 1, 2, 3$ and $\varepsilon > 0$. Assume that $(u_0, u_1) \in (W^{\frac{n}{2}+\varepsilon,1} \cap W^{\frac{n}{2}+\varepsilon,\infty}) \times (L^1 \cap L^\infty)$. Then, there exists a unique solution $u \in C([0, \infty); L^1 \cap L^\infty)$ to problem* (4.1) *such that*

$$\|u(t, \cdot)\|_{L^q(\mathbb{R}^n)} \leq C(1 + t)^{-\frac{n}{2}(1-\frac{1}{q})}, \tag{4.2}$$

$$\|u(t, \cdot) - \tilde{M}G_t\|_{L^q(\mathbb{R}^n)} = o(t^{-\frac{n}{2}(1-\frac{1}{q})}) \quad (t \to \infty) \tag{4.3}$$

for $q \in [1, \infty]$, where $\tilde{M} = \displaystyle\int_{\mathbb{R}^n} (u_0(y) + u_1(y))dy$.

4.1. Decay estimates for "localized" evolution operators.

In this subsection, we prepare several decay properties of the evolution operators.

Lemma 4.2. *Let $n = 1, 2, 3$, $1 \leq r \leq q \leq \infty$. Then there exists a constant $C > 0$ such that*

$$\|K_{jL}(t)g\|_q \leq C(1 + t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|g\|_r \tag{4.4}$$

for $j = 0, 1$.

Lemma 4.3. *Let $n = 1, 2, 3$, $\varepsilon > 0$ and $1 \leq r \leq q \leq \infty$. Then there exists a constant $C > 0$ such that*

$$\|K_{0H}(t)g\|_q \leq Ce^{-\frac{t}{2}}\||\nabla_x|^{\frac{n}{2}+\varepsilon}g\|_q, \tag{4.5}$$

$$\|K_{1H}(t)g\|_q \leq \begin{cases} Ce^{-\frac{t}{2}}\|g\|_r \text{ for } n = 1, \\ Ce^{-\frac{t}{2}}\|g\|_q \text{ for } n = 2, 3, \end{cases} \tag{4.6}$$

and

$$\|K_{jM}(t)g\|_q \leq Ce^{-\frac{t}{2}}\|g\|_r \text{ for } j = 0, 1. \tag{4.7}$$

Proof of Lemma 4.2. To show (4.4), it is sufficient to show that

$$\|K_{jL}(t)g\|_\infty \leq C(1 + t)^{-\frac{n}{2}}\|g\|_1, \tag{4.8}$$

$$\|K_{jL}(t)g\|_q \leq C\|g\|_q \tag{4.9}$$

for $1 \leq q \leq \infty$. Indeed, once we have (4.8) and (4.9), the Riesz-Thorin complex interpolation theorem yields (4.4). So, we first show (4.8). By the Hausdorff-Young inequality and (2.5), we see that

$$\|K_{jL}(t)g\|_\infty \le C\|\mathcal{K}_{jL}(t,\xi)\hat{g}\|_1 \le \|\mathcal{K}_{jL}(t)\|_1\|\hat{g}\|_\infty$$
$$\le \|e^{-(1+t)|\xi|^2}\|_1\|g\|_1 = C(1+t)^{-\frac{n}{2}}\|g\|_1,$$

which show the desired estimate (4.8). Next, we prove (4.9) by applying (2.2). Then by using (3.9) - (3.11) and (2.5), we can assert the upper bounds of $\|\nabla_\xi^k \mathcal{K}_{jL}(t)\|_2$ for $k = 0, 1, 2$ as follows:

$$\|\nabla_\xi^k \mathcal{K}_{jL}(t)\|_2 \le C(1+t)^{-\frac{n}{4}+\frac{k}{2}}. \tag{4.10}$$

Therefore for $n = 1$, we apply (4.10) with $k = 0, 1$ and (2.2) with $s = 1$ to have

$$\begin{aligned} M_\infty(\mathcal{K}_{jL}(t)) &\le C\|\mathcal{K}_{jL}(t)\|_2^{1-\frac{1}{2}}\|\mathcal{K}_{jL}(t)\|_{\dot{H}^1}^{\frac{1}{2}} \\ &\le C\|\mathcal{K}_{jL}(t)\|_2^{1-\frac{1}{2}}\|\nabla_\xi \mathcal{K}_{jL}(t)\|_2^{\frac{1}{2}} \\ &\le C(1+t)^{-\frac{1}{4}}(1+t)^{-\frac{1}{4}+\frac{1}{2}} \le C. \end{aligned} \tag{4.11}$$

On the other hand, for $n = 2, 3$, we use (4.10) with $k = 0, 2$ and (2.2) with $s = 2$ to see

$$\begin{aligned} M_\infty(\mathcal{K}_{jL}(t)) &\le C\|\mathcal{K}_{jL}(t)\|_2^{1-\frac{n}{4}}\|\mathcal{K}_{jL}(t)\|_{\dot{H}^2}^{\frac{n}{4}} \\ &\le C\|\mathcal{K}_{jL}(t)\|_2^{1-\frac{n}{4}}\|\nabla_\xi^2 \mathcal{K}_{jL}(t)\|_2^{\frac{n}{4}} \\ &\le C(1+t)^{-\frac{n}{4}(1-\frac{n}{4})}(1+t)^{\frac{n}{4}(-\frac{n}{4}+1)} \le C. \end{aligned} \tag{4.12}$$

By combining (4.11), (4.12) and (2.1) one can obtain

$$M_q(\mathcal{K}_{jL}(t)) \le M_\infty(\mathcal{K}_{jL}(t)) \le C$$

for $1 \le q \le \infty$, which proves the desired estimate (4.9) by the definition of M_q. □

