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A GENERALIZATION OF FUNCTIONAL LIMIT THEOREMS ON THE RIEMANN ZETA PROCESS

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Abstract

$\zeta(\cdot)$ being the Riemann zeta function, $\zeta_\sigma(t) := \frac{\zeta(\sigma+it)}{\zeta(\sigma)}$ is, for $\sigma > 1$, a characteristic function of some infinitely divisible distribution μ_σ. A process with time parameter σ having μ_σ as its marginal at time σ is called a Riemann zeta process. Ehm [2] has found a functional limit theorem on this process being a backwards Lévy process. In this paper, we replace $\zeta(\cdot)$ with a Dirichlet series $\eta(\cdot; a)$ generated by a nonnegative, completely multiplicative arithmetical function $a(\cdot)$ satisfying (3), (4) and (5) below, and derive the same type of functional limit theorem as Ehm on the process corresponding to $\eta(\cdot; a)$ and being a backwards Lévy process.

Introduction

Let $\zeta(\cdot)$ be the Riemann zeta function. Then $\zeta_\sigma(t) := \frac{\zeta(\sigma+it)}{\zeta(\sigma)}$ is, for $\sigma > 1$, a characteristic function of some infinitely divisible distribution μ_σ. This μ_σ is called the Riemann zeta distribution indexed by parameter σ. We are interested in a (stochastic) process with time parameter σ whose marginal distribution at time σ is μ_σ. Such a process is called a Riemann zeta (stochastic) process.

Ehm [2] has constructed this process so as to be a backwards Lévy process, and found a functional limit theorem on the process.

In this paper, we generalize the setting of Ehm. We replace $\zeta(s)$ with a Dirichlet series $\eta(s; a) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$, where $a(\cdot)$ is a nonnegative, completely multiplicative arithmetical function satisfying (3), (4) and (5) below, and then derive the same type of functional limit theorem as Ehm on the process $\left(-Z(\sigma; a)\right)_{1<\sigma<\infty}$ corresponding to $\eta(\cdot; a)$ and being a backwards Lévy process, which is shortly called the $\eta(\cdot; a)$-process.

In Section 1, we review Ehm's result. In Section 2, we state our main result (cf. Theorem 1) and prove it, and in Section 3 give some examples of $a(\cdot)$.

In Section 4, we generalize $a(\cdot)$ more, and then investigate limit theorems on $Z(\sigma; a)$ as $\sigma \searrow 1$ (cf. Theorems 2 ~ 4).

1. Review of Ehm's result

1.1. Riemann zeta distribution. The Riemann zeta function $\zeta(\cdot)$ has two representations:

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$$\zeta(s) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}, \\ \displaystyle\prod_{p:\text{prime}} \frac{1}{1 - \frac{1}{p^s}}. \end{cases}$$

Here $s = \sigma + it$, $\sigma > 1$, $t \in \mathbb{R}$. The former is a Dirichlet series representation and the latter is an Euler product representation. For fixed $\sigma > 1$, $\zeta(\sigma + i\cdot)$ is positive definite as a function of $\mathbb{R}$. In other words, $\zeta_\sigma(t) = \frac{\zeta(\sigma+it)}{\zeta(\sigma)}$ is a characteristic function of $\mu_\sigma := \sum_{n=1}^{\infty} \frac{1}{\zeta(\sigma)n^\sigma}\delta_{\log\frac{1}{n}}$. Indeed, by the former representation,

$$\zeta_\sigma(t) = \frac{1}{\zeta(\sigma)} \sum_{n=1}^{\infty} \frac{e^{i(\log\frac{1}{n})t}}{n^\sigma} = \int_{\mathbb{R}} e^{ixt} \sum_{n=1}^{\infty} \frac{1}{\zeta(\sigma)n^\sigma}\delta_{\log\frac{1}{n}}(dx) = \widehat{\mu_\sigma}(t).$$

μ_σ is called a Riemann zeta distribution with parameter σ. Furthermore, it is easy to see that μ_σ is an infinitely divisible distribution: By the latter representation and

$$(1) \qquad 1 + z = \exp\left\{\int_0^1 \frac{z}{1+zs} ds\right\}, \quad z \in \mathbb{C} \setminus (-\infty, -1],$$

it is checked that

$$\begin{aligned}
\zeta_\sigma(t) &= \prod_{p:\text{prime}} \frac{1 - \frac{1}{p^\sigma}}{1 - \frac{1}{p^{\sigma+it}}} \\
&= \prod_{p:\text{prime}} \exp\left\{\int_0^1 \frac{-\frac{1}{p^\sigma}}{1 - \frac{s}{p^\sigma}} ds - \int_0^1 \frac{-\frac{1}{p^{\sigma+it}}}{1 - \frac{s}{p^{\sigma+it}}} ds\right\} \\
&= \prod_{p:\text{prime}} \exp\left\{\int_0^1 \sum_{n=1}^{\infty} \frac{s^{n-1}}{p^{n\sigma}} (e^{i(\log\frac{1}{p})nt} - 1) ds\right\} \\
&= \exp\left\{\int_{\mathbb{R}\setminus\{0\}} (e^{ixt} - 1)\nu_\sigma(dx)\right\},
\end{aligned}$$

where

$$\nu_\sigma(dx) := \sum_{p:\text{prime}} \sum_{n=1}^{\infty} \frac{1}{np^{n\sigma}} \delta_{n\log\frac{1}{p}}(dx), \quad x \in \mathbb{R} \setminus \{0\}$$

is a Lévy measure.

$\zeta(\cdot)$ is extended meromorphically to the whole complex plane with only a simple pole at 1 with residue 1. Thus, asymptotically

$$(s-1)\zeta(s) = 1 + O(|s-1|) \quad \text{as } s \to 1.$$

By this, we easily have the following limit theorem for μ_σ as $\sigma \searrow 1$:

Claim 1. *As $\sigma \searrow 1$,*

$$\mu_\sigma\left(\frac{-1}{\sigma - 1} dx\right) \to \mathbf{1}_{(0,\infty)}(x)e^{-x}dx \text{ (= the exponential distribution with parameter 1).}$$

Proof. For $^\forall t \in \mathbb{R}$,

$$\begin{aligned}\int_{\mathbb{R}} e^{itx}\mu_\sigma\Big(\frac{-1}{\sigma-1}dx\Big) &= \int_{\mathbb{R}} e^{it\left(-(\sigma-1)y\right)}\mu_\sigma(dy)\\ &= \zeta_\sigma(-t(\sigma-1))\\ &= \frac{\zeta(\sigma+i(-t(\sigma-1)))}{\zeta(\sigma)}\\ &\to \frac{1}{1-it} = \int_0^\infty e^{itx}e^{-x}dx \quad \text{as } \sigma \searrow 1.\end{aligned}$$

□

1.2. Riemann zeta process. A process with time parameter $\sigma \in (1,\infty)$ having μ_σ as its marginal at σ is called a Riemann zeta process. Following Ehm [2], we construct the process so as to be a backwards Lévy process.

DEFINITION 1. A process $(Y(u))_{0\le u<1}$ on some probability space $(\Omega,\mathcal{F},P)$ is called a geometric process if the following (a) ~ (d) hold:

(a) For each $u \in [0,1)$, $Y(u) \in \{0,1,2,\ldots\}$. Especially $Y(0)=0$.
(b) $[0,1) \ni u \mapsto Y(u) \in \mathbb{R}$ is right-continuous and non-decreasing.
(c) $(Y(u))_{0\le u<1}$ is a Lévy process, i.e., for every $0<u_0<u_1<\cdots<u_n<1$,

$$Y(u_0),\ Y(u_1)-Y(u_0),\ \ldots,\ Y(u_n)-Y(u_{n-1}) \text{ are independent,}$$

and, for each $u \in (0,1)$, $Y(u) = Y(u-)$ a.s.
(d) For each $0 \le u < v < 1$,

$$E[e^{it(Y(v)-Y(u))}] = \frac{1-ue^{it}}{1-u}\frac{1-v}{1-ve^{it}}.$$

In particular

$$E[e^{itY(u)}] = \frac{1-u}{1-ue^{it}}.$$

Thus $P(Y(u)=n) = u^n(1-u)$, $n \in \{0,1,2,\ldots\}$, in other words, $Y(u)$ is geometrically distributed with parameter $1-u$.

DEFINITION 2. Let $\{Y_p\}_{p:\text{prime}}$ be a sequence of independent geometric processes on some $(\Omega,\mathcal{F},P)$. Then we define

$$Z(\sigma) := \sum_{p:\text{prime}} Y_p(p^{-\sigma})\log p, \quad \sigma \in (1,\infty).$$

Claim 2. *$(-Z(\sigma))_{1<\sigma<\infty}$ is a Riemann zeta process, and a backwards Lévy process. This means the following:*
(i) $(1,\infty) \ni \sigma \mapsto Z(\sigma) \in [0,\infty)$ *is left-continuous and non-increasing;*
(ii) *For $\sigma > 1$, $Z(\sigma+) = Z(\sigma)$ a.s., $Z(1+) = \infty$ a.s. and $Z(\infty) = 0$;*
(iii) *For every $\infty > \sigma_0 > \sigma_1 > \cdots > \sigma_n > 1$,*

$$Z(\sigma_0),\ Z(\sigma_1)-Z(\sigma_0),\ \ldots,\ Z(\sigma_n)-Z(\sigma_{n-1}) \text{ are independent.}$$

Proof. Claim 2 is contained in Claim 7 below. So the proof of Claim 2 is omitted. □

1.3. Ehm's functional limit theorem. Claim 1 can be restated in terms of $Z(\sigma)$:

Claim 3. *As* $\sigma \searrow 1$,

$$\text{the distribution of } (\sigma - 1)Z(\sigma) \to \text{the exponential distribution with parameter 1.}$$

Proof. As $\sigma \searrow 1$,

$$\begin{aligned} P((\sigma - 1)Z(\sigma) \in dx) &= \mu_\sigma\Big(\frac{-1}{\sigma - 1}dx\Big) \\ &\to \text{the exponential distribution with parameter 1.} \end{aligned}$$

□

This limit theorem is generalized as a functional limit theorem:

Fact 1 (cf. [2]). *Let* $\varphi : (1, \infty) \to (0, \infty)$ *be a* C^1*, strictly decreasing function such that*

$$\varphi(1+) = \infty,\ \varphi(\infty) = 0,$$
$$\varphi(\sigma) \sim \frac{1}{(\sigma - 1)^2} \quad \text{as } \sigma \searrow 1.$$

Let $N(dsdu)$ *be a Poisson random measure on* $(0, \infty) \times (0, \infty)$ *with mean measure*

$$n(dsdu) := \frac{1}{2}e^{-u/\sqrt{s}}s^{-3/2}dsdu, \quad s, u > 0.$$

Then the following holds:

$$\Big(\frac{1}{\sqrt{T}}Z(\varphi^{-1}(Tt))\Big)_{t\ge0} \xrightarrow{\mathcal{D}} \Big(\int_0^{t+}\int_{(0,\infty)} uN(dsdu)\Big)_{t\ge0}$$
$$\text{in } D([0, \infty) \to \mathbb{R}) \quad \text{as } T \to \infty.$$

Here $D([0, \infty) \to \mathbb{R})$ is the space of all real functions on $[0, \infty)$ that are right-continuous and have left-hand limits. This space is endowed with the J_1-topology (cf. [8, 1]), so that it becomes a Polish space. $(\frac{1}{\sqrt{T}}Z(\varphi^{-1}(Tt)))_{t\ge0}$ and $(\int_0^{t+}\int_{(0,\infty)} uN(dsdu))_{t\ge0}$ are random elements of $D([0, \infty) \to \mathbb{R})$, that is, they are $D([0, \infty) \to \mathbb{R})$-valued random variables. In other words, almost all samples $t \mapsto \frac{1}{\sqrt{T}}Z(\varphi^{-1}(Tt))$ and $t \mapsto \int_0^{t+}\int_{(0,\infty)} uN(dsdu)$ belong to $D([0, \infty) \to \mathbb{R})$, and for each $t \in [0, \infty)$, $\frac{1}{\sqrt{T}}Z(\varphi^{-1}(Tt))$ and $\int_0^{t+}\int_{(0,\infty)} uN(dsdu)$ are real random variables. The convergence above denotes the weak convergence of the distribution of $(\frac{1}{\sqrt{T}}Z(\varphi^{-1}(Tt)))_{t\ge0}$ to that of $(\int_0^{t+}\int_{(0,\infty)} uN(ds\,du))_{t\ge0}$.

