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REGULARITY OF SCHRÖDINGER'S FUNCTIONAL EQUATION AND MEAN FIELD PDES FOR H-PATH PROCESSES

Dedicated to Professor Masayoshi Takeda on the occasion of his sixtieth birthday

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Abstract

We show that the solution of Schrödinger's functional equation is measurable in space, kernel and marginals. As an application, we show that the drift vector of the h-path process with given two end point marginals is a measurable function of space, time and marginal at each time. In particular, we show that the coefficients of mean field PDE systems which the marginals satisfy are measurable functions of space, time and marginal.

1. Introduction

E. Schrödinger considered a probabilistic problem from which he obtained the so-called Schrödinger's functional equation (see section 7 in [24] and also [3, 23]). We describe Schrödinger's functional equation. Let S be a σ-compact metric space, let $C(S \times S)$ denote the space of all continuous functions on $S \times S$ with the topology induced by the uniform convergence on every compact subset of S and let $\mathcal{P}(S)$ denote the space of all Borel probability measures on S with the strong topology. Fix a positive function $q \in C(S \times S)$. Schrödinger's functional equation can be described as follows. For $\mu_1, \mu_2 \in \mathcal{P}(S)$, find a product measure $\nu_1(dx_1)\nu_2(dx_2)$ of nonnegative σ-finite Borel measures on S for which the following holds:

$$\begin{cases} \mu_1(dx_1) = \nu_1(dx_1) \displaystyle\int_S q(x_1, x_2)\nu_2(dx_2), \\ \mu_2(dx_2) = \nu_2(dx_2) \displaystyle\int_S q(x_1, x_2)\nu_1(dx_1) \end{cases} \tag{1.1}$$

It is known that (1.1) has the unique solution (see [6, 12] and also [4, 10]).

$$u_i(x_i) := \log\left(\int_S q(x_1, x_2)\nu_j(dx_j)\right), \quad i, j = 1, 2, i \neq j. \tag{1.2}$$

Then $\exp(u_1(x))$ and $\exp(u_2(x))$ are positive and

$$\mu_i(dx) = \exp(u_i(x))\nu_i(dx), \quad i = 1, 2. \tag{1.3}$$

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(1.1) can be rewritten as follows: for $i, j = 1, 2$, $i \neq j$,

$$\exp(u_i(x_i)) = \int_S q(x_1, x_2) \exp(-u_j(x_j))\mu_j(dx_j), \quad \mu_i(dx_i) - a.s.. \tag{1.4}$$

In particular, Schrödinger's problem (1.1) is equivalent to finding a function $u_1(x_1) + u_2(x_2)$ for which (1.4) holds. Since $\nu_1(dx_1)\nu_2(dx_2)$ is the unique solution of (1.1), it is a functional of μ_1, μ_2 and q. Since it is a product measure, ν_1 and ν_2 are also functionals of μ_1, μ_2 and q (see the proof of Corollary 2.1 in section 3):

$$\nu_i(dx) = \nu_i(dx; q, \mu_1, \mu_2), \quad u_i(x) = u_i(x; q, \mu_1, \mu_2), \quad i = 1, 2. \tag{1.5}$$

This does not imply the uniqueness of ν_1 and ν_2. Indeed, for $C > 0$,

$$\nu_1\nu_2 = C\nu_1 \cdot C^{-1}\nu_2.$$

Let $\{A_n\}_{n\geq 1}$ be a nondecreasing sequence of compact subsets of S such that $S = \cup_{n\geq 1}A_n$. $A_1 := S$ when S is compact. We assume that the following holds so that ν_i, u_i, $i = 1, 2$ are unique:

$$\nu_1(A_{n_0(\mu_1,\mu_2)}) = \nu_2(A_{n_0(\mu_1,\mu_2)}), \tag{1.6}$$

where $n_0(\mu_1, \mu_2) := \min\{n \geq 1 | \mu_1(A_n)\mu_2(A_n) > 0\}$.

Let $\mathcal{M}(S)$ denote the space of all Radon measures on S. In this paper we denote by a Radon measure a locally finite and inner regular Borel measure. It is known that a locally finite and σ-finite Borel measure on a σ-compact metric space is a Radon measure in our sense (see e.g., p. 901, Prop. 32.3.4 in [11]).

In Theorem 2.1, we show that if S is compact, then the following are strongly continuous:

$$\nu_i(dx; \cdot, \cdot, \cdot) : C(S \times S) \times \mathcal{P}(S) \times \mathcal{P}(S) \mapsto \mathcal{M}(S),$$

$$u_i : C(S \times S) \times \mathcal{P}(S) \times \mathcal{P}(S) \mapsto C(S),$$

and $u_i \in C(S \times C(S \times S) \times \mathcal{P}(S) \times \mathcal{P}(S))$. In Corollary 2.1, we also show that if S is σ-compact, then the following are weakly Borel measurable and Borel measurable, respectively:

$$\nu_i(dx; \cdot, \cdot, \cdot) : C(S \times S) \times \mathcal{P}(S) \times \mathcal{P}(S) \mapsto \mathcal{M}(S),$$

$$u_i : S \times C(S \times S) \times \mathcal{P}(S) \times \mathcal{P}(S) \mapsto \mathbf{R} \cup \{\infty\}.$$

As an application of this measurability result, we show that the coefficients of the mean field PDE system which the marginal distributions of the h-path process with given two end point marginals satisfy are measurable functions of space, time and marginal. To describe the problem more precisely, we introduce Jamison's result on SDEs for the h-path process with given two end point marginals. We first describe assumptions and then state Jamison's results.