Proof of Lemma 4.3. Firstly, we remark that (4.5) and (4.6) can be derived by the same idea. Hence we only check (4.6). As in the proof of Lemma 4.2, we only need to show

$$\|K_{1H}(t)g\|_\infty \le Ce^{-t}\|g\|_1, \tag{4.13}$$

for $n = 1$ and

$$\|K_{1H}(t)g\|_q \le Ce^{-\frac{t}{2}}\|g\|_q, \tag{4.14}$$

for $1 \le q \le \infty$ and $n = 1, 2, 3$. For $n = 1$, the Hausdorff-Young inequality and (3.27) yield

$$\|K_{1H}(t)g\|_\infty \le \|\mathcal{K}_{1H}(t,\xi)\hat{g}\|_1 \le Ce^{-t}\|\,|\xi|^{-2}\chi_H\|_1\|\hat{g}\|_\infty \le Ce^{-t}\|g\|_1,$$

since $|\xi|^{-2}\chi_H \in L^1(\mathbb{R})$, which is the desired estimate (4.13). In order to show (4.14), we again apply the same argument as (4.9). Indeed, by (3.27) - (3.29), we see

$$\|\nabla_\xi^k \mathcal{K}_{1H}(t,\xi)\|_2 \le Ce^{-\frac{t}{2}} \tag{4.15}$$

for $k = 0, 1, 2$. Here we have just used the fact that

$$|\xi|^{-2}(\chi_H + |\chi_H'| + |\chi_H''|) \in L^2(\mathbb{R}^n)$$

for $n = 1, 2, 3$. Therefore, we apply (4.15) with $k = 0, 1$, (2.1) and (2.2) with $s = 1$ to have

$$M_q(\mathcal{K}_{1H}(t)) \le M_\infty(\mathcal{K}_{1H}(t)) \le C\|\mathcal{K}_{1H}(t)\|_2^{1-\frac{1}{2}}\|\mathcal{K}_{1H}(t)\|_{\dot{H}^1}^{\frac{1}{2}} \le C\|\mathcal{K}_{1H}(t)\|_2^{1-\frac{1}{2}}\|\nabla_\xi\mathcal{K}_{1H}(t)\|_2^{\frac{1}{2}} \le Ce^{-\frac{t}{2}}, \tag{4.16}$$

for the case $n = 1$. When $n = 2, 3$, by (4.15) with $k = 0, 2$, (2.1) and (2.2) with $s = 2$ one can find that

$$M_q(\mathcal{K}_{1H}(t)) \le M_\infty(\mathcal{K}_{1H}(t)) \le C\|\mathcal{K}_{1H}(t)\|_2^{1-\frac{n}{4}}\|\mathcal{K}_{1H}(t)\|_{\dot{H}^2}^{\frac{n}{4}} \le C\|\mathcal{K}_{1H}(t)\|_2^{1-\frac{n}{4}}\|\nabla_\xi^2\mathcal{K}_{1H}(t)\|_2^{\frac{n}{4}} \le Ce^{-\frac{t}{2}}. \tag{4.17}$$

By the definition of M_q, with the help of (4.16) and (4.17), we obtain the desired estimate (4.14) for $n = 1, 2, 3$.

Finally, we check (4.7). The proof of (4.7) is immediate. Indeed, we now apply the argument for (4.4), with (4.10) replaced by (3.22) to obtain (4.7), and the proof of Lemma 4.3 is now complete. □

4.2. Asymptotic behavior of the low frequency part. In this subsection, we state that the evolution operators $\mathcal{K}_{jL}(t)g$ for $j = 0, 1$ are well-approximated by the solution of the heat equation.

Lemma 4.4. *Let $n = 1, 2, 3$, $1 \le r \le q \le \infty$. Then there exists a constant $C > 0$ such that*

$$\|K_{jL}(t)g - e^{t\Delta}(\check{\chi}_L * g)\|_q \le C(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})-1}\|g\|_r \tag{4.18}$$

for $j = 0, 1$.

Proof. For the proof, we again apply the similar argument to the proof of Lemma 4.2. Namely, we claim that

$$\|K_{jL}(t)g - e^{t\Delta}(\check{\chi}_L * g)\|_\infty \le C(1+t)^{-\frac{n}{2}-1}\|g\|_1, \tag{4.19}$$

$$\|K_{jL}(t)g - e^{t\Delta}(\check{\chi}_L * g)\|_q \le C(1+t)^{-1}\|g\|_q, \tag{4.20}$$

for $1 \le q \le \infty$. Here we recall that (4.19), (4.20) and the Riesz-Thorin interpolation theorem show (4.18). Therefore it suffices to prove (4.19) and (4.20) in order to get (4.18).

We first show (4.19). The Hausdorff - Young inequality, (3.14) and (2.5) with $k = 2$ and $r = 1$ show

$$\begin{aligned}\|K_{jL}(t)g - e^{t\Delta}(\check{\chi}_L * g)\|_\infty &\le C\|(\mathcal{K}_{jL}(t) - e^{-t|\xi|^2}\chi_L)\hat{g}\|_1 \\ &\le C\|\mathcal{K}_{jL}(t) - e^{-t|\xi|^2}\chi_L\|_1\|\hat{g}\|_\infty \\ &\le C\|\,|\xi|^2 e^{-(1+t)|\xi|^2}\chi_L\|_1\|g\|_1 \le C(1+t)^{-\frac{n}{2}-1}\|g\|_1,\end{aligned}$$

which is the desired estimate (4.19).

Next, we prove (4.20). Observing (3.14) - (3.16) and (2.5), we get

$$\|\nabla_\xi^k(\mathcal{K}_{jL}(t) - e^{-t|\xi|^2}\chi_L)\|_2 \le C(1+t)^{-\frac{n}{4}-1+\frac{k}{2}} \tag{4.21}$$

for $k = 0, 1, 2$.