The statement of this fact is different from that of Ehm [2]. Suited to our theorem stated in Section 2, the above fact has been presented.

2. Our main result

2.1. Completely multiplicative arithmetical function $a(\cdot)$. If an arithmetical function $a : \mathbb{N} \to \mathbb{R}$ satisfies

$$a(mn) = a(m)a(n), \quad {}^{\forall}m, {}^{\forall}n \in \mathbb{N}, \tag{2}$$

then $a(\cdot)$ is said to be completely multiplicative. $a(\cdot) = 0$ (i.e., $a(n) = 0$ $({}^{\forall}n \in \mathbb{N})$) and $a(\cdot) = 1$ (i.e., $a(n) = 1$ $({}^{\forall}n \in \mathbb{N})$) are clearly such arithmetical functions. For a completely multiplicative $a(\cdot)$, it should be remarked that

$$a(\cdot) \neq 0 \ \text{(i.e., } {}^{\exists}n_0 \in \mathbb{N} \ \text{ s.t. } \ a(n_0) \neq 0) \underset{\text{iff}}{\Leftrightarrow} a(1) = 1,$$

in other words, $a(\cdot) = 0 \underset{\text{iff}}{\Leftrightarrow} a(1) = 0$. Since $a(\cdot) = 0$ is too trivial, it is excluded from completely multiplicative companions. Thus, from now on $a : \mathbb{N} \to \mathbb{R}$ is called completely multiplicative if $a(1) = 1$ and (2) is satisfied. In this case, if $n = \prod_p p^{\alpha_p(n)}$ is the prime factorization of $n \in \mathbb{N}$, where

$$\alpha_p(n) = \max\{m \in \{0, 1, 2, \ldots\}; p^m \mid n\},$$

then

$$a(n) = \prod_p a(p)^{\alpha_p(n)}.$$

Here let $x^0 = 1$ for $x \in \mathbb{R}$. Thus, the value of $a(\cdot)$ is completely determined by that of $(a(p))_{p:\text{prime}}$.

In the following, let $a : \mathbb{N} \to [0, \infty)$ be a completely multiplicative arithmetical function[1] such that

$$\sup_p a(p) < \infty, \ \sup_p \frac{a(p)}{p} \leq 1, \tag{3}$$

$$ {}^{\exists}\tau \geq 0 \ \text{ s.t. } \sum_{p \leq x} \frac{a(p)\log p}{p} = (\tau + o(1))\log x \quad \text{as } x \to \infty, \tag{4}$$

$$\tau + \#\{p; a(p) = p\} > 0. \tag{5}$$

Note that $0 \leq \#\{p; a(p) = p\} < \infty$ since $\sup_p a(p) < \infty$. When $\tau > 0$ in (4), (5) holds automatically. When $\tau = 0$ in (4), (5) becomes $\#\{p; a(p) = p\} > 0$. By Mertens' first theorem:

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1) \quad \text{as } x \to \infty \tag{6}$$

(cf. [5, Theorem 425] or [10, Chapter I.1, Theorem 7]), $a(\cdot) = 1$ is a typical example. In Section 3, we will give some other examples.

In what follows up to the end of Section 2, let us fix such an arithmetical function $a(\cdot)$.

2.2. Presentation of Theorem 1. To state our main result – Theorem 1, we need some definitions:

[1]For simplicity, we restrict completely multiplicative arithmetical functions appearing in this paper to be nonnegative.

Definition 3. For $s=\sigma+it$ ($\sigma>1$, $t\in\mathbb{R}$), we define

$$\eta(s;a):=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}=\prod_p\frac{1}{1-\frac{a(p)}{p^s}}.$$

By virtue of Claims 4 and 5 below, this is well-defined. When $a(\cdot)=1$, $\eta(\cdot;1)=\zeta(\cdot)$!

Definition 4. For $\sigma\in(1,\infty)$, we define

$$\eta_\sigma(t;a):=\frac{\eta(\sigma+it;a)}{\eta(\sigma;a)},\quad t\in\mathbb{R}.$$

If $\mu_\sigma(dx;a)$ and $\nu_\sigma(dx;a)$ are a 1-dimensional probability measure and a Lévy measure, respectively, defined by

$$\mu_\sigma(dx;a):=\sum_{n=1}^{\infty}\frac{a(n)}{\eta(\sigma;a)n^\sigma}\delta_{\log\frac{1}{n}}(dx),$$
$$\nu_\sigma(dx;a):=\sum_p\sum_{n=1}^{\infty}\frac{a(p)^n}{np^{n\sigma}}\delta_{n\log\frac{1}{p}}(dx),$$

then

$$\eta_\sigma(t;a)=\widehat{\mu_\sigma(\cdot;a)}(t)=\exp\Big\{\int_{\mathbb{R}\setminus\{0\}}(e^{ixt}-1)\nu_\sigma(dx;a)\Big\}.$$

$\mu_\sigma(\cdot;a)$ is called the $\eta(\cdot;a)$-distribution with parameter σ.

Definition 5. Let $\{Y_p\}_p$ be a sequence of independent geometric processes on some $(\Omega,\mathcal{F},P)$. Then we define

$$Z(\sigma;a):=\sum_p Y_p\Big(\frac{a(p)}{p^\sigma}\Big)\log p,\quad \sigma\in(1,\infty).$$

By Claim 7 below, $(-Z(\sigma;a))_{1<\sigma<\infty}$ is a backwards Lévy process whose marginal distribution at σ is $\mu_\sigma(\cdot;a)$. Thus, by imitating $(-Z(\sigma))_{1<\sigma<\infty}$, this is called an $\eta(\cdot;a)$-process.

Our main result is the following:

Theorem 1. *Let $\varphi:(1,\infty)\to(0,\infty)$ be a C^1, strictly decreasing function such that*

$$\varphi(1+)=\infty,\ \varphi(\infty)=0, \tag{7}$$

$$\varphi(\sigma)\sim\frac{1}{(\sigma-1)^2}\quad \textit{as } \sigma\searrow 1. \tag{8}$$

Let $\rho:=\tau+\#\{p;a(p)=p\}>0$ (cf. Claim 6 *below), and $N^{(\rho)}(dsdu)$ be a Poisson random measure on $(0,\infty)\times(0,\infty)$ with mean measure*

$$n^{(\rho)}(dsdu):=\frac{\rho}{2}e^{-\frac{u}{\sqrt{s}}}s^{-\frac{3}{2}}dsdu,\quad s,u>0. \tag{9}$$

Then the following holds:

$$\Big(\frac{1}{\sqrt{T}}Z(\varphi^{-1}(Tt);a)\Big)_{t\geq0}\xrightarrow{\mathcal{D}}\Big(\int_0^{t+}\int_{(0,\infty)}uN^{(\rho)}(dsdu)\Big)_{t\geq0}$$

$$in \ D([0,\infty)\to\mathbb{R}) \ \ as \ T\to\infty.$$

Remark 1. In the definition of $(Z(\sigma))_{1<\sigma<\infty}$ (cf. Definition 2), we replaced $p^{-\sigma}$ with $\frac{a(p)}{p^{\sigma}}$ and obtained a functional limit theorem on the resultant process $(Z(\sigma;a))_{1<\sigma<\infty}$. Since $(Z(\sigma;a))_{1<\sigma<\infty}$ is a process defined from $\eta(\cdot;a)$, we could even say that this functional limit theorem comes from a topic of the number theory. As a different generalization of [2], Ehm [3] replaced $\log p$ with more general coefficient c_p and considered a functional limit theorem on the resultant process. In this case, though this process is of zeta type, its functional limit theorem is no longer related to the number theory.

2.3. Some claims. To make definitions given in the preceding subsection meaningful, we here present some claims:

Claim 4. (i) $\sum_{p\le x}\frac{a(p)}{p}=(\tau+o(1))\log\log x$ *as* $x\to\infty$. *When* $\tau>0$, $\sum_p\frac{a(p)}{p}=\infty$.
(ii) *For* $\sigma>1$ *and* $t\in\mathbb{R}$, $\prod_p\frac{1}{1-\frac{a(p)}{p^{\sigma+it}}}$ *is convergent. That is,* $\prod_{p\le x}\frac{1}{1-\frac{a(p)}{p^{\sigma+it}}}$ *is convergent as* $x\to\infty$. *As* $\sigma\searrow 1$, $\prod_p\frac{1}{1-\frac{a(p)}{p^{\sigma}}}\nearrow\infty$.

Proof. (i) For simplicity, let

$$C(x):=\sum_{p\le x}\frac{a(p)\log p}{p},\quad x\in\mathbb{R}. \tag{10}$$

$C(\cdot)$ is non-decreasing, right-continuous, $C(x)=0$ $(^{\forall}x<2)$, and

$$C(dt)=\sum_p\frac{a(p)\log p}{p}\delta_p(dt). \tag{11}$$

If, for $x>1$, we set

$$\delta(x):=\frac{C(x)}{\log x}-\tau, \tag{12}$$

then $\delta(\cdot)$ is of bounded variation on every bounded closed interval of $(1,\infty)$, and by (4),

$$\lim_{x\to\infty}\delta(x)=0. \tag{13}$$

(11) and $C(t)=\tau\log t+\delta(t)\log t$ tell us that for $x\ge 3$,

$$\begin{aligned}\sum_{p\le x}\frac{a(p)}{p}&=\sum_{p\le x}\frac{1}{\log p}\frac{a(p)\log p}{p}\\&=\int_{(2-\varepsilon,x]}\frac{C(dt)}{\log t}\quad[\text{where } 0<\varepsilon<1]\\&=\tau\int_{2-\varepsilon}^{x}\frac{dt}{t\log t}+\int_{2-\varepsilon}^{x}\frac{\delta(t)}{t\log t}dt+\int_{(2-\varepsilon,x]}\delta(dt)\\&=\tau\int_{2-\varepsilon}^{x}\frac{dt}{t\log t}+\int_{2-\varepsilon}^{x}\frac{\delta(t)}{t\log t}dt+\delta(x)+\tau.\end{aligned}$$

By letting $\varepsilon\searrow 0$,

$$\sum_{p\le x}\frac{a(p)}{p}=\tau\int_2^x\frac{dt}{t\log t}+\int_2^x\frac{\delta(t)}{t\log t}dt+\tau+\delta(x)$$

$$= \log\log x\Big(\tau + \frac{\tau(-\log\log 2+1)}{\log\log x} + \frac{\delta(x)}{\log\log x} + \int_{\frac{\log\log 2}{\log\log x}}^{1} \delta(e^{e^{(\log\log x)u}})du\Big)$$

$$\text{[by the change of variables } \tfrac{\log\log t}{\log\log x} = u\text{].}$$

Since, by (13) and the bounded convergence theorem,

$$\lim_{x\to\infty} \int_{\frac{\log\log 2}{\log\log x}}^{1} \delta(e^{e^{(\log\log x)u}})du = 0,$$

we have

$$\sum_{p\le x} \frac{a(p)}{p} = (\log\log x)(\tau + o(1)) \quad \text{as } x\to\infty.$$

(ii) First, (1) is rewritten as

$$1+z = e^z \exp\Big\{-z^2 \int_0^1 \frac{s}{1+zs} ds\Big\}, \quad z \in \mathbb{C}\setminus(-\infty,-1]. \tag{14}$$

Let $\sigma > 1$ and $t \in \mathbb{R}$. Since, by (3),

$$\Big|\frac{a(p)}{p^{\sigma+it}}\Big| = \frac{1}{p^{\sigma-1}}\frac{a(p)}{p} \le \frac{1}{p^{\sigma-1}} \le \frac{1}{2^{\sigma-1}} < 1, \tag{15}$$