(A1.1) $d \geq 1$ and $\sigma(t, x) = (\sigma^{ij}(t, x))_{i,j=1}^d$, $(t, x) \in [0, 1] \times \mathbf{R}^d$, is a $d \times d$-matrix. $a(t, x) := \sigma(t, x)\sigma(t, x)^*$, $(t, x) \in [0, 1] \times \mathbf{R}^d$, is uniformly positive definite, bounded, once continuously differentiable and uniformly Hölder continuous. $D_x a(t, x)$ is bounded and the first derivatives of $a(t, x)$ are uniformly Hölder continuous with respect to x.

(A1.2) $b(t, x) : [0, 1] \times \mathbf{R}^d \mapsto \mathbf{R}^d$ is bounded, continuous and uniformly Hölder continuous

with respect to x.

Theorem 1.1 ([13], p. 330)**.** *Suppose that* (A1.1) *and* (A1.2) *hold. Then for any* $P_0 \in \mathcal{P}(\mathbf{R}^d)$*, the following SDE has the unique weak solution with a positive continuous transition probability density* $p(t, x; s, y)$*,* $0 \le t < s \le 1$*,* $x, y \in \mathbf{R}^d$*:*

$$\begin{aligned} dX(t) &= b(t, X(t))dt + \sigma(t, X(t))dW(t), \quad 0 < t < 1, \\ PX(0)^{-1} &= P_0. \end{aligned} \tag{1.7}$$

Here $W(t)$ *denotes a d-dimensional* $\sigma[X(s); 0 \le s \le t]$*-Brownian motion. Besides, for any* $\mu_1, \mu_2 \in \mathcal{P}(\mathbf{R}^d)$*, and the solution* ν_2 *of (1.1) with* S *and* $q(x_1, x_2)$ *respectively replaced by* $\mathbf{R}^d$ *and* $p(0, x_1; 1, x_2)$*,*

$$h(t, x) := \int_{\mathbf{R}^d} p(t, x; 1, x_2)\nu_2(dx_2) \in C^{1,2}([0, 1) \times \mathbf{R}^d), \tag{1.8}$$

$$\left(\frac{\partial}{\partial t} + \mathcal{A}_t\right) h(t, x) = 0, \quad (t, x) \in [0, 1) \times \mathbf{R}^d. \tag{1.9}$$

Here

$$\mathcal{A}_t := \frac{1}{2} Trace(a(t, x)D_x^2) + \langle b(t, x), D_x \rangle.$$

Theorem 1.2 (Markovian reciprocal process)**.** ([13], Theorem 2) *Suppose that* (A1.1) *and* (A1.2) *hold. Then for any* $P_0, P_1 \in \mathcal{P}(\mathbf{R}^d)$ *for which* $P_1(dy) \ll dy$*, there exists the unique weak solution to the following SDE:*

$$\begin{aligned} &dX(t) = \{a(t, X(t))D_x \log h(t, X(t)) + b(t, X(t))\}dt + \sigma(t, X(t))dW(t), \quad 0 < t < 1, \\ &PX(t)^{-1} = P_t, \quad t = 0, 1. \end{aligned} \tag{1.10}$$

Here, to define $h(t, x)$*, we consider (1.1) with* $\mu_1, \mu_2, q(x_1, x_2)$ *and* S *respectively replaced by* $P_0, P_1, p(0, x_1; 1, x_2)$ *and* $\mathbf{R}^d$*.* $W(t)$ *also denotes a d-dimensional* $\sigma[X(s); 0 \le s \le t]$*-Brownian motion. Besides,*

$$PX(t)^{-1}(dx) = \left(\int_{\mathbf{R}^d} \nu_1(dx_1)p(0, x_1; t, x)\right) h(t, x)dx, \quad 0 \le t \le 1, \tag{1.11}$$

where

$$\int_{\mathbf{R}^d} \nu_1(dx_1)p(0, x_1; 0, x)dx := \nu_1(dx),$$

$$h(1, x) = \int_{\mathbf{R}^d} \nu_2(dx_2)p(1, x; 1, x_2) := \frac{\nu_2(dx)}{dx}.$$

Remark 1.1. Replace S by $\mathbf{R}^d$ in (1.1). Then the following holds (see (1.2), (1.3), (1.8) and (1.11)): for $x \in \mathbf{R}^d$,

$$h(t, x) := \begin{cases} \exp\{u_1(x; p(t, \cdot; 1, \cdot), PX(t)^{-1}, P_1)\}, & t \in [0, 1), \\ \dfrac{\nu_2(dx; p(0, \cdot; t, \cdot), P_0, PX(t)^{-1})}{dx} \\ = \exp\{-u_2(x; p(0, \cdot; t, \cdot), P_0, PX(t)^{-1})\}\dfrac{PX(t)^{-1}(dx)}{dx}, & t \in (0, 1]. \end{cases} \tag{1.12}$$