In order to check (4.20) for the case $n = 1$, we apply (2.2) with $s = 1$ and (4.21) with $k = 0, 1$ to get

$$\begin{aligned} M_\infty(\mathcal{K}_{jL}(t) - e^{-t|\xi|^2}\chi_L) &\le C\|\mathcal{K}_{jL}(t) - e^{-t|\xi|^2}\chi_L\|_2^{1-\frac{1}{2}}\|\mathcal{K}_{jL}(t) - e^{-t|\xi|^2}\chi_L\|_{\dot{H}^1}^{\frac{1}{2}} \\ &\le C\|\mathcal{K}_{jL}(t) - e^{-t|\xi|^2}\chi_L\|_2^{1-\frac{1}{2}}\|\nabla_\xi(\mathcal{K}_{jL}(t) - e^{-t|\xi|^2}\chi_L)\|_2^{\frac{1}{2}} \\ &\le C(1+t)^{\frac{1}{2}(-\frac{1}{4}-1)}(1+t)^{\frac{1}{2}(-\frac{1}{4}-\frac{1}{2})} \le C(1+t)^{-1}. \end{aligned} \tag{4.22}$$

Namely, we have arrived at (4.20) with $n = 1$ since combining (2.1) and (4.22) gives (4.20).

In the case when $n = 2, 3$, we use (4.21) with $k = 0, 2$ and (2.2) with $s = 2$ to obtain

$$\begin{aligned} M_\infty(\mathcal{K}_{jL}(t) - e^{-t|\xi|^2}\chi_L) &\le C\|\mathcal{K}_{jL}(t) - e^{-t|\xi|^2}\chi_L\|_2^{1-\frac{n}{4}}\|\mathcal{K}_{jL}(t) - e^{-t|\xi|^2}\chi_L\|_{\dot{H}^2}^{\frac{n}{4}} \\ &\le C\|\mathcal{K}_{jL}(t) - e^{-t|\xi|^2}\chi_L\|_2^{1-\frac{n}{4}}\|\nabla_\xi^2(\mathcal{K}_{jL}(t) - e^{-t|\xi|^2}\chi_L)\|_2^{\frac{n}{4}} \\ &\le C(1+t)^{(-\frac{n}{4}-1)(1-\frac{n}{4})}(1+t)^{-\frac{n}{4}\frac{n}{4}} = C(1+t)^{-1}. \end{aligned}$$

That is, $M_q(\mathcal{K}_{jL}(t) - e^{-t|\xi|^2}\chi_L) \le M_\infty(\mathcal{K}_{jL}(t) - e^{-t|\xi|^2}\chi_L) \le C(1+t)^{-1}$ for $1 \le q \le \infty$ by (2.1) again. This shows (4.10) with $n = 2, 3$, which proves Lemma 4.4. □

4.3. Proof of Proposition 4.1. In this subsection, we shall prove Proposition 4.1.

We start with the observation that the results obtained in previous subsections guarantee the decay property and large time behavior of the evolution operators $K_0(t)$ and $K_1(t)$.

Corollary 4.5. *Let $n = 1, 2, 3$, $\varepsilon > 0$ and $1 \le r \le q \le \infty$. Then there exists a constant $C > 0$ such that*

$$\|K_0(t)g\|_q \le C(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|g\|_r + Ce^{-\frac{t}{2}}\|\,|\nabla_x|^{\frac{n}{2}+\varepsilon}g\|_q, \tag{4.23}$$

$$\|K_1(t)g\|_q \le C(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|g\|_r + Ce^{-\frac{t}{2}}\|g\|_q, \tag{4.24}$$

$$\|(K_0(t) - e^{t\Delta})g\|_q \le C(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})-1}\|g\|_r + Ce^{-\frac{t}{2}}\|\,|\nabla_x|^{\frac{n}{2}+\varepsilon}g\|_q, \tag{4.25}$$

$$\|(K_1(t) - e^{t\Delta})g\|_q \le C(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})-1}\|g\|_r + Ce^{-\frac{t}{2}}\|g\|_q. \tag{4.26}$$

Remark 4.6. We note that under the statement above for $n = 1$, we see that

$$\begin{aligned} &\|K_1(t)g\|_q \le C(1+t)^{-\frac{1}{2}(\frac{1}{r}-\frac{1}{q})}\|g\|_r, \\ &\|(K_1(t) - e^{t\Delta})g\|_q \le C(1+t)^{-\frac{1}{2}(\frac{1}{r}-\frac{1}{q})-1}\|g\|_r, \end{aligned}$$

since $Ce^{-\frac{t}{2}}\|g\|_r$ is estimated by $C(1+t)^{-\frac{1}{2}(\frac{1}{r}-\frac{1}{q})-1}\|g\|_r$. The same reasoning can be applied to the case $q = r$, namely,

$$\|K_1(t)g\|_q \le \|g\|_q, \tag{4.27}$$

$$\|(K_1(t) - e^{t\Delta})g\|_q \le C(1+t)^{-1}\|g\|_q. \tag{4.28}$$

Proof. The proof of the estimates (4.23) - (4.26) is similar. Here we only show the proof of (4.23). Combining (4.4) with $j = 0$, (4.5) and (4.7) with $j = 0$, and the definition of the localized operators, we see that

$$\|K_0(t)g\|_q \le \sum_{k=L,M,H}\|K_{0K}(t)g\|_q$$

$$\le C(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|g\|_r + Ce^{-\frac{t}{2}}\|g\|_r + Ce^{-\frac{t}{2}}\| \, |\nabla_x|^{\frac{n}{2}+\varepsilon} g\|_q$$
$$\le C(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|g\|_r + Ce^{-\frac{t}{2}}\| \, |\nabla_x|^{\frac{n}{2}+\varepsilon} g\|_q,$$

which show the desired estimate (4.23). This completes the proof of Corollary 4.5. □

By combining (4.25), (4.26) and (2.4), we can assert the approximation formula of the evolution operators $K_0(t)$ and $K_1(t)$ in terms of the heat kernel for large t.