(14) implies that

$$\frac{1}{1-\frac{a(p)}{p^{\sigma+it}}} = e^{\frac{a(p)}{p^{\sigma+it}}} \exp\Big\{\frac{a(p)^2}{p^{2\sigma+i2t}} \int_0^1 \frac{s}{1-\frac{a(p)}{p^{\sigma+it}}s} ds\Big\}. \tag{16}$$

Multiplication in $p \le x$ yields that

$$\prod_{p\le x} \frac{1}{1-\frac{a(p)}{p^{\sigma+it}}} = \exp\Big\{\sum_{p\le x}\frac{a(p)}{p^{\sigma+it}}\Big\} \exp\Big\{\sum_{p\le x}\frac{a(p)^2}{p^{2\sigma+i2t}} \int_0^1 \frac{s}{1-\frac{a(p)}{p^{\sigma+it}}s} ds\Big\}.$$

Here, by noting that

$$\begin{aligned}\sum_p \Big|\frac{a(p)^2}{p^{2\sigma+i2t}} \int_0^1 \frac{s}{1-\frac{a(p)}{p^{\sigma+it}}s} ds\Big| &\le \sum_p \frac{a(p)^2}{p^{2\sigma}} \int_0^1 \frac{s}{1-|\frac{a(p)}{p^{\sigma+it}}s|} ds \\ &\le \frac{(\sup_p a(p))^2}{2(1-\frac{1}{2^{\sigma-1}})} \sum_p \frac{1}{p^{2\sigma}} \ \text{[cf. (15)]} < \infty,\end{aligned}$$

the convergence of $\prod_p \frac{1}{1-\frac{a(p)}{p^{\sigma+it}}}$ is reduced to that of $\sum_p \frac{a(p)}{p^{\sigma+it}}$. Since

$$\sum_p \Big|\frac{a(p)}{p^{\sigma+it}}\Big| = \sum_p \frac{a(p)}{p^\sigma} \le \sup_p a(p) \sum_p \frac{1}{p^\sigma} \le \sup_p a(p) \sum_n \frac{1}{n^\sigma} < \infty,$$

$\prod_p \frac{1}{1-\frac{a(p)}{p^{\sigma+it}}}$ is convergent.

Next we check the divergence of $\prod_p \frac{1}{1-\frac{a(p)}{p^\sigma}}$ as $\sigma \searrow 1$. First

$$\prod_p \frac{1}{1-\frac{a(p)}{p^\sigma}} = \Big(\prod_{p;a(p)=p} \frac{1}{1-\frac{1}{p^{\sigma-1}}}\Big)\Big(\prod_{p;a(p)<p} \frac{1}{1-\frac{a(p)}{p^\sigma}}\Big)$$

$$= \Big(\prod_{p;a(p)=p} \frac{1}{1 - \frac{1}{p^{\sigma-1}}} \Big) \exp\Big\{ \sum_p \frac{a(p)}{p^\sigma} \Big\} \exp\Big\{ - \sum_{p;a(p)=p} \frac{1}{p^{\sigma-1}} \Big\}$$
$$\times \exp\Big\{ \sum_{p;a(p)<p} \frac{a(p)^2}{p^{2\sigma}} \int_0^1 \frac{s}{1 - \frac{a(p)}{p^\sigma} s} ds \Big\}.$$

Here note that

$$\sup_{q;a(q)<q} \frac{a(q)}{q} < 1. \tag{17}$$

This tells us that for p with $a(p) < p$,

$$-\frac{a(p)}{p^\sigma} = -\frac{a(p)}{p} \frac{1}{p^{\sigma-1}} \geq -\Big(\sup_{q;a(q)<q} \frac{a(q)}{q} \Big) \frac{1}{p^{\sigma-1}} > -\Big(\sup_{q;a(q)<q} \frac{a(q)}{q} \Big) > -1, \tag{18}$$

so that

$$\sum_{p;a(p)<p} \frac{a(p)^2}{p^{2\sigma}} \int_0^1 \frac{s}{1 - \frac{a(p)}{p^\sigma} s} ds \leq \frac{(\sup_{q;a(q)<q} a(q))^2}{2(1 - \sup_{q;a(q)<q} \frac{a(q)}{q})} \sum_p \frac{1}{p^2} < \infty.$$

Thus

$$\lim_{\sigma \searrow 1} \exp\Big\{ \sum_{p;a(p)<p} \frac{a(p)^2}{p^{2\sigma}} \int_0^1 \frac{s}{1 - \frac{a(p)}{p^\sigma} s} ds \Big\} = \exp\Big\{ \sum_{p;a(p)<p} \frac{a(p)^2}{p^2} \int_0^1 \frac{s}{1 - \frac{a(p)}{p} s} ds \Big\} < \infty.$$

Clearly

$$\lim_{\sigma \searrow 1} \exp\Big\{ - \sum_{p;a(p)=p} \frac{1}{p^{\sigma-1}} \Big\} = e^{-\#\{p;a(p)=p\}} < \infty.$$

When $\tau > 0$,

$$\lim_{\sigma \searrow 1} \exp\Big\{ \sum_p \frac{a(p)}{p^\sigma} \Big\} = \exp\Big\{ \sum_p \frac{a(p)}{p} \Big\} = \infty$$

by (i). When $\tau = 0$,

$$\lim_{\sigma \searrow 1} \prod_{p;a(p)=p} \frac{1}{1 - \frac{1}{p^{\sigma-1}}} = \infty$$

since $\{p; a(p) = p\} \neq \emptyset$. Therefore, putting all together, we have

$$\lim_{\sigma \searrow 1} \prod_p \frac{1}{1 - \frac{a(p)}{p^\sigma}} = \infty.$$

□

Claim 5. *For $\sigma > 1$ and $t \in \mathbb{R}$, $\sum_{n=1}^\infty \frac{a(n)}{n^{\sigma+it}}$ is absolutely convergent, and coincides with $\prod_p \frac{1}{1-\frac{a(p)}{p^{\sigma+it}}}$.*

Proof. Fix $\sigma > 1$ and $t \in \mathbb{R}$. Let p_j be the jth prime number. Note that

$$\mathbb{N}_L := \{p_1^{\alpha_1} \cdots p_L^{\alpha_L}; 0 \leq \alpha_1, \ldots, \alpha_L \leq L\} \nearrow \mathbb{N} \quad \text{as } L \to \infty.$$

By the completely multiplicative property of $a(\cdot)$,

$$\begin{aligned}
\sum_{n\in\mathbb{N}_L}\frac{a(n)}{n^{\sigma+it}} &= \sum_{0\le\alpha_1,\dots,\alpha_L\le L}\frac{a(p_1^{\alpha_1}\cdots p_L^{\alpha_L})}{(p_1^{\alpha_1}\cdots p_L^{\alpha_L})^{\sigma+it}}\\
&= \sum_{0\le\alpha_1,\dots,\alpha_L\le L}\frac{a(p_1)^{\alpha_1}}{(p_1^{\sigma+it})^{\alpha_1}}\times\cdots\times\frac{a(p_L)^{\alpha_L}}{(p_L^{\sigma+it})^{\alpha_L}}\\
&= \frac{1-(\frac{a(p_1)}{p_1^{\sigma+it}})^{L+1}}{1-\frac{a(p_1)}{p_1^{\sigma+it}}}\times\cdots\times\frac{1-(\frac{a(p_L)}{p_L^{\sigma+it}})^{L+1}}{1-\frac{a(p_L)}{p_L^{\sigma+it}}}\quad\text{[cf. (15)]}\\
&= \Big(\prod_{j=1}^{L}\frac{1}{1-\frac{a(p_j)}{p_j^{\sigma+it}}}\Big)\prod_{j=1}^{L}\Big(1-\Big(\frac{a(p_j)}{p_j^{\sigma+it}}\Big)^{L+1}\Big).
\end{aligned}$$

By (1),

$$\prod_{j=1}^{L}\Big(1-\Big(\frac{a(p_j)}{p_j^{\sigma+it}}\Big)^{L+1}\Big) = \exp\Big\{\sum_{j=1}^{L}\Big(-\Big(\frac{a(p_j)}{p_j^{\sigma+it}}\Big)^{L+1}\Big)\int_0^1\frac{1}{1-(\frac{a(p_j)}{p_j^{\sigma+it}})^{L+1}s}ds\Big\}.$$

Since

$$\begin{aligned}
&\Big|\sum_{j=1}^{L}\Big(-\Big(\frac{a(p_j)}{p_j^{\sigma+it}}\Big)^{L+1}\Big)\int_0^1\frac{1}{1-(\frac{a(p_j)}{p_j^{\sigma+it}})^{L+1}s}ds\Big|\\
&\le L\Big(\frac{1}{2^{\sigma-1}}\Big)^{L+1}\frac{1}{1-(\frac{1}{2^{\sigma-1}})^{L+1}}\quad\text{[cf. (15)]}\to 0\quad\text{as } L\to\infty,
\end{aligned}$$

we have

$$\lim_{L\to\infty}\prod_{j=1}^{L}\Big(1-\Big(\frac{a(p_j)}{p_j^{\sigma+it}}\Big)^{L+1}\Big) = 1,$$

which implies

$$\lim_{L\to\infty}\sum_{n\in\mathbb{N}_L}\frac{a(n)}{n^{\sigma+it}} = \prod_p\frac{1}{1-\frac{a(p)}{p^{\sigma+it}}}.$$

When $t = 0$, the monotone convergence theorem tells us that

$$\sum_{n=1}^{\infty}\frac{a(n)}{n^{\sigma}} = \sum_{n\in\mathbb{N}}\frac{a(n)}{n^{\sigma}} = \lim_{L\to\infty}\sum_{n\in\mathbb{N}_L}\frac{a(n)}{n^{\sigma}} = \prod_p\frac{1}{1-\frac{a(p)}{p^{\sigma}}} < \infty.$$

This shows the absolute convergence of $\sum_{n=1}^{\infty}\frac{a(n)}{n^{\sigma+it}}$, so that

$$\sum_{n=1}^{\infty}\frac{a(n)}{n^{\sigma+it}} = \sum_{n\in\mathbb{N}}\frac{a(n)}{n^{\sigma+it}} = \lim_{L\to\infty}\sum_{n\in\mathbb{N}_L}\frac{a(n)}{n^{\sigma+it}} = \prod_p\frac{1}{1-\frac{a(p)}{p^{\sigma+it}}}$$

is obtained. □

Claim 6. $-\dfrac{\eta'(\sigma;a)}{\eta(\sigma;a)}\sim\dfrac{\rho}{\sigma-1}$ *as* $\sigma\searrow 1$, *where* $\rho = \tau + \#\{p; a(p) = p\} > 0$ *(cf. (5)).*

Proof. It is divided into 5 steps.