As an application of Corollary 2.1 in section 2, we show that

$$U(t,x,P) := \begin{cases} u_1(x; p(t,\cdot\,;1,\cdot), P, P_1), & t \in [0,1), \\ \log\left(\dfrac{\nu_2(dx; p(0,\cdot\,;t,\cdot), P_0, P)}{dx}\right), & t = 1 \end{cases} \tag{1.13}$$

is a Borel measurable function from $[0,1] \times \mathbf{R}^d \times \mathcal{P}(\mathbf{R}^d)$ to $\mathbf{R}$ (see Corollary 2.2). Theorems 1.1 and 1.2 and (1.12)-(1.13) imply that if $P_1(dy) \ll dy$, then $p(t,x)dx := PX(t)^{-1}(dx)$ satisfies the following mean field PDE system (see [1, 2, 5, 14] and the references therein for the mean field games and the master equations). For any $f \in C_b^2(\mathbf{R}^d)$ and $t \in (0,1]$,

$$\begin{aligned} &\int_{\mathbf{R}^d} f(x)p(t,x)dx - \int_{\mathbf{R}^d} f(x)P_0(dx) \\ &= \int_0^t ds \int_{\mathbf{R}^d} (\mathcal{A}_s f(x) + \langle a(s,x)D_x U(s,x,PX(s)^{-1}), Df(x)\rangle)p(s,x)dx, \end{aligned} \tag{1.14}$$

and for $(t,x) \in (0,1) \times \mathbf{R}^d$,

$$\begin{aligned} 0 &= \frac{\partial U(t,x,PX(t)^{-1})}{\partial t} + \mathcal{A}_t U(t,x,PX(t)^{-1}) \\ &\quad + \frac{1}{2}\langle a(t,x)D_x U(t,x,PX(t)^{-1}), D_x U(t,x,PX(t)^{-1})\rangle, \end{aligned} \tag{1.15}$$

$$U(1,x,PX(1)^{-1}) = \log\left(\frac{\nu_2(dx; p(0,\cdot\,;1,\cdot), P_0, P_1)}{dx}\right).$$

Here we consider $U(t,x,PX(t)^{-1})$ as a function of (t,x).

Let $\gamma(t;\omega)$ denote a progressively measurable $\mathbf{R}^d$-valued stochastic process on some filtered probability space and consider the following SDE in a weak sense:

$$dX^\gamma(t) = \{\gamma(t;\omega) + b(t,X^\gamma(t))\}dt + \sigma(t,X^\gamma(t))dW(t), \tag{1.16}$$

provided it exists (see e.g. [8]). Here $W(t)$ denotes a d-dimensional Brownian motion defined on the same filtered probability space as $\gamma(t;\omega)$.

It is also known that the h-path process with given two end point marginals is the unique minimizer of the following stochastic optimal control problem (see [7, 9], [15]-[22], [25], [26] and the references therein for recent progress, especially for stochastic optimal transport).

Theorem 1.3 ([7], [21], [26])**.** *Suppose that* (A1.1) *and* (A1.2) *hold. Then for any* $P_0, P_1 \in \mathcal{P}(\mathbf{R}^d)$ *for which* $P_1(dy) \ll dy$, $\gamma(t;\omega) = a(t,X^\gamma(t))D_x \log h(t,X^\gamma(t))$ *is the unique minimizer of the following:*

$$\begin{aligned} &V(P_0,P_1) \\ &:= \inf\left\{E\left[\int_0^1 \frac{1}{2}|\sigma(t,X^\gamma(t))^{-1}\gamma(t)|^2 dt\right]\middle| PX^\gamma(t)^{-1} = P_t, t = 0,1\right\} \\ &= \int_{\mathbf{R}^d} \log h(1,x)P_1(dx) - \int_{\mathbf{R}^d} \log h(0,x)P_0(dx), \end{aligned} \tag{1.17}$$

provided it is finite (see (1.10) *for notation).*

Remark 1.2. A sufficient condition for the finiteness of $V(P_0, P_1)$ is given in [20] for more general problems.

Schrödinger's functional equation (1.1) with $q(x_1, x_2)$ and S respectively replaced by $p(0, x_1; 1, x_2)$ and $\mathbf{R}^d$ is equivalent to the Euler equation for $V(P_0, P_1)$. We state and prove it for readers' convenience since we could not find any literature (see Proposition 2.1).

In section 2 we state our main results and prove them in section 3.

2. Main results

In this section we state our main results. We first describe assumptions.

(A2.1) S is a compact metric space.

(A2.2) $q \in C(S \times S; (0, \infty))$.

(A2.1)' S is a σ-compact metric space.

For a metric space X and $\mu \in \mathcal{M}(X)$,

$$\|\mu\| := \sup\left\{\int_X \phi(x)\mu(dx)\middle|\phi \in C(X), \|\phi\|_\infty \le 1\right\} \in [0, \infty], \tag{2.1}$$

where for $f \in C(X)$,

$$\|f\|_\infty := \sup_{x\in X} |f(x)|. \tag{2.2}$$

When S is compact, we have the continuity results on ν_i, u_i in (1.5) (Recall (1.6)).