Corollary 4.7. *Let $n = 1, 2, 3$, $\varepsilon > 0$ and $(g_0, g_1) \in (W^{\frac{n}{2}+\varepsilon,1} \cap W^{\frac{n}{2}+\varepsilon,q}) \times (L^1 \cap L^q)$. Then it is true that*

$$\|K_j(t)g_j - m_j G_t\|_q = o(t^{-\frac{n}{2}(1-\frac{1}{q})}), \tag{4.29}$$

as $t \to \infty$ for $j = 0, 1$, where $m_j = \int_{\mathbb{R}^n} g_j(y)dy$.

Proof. For $j = 0$, we apply (4.25) and (2.4) to get

$$t^{\frac{n}{2}(1-\frac{1}{q})}\|K_0(t)g_0 - m_0 G_t\|_q$$
$$\le t^{\frac{n}{2}(1-\frac{1}{q})}\|(K_0(t) - e^{t\Delta})g_0\|_q + t^{\frac{n}{2}(1-\frac{1}{q})}\|e^{t\Delta}g_0 - m_0 G_t\|_q$$
$$\le C(1+t)^{-1}\|g_0\|_1 + Ce^{-\frac{t}{2}}\| \, |\nabla_x|^{\frac{n}{2}+\varepsilon} g\|_q + t^{\frac{n}{2}(1-\frac{1}{q})}\|e^{t\Delta}g_0 - m_0 G_t\|_q$$
$$\to 0$$

as $t \to \infty$, which is the desired estimate (4.29) with $j = 0$. We now apply this argument with (4.25) replaced by (4.26), to obtain the estimate (4.29) with $j = 1$, and Corollary 4.7 now follows. □

Now, we are in a position to prove Proposition 4.1 by combining Corollaries 4.5 and 4.7. Proof of Proposition 4.1. We recall that the solution to (4.1) is expressed as $u(t, \cdot) = K_0(t)u_0 + K_1(t)u_1$. Then it follows from (4.23) and (4.24) with $r = 1$,

$$\|u(t)\|_q \le \|K_0(t)u_0\|_q + \|K_1(t)u_1\|_q \le C(1+t)^{-\frac{n}{2}(1-\frac{1}{q})},$$

which is the desired estimate (4.2). Also we see at once (4.3). Indeed, (4.25), (4.26) with $r = 1$ and (4.29) give

$$\begin{aligned}\|(u(t,\cdot) - \tilde{M}G_t)\|_q &\le \|(K_0(t) - e^{t\Delta})u_0\|_q + \|(K_1(t) - e^{t\Delta})u_1\|_q \\ &\quad + \|(e^{t\Delta}(u_0 + u_1) - \tilde{M}G_t\|_q \\ &\le C(1+t)^{-\frac{n}{2}(1-\frac{1}{q})-1} + o(t^{-\frac{n}{2}(1-\frac{1}{q})})\end{aligned}$$

as $t \to \infty$, which is the desired estimate (4.3). This proves Proposition 4.1. □

5. Existence of global solutions

This section is devoted to the proof of Theorem 1.1. Here we prepare some notation, which will be used soon. We define the closed subspace of $C([0, \infty); L^1 \cap L^\infty)$ as

$$X := \{u \in C([0, \infty); L^1 \cap L^\infty); \|u\|_X \le M\},$$

where

$$\|u\|_X := \sup_{t\geq 0}\{\|u(t)\|_1 + (1+t)^{\frac{n}{2}}\|u(t)\|_\infty\}$$

and $M > 0$ will be determined later. We also introduce the mapping Φ on X by

$$\Phi[u](t) := K_0(t)u_0 + K_1(t)u_1 + \int_0^t K_1(t-\tau)f(u)(\tau)d\tau. \tag{5.1}$$

For simplicity of notation, we denote the integral term of (5.1) by $I[u](t)$:

$$I[u](t) := \int_0^t K_1(t-\tau)f(u)(\tau)d\tau. \tag{5.2}$$

In this situation, we claim that

$$\|\Phi[u]\|_X \leq M \tag{5.3}$$

for all $u \in X$ and

$$\|\Phi[u] - \Phi[v]\|_X \leq \frac{1}{2}\|u - v\|_X \tag{5.4}$$

for all $u, v \in X$. For the proof of Theorem 1.1, it suffices to show (5.3) and (5.4). Indeed, once we have (5.3) and (5.4), we see that Φ is a contraction mapping on X. Therefore it is immediate from the Banach fixed point theorem that Φ has a unique fixed point in X. Namely, there exists a unique global solution $u = \Phi[u]$ in X and Theorem 1.1 can be proved. We remark that the linear solution $K_0(t)u_0 + K_1(t)u_1$ is estimated suitably by linear estimates stated in Proposition 4.1. In what follows, we concentrate on estimates for $I[u](t)$ defined by (5.2). Firstly we prepare several estimates of the norms for $f(u)$ and $f(u) - f(v)$, which will be used below.