1° Since, by Definition 3,

$$\log\eta(\sigma;a) = -\sum_p \log\Bigl(1-\frac{a(p)}{p^\sigma}\Bigr),$$

differentiating in σ yields that

$$\begin{aligned}
-\frac{\eta'(\sigma;a)}{\eta(\sigma;a)} &= \sum_p \frac{-\frac{a(p)}{p^\sigma}\log\frac{1}{p}}{1-\frac{a(p)}{p^\sigma}}\\
&= \sum_p \frac{a(p)}{p^\sigma}(\log p)\Biggl(1+\frac{\frac{a(p)}{p^\sigma}}{1-\frac{a(p)}{p^\sigma}}\Biggr)\\
&= \sum_p \frac{a(p)}{p^\sigma}\log p\\
&\quad + \sum_{p;a(p)=p}\frac{\log p}{p^{2(\sigma-1)}}\frac{1}{1-\frac{1}{p^{\sigma-1}}} + \sum_{p;a(p)<p}\frac{a(p)^2}{p^{2\sigma}}(\log p)\frac{1}{1-\frac{a(p)}{p^\sigma}}\\
&=: \text{the first term} + \text{the second term} + \text{the third term}.
\end{aligned}$$

2° Clearly

$$\begin{aligned}
(\sigma-1)\times\text{the second term} &= \sum_{p;a(p)=p}\frac{1}{p^{\sigma-1}}\frac{(\sigma-1)\log p}{e^{(\sigma-1)\log p}-1}\\
&\to \sum_{p;a(p)=p} 1 = \#\{p;a(p)=p\} \quad \text{as } \sigma\searrow 1.
\end{aligned}$$

3° By (11),

$$\begin{aligned}
\text{the first term} &= \sum_p \frac{1}{p^{\sigma-1}}\frac{a(p)\log p}{p}\\
&= \int_{(2-\varepsilon,\infty)}\frac{C(dx)}{x^{\sigma-1}} \quad [\text{where } 0<\varepsilon<1]\\
&= \tau\int_{2-\varepsilon}^\infty\frac{dx}{x^\sigma} + \int_{2-\varepsilon}^\infty\frac{\delta(x)}{x^\sigma}dx + \int_{(2-\varepsilon,\infty)}\frac{\log x}{x^{\sigma-1}}\delta(dx)\\
&= \tau\int_{2-\varepsilon}^\infty\frac{dx}{x^\sigma} + \int_{2-\varepsilon}^\infty\frac{\delta(x)}{x^\sigma}dx\\
&\quad + \int_{(2-\varepsilon,\infty)}\Bigl(d\Bigl(\frac{\log x}{x^{\sigma-1}}\delta(x)\Bigr)-\delta(x)d\bigl(x^{-\sigma+1}\log x\bigr)\Bigr)\\
&\qquad [\text{by integration by parts}]\\
&= \tau\Bigl(\int_{2-\varepsilon}^\infty\frac{dx}{x^\sigma}+\frac{\log(2-\varepsilon)}{(2-\varepsilon)^{\sigma-1}}\Bigr) + (\sigma-1)\int_{2-\varepsilon}^\infty\delta(x)\frac{\log x}{x^\sigma}dx.
\end{aligned}$$

Letting $\varepsilon\searrow 0$ yields that

$$\text{the first term} = \tau\Bigl(\int_2^\infty\frac{dx}{x^\sigma}+\frac{\log 2}{2^{\sigma-1}}\Bigr) + (\sigma-1)\int_2^\infty\delta(x)\frac{\log x}{x^\sigma}dx$$

$$= \frac{1}{\sigma-1}\Big\{\frac{\tau}{2^{\sigma-1}}(1+(\sigma-1)\log 2) + \int_{(\sigma-1)\log 2}^{\infty} \delta(e^{\frac{z}{\sigma-1}})e^{-z}z dz\Big\}$$

[by the change of variables $(\sigma-1)\log x = z$].

Since, by (13) and the Lebesgue convergence theorem,

$$\lim_{\sigma \searrow 1} \int_{(\sigma-1)\log 2}^{\infty} \delta(e^{\frac{z}{\sigma-1}})e^{-z}z dz = 0,$$

we have

$$(\sigma-1) \times \text{the first term} \to \tau \quad \text{as } \sigma \searrow 1.$$

4° Since, by (17),

$$\text{the third term} \le \Big(\sum_{p;a(p)<p} \frac{a(p)^2}{p^2}(\log p)\Big)\frac{1}{1-\sup_{q;a(q)<q}\frac{a(q)}{q}},$$

we have

$$(\sigma-1) \times \text{the third term} \to 0 \quad \text{as } \sigma \searrow 1.$$

5° By putting 1° ∼ 4° together,

$$(\sigma-1) \times \Big(-\frac{\eta'(\sigma;a)}{\eta(\sigma;a)}\Big) \to \tau + \#\{p; a(p) = p\} = \rho \quad \text{as } \sigma \searrow 1.$$

□

Claim 7. *$(-Z(\sigma;a))_{1<\sigma<\infty}$ is a backwards Lévy process whose marginal distribution at σ is $\mu_\sigma(\cdot;a)$.*

Proof. It is divided into 3 steps.

1° Since, for $\sigma > 1$,

$$\sum_p P\Big(Y_p\Big(\frac{a(p)}{p^\sigma}\Big) > 0\Big) = \sum_p \frac{a(p)}{p^\sigma} < \infty,$$

Borel-Cantelli's first lemma tells us that

$$P\Big({}^\exists p_0\text{: prime s.t. } Y_p\Big(\frac{a(p)}{p^\sigma}\Big) = 0, \ {}^\forall p > p_0\Big) = 1.$$

This implies that

$$\sum_p Y_p\Big(\frac{a(p)}{p^\sigma}\Big)\log p \text{ is a finite sum a.s.}$$

Thus $Z(\sigma;a)$ is well-defined.

2° Since, for each prime p, $(1,\infty) \ni \sigma \mapsto Y_p(\frac{a(p)}{p^\sigma}) \in \{0,1,2,\ldots\}$ is left-continuous and non-increasing, so is $(1,\infty) \ni \sigma \mapsto Z(\sigma;a) \in [0,\infty)$.

Since $\frac{a(p)}{p^{\sigma'}} \nearrow \frac{a(p)}{p^\sigma}$ as $\sigma' \searrow \sigma$ (>1) $({}^\forall p; a(p) > 0)$,

$$Y_p\Big(\frac{a(p)}{p^{\sigma'}}\Big) \nearrow Y_p\Big(\frac{a(p)}{p^\sigma}-\Big) = Y_p\Big(\frac{a(p)}{p^\sigma}\Big) \quad \text{a.s. } ({}^\forall p; a(p) > 0).$$

By the monotone convergence theorem,

$$Z(\sigma';a) \nearrow \sum_p Y_p\Big(\frac{a(p)}{p^\sigma}\Big)\log p \quad \text{a.s.}$$

Thus, in the case where $\sigma > 1$, $Z(\sigma+;a) = Z(\sigma;a)$ a.s.

For $\sigma' > 1$,

$$\begin{aligned}
Z(\sigma';a) &= \sum_p Y_p\Big(\frac{a(p)}{p^{\sigma'}}\Big)\log p \\
&= \sum_{p;a(p)=p} Y_p\Big(\frac{1}{p^{\sigma'-1}}\Big)\log p + \sum_{p;a(p)<p} Y_p\Big(\frac{a(p)}{p^{\sigma'}}\Big)\log p \\
&=: \text{the first term} + \text{the second term}.
\end{aligned}$$

In the same way as above,

$$\lim_{\sigma'\searrow 1} \text{the second term} = \sum_{p;a(p)<p} Y_p\Big(\frac{a(p)}{p}\Big)\log p \quad \text{a.s.}$$

When $\tau > 0$, Claim 4(i) tells us that

$$\sum_{p;a(p)<p} P\Big(Y_p\Big(\frac{a(p)}{p}\Big) > 0\Big) = \sum_{p;a(p)<p} \frac{a(p)}{p} = \infty.$$

By the independence of $\{Y_p\}_p$ and Borel-Cantelli's second lemma,

$$\lim_{\sigma'\searrow 1} \text{the second term} = \infty \quad \text{a.s.}$$

When $\tau = 0$, $\{p; a(p) = p\} \neq \emptyset$. By noting that $Y_p(1-) = \infty$ a.s.,

$$\lim_{\sigma'\searrow 1} \text{the first term} = \infty \quad \text{a.s.}$$

Thus $Z(1+;a) = \infty$ a.s.

As $\sigma \nearrow \infty$, $\frac{a(p)}{p^\sigma} \searrow 0$, and hence $Y_p(\frac{a(p)}{p^\sigma}) \searrow Y_p(0) = 0$. By the Lebesgue convergence theorem, $Z(\infty;a) = 0$.

From Definition 1(c) and the independence of $\{Y_p\}_p$, it follows that for every $\infty > \sigma_0 > \sigma_1 > \cdots > \sigma_n > 1$,

$$Z(\sigma_0;a), Z(\sigma_1;a) - Z(\sigma_0;a), \dots, Z(\sigma_n;a) - Z(\sigma_{n-1};a) \text{ are independent.}$$

3° For $t \in \mathbb{R}$,

$$\begin{aligned}
E[e^{it(-Z(\sigma;a))}] &= \prod_p E\Big[e^{it(\log\frac{1}{p})Y_p\left(\frac{a(p)}{p^\sigma}\right)}\Big] \\
&= \prod_p \frac{1-\frac{a(p)}{p^\sigma}}{1-\frac{a(p)}{p^{\sigma+it}}} \\
&= \frac{\eta(\sigma+it;a)}{\eta(\sigma;a)} = \eta_\sigma(t;a) = \int_{\mathbb{R}} e^{itx}\mu_\sigma(dx;a).
\end{aligned}$$

□

2.4. Proof and corollary of Theorem 1. We are now in position to prove Theorem 1.

Proof of Theorem 1. Let us fix the $\varphi(\cdot)$ in Theorem 1. The proof is divided into 2 steps.

1° Fix $T > 0$. By Claim 7, $(X_T(t; a) := \frac{1}{\sqrt{T}} Z(\varphi^{-1}(Tt); a))_{t \geq 0}$ is, in the usual sense, a Lévy process with increasing paths (cf. Itô [6]). Here we set $X_T(0; a) := 0$ by $\varphi^{-1}(0) = \varphi^{-1}(0+) = \infty$ and $Z(\infty; a) = 0$.

Let $N_T(dsdu)$ be a Poisson random measure on $(0, \infty) \times (0, \infty)$ defined by $X_T(\cdot; a)$:

$$N_T(A) := \#\Big\{t > 0; (t, X_T(t; a) - X_T(t-; a)) \in A\Big\}, \quad A \in \mathcal{B}((0, \infty) \times (0, \infty)).$$

Then, the Lévy-Itô decomposition of $X_T(\cdot; a)$ is given as

$$X_T(t; a) = \int_0^{t+} \int_{(0,\infty)} u N_T(dsdu), \quad t \geq 0.$$

And, a mean measure $n_T(dsdu)$ of $N_T(dsdu)$ is given as

$$n_T(dsdu) = \sum_p \sum_{n=1}^{\infty} (a(p) p^{-\varphi^{-1}(Ts)})^n (\log p) \frac{-T}{\varphi'(\varphi^{-1}(Ts))} ds\, \delta_{\frac{1}{\sqrt{T}} n \log p}(du). \tag{19}$$

Proof. Temporarily let $n'_T(dsdu)$ be a right-hand side of (19). Clearly $n'_T(dsdu)$ is a measure on $(0, \infty) \times (0, \infty)$, and

$$\begin{aligned}
\iint_{\substack{0<s\leq t\\ u>0}} u n'_T(dsdu) &= \sum_p \sum_{n=1}^{\infty} \int_0^t (a(p) p^{-\varphi^{-1}(Ts)})^n (\log p) \frac{-T}{\varphi'(\varphi^{-1}(Ts))} \frac{1}{\sqrt{T}} n (\log p)\, ds \\
&= \frac{1}{\sqrt{T}} \sum_p \sum_{n=1}^{\infty} \int_{\varphi^{-1}(Tt)}^{\infty} (a(p) p^{-r})^n n (\log p)^2 dr \\
&\qquad [\text{by the change of variables } \varphi^{-1}(Ts) = r] \\
&= \frac{1}{\sqrt{T}} \sum_p \Big(\sum_{n=1}^{\infty} (a(p) p^{-\varphi^{-1}(Tt)})^n \Big) \log p \\
&= \frac{1}{\sqrt{T}} \sum_p \frac{\frac{a(p)}{p^{\varphi^{-1}(Tt)}} \log p}{1 - \frac{a(p)}{p^{\varphi^{-1}(Tt)}}} = \frac{1}{\sqrt{T}} \left(- \frac{\eta'(\sigma; a)}{\eta(\sigma; a)} \right) \Bigg|_{\sigma = \varphi^{-1}(Tt)} < \infty.
\end{aligned}$$

By this and $\iint_{\substack{0<s\leq t\\ u>0}} (u \wedge 1) n_T(dsdu) < \infty$, it suffices to check that for t and $\lambda > 0$,

$$E[e^{-\lambda X_T(t;a)}] = \exp\Big\{ - \iint_{\substack{0<s\leq t\\ u>0}} (1 - e^{-\lambda u}) n'_T(dsdu) \Big\}. \tag{20}$$