Theorem 2.1. *Suppose that* (A2.1) *and* (A2.2) *hold. Suppose also that* $\mu_{i,n}$, $\mu_i \in \mathcal{P}(S)$, $q_n \in C(S \times S; (0, \infty))$, $i = 1, 2$, $n \ge 1$ *and*

$$\lim_{n\to\infty} (\|\mu_{1,n}\mu_{2,n} - \mu_1\mu_2\| + \|q_n - q\|_\infty) = 0. \tag{2.3}$$

Then

$$\begin{aligned} \lim_{n\to\infty} \|&\nu_1(\cdot; q_n, \mu_{1,n}, \mu_{2,n})\nu_2(\cdot; q_n, \mu_{1,n}, \mu_{2,n}) \\ &-\nu_1(\cdot; q, \mu_1, \mu_2)\nu_2(\cdot; q, \mu_1, \mu_2)\| = 0, \end{aligned} \tag{2.4}$$

$$\lim_{n\to\infty} \sum_{i=1}^{2} \|u_i(\cdot; q_n, \mu_{1,n}, \mu_{2,n}) - u_i(\cdot; q, \mu_1, \mu_2)\|_\infty = 0. \tag{2.5}$$

Besides, for $i = 1, 2$, *and* $\{x_n\}_{n\ge1} \subset S$ *which converges, as* $n \to \infty$, *to* $x \in S$,

$$\lim_{n\to\infty} u_i(x_n; q_n, \mu_{1,n}, \mu_{2,n}) = u_i(x; q, \mu_1, \mu_2). \tag{2.6}$$

When S is σ-compact, we only have the Borel measurability results on ν_i, u_i in (1.5).

Corollary 2.1. *Suppose that* (A2.1)' *and* (A2.2) *hold. Then the following are Borel measurable: for* $i = 1, 2$,

$$\int_S f(x)\nu_i(dx; \cdot, \cdot, \cdot) : C(S \times S) \times \mathcal{P}(S) \times \mathcal{P}(S) \mapsto \mathbf{R}, \quad f \in C_0(S),$$

$$u_i : S \times C(S \times S) \times \mathcal{P}(S) \times \mathcal{P}(S) \mapsto \mathbf{R} \cup \{\infty\}.$$

As an application of Corollary 2.1, we obtain the following.

Corollary 2.2. *Suppose that* (A1.1) *and* (A1.2) *hold. Then* $U(t, x, P)$ *in* (1.13) *is a Borel measurable function from* $[0, 1] \times \mathbf{R}^d \times \mathcal{P}(\mathbf{R}^d)$ *to* $\mathbf{R}$. *In particular,* (1.14)-(1.15) *hold.*

For $P_0 \in \mathcal{P}(\mathbf{R}^d)$ and Borel measurable $f : \mathbf{R}^d \mapsto \mathbf{R}$,

$$V^*_{P_0}(f) := \sup\left\{\int_{\mathbf{R}^d} f(x)P(dx) - V(P_0, P) : P \in \mathcal{P}(\mathbf{R}^d)\right\} \tag{2.7}$$

(see (1.17) for notation). Then since $P \mapsto V(P_0, P)$ is convex, lower semicontinuous and $\not\equiv \infty$, for $P \in \mathcal{P}(\mathbf{R}^d)$,

$$V(P_0, P) = \sup\left\{\int_{\mathbf{R}^d} f(x)P(dx) - V^*_{P_0}(f) : f \in C_b(\mathbf{R}^d)\right\} \in [0, \infty] \tag{2.8}$$

(see [18, 19, 21, 25] and the references therein). The following gives the variational meaning to Schrödinger's functional equation.

Proposition 2.1. *Suppose that* (A1.1) *and* (A1.2) *hold. Then for any* $P_0, P_1 \in \mathcal{P}(\mathbf{R}^d)$ *for which* $P_1(dy) \ll dy$ *and for which* $V(P_0, P_1)$ *is finite, Schrödinger's functional equation* (1.1) *with* μ_1, μ_2 *and* $q(x_1, x_2)$ *respectively replaced by* P_0, P_1 *and* $p(0, x_1; 1, x_2)$ *is equivalent to the following:*

$$P_1(dy) = \frac{\delta V^*_{P_0}(\log h(1, \cdot))}{\delta f}(dy). \tag{2.9}$$

Here $\dfrac{\delta V^*_{P_0}(f)}{\delta f}$ *denotes the Gâteaux derivative of* $V^*_{P_0}(f)$.

3. Proof of main results

In this section we state and prove lemmas and prove our main results.

$$\begin{aligned} m_q &:= \inf\{q(x_1, x_2) | x_1, x_2 \in S\}, \\ M_q &:= \sup\{q(x_1, x_2) | x_1, x_2 \in S\}. \end{aligned} \tag{3.1}$$

The following two lemmas are proved in [4].