By using the mean value theorem, we can see that there exists $\theta \in [0, 1]$ such that

$$f(u) - f(v) = f'(\theta u + (1-\theta)v)(u - v).$$

Therefore, by noting the definition of $\|\cdot\|_X$, we arrive at the estimate

$$\begin{aligned} \|f(u) - f(v)\|_1 &\leq \|f'(\theta u + (1-\theta)v)\|_\infty \|u - v\|_1 \\ &\leq C\|\theta u + (1-\theta)v\|_\infty^{p-1}\|u - v\|_1 \\ &\leq C(\|u\|_\infty^{p-1} + \|v\|_\infty^{p-1})\|u - v\|_1 \\ &\leq C(1+\tau)^{-\frac{n}{2}(p-1)}(\|u\|_X^{p-1} + \|v\|_X^{p-1})\|u - v\|_X \\ &\leq C(1+\tau)^{-\frac{n}{2}(p-1)}M^{p-1}\|u - v\|_X \end{aligned} \tag{5.5}$$

for $u, v \in X$. By the similar way, we have

$$\begin{aligned} \|f(u) - f(v)\|_\infty &\leq C(\|u\|_\infty^{p-1} + \|v\|_\infty^{p-1})\|u - v\|_\infty \\ &\leq C(1+\tau)^{-\frac{np}{2}}M^{p-1}\|u - v\|_X \end{aligned} \tag{5.6}$$

for $u, v \in X$. If we take $v = 0$ in (5.5) and (5.6), and if we recall $\|u\|_X \leq M$, we easily see that

$$\begin{aligned} &\|f(u)\|_1 \leq C(1+\tau)^{-\frac{n}{2}(p-1)}M^p, \\ &\|f(u)\|_\infty \leq C(1+\tau)^{-\frac{np}{2}}M^p \end{aligned} \tag{5.7}$$

for $u \in X$.

Now, by using the above estimates in (5.7), let us derive the estimate of $\|I[u](t)\|_1$ for $n = 1, 2, 3$.

To begin with, we apply (4.27) with $q = 1$, (5.8), (2.4) and (2.5) to have

$$\begin{aligned} \|I[u](t)\|_1 &\le \int_0^t \|K_1(t-\tau)f(u)\|_1\,d\tau \le C\int_0^t \|f(u)\|_1\,d\tau \\ &\le C\|u\|_X^p \int_0^t (1+\tau)^{-\frac{n}{2}(p-1)}d\tau \le CM^p, \end{aligned} \tag{5.8}$$

since $-\dfrac{n}{2}(p-1) < -1$ for $p > 1 + \dfrac{2}{n}$.

Secondly by the similar way to (5.8), we calculate $\|I[u](t) - I[v](t)\|_1$ as follows:

$$\begin{aligned} \|I[u](t) - I[v](t)\|_1 &\le \int_0^t \|K_1(t-\tau)(f(u)-f(v))\|_1\,d\tau \\ &\le C\int_0^t \|f(u)-f(v)\|_1\,d\tau \\ &\le CM^{p-1}\|u-v\|_X \int_0^t (1+\tau)^{-\frac{n}{2}(p-1)}d\tau \\ &\le CM^{p-1}\|u-v\|_X, \end{aligned} \tag{5.9}$$

for $u, v \in X$, where we have just used (5.5) and (5.6).

For the proof of Theorem 1.1, it still remains to get the estimates for $\|\Phi[u](t)\|_\infty$ and $\|\Phi[u](t) - \Phi[v](t)\|_\infty$.

Now, in order to obtain the estimate for $\|\Phi[u](t)\|_\infty$, we split the nonlinear term into two parts:

$$\begin{aligned} \|I[u](t)\|_\infty &\le \int_0^{\frac{t}{2}} \|K_1(t-\tau)f(u)\|_\infty\,d\tau + \int_{\frac{t}{2}}^t \|K_1(t-\tau)f(u)\|_\infty\,d\tau \\ &=: J_1(t) + J_2(t). \end{aligned} \tag{5.10}$$

To obtain the estimate of $J_1(t)$, we apply (4.24) with $q = \infty$ and $r = 1$ and (5.7) to have

$$\begin{aligned} J_1(t) &\le C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{2}}\,\|f(u)\|_1\,d\tau + C\int_0^{\frac{t}{2}} e^{-\frac{t-\tau}{2}}\,\|f(u)\|_\infty\,d\tau \\ &\le C(1+t)^{-\frac{n}{2}}\int_0^{\frac{t}{2}} (1+\tau)^{-\frac{n}{2}(p-1)}d\tau M^p + Ce^{-\frac{1}{2}t}\int_0^{\frac{t}{2}} (1+\tau)^{-\frac{np}{2}}d\tau M^p \\ &\le C(1+t)^{-\frac{n}{2}}M^p, \end{aligned} \tag{5.11}$$

where we have used the fact that $-\dfrac{n}{2}(p-1) < -1$.

For the term $J_2(t)$, by using (4.27) with $q = \infty$ and (5.7) we obtain

$$J_2(t) \le C\int_{\frac{t}{2}}^t \|f(u)\|_\infty\,d\tau \le C\int_{\frac{t}{2}}^t (1+\tau)^{-\frac{np}{2}}d\tau M^p \le C(1+t)^{-\frac{np}{2}+1}M^p, \tag{5.12}$$

where we remark that the power in the right hand side $-\dfrac{np}{2} + 1$ is strictly smaller than $-\dfrac{n}{2}$ since $-\dfrac{np}{2} + 1 = -\dfrac{n}{2}(p-1) + 1 - \dfrac{n}{2}$ and $-\dfrac{n}{2}(p-1) < -1$. By combining (5.10) - (5.12), we arrive at

$$\|I[u](t)\|_\infty \le J_1(t) + J_2(t) \le C(1+t)^{-\frac{n}{2}} M^p. \tag{5.13}$$