For, let

$$X_T(t; a) = m(t) + \int_0^{t+} \int_{(0,\infty)} u N_T(dsdu)$$

be the Lévy-Itô decomposition of $X_T(\cdot; a)$, where $m(\cdot)$ is a deterministic, continuous and non-decreasing process with $m(0) = 0$. Then

$$E[e^{-\lambda X_T(t;a)}] = \exp\Big\{-\lambda m(t) - \iint_{\substack{0<s\le t\\ u>0}} (1-e^{-\lambda u}) n_T(dsdu)\Big\}.$$

If, moreover, (20) holds, then

$$\begin{aligned} m(t) &= \iint_{\substack{0<s\le t\\ u>0}} \frac{1-e^{-\lambda u}}{\lambda} n'_T(dsdu) - \iint_{\substack{0<s\le t\\ u>0}} \frac{1-e^{-\lambda u}}{\lambda} n_T(dsdu) \\ &\to 0 \quad \text{as } \lambda\to\infty. \end{aligned}$$

This convergence immediately follows from the Lebesgue convergence theorem since $0 < \frac{1-e^{-\lambda u}}{\lambda} \le \frac{1}{\lambda}((\lambda u)\wedge 1) = u \wedge (\frac{1}{\lambda}) \le u\wedge 1$ ($^\forall \lambda \ge 1$), $\iint_{\substack{0<s\le t\\ u>0}} (u\wedge 1) n'_T(dsdu) < \infty$, $\iint_{\substack{0<s\le t\\ u>0}} (u \wedge 1)\, n_T(dsdu) < \infty$ and $\lim_{\lambda\to\infty} \frac{1-e^{-\lambda u}}{\lambda} = 0$ ($^\forall u > 0$). This implies that

$$X_T(t;a) = \int_0^{t+}\int_{(0,\infty)} uN_T(dsdu), \quad n_T(dsdu) = n'_T(dsdu).$$

(20) is shown in the following way: By Definition 5, Definition 1(d) and (1),

$$\begin{aligned} E[e^{-\lambda X_T(t;a)}] &= \prod_p E\Big[e^{-\frac{\lambda}{\sqrt{T}}(\log p) Y_p\left(a(p)p^{-\varphi^{-1}(Tt)}\right)}\Big] \\ &= \prod_p \frac{1-a(p)p^{-\varphi^{-1}(Tt)}}{1-a(p)p^{-\varphi^{-1}(Tt)}p^{-\frac{\lambda}{\sqrt{T}}}} \\ &= \prod_p \exp\Big\{\int_0^1 \frac{-a(p)p^{-\varphi^{-1}(Tt)}}{1-a(p)p^{-\varphi^{-1}(Tt)}s} ds - \int_0^1 \frac{-a(p)p^{-\varphi^{-1}(Tt)}p^{-\frac{\lambda}{\sqrt{T}}}}{1-a(p)p^{-\varphi^{-1}(Tt)}p^{-\frac{\lambda}{\sqrt{T}}}s} ds\Big\} \\ &= \prod_p \exp\Big\{\int_0^t \frac{-a(p)p^{-\varphi^{-1}(Tt)}}{1-a(p)p^{-\varphi^{-1}(Tr)}} p^{\varphi^{-1}(Tt)} p^{-\varphi^{-1}(Tr)} (-\log p) \frac{Tdr}{\varphi'(\varphi^{-1}(Tr))} \\ &\qquad + \int_0^t \frac{a(p)p^{-\varphi^{-1}(Tt)}p^{-\frac{\lambda}{\sqrt{T}}}}{1-a(p)p^{-\varphi^{-1}(Tr)}p^{-\frac{\lambda}{\sqrt{T}}}} p^{\varphi^{-1}(Tt)} p^{-\varphi^{-1}(Tr)} (-\log p) \frac{Tdr}{\varphi'(\varphi^{-1}(Tr))}\Big\} \\ &\qquad \Big[\text{by the change of variables } r = \tfrac{1}{T}\varphi\big(\varphi^{-1}(Tt) + \tfrac{\log\frac{1}{s}}{\log p}\big)\Big] \\ &= \prod_p \exp\Big\{\int_0^t \sum_{n=1}^\infty (a(p)p^{-\varphi^{-1}(Tr)})^n (-1 + e^{-\lambda\frac{n}{\sqrt{T}}\log p})(\log p) \frac{-T}{\varphi'(\varphi^{-1}(Tr))} dr\Big\} \\ &= \exp\Big\{-\iint_{\substack{0<s\le t\\ u>0}} (1-e^{-\lambda u}) n'_T(dsdu)\Big\}. \end{aligned}$$

□

2° By Claims 8 and 9 below,

$$n_T(dsdu) \to n^{(\rho)}(dsdu) \quad \text{vaguely} \quad \text{as } T\to\infty,$$

$$\lim_{k\to\infty}\limsup_{T\to\infty} \iint_{\substack{0<s\le t\\ u\in(0,\infty)\setminus[\frac{1}{k},k]}} u n_T(dsdu) = 0, \quad ^\forall t > 0.$$

Applying the general theory of Kasahara-Watanabe [7], we have

$$\left(\int_0^{t+}\int_{(0,\infty)} uN_T(dsdu)\right)_{t\geq 0} \xrightarrow{\mathcal{D}} \left(\int_0^{t+}\int_{(0,\infty)} uN^{(\rho)}(dsdu)\right)_{t\geq 0}$$
$$\text{in } D([0,\infty)\to\mathbb{R}) \quad \text{as } T\to\infty,$$

which is the assertion of the theorem. □

As for Claims 8 and 9, we begin with the following lemma:

Lemma 1. *For $\lambda \geq 0$ and $t > 0$,*

$$\lim_{T\to\infty} \iint_{\substack{0<s\leq t\\ u>0}} e^{-\lambda u} un_T(dsdu) = \frac{\rho}{\lambda + \frac{1}{\sqrt{t}}} = \iint_{\substack{0<s\leq t\\ u>0}} e^{-\lambda u} un^{(\rho)}(dsdu).$$

Proof. Fix $\lambda \geq 0$ and $t > 0$. By (19),

$$\begin{aligned}
&\iint_{\substack{0<s\leq t\\ u>0}} e^{-\lambda u} un_T(dsdu)\\
&= \sum_p \sum_{n=1}^{\infty} \int_0^t e^{-\lambda \frac{1}{\sqrt{T}} n\log p} \frac{1}{\sqrt{T}} n(\log p)\big(a(p)p^{-\varphi^{-1}(Ts)}\big)^n (\log p) \frac{-T}{\varphi'(\varphi^{-1}(Ts))} ds\\
&= \frac{1}{\sqrt{T}} \sum_p (\log p) \sum_{n=1}^{\infty} \big(a(p)p^{-\varphi^{-1}(Tt)-\frac{\lambda}{\sqrt{T}}}\big)^n\\
&= \frac{1}{\sqrt{T}} \sum_p \frac{\frac{a(p)}{p^{\varphi^{-1}(Tt)+\frac{\lambda}{\sqrt{T}}}}\log p}{1-\frac{a(p)}{p^{\varphi^{-1}(Tt)+\frac{\lambda}{\sqrt{T}}}}} = \frac{1}{\sqrt{T}}\left(-\frac{\eta'(\sigma;a)}{\eta(\sigma;a)}\right)\Bigg|_{\sigma=\varphi^{-1}(Tt)+\frac{\lambda}{\sqrt{T}}}.
\end{aligned}$$

Since $\varphi^{-1}(Tt) \searrow 1$ as $T\to\infty$, and thus $\sigma = \varphi^{-1}(Tt) + \frac{\lambda}{\sqrt{T}} \searrow 1$, Claim 6 tells us that

$$(\sigma-1)\left(-\frac{\eta'(\sigma;a)}{\eta(\sigma;a)}\right) \to \rho.$$

On the other hand, by (8),

$$\varphi^{-1}(Tt) - 1 \sim \frac{1}{\sqrt{\varphi(\varphi^{-1}(Tt))}} = \frac{1}{\sqrt{T}\sqrt{t}},$$

and thus

$$\sqrt{T}(\sigma-1) = \sqrt{T}\Big(\varphi^{-1}(Tt) + \frac{\lambda}{\sqrt{T}} - 1\Big) = \lambda + \sqrt{T}(\varphi^{-1}(Tt)-1) \to \lambda + \frac{1}{\sqrt{t}}.$$

Combining two convergences above, we have

$$\frac{1}{\sqrt{T}}\left(-\frac{\eta'(\sigma;a)}{\eta(\sigma;a)}\right) = \frac{1}{\sqrt{T}(\sigma-1)}\cdot(\sigma-1)\left(-\frac{\eta'(\sigma;a)}{\eta(\sigma;a)}\right) \to \frac{\rho}{\lambda+\frac{1}{\sqrt{t}}}.$$

Next, by (9),

$$\iint_{\substack{0<s\le t\\ u>0}} e^{-\lambda u} u n^{(\rho)}(dsdu) = \frac{\rho}{2}\int_0^t s^{-\frac{3}{2}} ds \int_0^\infty u e^{-\left(\frac{1}{\sqrt{s}}+\lambda\right)u} du$$

$$= \frac{\rho}{2}\int_0^t s^{-\frac{3}{2}} \frac{ds}{(\frac{1}{\sqrt{s}}+\lambda)^2}$$

$$\left[\text{since } \int_0^\infty u e^{-\mu u} du = \frac{1}{\mu^2} \ (\mu > 0)\right]$$

$$= \frac{\rho}{\lambda + \frac{1}{\sqrt{t}}}.$$

□

Claim 8. *As $T \to \infty$, $n_T(dsdu) \to n^{(\rho)}(dsdu)$ vaguely. That is, for $^\forall j \in C_c([0,\infty)\times(0,\infty))$,*

$$\iint_{(0,\infty)\times(0,\infty)} j(s,u) n_T(dsdu) \to \iint_{(0,\infty)\times(0,\infty)} j(s,u) n^{(\rho)}(dsdu).$$

Here $C_c([0,\infty)\times(0,\infty))$ is the set of all real-valued continuous functions on $[0,\infty)\times(0,\infty)$ *with compact support.*

Proof. It is divided into 6 steps.

1° For $\lambda \ge 0$, let $f_\lambda(u) = e^{-\lambda u}$. Then

$$f_\lambda \in C_\infty([0,\infty)) \ \text{ if } \ \lambda > 0, \quad f_0 = 1 \in C_b([0,\infty)).$$

Here

$C_b([0,\infty))$ = the set of all real-valued, bounded, continuous functions on $[0,\infty)$,

$C_\infty([0,\infty)) = \{f \in C_b([0,\infty)); \lim_{u\to\infty} f(u) = 0\}$,

$C_c([0,\infty)) = \{f \in C_b([0,\infty)); \operatorname{supp} f \text{ is compact}\}$.

Let $[0,\infty]$ be the one-point compactification of $[0,\infty)$. If, at point ∞, we define

$$f_\lambda(\infty) := \begin{cases} 0 & \text{if } \lambda > 0, \\ 1 & \text{if } \lambda = 0, \end{cases}$$

then $f_\lambda \in C([0,\infty])$[2]. Letting $\mathcal{A} \subset C([0,\infty])$ be the set of all linear combinations of f_λ, $\lambda \ge 0$, we can check that

- $\mathcal{A}$ is an algebra,
- $\mathcal{A}$ separates points on $[0,\infty]$,
- $\mathcal{A}$ vanishes at no point of $[0,\infty]$.

Thus, by the Stone-Weierstrass theorem (cf. [9, Theorem 7.32]), $\overline{\mathcal{A}} = C([0,\infty])$. Particularly, for $^\forall f \in C_c([0,\infty))$ and $^\forall\varepsilon > 0$,

$$^\exists g \in \mathcal{A} \ \text{ s.t. } \sup_{0\le u<\infty} |f(u) - g(u)| < \varepsilon.$$

[2] The extension of f_λ to $[0,\infty]$ is denoted by the same symbol f_λ.