Lemma 3.1 ([4], p. 194)**.** *Suppose that* (A2.1) *and* (A2.2) *hold. Then, for any* $\mu_1, \mu_2 \in \mathcal{P}(S)$, *there exists a unique pair of nonnegative finite measures* ν_1, ν_2 *on* S *for which* (1.1) *and the following holds:*

$$\frac{1}{\sqrt{M_q}} \le \nu_1(S) = \nu_2(S) \le \frac{1}{\sqrt{m_q}}, \tag{3.2}$$

$$\frac{m_q}{\sqrt{M_q}} \le \exp(u_i(x)) \le \frac{M_q}{\sqrt{m_q}}, \quad x \in S, i = 1, 2 \tag{3.3}$$

(see (1.2) *for notation).*

Lemma 3.2 ([4], section 7)**.** *Suppose that* (A2.1) *and* (A2.2) *hold. Then, there exists a function* $c(a, b)$ *which is nonincreasing in* a *and nondecreasing in* b *such that for any* $\mu_i, \tilde{\mu}_i \in \mathcal{P}(S)$, $i = 1, 2$,

$$\|\nu_1\nu_2 - \tilde{\nu}_1\tilde{\nu}_2\| \le c(m_q, M_q)\|\mu_1\mu_2 - \tilde{\mu}_1\tilde{\mu}_2\|^{\frac{1}{2}}. \tag{3.4}$$

Here $\tilde{\nu}_i(dx) := \nu_i(dx; q, \tilde{\mu}_1, \tilde{\mu}_2)$, $i = 1, 2$ *(see* (1.5) *and* (2.1) *for notation).*

The following lemma can be proved by Lemma 3.1.

Lemma 3.3. *Suppose that* (A2.1) *and* (A2.2) *hold and that* $q_n \in C(S \times S; (0, \infty))$, $n \geq 1$ *and*

$$\lim_{n\to\infty} \|q_n - q\|_\infty = 0 \tag{3.5}$$

(see (2.2) *for notation). Then, for any* $\mu_i \in \mathcal{P}(S)$, $i = 1, 2$,

$$\lim_{n\to\infty} \|\nu_{n,1}\nu_{n,2} - \nu_1\nu_2\| = 0, \tag{3.6}$$

where $\nu_{n,i}(dx) := \nu_i(dx; q_n, \mu_1, \mu_2)$ *(see* (1.5) *and* (2.1) *for notation).*

Proof. $u_{n,i}(x) := u_i(x; q_n, \mu_1, \mu_2)$. Then, from (1.2)-(1.3),

$$\begin{aligned} u_{n,i}(x_i) &= \log\left(\int_S q_n(x_1, x_2)\nu_{n,j}(dx_j)\right), \quad i, j = 1, 2, i \neq j, \\ \nu_{n,i}(dx) &= \exp(-u_{n,i}(x))\mu_i(dx), \quad i = 1, 2. \end{aligned} \tag{3.7}$$

For $i = 1, 2$, $\left\{\dfrac{\nu_{n,i}(dx)}{\nu_{n,i}(S)}\right\}_{n\geq 1}$ is a tight family of probability measures and $\{\nu_{n,i}(S)\}_{n\geq 1}$ is bounded from above and below by (3.2). In particular, there exist $\{s(n)\}_{n\geq 1}$ and a finite measure $\overline{\nu}_i$ such that $\nu_{s(n),i}$ weakly converges, as $n \to \infty$, to $\overline{\nu}_i$. From construction, (3.2) with ν_i replaced by $\overline{\nu}_i$ also holds.

$$\overline{u}_i(x_i) := \log\left(\int_S q(x_1, x_2)\overline{\nu}_j(dx_j)\right), \quad i, j = 1, 2, i \neq j. \tag{3.8}$$

Then for $i = 1, 2$,

$$\overline{\nu}_i(dx) = \exp(-\overline{u}_i(x))\mu_i(dx). \tag{3.9}$$

Indeed, from (3.7),

$$\nu_{s(n),i}(dx) - \exp(-\overline{u}_i(x))\mu_i(dx) = (\exp(-u_{s(n),i}(x)) - \exp(-\overline{u}_i(x)))\mu_i(dx).$$

For $i, j = 1, 2, i \neq j$ and $x_i \in S$,

$$\begin{aligned} |\exp(u_{s(n),i}(x_i)) - \exp(\overline{u}_i(x_i))| &\leq \left|\int_S (q_{s(n)}(x_1, x_2) - q(x_1, x_2))\nu_{s(n),j}(dx_j)\right| \\ &\quad + \left|\int_S q(x_1, x_2)(\nu_{s(n),j}(dx_j) - \overline{\nu}_j(dx_j))\right| \\ &\leq \|q_{s(n)} - q\|_\infty \times \nu_{s(n),j}(S) \\ &\quad + \left|\int_S q(x_1, x_2)(\nu_{s(n),j}(dx_j) - \overline{\nu}_j(dx_j))\right| \to 0, \quad n \to \infty, \end{aligned}$$

from (3.2) and (3.5). From (3.3),

$$\exp(-u_{s(n),i}(x_i)) \leq \frac{\sqrt{M_{q_{s(n)}}}}{m_{q_{s(n)}}} \to \frac{\sqrt{M_q}}{m_q}, \quad n \to \infty, i = 1, 2.$$

In particular, the bounded convergence theorem implies that (3.9) is true.