Next, we estimate $\|\Phi[u](t) - \Phi[v](t)\|_\infty$. Again, we divide $\|I[u](t) - I[v](t)\|_\infty$ into two parts:

$$\begin{aligned} \|I[u](t) - I[v](t)\|_\infty &\le \int_0^{\frac{t}{2}} \|K_1(t-\tau)(f(u) - f(v))\|_\infty \, d\tau \\ &\quad + \int_{\frac{t}{2}}^{t} \|K_1(t-\tau)(f(u) - f(v))\|_\infty \, d\tau \\ &=: J_3(t) + J_4(t). \end{aligned} \tag{5.14}$$

As in the proof of (5.11), we can deduce that

$$\begin{aligned} J_3(t) &\le C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{2}} \|f(u) - f(v)\|_1 \, d\tau \\ &\quad + C \int_0^{\frac{t}{2}} e^{-\frac{t-\tau}{2}} \|f(u) - f(v)\|_\infty \, d\tau \\ &\le C(1+t)^{-\frac{n}{2}} \int_0^{\frac{t}{2}} (1+\tau)^{-\frac{n}{2}(p-1)} d\tau M^{p-1} \|u - v\|_X \\ &\quad + Ce^{-\frac{1}{2}t} \int_0^{\frac{t}{2}} (1+\tau)^{-\frac{np}{2}} d\tau M^{p-1} \|u - v\|_X \\ &\le C(1+t)^{-\frac{n}{2}} M^{p-1} \|u - v\|_X, \end{aligned} \tag{5.15}$$

where we have used the fact that $-\dfrac{np}{2} + 1 < -\dfrac{n}{2}$ again. In the same manner as (5.12), we can get

$$\begin{aligned} J_4(t) &\le C \int_{\frac{t}{2}}^{t} \|f(u) - f(v)\|_\infty \, d\tau \\ &\le C \int_{\frac{t}{2}}^{t} (1+\tau)^{-\frac{np}{2}} d\tau M^{p-1} \|u - v\|_X \\ &\le C(1+t)^{-\frac{np}{2}+1} M^{p-1} \|u - v\|_X. \end{aligned} \tag{5.16}$$

Thus, (5.14) - (5.16) yield

$$\|I[u](t) - I[v](t)\|_\infty \le J_3(t) + J_4(t) \le C(1+t)^{-\frac{n}{2}} M^{p-1} \|u - v\|_X. \tag{5.17}$$

By (4.23), (4.24), (5.8) and (5.13), we deduce that

$$\begin{aligned} \|\Phi[u]\|_X &\le \|K_0(t)u_0 + K_1(t)u_1\|_X + \|I[u]\|_X \\ &\le C_0(\|u_0\|_{W^{\frac{n}{2}+\varepsilon,1} \cap W^{\frac{n}{2}+\varepsilon,\infty}} + \|u_1\|_{L^1 \cap L^\infty}) + C_1 M^p \end{aligned} \tag{5.18}$$

for some $C_0 > 0$ and $C_1 > 0$.

Similar arguments can be applied to $\|\Phi[u] - \Phi[v]\|_X$ by using (5.9) and (5.17), and then one can assert that

$$\|\Phi[u] - \Phi[v]\|_X \le \|I[u] - I[v]\|_X \le C_2 M^{p-1} \|u - v\|_X \tag{5.19}$$

for some $C_2 > 0$. By choosing $\|u_0\|_{W^{\frac{n}{2}+\varepsilon,1}\cap W^{\frac{n}{2}+\varepsilon,\infty}} + \|u_1\|_{L^1\cap L^\infty}$ sufficiently small, we can make sure the validity of the inequality such as

$$C_1 M^p < \frac{1}{2}M, \quad C_2 M^{p-1} < \frac{1}{2}, \tag{5.20}$$

because of the relation $M = 2C_0(\|u_0\|_{W^{\frac{n}{2}+\varepsilon,1}\cap W^{\frac{n}{2}+\varepsilon,\infty}} + \|u_1\|_{L^1\cap L^\infty})$. By combining (5.18), (5.19) and (5.20) one has the desired estimates (5.3) and (5.4), and the proof is now complete.

6. Asymptotic behavior of the solution

In this section, we show the proof of Theorem 1.2. For the proof of Theorem 1.2, we prepare slightly general setting. Here, we introduce the function $F = F(t,x) \in L^1(0,\infty; L^1(\mathbb{R}^n))$ satisfying

$$\|F(t)\|_q \le C(1+t)^{-\frac{n}{2}(p-1)-\frac{n}{2}(1-\frac{1}{q})}, \tag{6.1}$$

for $1 \le q \le \infty$ and $p > 1 + \dfrac{2}{n}$. We can now formulate our main statement in this section.

Proposition 6.1. *Let $n \ge 1$ and $p > 1 + \dfrac{2}{n}$, and assume* (6.1). *Then it holds that*

$$\left\| \left(\int_0^t K_1(t-\tau)F(\tau)d\tau - \int_0^\infty \int_{\mathbb{R}^n} F(\tau,y)dyd\tau \cdot G_t(x) \right) \right\|_q = o(t^{-\frac{n}{2}(1-\frac{1}{q})}) \tag{6.2}$$

as $t \to \infty$.