2° For $^\forall t > 0$ and $^\forall f \in C_c([0,\infty))$,

$$\lim_{T\to\infty} \iint_{\substack{0<s\le t\\ u>0}} f(u) u n_T(dsdu) = \iint_{\substack{0<s\le t\\ u>0}} f(u) u n^{(\rho)}(dsdu).$$

Proof. Fix $t > 0$ and $f \in C_c([0,\infty))$. By 1°, f can be approximated by a sequence $\{g_k\}$ of $\mathcal{A}$. Since, by Lemma 1,

$$\lim_{T\to\infty} \iint_{\substack{0<s\le t\\ u>0}} g_k(u) u n_T(dsdu) = \iint_{\substack{0<s\le t\\ u>0}} g_k(u) u n^{(\rho)}(dsdu),$$

it follows in a routine way that

$$\lim_{T\to\infty} \iint_{\substack{0<s\le t\\ u>0}} f(u) u n_T(dsdu) = \iint_{\substack{0<s\le t\\ u>0}} f(u) u n^{(\rho)}(dsdu).$$

□

3° For $^\forall t > 0$,

$$\lim_{\lambda\searrow 0} \limsup_{T\to\infty} \iint_{\substack{0<s\le t\\ u\ge \frac{1}{\lambda}}} u n_T(dsdu) = 0.$$

Proof. Fix $t > 0$. Noting that for $\lambda > 0$ and $u > 0$,

$$1 - e^{-\lambda u} = \int_0^u (-e^{-\lambda v})' dv = \int_{(0,\infty)} \mathbf{1}_{v\le u} \lambda e^{-\lambda v} dv,$$

we obtain the following lower estimate:

$$\begin{aligned}
&\iint_{\substack{0<s\le t\\ u>0}} u n_T(dsdu) - \iint_{\substack{0<s\le t\\ u>0}} e^{-\lambda u} u n_T(dsdu)\\
&= \iint_{\substack{0<s\le t\\ u>0}} (1 - e^{-\lambda u}) u n_T(dsdu)\\
&= \iint_{\substack{0<s\le t\\ u>0}} u n_T(dsdu) \int_{(0,\infty)} \mathbf{1}_{v\le u} \lambda e^{-\lambda v} dv\\
&= \int_{(0,\infty)} \lambda e^{-\lambda v} dv \iint_{\substack{0<s\le t\\ u\ge v}} u n_T(dsdu)\\
&= \int_{(0,\infty)} e^{-w} dw \iint_{\substack{0<s\le t\\ u\ge \frac{w}{\lambda}}} u n_T(dsdu) \quad \text{[by the change of variables } \lambda v = w\text{]}\\
&\ge \int_0^1 e^{-w} dw \iint_{\substack{0<s\le t\\ u\ge \frac{1}{\lambda}}} u n_T(dsdu) \quad \text{[since } [\tfrac{w}{\lambda},\infty) \supset [\tfrac{1}{\lambda},\infty) \text{ for } 0 < w \le 1\text{]}
\end{aligned}$$

$$= (1 - e^{-1}) \iint_{\substack{0<s\le t \\ u\ge\frac{1}{\lambda}}} un_T(dsdu).$$

By Lemma 1,

$$\limsup_{T\to\infty} \iint_{\substack{0<s\le t \\ u\ge\frac{1}{\lambda}}} un_T(dsdu) \le \frac{1}{1-e^{-1}} \lim_{T\to\infty}\left(\iint_{\substack{0<s\le t \\ u>0}} un_T(dsdu) - \iint_{\substack{0<s\le t \\ u>0}} e^{-\lambda u} un_T(dsdu)\right)$$

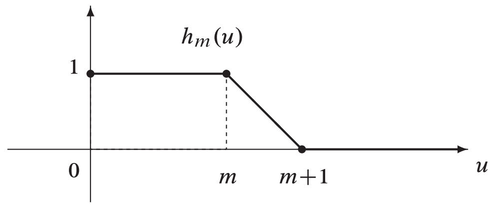
$$= \frac{\rho}{1-e^{-1}} \frac{\lambda t}{1+\lambda\sqrt{t}} \to 0 \quad \text{as } \lambda \searrow 0.$$

□

4° For ${}^\forall t > 0$ and ${}^\forall f \in C_{\mathrm{b}}([0,\infty))$,

$$\lim_{T\to\infty} \iint_{\substack{0<s\le t \\ u>0}} f(u) un_T(dsdu) = \iint_{\substack{0<s\le t \\ u>0}} f(u) un^{(\rho)}(dsdu).$$

Proof. Fix $t > 0$ and $f \in C_{\mathrm{b}}([0,\infty))$. For each $m \in \mathbb{N}$, set $h_m \in C_{\mathrm{c}}([0,\infty))$ by



Note that $f \cdot h_m \in C_{\mathrm{c}}([0,\infty))$ and

(21) $$\left| f(u) - (f \cdot h_m)(u) \right| = |f(u)|(1 - h_m(u)) \le \|f\|_\infty \mathbf{1}_{[m,\infty)}(u).$$

By 2°,

$$\lim_{T\to\infty} \iint_{\substack{0<s\le t \\ u>0}} (f \cdot h_m)(u) un_T(dsdu) = \iint_{\substack{0<s\le t \\ u>0}} (f \cdot h_m)(u) un^{(\rho)}(dsdu).$$

Also, by (21) and 3°,

$$\limsup_{T\to\infty} \iint_{\substack{0<s\le t \\ u>0}} \left|(f \cdot h_m)(u) - f(u)\right| un_T(dsdu) \to 0 \quad \text{as } m \to \infty.$$

Thus, it follows in the usual way that

$$\lim_{T\to\infty} \iint_{\substack{0<s\le t \\ u>0}} f(u) un_T(dsdu) = \iint_{\substack{0<s\le t \\ u>0}} f(u) un^{(\rho)}(dsdu).$$

□

5° For ${}^\forall h \in C_{\mathrm{c}}([0,\infty) \times [0,\infty))$,

$$\lim_{T\to\infty} \iint_{(0,\infty)\times(0,\infty)} h(s,u) un_T(dsdu) = \iint_{(0,\infty)\times(0,\infty)} h(s,u) un^{(\rho)}(dsdu).$$

Proof. Fix $h \in C_c([0,\infty)\times[0,\infty))$. Since h is uniformly continuous on $[0,\infty)\times[0,\infty)$,

$$\forall\varepsilon > 0,\ \exists\delta > 0 \ \text{ s.t. } \ |s-s'|<\delta,\ |u-u'|<\delta \Rightarrow \big|h(s,u)-h(s',u')\big| < \varepsilon. \tag{22}$$

Also, since $\operatorname{supp} h$ is compact,

$$\exists t > 0 \ \text{ s.t. } \ \operatorname{supp} h \subset [0,t]\times[0,t].$$

Take a large $n \in \mathbb{N}$ such that $\frac{t}{n} < \delta$, and rewrite

$$\begin{aligned}
\iint_{(0,\infty)\times(0,\infty)} h(s,u)un_T(dsdu) &= \iint_{(0,t]\times(0,t]} h(s,u)un_T(dsdu)\\
&= \sum_{k=1}^{n} \iint_{\substack{\frac{k-1}{n}t<s\le\frac{k}{n}t\\ u>0}} h\Big(\frac{k-1}{n}t,u\Big)un_T(dsdu)\\
&\quad+ \sum_{k=1}^{n} \iint_{(\frac{k-1}{n}t,\frac{k}{n}t]\times(0,t]} \Big(h(s,u)-h\Big(\frac{k-1}{n}t,u\Big)\Big)un_T(dsdu),
\end{aligned}$$

$$\begin{aligned}
\iint_{(0,\infty)\times(0,\infty)} h(s,u)un^{(\rho)}(dsdu) &= \sum_{k=1}^{n} \iint_{\substack{\frac{k-1}{n}t<s\le\frac{k}{n}t\\ u>0}} h\Big(\frac{k-1}{n}t,u\Big)un^{(\rho)}(dsdu)\\
&\quad+ \sum_{k=1}^{n} \iint_{(\frac{k-1}{n}t,\frac{k}{n}t]\times(0,t]} \Big(h(s,u)-h\Big(\frac{k-1}{n}t,u\Big)\Big)un^{(\rho)}(dsdu).
\end{aligned}$$

Then, by (22),

$$\begin{aligned}
&\Big|\iint_{(0,\infty)\times(0,\infty)} h(s,u)un_T(dsdu) - \iint_{(0,\infty)\times(0,\infty)} h(s,u)un^{(\rho)}(dsdu)\Big|\\
&\le \Big|\sum_{k=1}^{n}\Big(\iint_{\substack{0<s\le\frac{k}{n}t\\ u>0}} h\Big(\frac{k-1}{n}t,u\Big)un_T(dsdu) - \iint_{\substack{0<s\le\frac{k}{n}t\\ u>0}} h\Big(\frac{k-1}{n}t,u\Big)un^{(\rho)}(dsdu)\\
&\qquad -\Big(\iint_{\substack{0<s\le\frac{k-1}{n}t\\ u>0}} h\Big(\frac{k-1}{n}t,u\Big)un_T(dsdu) - \iint_{\substack{0<s\le\frac{k-1}{n}t\\ u>0}} h\Big(\frac{k-1}{n}t,u\Big)un^{(\rho)}(dsdu)\Big)\Big)\Big|\\
&\quad+\sum_{k=1}^{n}\Big(\iint_{(\frac{k-1}{n}t,\frac{k}{n}t]\times(0,t]} \Big|h(s,u)-h\Big(\frac{k-1}{n}t,u\Big)\Big|un_T(dsdu)\\
&\qquad+\iint_{(\frac{k-1}{n}t,\frac{k}{n}t]\times(0,t]} \Big|h(s,u)-h\Big(\frac{k-1}{n}t,u\Big)\Big|un^{(\rho)}(dsdu)\Big)\\
&\le \Big|\sum_{k=1}^{n}\Big(\iint_{\substack{0<s\le\frac{k}{n}t\\ u>0}} h\Big(\frac{k-1}{n}t,u\Big)un_T(dsdu) - \iint_{\substack{0<s\le\frac{k}{n}t\\ u>0}} h\Big(\frac{k-1}{n}t,u\Big)un^{(\rho)}(dsdu)
\end{aligned}$$

$$-\Bigg(\iint_{\substack{0<s\le\frac{k-1}{n}t\\ u>0}} h\Big(\frac{k-1}{n}t,u\Big)un_T(dsdu)-\iint_{\substack{0<s\le\frac{k-1}{n}t\\ u>0}} h\Big(\frac{k-1}{n}t,u\Big)un^{(\rho)}(dsdu)\Bigg)\Bigg|$$

$$+\,\varepsilon\Bigg(\iint_{\substack{0<s\le t\\ u>0}} un_T(dsdu)+\iint_{\substack{0<s\le t\\ u>0}} un^{(\rho)}(dsdu)\Bigg)$$

=: the first term + the second term.