From (3.8)-(3.9),

$$\begin{aligned}(3.10)\qquad \mu_i(dx_i) &= \exp(-\overline{u}_i(x_i))\mu_i(dx_i)\exp(\overline{u}_i(x_i)) \\ &= \overline{\nu}_i(dx_i)\int_S q(x_1,x_2)\overline{\nu}_j(dx_j), \quad i,j=1,2, i\neq j.\end{aligned}$$

The uniqueness of the solution to (1.1) implies that

$$(3.11)\qquad \overline{\nu}_i(dx) = \nu_i(dx), \quad i=1,2$$

since (3.2) hold for both of $\overline{\nu}_i$ and ν_i. Since the above method applies for any subsequence of $\{q_n\}_{n\geq 1}$, the discussion in (3.9) implies that the following holds:

$$(3.12)\qquad \lim_{n\to\infty}\|\nu_{n,i}-\nu_i\| = 0, \quad i=1,2.$$

(3.2) and (3.12) completes the proof. □

We prove Theorem 2.1 by Lemmas 3.1-3.3.

Proof of Theorem 2.1. Lemmas 3.2 and 3.3 imply (2.4). We prove (2.5). Without loss of generality, we only have to consider the case when $i=1$. For sufficiently large $n\geq 1$,

$$\begin{aligned}(3.13)\qquad &\|u_1(\cdot\,;q_n,\mu_{1,n},\mu_{2,n})-u_1(\cdot\,;q,\mu_1,\mu_2)\|_\infty \\ \leq\ & -\log\left\{1-\frac{\sqrt{M_q}}{m_q}\left(\frac{\|q_n-q\|_\infty}{\sqrt{m_{q_n}}}+\|q\|_\infty\cdot\|\nu_2(\cdot\,;q_n,\mu_{1,n},\mu_{2,n})-\nu_2(\cdot\,;q,\mu_1,\mu_2)\|\right)\right\} \\ \to\ & 0, \quad n\to\infty.\end{aligned}$$

We prove (3.13). For $x\in S$,

$$\begin{aligned}&u_1(x;q_n,\mu_{1,n},\mu_{2,n})-u_1(x;q,\mu_1,\mu_2) \\ =\ & \log\left(1+\frac{\int_S q_n(x,x_2)\nu_2(dx_2;q_n,\mu_{1,n},\mu_{2,n})-\int_S q(x,x_2)\nu_2(dx_2;q,\mu_1,\mu_2)}{\int_S q(x,x_2)\nu_2(dx_2;q,\mu_1,\mu_2)}\right),\end{aligned}$$

$$\begin{aligned}&\left|\frac{\int_S q_n(x,x_2)\nu_2(dx_2;q_n,\mu_{1,n},\mu_{2,n})-\int_S q(x,x_2)\nu_2(dx_2;q,\mu_1,\mu_2)}{\int_S q(x,x_2)\nu_2(dx_2;q,\mu_1,\mu_2)}\right| \\ \leq\ & \frac{1}{\int_S q(x,x_2)\nu_2(dx_2;q,\mu_1,\mu_2)}\left\{\left|\int_S (q_n(x,x_2)-q(x,x_2))\nu_2(dx_2;q_n,\mu_{1,n},\mu_{2,n})\right|\right. \\ &\qquad \left.+\left|\int_S q(x,x_2)(\nu_2(dx_2;q_n,\mu_{1,n},\mu_{2,n})-\nu_2(dx_2;q,\mu_1,\mu_2))\right|\right\} \\ \leq\ & \frac{\sqrt{M_q}}{m_q}\left(\frac{\|q_n-q\|_\infty}{\sqrt{m_{q_n}}}+\|q\|_\infty\cdot\|\nu_2(\cdot\,;q_n,\mu_{1,n},\mu_{2,n})-\nu_2(\cdot\,;q,\mu_1,\mu_2)\|\right)\end{aligned}$$

from (3.2)-(3.3). The following also holds:

$$\lim_{n\to\infty}\|\nu_2(\cdot\,;q_n,\mu_{1,n},\mu_{2,n})-\nu_2(\cdot\,;q,\mu_1,\mu_2)\| = 0.$$

Indeed, for $f\in C_b(S)$ for which $\|f\|_\infty\leq 1$, from (2.4) and (3.2),

$$\int_S f(x)(\nu_2(dx;q_n,\mu_{1,n},\mu_{2,n})-\nu_2(dx;q,\mu_1,\mu_2))$$

$$
\begin{aligned}
&= \frac{1}{\nu_1(S\,;q_n,\mu_{1,n},\mu_{2,n})}\int_S f(x)\Big\{(\nu_1(S\,;q_n,\mu_{1,n},\mu_{2,n})\nu_2(dx;q_n,\mu_{1,n},\mu_{2,n}) \\
&\qquad -\nu_1(S\,;q,\mu_1,\mu_2)\nu_2(dx;q,\mu_1,\mu_2)) \\
&\qquad + (\nu_1(S\,;q,\mu_1,\mu_2)-\nu_1(S\,;q_n,\mu_{1,n},\mu_{2,n}))\,\nu_2(dx;q,\mu_1,\mu_2)\Big\} \\
&\le \sqrt{M_{q_n}}\Big\{\|\nu_1(\cdot\,;q_n,\mu_{1,n},\mu_{2,n})\nu_2(\cdot\,;q_n,\mu_{1,n},\mu_{2,n})-\nu_1(\cdot\,;q,\mu_1,\mu_2)\nu_2(\cdot\,;q,\mu_1,\mu_2)\| \\
&\qquad +\Big|\sqrt{\nu_1(S\,;q,\mu_1,\mu_2)\nu_2(S\,;q,\mu_1,\mu_2)}-\sqrt{\nu_1(S\,;q_n,\mu_{1,n},\mu_{2,n})\nu_2(S\,;q_n,\mu_{1,n},\mu_{2,n})}\Big|\frac{1}{\sqrt{m_q}}\Big\} \\
&\to 0, \quad n\to\infty.
\end{aligned}
$$