As a first step of the proof of Proposition 6.1, we split the integral terms into five parts. Namely, we see that

$$\begin{aligned}
&\int_0^t K_1(t-\tau)F(\tau)d\tau - \int_0^\infty \int_{\mathbb{R}^n} F(\tau,y)dyd\tau \cdot G_t(x) \\
&= \int_0^{\frac{t}{2}} (K_1(t-\tau) - e^{(t-\tau)\Delta})F(\tau)d\tau + \int_{\frac{t}{2}}^t K_1(t-\tau)F(\tau)d\tau \\
&+ \int_0^{\frac{t}{2}} (e^{(t-\tau)\Delta} - e^{t\Delta})F(\tau)d\tau + \int_0^{\frac{t}{2}} \left(e^{t\Delta}F(\tau) - \int_{\mathbb{R}^n} F(\tau,y)dy \cdot G_t(x) \right) d\tau \\
&- \int_{\frac{t}{2}}^\infty \int_{\mathbb{R}^n} F(\tau,y)dyd\tau \cdot G_t(x),
\end{aligned}$$

and here we set each terms as follows:

$$\begin{aligned}
A_1(t) &:= \int_0^{\frac{t}{2}} (K_1(t-\tau) - e^{(t-\tau)\Delta})F(\tau)d\tau, \\
A_2(t) &:= \int_{\frac{t}{2}}^t K_1(t-\tau)F(\tau)d\tau, \ A_3(t) := \int_0^{\frac{t}{2}} (e^{(t-\tau)\Delta} - e^{t\Delta})F(\tau)d\tau, \\
A_4(t) &:= \int_0^{\frac{t}{2}} \left(e^{t\Delta}F(\tau) - \int_{\mathbb{R}^n} F(\tau,y)dy \cdot G_t(x) \right) d\tau \\
A_5(t) &:= -\int_{\frac{t}{2}}^\infty \int_{\mathbb{R}^n} F(\tau,y)dyd\tau \cdot G_t(x).
\end{aligned}$$

In what follows, we estimate each $A_j(t)$ for $j = 1, \cdots, 5$, respectively.

Lemma 6.2. *Under the same assumptions as in* Proposition 6.1*, there exists a constant* $C > 0$ *such that*

$$\|A_1(t)\|_q \le C(1+t)^{-\frac{n}{2}(1-\frac{1}{q})-1}, \tag{6.3}$$

$$\|A_j(t)\|_q \le Ct^{-\frac{n}{2}(1-\frac{1}{q})-\frac{n}{2}(p-1)+1} \quad (j = 2, 5), \tag{6.4}$$

$$\|A_3(t)\|_q \le \begin{cases} Ct^{-\frac{n}{2}(1-\frac{1}{q})-1}\log(2+t), & p \ge 1+\frac{4}{n}, \\ Ct^{-\frac{n}{2}(1-\frac{1}{q})-\frac{n}{2}(p-1)+1}, & 1+\frac{2}{n} < p < 1+\frac{4}{n}, \end{cases} \tag{6.5}$$

$$\|A_4(t)\|_q = o\Big(t^{-\frac{n}{2}(1-\frac{1}{q})}\Big), \tag{6.6}$$

as $t \to \infty$ *for* $1 \le q \le \infty$.

Proof. First, we show (6.3). By (4.26) with $r = 1$ and (6.1) we see that

$$\begin{aligned}
\|A_1(t)\|_q &\le \int_0^{\frac{t}{2}} \|(K_1(t-\tau) - e^{(t-\tau)\Delta})F(\tau)\|_q d\tau \\
&\le C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{2}(1-\frac{1}{q})-1}\|F(\tau)\|_1 d\tau + C\int_0^{\frac{t}{2}} e^{-\frac{t-\tau}{2}}\|F(\tau)\|_q d\tau \\
&\le C(1+t)^{-\frac{n}{2}(1-\frac{1}{q})-1}\int_0^{\frac{t}{2}} (1+\tau)^{-\frac{n}{2}(p-1)} d\tau \\
&+ Ce^{-\frac{t}{2}}\int_0^{\frac{t}{2}} (1+\tau)^{-\frac{n}{2}(p-1)-\frac{n}{2}(1-\frac{1}{q})} d\tau \\
&\le C(1+t)^{-\frac{n}{2}(1-\frac{1}{q})-1},
\end{aligned}$$

which is the desired estimate (6.3). Next, we show (6.4) wit $j = 2$. By (4.27) and (6.1), we see that

$$\begin{aligned}
\|A_2(t)\|_q &\le \int_{\frac{t}{2}}^{t} \|K_1(t-\tau)F(\tau)\|_q d\tau \le C\int_{\frac{t}{2}}^{t} \|F(\tau)\|_q d\tau \\
&\le C\int_{\frac{t}{2}}^{t} (1+\tau)^{-\frac{n}{2}(1-\frac{1}{q})-\frac{n}{2}(p-1)} d\tau \\
&\le C(1+t)^{-\frac{n}{2}(1-\frac{1}{q})-\frac{n}{2}(p-1)+1},
\end{aligned}$$

which is the desired estimate (6.4) with $j = 2$.

Thirdly, we show (6.4) with $j = 5$. By the combination of (6.1) and the direct computation, we get

$$\begin{aligned}
\|A_5(t)\|_q &\le \int_{\frac{t}{2}}^{\infty} \|F(\tau)\|_1 d\tau \|G_t\|_q \\
&\le \int_{\frac{t}{2}}^{\infty} (1+\tau)^{-\frac{n}{2}(p-1)} d\tau \|G_t\|_q \le Ct^{-\frac{n}{2}(1-\frac{1}{q})-\frac{n}{2}(p-1)+1},
\end{aligned}$$

which is the desired estimate (6.4) with $j = 5$.