By 4°, $\lim_{T\to\infty}$ the first term $=0$ since $h(\frac{k-1}{n}t,\cdot)\in C_{\mathrm{b}}([0,\infty))$, and by Lemma 1, $\lim_{T\to\infty}$ the second term $=\varepsilon\frac{2\rho}{1/\sqrt{t}}\to 0$ as $\varepsilon\searrow 0$. Thus we have the assertion of 5°. □

6° For $^\forall j\in C_{\mathrm{c}}([0,\infty)\times(0,\infty))$, set

$$h(s,u):=\begin{cases}\dfrac{1}{u}j(s,u) & \text{if } u>0,\\ 0 & \text{if } u=0.\end{cases}$$

Then $h\in C_{\mathrm{c}}([0,\infty)\times[0,\infty))$ and $j(s,u)=h(s,u)u$. By 5°,

$$\begin{aligned}\lim_{T\to\infty}\iint_{(0,\infty)\times(0,\infty)} j(s,u)n_T(dsdu)&=\lim_{T\to\infty}\iint_{(0,\infty)\times(0,\infty)} h(s,u)un_T(dsdu)\\ &=\iint_{(0,\infty)\times(0,\infty)} h(s,u)un^{(\rho)}(dsdu)\\ &=\iint_{(0,\infty)\times(0,\infty)} j(s,u)n^{(\rho)}(dsdu).\end{aligned}$$

□

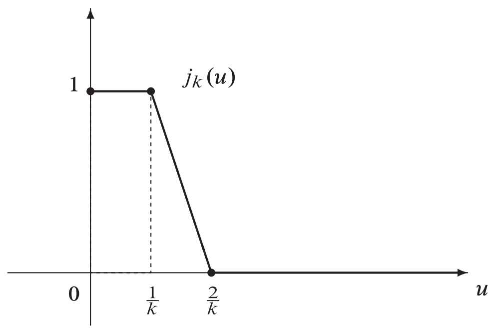
Claim 9. *For $^\forall t>0$,*

$$\lim_{k\to\infty}\limsup_{T\to\infty}\iint_{\substack{0<s\le t\\ u\in(0,\infty)\setminus[\frac{1}{k},k]}} un_T(dsdu)=0.$$

Proof. Fix $t>0$. First

$$\begin{aligned}\iint_{\substack{0<s\le t\\ u\in(0,\infty)\setminus[\frac{1}{k},k]}} un_T(dsdu)&=\iint_{\substack{0<s\le t\\ 0<u<\frac{1}{k}}} un_T(dsdu)+\iint_{\substack{0<s\le t\\ u>k}} un_T(dsdu)\\ &\le\iint_{\substack{0<s\le t\\ u>0}} j_k(u)un_T(dsdu)+\iint_{\substack{0<s\le t\\ u\ge\frac{1}{\frac{1}{k}}}} un_T(dsdu).\end{aligned}$$

Here $j_k\in C_{\mathrm{c}}([0,\infty))$ is as follows:



By 2° and 3° in the proof of Claim 8,

$$\limsup_{T\to\infty} \iint_{\substack{0<s\le t \\ u\in(0,\infty)\setminus[\frac{1}{k},k]}} un_T(dsdu)$$

$$\le \iint_{\substack{0<s\le t \\ u>0}} j_k(u)un^{(\rho)}(dsdu) + \limsup_{T\to\infty} \iint_{\substack{0<s\le t \\ u\ge \frac{1}{\frac{1}{k}}}} un_T(dsdu) \to 0 \quad \text{as } k\to\infty.$$

□

Corollary 1. *For each $t>0$,*

$$\text{the distribution of } \frac{1}{\sqrt{T}} Z(\varphi^{-1}(Tt);a) \to \mathbf{1}_{(0,\infty)}(x)\frac{1}{\Gamma(\rho)}(\sqrt{t})^{-\rho}x^{\rho-1}e^{-\frac{x}{\sqrt{t}}}dx \quad \textit{as } T\to\infty.$$

(The limiting distribution is the gamma distribution with parameters ρ, $\sqrt{t}$.) In particular, letting $t=1$ and $\sigma=\varphi^{-1}(T)$ and then noting that

$$T\to\infty \Leftrightarrow \sigma\searrow 1, \quad (\sigma-1)^2\varphi(\sigma)\to 1 \quad \textit{as } \sigma\searrow 1$$

tell us that

$$\text{the distribution of } (\sigma-1)Z(\sigma;a) \to \mathbf{1}_{(0,\infty)}(x)\frac{1}{\Gamma(\rho)}x^{\rho-1}e^{-x}dx \quad \textit{as } \sigma\searrow 1.$$

Proof. Fix $\lambda>0$ and $t>0$. By Theorem 1,

$$\lim_{T\to\infty} E[e^{-\lambda\frac{1}{\sqrt{T}}Z(\varphi^{-1}(Tt);a)}] = E\Big[e^{-\lambda\int_0^{t+}\int_{(0,\infty)} uN^{(\rho)}(dsdu)}\Big]$$

$$= \exp\Big\{-\iint_{\substack{0<s\le t \\ u>0}} (1-e^{-\lambda u})n^{(\rho)}(dsdu)\Big\}.$$

Here, by Lemma 1,

$$\iint_{\substack{0<s\le t \\ u>0}} (1-e^{-\lambda u})n^{(\rho)}(dsdu) = \iint_{\substack{0<s\le t \\ u>0}} \int_0^1 e^{-a\lambda u}\lambda u da\, n^{(\rho)}(dsdu)$$

$$= \lambda\int_0^1 \frac{\rho}{a\lambda+\frac{1}{\sqrt{t}}}da = \rho\log(1+\sqrt{t}\lambda).$$

Substituting this into the last right-hand side of the preceding expression, we have

$$\begin{aligned}
\lim_{T\to\infty} E[e^{-\lambda\frac{1}{\sqrt{T}}Z(\varphi^{-1}(Tt);a)}] &= \exp\{-\rho\log(1+\sqrt{t}\lambda)\} \\
&= (1+\sqrt{t}\lambda)^{-\rho} \\
&= \frac{1}{\Gamma(\rho)}\int_0^\infty e^{-(1+\sqrt{t}\lambda)y}y^{\rho-1}dy \\
&\qquad \left[\text{since } \mu^{-\rho} = \tfrac{1}{\Gamma(\rho)}\textstyle\int_0^\infty e^{-\mu y}y^{\rho-1}dy \ \ (\mu>0)\right] \\
&= \frac{1}{\Gamma(\rho)}\int_0^\infty e^{-\lambda x}(\sqrt{t})^{-\rho}x^{\rho-1}e^{-\frac{x}{\sqrt{t}}}dx \\
&\qquad [\text{by the change of variables } x=\sqrt{t}y],
\end{aligned}$$

which shows the assertion of the corollary. □

3. Examples of arithmetical function $a(\cdot)$

Example 1. Let a sequence $(a(p))_p$ be nonnegative, i.e., $a(p) \geq 0$ $({}^\forall p)$. If $a(p) \to c \in [0,\infty)$ as $p \to \infty$, then

$$\sum_{p\leq x}\frac{a(p)\log p}{p} = (c+o(1))\log x \quad \text{as } x\to\infty.$$

Thus the condition (4) holds with $\tau = c$.

Proof. By Mertens' first theorem (6),

$$\begin{aligned}
&\left|\frac{1}{\log x}\sum_{p\leq x}\frac{a(p)\log p}{p} - c\right| \\
&= \left|\frac{1}{\log x}\sum_{p\leq x}\frac{c\log p}{p} + \frac{1}{\log x}\sum_{p\leq x}\frac{(a(p)-c)\log p}{p} - c\right| \\
&= \Bigg|c\frac{\log x + O(1)}{\log x} - c \\
&\qquad + \frac{1}{\log x}\sum_{p\leq y}\frac{(a(p)-c)\log p}{p} + \frac{1}{\log x}\sum_{y<p\leq x}\frac{(a(p)-c)\log p}{p}\Bigg| \\
&\qquad [\text{where we fix } y\in(2,x) \text{ arbitrarily}] \\
&\leq \frac{c|O(1)|}{\log x} + \frac{1}{\log x}\left|\sum_{p\leq y}\frac{(a(p)-c)\log p}{p}\right| + \left(\sup_{p>y}|a(p)-c|\right)\left(1+\frac{|O(1)|}{\log x}\right).
\end{aligned}$$

By letting $x\to\infty$,

$$\limsup_{x\to\infty}\left|\frac{1}{\log x}\sum_{p\leq x}\frac{a(p)\log p}{p} - c\right| \leq \sup_{p>y}|a(p)-c| \to 0 \quad \text{as } y\to\infty.$$

This shows the assertion of Example 1. □

EXAMPLE 2. For coprime $a, m \in \mathbb{N}$, we set $E_{a,m} := \{p; p \equiv a \pmod m\}$. Then

$$\sum_{p \le x} \frac{\mathbf{1}_{E_{a,m}}(p) \log p}{p} = \Big(\frac{1}{\phi(m)} + o(1)\Big) \log x \quad \text{as } x \to \infty,$$

where $\phi(\cdot)$ is Euler's function. Thus, $(\mathbf{1}_{E_{a,m}}(p))_p$ satisfies the condition (4) with $\tau = \frac{1}{\phi(m)}$.

Proof. We use the prime number theorem for arithmetic progressions in the following form:

$$\vartheta_{a,m}(x) := \sum_{p \le x} \mathbf{1}_{E_{a,m}}(p) \log p = \Big(\frac{1}{\phi(m)} + o(1)\Big)x \quad \text{as } x \to \infty. \tag{23}$$

$\vartheta_{a,m}(\cdot)$ is non-decreasing, right-continuous and $\vartheta_{a,m}(t) = 0$ $({}^\forall t < 2)$. Noting that

$$\vartheta_{a,m}(dt) = \sum_p \mathbf{1}_{E_{a,m}}(p)(\log p)\delta_p(dt),$$

we compute that for $0 < \varepsilon < 1$,

$$\begin{aligned}
\sum_{p \le x} \frac{\mathbf{1}_{E_{a,m}}(p) \log p}{p} &= \int_{(2-\varepsilon, x]} \frac{1}{t} \vartheta_{a,m}(dt) \\
&= \int_{(2-\varepsilon, x]} \frac{1}{t} d\Big(\frac{t}{\phi(m)} + t\delta(t)\Big) \\
&\qquad [\text{where } \delta(t) := \vartheta_{a,m}(t)/t - 1/\phi(m) \ \ (t \ge 1)] \\
&= \frac{1}{\phi(m)} \int_{2-\varepsilon}^x \frac{dt}{t} + \int_{2-\varepsilon}^x \frac{\delta(t)}{t} dt + \delta(x) + \frac{1}{\phi(m)}.
\end{aligned}$$

By letting $\varepsilon \searrow 0$,

$$\begin{aligned}
\sum_{p \le x} \frac{\mathbf{1}_{E_{a,m}}(p) \log p}{p} &= \frac{1}{\phi(m)} \int_2^x \frac{dt}{t} + \int_2^x \frac{\delta(t)}{t} dt + \delta(x) + \frac{1}{\phi(m)} \\
&= (\log x)\Big(\frac{1}{\phi(m)}\Big(1 + \frac{-\log 2 + 1}{\log x}\Big) + \int_{\frac{\log 2}{\log x}}^1 \delta(x^r) dr + \frac{\delta(x)}{\log x}\Big) \\
&\qquad [\text{by the change of variables } \tfrac{\log t}{\log x} = r].
\end{aligned}$$

Since, from $\lim_{x \to \infty} \delta(x) = 0$ (cf. (23)) and the bounded convergence theorem,

$$\lim_{x \to \infty} \int_{\frac{\log 2}{\log x}}^1 \delta(x^r) dr = 0,$$

we have

$$\sum_{p \le x} \frac{\mathbf{1}_{E_{a,m}}(p) \log p}{p} = \Big(\frac{1}{\phi(m)} + o(1)\Big) \log x \quad \text{as } x \to \infty.$$

□

EXAMPLE 3. If a sequence $(a(p))_p$ with $0 \le a(p) \le 1$ $({}^\forall p)$ satisfies that $\sum_p \frac{1-a(p)}{p} < \infty$, then

$$\sum_{p\le x}\frac{a(p)\log p}{p}=(1+o(1))\log x\quad\text{as } x\to\infty.$$

Thus the condition (4) holds with $\tau=1$.

Proof. For simplicity,

$$D(x):=\sum_{p\le x}\frac{1-a(p)}{p},\quad x\in\mathbb{R}.$$

$D(\cdot)$ is non-decreasing, right-continuous, $D(\infty)<\infty$ and $D(x)=0$ $(^\forall x<2)$. Since, by Mertens' first theorem (6),

$$\begin{aligned}\sum_{p\le x}\frac{a(p)\log p}{p}&=\sum_{p\le x}\frac{\log p}{p}-\sum_{p\le x}\frac{1-a(p)}{p}\log p\\&=\log x+O(1)-\sum_{p\le x}\frac{1-a(p)}{p}\log p,\end{aligned}$$

it suffices to show that

$$\sum_{p\le x}\frac{1-a(p)}{p}\log p=o(\log x).$$

$D(dt)=\sum_p\frac{1-a(p)}{p}\delta_p(dt)$ and integration by parts tell us that for $x\ge 2$,

$$\begin{aligned}&\sum_{p\le x}\frac{1-a(p)}{p}\log p\\&=\int_{(2-\varepsilon,x]}(\log t)D(dt)\quad[\text{where } 0<\varepsilon<1]\\&=\int_{(2-\varepsilon,x]}\Big(d(D(t)\log t)-D(t)\frac{dt}{t}\Big)\\&=D(x)\log x-\int_1^x\frac{D(t)}{t}dt\quad[\text{since } D(t)=0\ \ (^\forall t<2)]\\&=(\log x)\int_0^1(D(x)-D(x^r))dr\quad\Big[\text{by the change of variables } \tfrac{\log t}{\log x}=r\Big].\end{aligned}$$

By noting that for each $r\in(0,1]$,

$$0\le D(x)-D(x^r)\le D(x)\le D(\infty)<\infty\ \ (^\forall x\ge 2),\quad \lim_{x\to\infty}(D(x)-D(x^r))=0,$$

it follows from the bounded convergence theorem that

$$\lim_{x\to\infty}\frac{1}{\log x}\sum_{p\le x}\frac{1-a(p)}{p}\log p=0.$$

□

REMARK 2. When $0\le a(p)\le 1$ $(^\forall p)$, there is no implication between the condition for $(a(p))_p$ in Example 1 and that in Example 3.