The following completes the proof of (3.13):

$$\log(1-|a|)\le\log(1+a)\le\log(1+|a|)\le-\log(1-|a|), \quad |a|<1.$$

We prove (2.6). From (2.5), we only have to prove the following: for $i=1,2$, and $\{x_n\}_{n\ge1}\subset S$ which converges to $x\in S$ as $n\to\infty$,

$$\lim_{n\to\infty} u_i(x_n;q,\mu_1,\mu_2)=u_i(x;q,\mu_1,\mu_2). \tag{3.14}$$

This can be proved by the bounded convergence theorem. □

For $\mu_1,\mu_2\in\mathcal{P}(S)$,

$$\mu_{i|n}(E):=\frac{\mu_i(E\cap A_n)}{\mu_i(A_n)}, \quad E\in\mathcal{B}(S), \quad n\ge n_0(\mu_1,\mu_2), i=1,2, \tag{3.15}$$

where $\mathcal{B}(S)$ denotes the Borel σ-field of S (see (1.6) for notation). When we replace X and S by A_n in (2.1)-(2.2) and (3.1), we use notations $\|\cdot\|_n$, $\|\cdot\|_{\infty,n}$, $m_{q,n}$ and $M_{q,n}$ instead of $\|\cdot\|$, $\|\cdot\|_\infty$, m_q and M_q, respectively. We use a similar convention when it is not confusing.

We introduce and prove two lemmas to prove Corollary 2.1.

Lemma 3.4. *Suppose that* (A2.1)' *and* (A2.2) *hold. Then, for any* $\mu_1,\mu_2\in\mathcal{P}(S)$ *and any* $k\ge n_0(\mu_1,\mu_2)$, *there exists a unique pair of nonnegative finite measures* $\nu_{1|k}$, $\nu_{2|k}$ *on* A_k *for which* Lemma 3.1 *with* S, m_q, M_q, μ_i, ν_i, u_i, $i=1,2$ *replaced by* A_k, $m_{q,k}$, $M_{q,k}$, $\mu_{i|k}$, $\nu_{i|k}$, $u_{i|k}$, $i=1,2$ *respectively holds. Suppose, in addition, that* $\mu_{i,n}\in\mathcal{P}(S)$, $q_n\in C(S\times S;(0,\infty))$, $i=1,2$, $n\ge1$ *and*

$$\lim_{n\to\infty}(\|\mu_{1,n}\mu_{2,n}-\mu_1\mu_2\|_k+\|q_n-q\|_{\infty,k})=0.$$

Then (2.4)-(2.6) *hold even if* ν_i, $\mu_{i,n}$, μ_i, $\|\cdot\|$, u_i, $\|\cdot\|_\infty$ *and* S *is replaced by* $\nu_{i|k}$, $\mu_{i,n|k}$, $\mu_{i|k}$, $\|\cdot\|_k$, $u_{i|k}$, $\|\cdot\|_{\infty,k}$ *and* A_k, *respectively.*

Proof. Theorem 2.1 and the following completes the proof:

$$\|\mu_{1,n|k}\mu_{2,n|k}-\mu_{1|k}\mu_{2|k}\|_k\le 2\frac{\|\mu_{1,n}\mu_{2,n}-\mu_1\mu_2\|_k}{\mu_1(A_k)\mu_2(A_k)}. \tag{3.16}$$

(3.16) is true, since

$$
\begin{aligned}
&\mu_{1,n|k}(dx_1)\mu_{2,n|k}(dx_2)-\mu_{1|k}(dx_1)\mu_{2|k}(dx_2) \\
&= \frac{1}{\mu_1(A_k)\mu_2(A_k)}\Big\{(\mu_{1,n}(dx_1)\mu_{2,n}(dx_2)-\mu_1(dx_1)\mu_2(dx_2))
\end{aligned}
$$

$$+\frac{\mu_1(A_k)\mu_2(A_k)-\mu_{1,n}(A_k)\mu_{2,n}(A_k)}{\mu_{1,n}(A_k)\mu_{2,n}(A_k)}\mu_{1,n}(dx_1)\mu_{2,n}(dx_2)\Big\}.$$

□

For any $\mu_1, \mu_2 \in \mathcal{P}(S)$ and any $n \geq n_0(\mu_1, \mu_2)$,

$$\mu^{(n)}(dx_1dx_2) := q(x_1, x_2)1_{A_n\times A_n}(x_1, x_2)\nu_{1|n}(dx_1)\nu_{2|n}(dx_2). \tag{3.17}$$

The following is known.

Lemma 3.5 ([12], Theorem 3.2)**.** *Suppose that* (A2.1)' *and* (A2.2) *hold. Then for any* $\mu_i \in \mathcal{P}(S)$*,* $i = 1, 2$*, there exists a unique solution* $\nu_1(dx_1)\nu_2(dx_2)$ *to* (1.1) *and* $\mu^{(n)}(dx_1dx_2)$ *weakly converges, as* $n \to \infty$*, to* $\mu(dx_1dx_2) := q(x_1, x_2)\nu_1(dx_1)\nu_2(dx_2)$*.*

By Lemmas 3.4 and 3.5, we prove Corollary 2.1.