Let us prove (6.5). To begin with, observe that there exists $\theta \in [0,1]$ such that

$$G_{t-\tau}(x-y) - G_t(x-y) = (-\tau)\partial_t G_{t-\theta\tau}(x-y),$$

because of the mean value theorem on t. Then, we can apply (2.3) with $\tilde{k} = 0$, $\ell = 1$ and $r = 1$ to have

$$\begin{aligned}
\|A_3(t)\|_q &\le \int_0^{\frac{t}{2}} \|(e^{(t-\tau)\Delta} - e^{t\Delta})F(\tau)\|_q d\tau \\
&= \int_0^{\frac{t}{2}} \tau \|\partial_t e^{(t-\theta\tau)\Delta} F(\tau)\|_q d\tau \\
&\le C\int_0^{\frac{t}{2}} \tau(t-\tau)^{-\frac{n}{2}(1-\frac{1}{q})-1} \|F(\tau)\|_1 d\tau \\
&\le Ct^{-\frac{n}{2}(1-\frac{1}{q})-1} \int_0^{\frac{t}{2}} \tau(1+\tau)^{-\frac{n}{2}(p-1)} d\tau \\
&\le \begin{cases} Ct^{-\frac{n}{2}(1-\frac{1}{q})-1}\log(2+t), & p \ge 1+\frac{4}{n}, \\ Ct^{-\frac{n}{2}(1-\frac{1}{q})-\frac{n}{2}(p-1)+1}, & 1+\frac{2}{n} < p < 1+\frac{4}{n}, \end{cases}
\end{aligned}$$

which implies (6.5).

Finally, we prove (6.6). To show the estimate for $A_4(t)$, we first divide the integrand into two parts:

$$\begin{aligned}
&\int_0^{\frac{t}{2}} \left(e^{t\Delta}F(\tau,x) - \int_{\mathbb{R}^n} F(\tau,y)dy \cdot G_t(x) \right) d\tau \\
&= \int_0^{\frac{t}{2}} \int_{|y|\le t^{\frac{1}{4}}} + \int_0^{\frac{t}{2}} \int_{|y|\ge t^{\frac{1}{4}}} (G_t(x-y) - G_t(x))F(\tau,y)dyd\tau =: A_{41}(t) + A_{42}(t).
\end{aligned} \tag{6.7}$$

In what follows, we estimate $A_{41}(t)$ and $A_{42}(t)$, respectively. For the estimate of $A_{41}(t)$, we apply the mean value theorem again on x to have

$$G_t(x-y) - G_t(x) = (-y)\cdot \nabla_x G_t(x-\tilde{\theta}y)$$

with some $\tilde{\theta} \in [0,1]$, where $\cdot$ denotes the standard Euclid inner product. Then we arrive at the estimate

$$\begin{aligned}
\|A_{41}(t)\|_q &\le \int_0^{\frac{t}{2}} \int_{|y|\le t^{\frac{1}{4}}} \|G_t(x-y) - G_t(x)\|_{L^q_x} |F(\tau,y)| dyd\tau \\
&= \int_0^{\frac{t}{2}} \int_{|y|\le t^{\frac{1}{4}}} \left\|(-y)\cdot\nabla_x G_t(x-\tilde{\theta}y)\right\|_{L^q_x} |F(\tau,y)|dyd\tau \\
&\le Ct^{-\frac{n}{2}(1-\frac{1}{q})-\frac{1}{2}+\frac{1}{4}} \int_0^{\frac{t}{2}} \|F(\tau)\|_1 d\tau \\
&\le Ct^{-\frac{n}{2}(1-\frac{1}{q})-\frac{1}{4}} \int_0^{\frac{t}{2}} (1+\tau)^{-\frac{n}{2}(p-1)} d\tau \le Ct^{-\frac{n}{2}(1-\frac{1}{q})-\frac{1}{4}},
\end{aligned} \tag{6.8}$$

by direct calculations. On the other hand, for the term $A_{42}(t)$, we recall the fact that $\int_0^\infty \int_{\mathbb{R}^n} |F(\tau,y)|dyd\tau < \infty$ implies

$$\lim_{t\to\infty}\int_0^\infty\int_{|y|\geq t^{\frac{1}{4}}}|F(\tau,y)|dyd\tau=0.$$

Thus we see that

$$\begin{aligned}\|A_{42}(t)\|_q &\leq \int_0^{\frac{t}{2}}\int_{|y|\geq t^{\frac{1}{4}}}(\|G_t(x-y)\|_{L_x^q}+\|G_t(x)\|_{L_x^q})|F(\tau,y)|dyd\tau\\ &\leq Ct^{-\frac{n}{2}(1-\frac{1}{q})}\int_0^\infty\int_{|y|\geq t^{\frac{1}{4}}}|F(\tau,y)|dyd\tau,\end{aligned}$$

so that

$$t^{\frac{n}{2}(1-\frac{1}{q})}\|A_{42}(t)\|_q\to 0 \tag{6.9}$$

as $t\to\infty$. Therefore, by combining (6.7), (6.8) and (6.9) one has

$$\|A_4(t)\|_q\leq\|A_{41}(t)\|_q+\|A_{42}(t)\|_q=o(t^{-\frac{n}{2}(1-\frac{1}{q})})$$

as $t\to\infty$, which is the desired estimate (6.6). We complete the proof of Lemma 6.2. □

Proof of Proposition 6.1. For $1\leq q\leq\infty$, Lemma 6.2 immediately yields (6.4). Indeed, from (6.2) and (6.3) - (6.6) it follows that

$$\begin{aligned}&t^{\frac{n}{2}(1-\frac{1}{q})}\left\|\left(\int_0^t K_1(t-\tau)F(\tau)d\tau-\int_0^\infty\int_{\mathbb{R}^n}F(\tau,y)dyd\tau\cdot G_t(x)\right)\right\|_q\\ &\leq Ct^{\frac{n}{2}(1-\frac{1}{q})}\sum_{j=1}^5\|A_j(t)\|_q\to 0\end{aligned}$$

as $t\to\infty$, which is the desired conclusion. □

Now we are in a position to prove Theorem 1.2. Proof of Theorem 1.2. From the proof of Theorem 1.1, we see that the nonlinear term $f(u)$ satisfies the condition (6.1). Then we can apply Proposition 6.1 to $F(\tau,y)=f(u(\tau,y))$, and the proof is now complete. □

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