Proof of Corollary 2.1. Without loss of generality, we only have to prove the case when $i = 1$. From Lemma 3.4, for any $n \geq 1$, $\int_{S\times S} f(x_1, x_2)\mu^{(n)}(dx_1dx_2)$ is continuous in (f, q, μ_1, μ_2) on the open set

$$C_b(S \times S) \times C(S \times S) \times \{(\mu_1, \mu_2) \in \mathcal{P}(S) \times \mathcal{P}(S) | n_0(\mu_1, \mu_2) \leq n\}$$

(see (1.6)). Notice that $n_0(\mu_1, \mu_2) \leq n$ if and only if $\mu_1(A_n)\mu_2(A_n) > 0$. From Lemma 3.5, $\int_{S\times S} f(x_1, x_2)\mu(dx_1dx_2)$ is measurable in (f, q, μ_1, μ_2). The following implies the first part of Corollary 2.1: for $f \in C_o(S)$ and $x \in \mathbb{R}$,

$$\begin{aligned}
&\left\{(q, \mu_1, \mu_2) \in C(S \times S) \times \mathcal{P}(S) \times \mathcal{P}(S) \,\middle|\, \int_S f(x_1)\nu_1(dx_1; q, \mu_1, \mu_2) < x\right\} \\
&= \bigcup_{k=1}^{\infty}\Bigg\{(q, \mu_1, \mu_2) \in C(S \times S) \times \mathcal{P}(S) \times \mathcal{P}(S) \,\Bigg|\, n_0(\mu_1, \mu_2) = k, \\
&\qquad\qquad \frac{\int_{S\times A_k} f(x_1)q(x_1, x_2)^{-1}\mu(dx_1dx_2; q, \mu_1, \mu_2)}{\sqrt{\int_{A_k\times A_k} q(x_1, x_2)^{-1}\mu(dx_1dx_2; q, \mu_1, \mu_2)}} < x\Bigg\}.
\end{aligned}$$

For any $x \in S$ and $\phi_n \in C_o(S)$ for which $0 \leq \phi_n \leq 1$ and $\phi_n(y) = 1, y \in A_n$, $\int_S \phi_n(x_2)q(x, x_2)\nu_2(dx_2)$ is measurable in (q, μ_1, μ_2) in the same way as above and is continuous in x. In particular, it is measurable in (x, q, μ_1, μ_2) and so is the following: by Fatou's lemma,

$$\exp(u_1(x)) = \lim_{n\to\infty}\int_S \phi_n(x_2)q(x, x_2)\nu_2(dx_2).$$

□

Corollary 2.1 immediately implies Corollary 2.2.

Proof of Corollary 2.2. Since $p(t, \cdot\,; 1, \cdot)$ is continuous on $[0, 1)$ from Theorem 1.1,

$$(t, x, P, P_1) \mapsto (x, p(t, \cdot\,; 1, \cdot), P, P_1)$$

is continuous on $[0, 1) \times \mathbf{R}^d \times \mathcal{P}(\mathbf{R}^d) \times \mathcal{P}(\mathbf{R}^d)$, which implies the measurability of $U(t, x, P)$. It is easy to see that (1.14) - (1.15) hold. □

We prove Proposition 2.1.

Proof of Proposition 2.1.

$$\frac{\delta V^*_{P_0}(\log h(1,\cdot))}{\delta f}(dy) = h(1,y)dy \int_{\mathbf{R}^d} p(0,x;1,y)\frac{P_0(dx)}{h(0,x)}. \tag{3.18}$$

Indeed, for any $\psi \in C_b(\mathbf{R}^d)$ and $\varepsilon \in \mathbf{R}$, instead of P_1, consider Schrödinger's problem (1.1) with S, μ_1, μ_2 and $q(x_1, x_2)$ respectively replaced by $\mathbf{R}^d, P_0, \mu^{\varepsilon\psi}$ and $q(0, x_1; 1, x_2)$, where for $f \in C_b(\mathbf{R}^d)$

$$\mu^f(dy) := h(1,y)\exp(f(y))dy \int_{\mathbf{R}^d} P_0(dx)\frac{p(0,x;1,y)}{\int_{\mathbf{R}^d} p(0,x;1,z)h(1,z)\exp(f(z))dz}$$

(see (1.1)-(1.4)). Then, from Theorem 1.3 and (2.8) (see e.g. [7, 26] and also [21]),

$$\begin{aligned} &V^*_{P_0}(\log h(1,\cdot) + \varepsilon\psi) \\ &= \int_{\mathbf{R}^d} \log\left(\int_{\mathbf{R}^d} p(0,x;1,y)h(1,y)\exp(\varepsilon\psi(y))\,dy\right)P_0(dx) \end{aligned}$$

(see (1.8)). This implies (3.18). From (1.8),

$$P_0(dx) = \left(\int_{\mathbf{R}^d} h(1,y)dy\,p(0,x;1,y)\right)\frac{P_0(dx)}{h(0,x)}. \tag{3.19}$$

(3.18) and (3.19) completes the proof. □

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