(i) Let a subset $E \subset \{p \text{ is prime}\}$ be such that[3] $\#E = \infty$, $\sum_{p\in E}\frac{1}{p} < \infty$. Then $(a(p) = 1 - \mathbf{1}_E(p))_p$ satisfies $\sum_p \frac{1-a(p)}{p} < \infty$, but $\limsup_p a(p) = 1$ and $\liminf_p a(p) = 0$. This $(a(p))_p$ is in Example 3, but not in Example 1.
(ii) Let $0 \le a(p) \le 1$ be such that $a(p) = 1 - \frac{1}{\log\log p}$ $(p \gg 1)$. Then, clearly $a(p) \to 1$, but $\sum_p \frac{1-a(p)}{p} = \infty$. This $(a(p))_p$ is in Example 1, but not in Example 3.

Proof. (i) Since E is an infinite set by assumption, $a(p) = 0$ i.o. Since $\sum_{p\notin E}\frac{1}{p} = \infty$ by $\sum_p \frac{1}{p} = \infty$ and assumption, and thus, since $\{p \text{ is prime}\}\setminus E$ is also an infinite set, $a(p) = 1$ i.o.
(ii) We use the prime number theorem in the following form (cf. (23)):

$$\vartheta(x) := \sum_{p\le x}\log p = (1+o(1))x \quad \text{as } x\to\infty.$$

$\vartheta(\cdot)$ is non-decreasing, right-continuous and $\vartheta(t) = 0$ $({}^\forall t < 2)$. $\delta(t) := \frac{\vartheta(t)}{t} - 1$ $(t > 0)$ is of bounded variation on every bounded closed interval of $(0,\infty)$, and $\lim_{t\to\infty}\delta(t) = 0$. Take a prime q_0 large enough such that $1 - a(p) = \frac{1}{\log\log p}$ $(p > q_0)$. Then

$$\begin{aligned}
\sum_{q_0<p\le x}\frac{1-a(p)}{p} &= \sum_{q_0<p\le x}\frac{\log p}{p\log p\log\log p}\\
&= \int_{(q_0,x]}\frac{\vartheta(dt)}{t\log t\log\log t}\\
&= \int_{(q_0,x]}\frac{1}{t\log t\log\log t}d(t+t\delta(t))\\
&= \int_{q_0}^{x}\frac{1}{(\log\log t)(\log t)}\frac{dt}{t} + \int_{q_0}^{x}\frac{\delta(t)}{(\log\log t)(\log t)}\frac{dt}{t}\\
&\quad + \int_{(q_0,x]}\frac{\delta(dt)}{\log t\log\log t}\\
&= \int_{q_0}^{x}\frac{1}{(\log\log t)(\log t)}\frac{dt}{t} + \int_{q_0}^{x}\frac{\delta(t)}{(\log\log t)(\log t)}\frac{dt}{t}\\
&\quad + \int_{(q_0,x]}\Big(d\Big(\frac{\delta(t)}{\log t\log\log t}\Big) - \delta(t)d\Big(\frac{1}{\log t\log\log t}\Big)\Big)\\
&\qquad \text{[by integration by parts]}\\
&= \int_{q_0}^{x}\frac{1}{(\log\log t)(\log t)}\frac{dt}{t} + \int_{q_0}^{x}\frac{\delta(t)}{(\log\log t)(\log t)}\frac{dt}{t}\\
&\quad + \int_{q_0}^{x}\frac{\delta(t)(1+\log\log t)}{(\log t\log\log t)^2}\frac{dt}{t}\\
&\quad + \frac{\delta(x)}{\log x\log\log x} - \frac{\delta(q_0)}{\log q_0\log\log q_0}\\
&= \log\log\log x\Big(1 - \frac{\log\log\log q_0}{\log\log\log x} + \int_{\frac{\log\log\log q_0}{\log\log\log x}}^{1}\delta(e^{e^{e^{v\log\log\log x}}})dv\Big)
\end{aligned}$$

[3] Such an E exists. For example, take a sequence $\{q_i\}_{i=1}^{\infty}$ of prime numbers such that $q_1 = 2$, $q_{i+1} > q_i^2$ $(i \ge 1)$, and set $E := \{q_i; i \ge 1\}$. Then this E clearly satisfies the above conditions.

$$
\begin{aligned}
&+\int_{\log\log q_0}^{\log\log x}\delta(e^{e^r})\Big(1+\frac{1}{r}\Big)e^{-r}\frac{dr}{r}\\
&+\frac{\delta(x)}{\log x\log\log x}-\frac{\delta(q_0)}{\log q_0\log\log q_0}\\
&\quad[\text{by the change of variables }\log\log\log t=u].
\end{aligned}
$$

This implies that

$$\frac{1}{\log\log\log x}\sum_{q_0<p\le x}\frac{1-a(p)}{p}\to 1\quad\text{as } x\to\infty,$$

and thus

$$\sum_p\frac{1-a(p)}{p}=\infty.$$

□

REMARK 3. $E_{a,m}$ is an infinite set. By Mertens' first theorem (6) and Example 2,

$$
\begin{aligned}
\sum_{p\le x}\frac{(1-\mathbf{1}_{E_{a,m}}(p))\log p}{p}&=\sum_{p\le x}\frac{\log p}{p}-\sum_{p\le x}\frac{\mathbf{1}_{E_{a,m}}(p)\log p}{p}\\
&=\Big(1-\frac{1}{\phi(m)}+o(1)\Big)\log x\quad\text{as } x\to\infty.
\end{aligned}
$$

Since $\phi(m)\ge 2$ for $m\ge 3$, and thus $1-\frac{1}{\phi(m)}>0$, $\{p \text{ is prime}\}\setminus E_{a,m}$ is also an infinite set. Therefore $\limsup_p \mathbf{1}_{E_{a,m}}(p)=1$ and $\liminf_p \mathbf{1}_{E_{a,m}}(p)=0$. This tells us that for $m\ge 3$, $(\mathbf{1}_{E_{a,m}}(p))_p$ is not in Example 1.

4. Behavior of $Z(\sigma;a)$ as $\sigma\searrow 1$ for more general $a(\cdot)$

Roughly speaking, the aim of this section is as follows:

In case $\tau+\#\{p;a(p)=p\}=0$ in (5), how does $Z(\sigma;a)$ behave as $\sigma\searrow 1$?

To this end, for a nonnegative, completely multiplicative arithmetical function $a(\cdot)$, we consider, instead of (3) and (4), the following conditions:

$$\sup_p a(p)<\infty,\ \sup_p\frac{a(p)}{p}<1, \tag{24}$$

$$\sum_p\frac{a(p)\log p}{p}=\infty. \tag{25}$$

In the case where $\tau+\#\{p;a(p)=p\}=0\Leftrightarrow\tau=0$ and $a(p)\ne p$ $({}^\forall p)$, (3) becomes (24). But (4) does not always imply (25). In this paper, let us consider this convenient condition for us.

We begin with the following claim, which states that Claim 5 is valid even under the slightly weak condition (24):

Claim 10. (i) *For* $\sigma \in \mathbb{R} \setminus \{1\}$,

$$\sum_p \frac{a(p)}{p^\sigma} \begin{cases} = \infty & \text{if } -\infty < \sigma < 1, \\ < \infty & \text{if } \sigma > 1. \end{cases}$$

(ii) *For* $\sigma > 1$ *and* $t \in \mathbb{R}$, $\prod_p \frac{1}{1-\frac{a(p)}{p^{\sigma+it}}}$ *is convergent and* $\sum_{n=1}^\infty \frac{a(n)}{n^{\sigma+it}}$ *is absolutely convergent, and these coincide with each other.*
(iii) *For* $\sigma \in \mathbb{R} \setminus \{1\}$,

$$\sum_{n=1}^\infty \frac{a(n)}{n^\sigma} \begin{cases} = \infty & \text{if } -\infty < \sigma < 1, \\ < \infty & \text{if } \sigma > 1. \end{cases}$$

For $\sigma = 1$,

$$\sum_{n=1}^\infty \frac{a(n)}{n} = \infty \underset{\text{iff}}{\Leftrightarrow} \sum_p \frac{a(p)}{p} = \infty.$$

Proof. (i) From an inequality $x > \log x$ $(x > 0)$, the following implication is seen: For $\sigma \in \mathbb{R}$ and prime p,

$$\begin{aligned} p^{1-\sigma} > \log p^{1-\sigma} = (1-\sigma)\log p \Rightarrow \frac{1}{p^\sigma} > (1-\sigma)\frac{\log p}{p} \\ \Rightarrow \frac{a(p)}{p^\sigma} \geq (1-\sigma)\frac{a(p)\log p}{p}. \end{aligned}$$

In the case where $\sigma \in (-\infty, 1)$, $1-\sigma > 0$ and (25) imply

$$\sum_p \frac{a(p)}{p^\sigma} \geq (1-\sigma)\sum_p \frac{a(p)\log p}{p} = \infty;$$

In the case where $\sigma \in (1, \infty)$, (24) implies

$$\sum_p \frac{a(p)}{p^\sigma} \leq \sum_p \frac{\sup_q a(q)}{p^\sigma} < \sup_q a(q) \sum_{n=1}^\infty \frac{1}{n^\sigma} < \infty.$$

(ii) Let $\sigma \geq 1$ and $t \in \mathbb{R}$. By (24),

$$\left|\frac{a(p)}{p^{\sigma+it}}\right| = \frac{a(p)}{p^\sigma} \leq \frac{a(p)}{p} \leq \sup_q \frac{a(q)}{q} < 1,$$

so that

$$\frac{1}{1-\frac{a(p)}{p^{\sigma+it}}} = e^{\frac{a(p)}{p^{\sigma+it}}} \exp\left\{\frac{a(p)^2}{p^{2\sigma+i2t}} \int_0^1 \frac{s}{1-\frac{a(p)}{p^{\sigma+it}}s} ds\right\}.$$

Multiplication in $p \leq x$ yields that

$$\prod_{p\leq x} \frac{1}{1-\frac{a(p)}{p^{\sigma+it}}} = \exp\left\{\sum_{p\leq x} \frac{a(p)}{p^{\sigma+it}}\right\} \exp\left\{\sum_{p\leq x} \frac{a(p)^2}{p^{2\sigma+i2t}} \int_0^1 \frac{s}{1-\frac{a(p)}{p^{\sigma+it}}s} ds\right\}.$$

Here, by noting that

$$\sum_p \left| \frac{a(p)^2}{p^{2\sigma+i2t}} \int_0^1 \frac{s}{1-\frac{a(p)}{p^{\sigma+it}}s} ds \right| \leq \frac{(\sup_q a(q))^2}{2(1-\sup_q \frac{a(q)}{q})} \sum_p \frac{1}{p^2} < \infty,$$

the convergence of $\prod_p \frac{1}{1-\frac{a(p)}{p^{\sigma+it}}}$ is reduced to that of $\sum_p \frac{a(p)}{p^{\sigma+it}}$. Since, in the case where $\sigma > 1$,

$$\sum_p \left| \frac{a(p)}{p^{\sigma+it}} \right| = \sum_p \frac{a(p)}{p^{\sigma}} < \infty$$

by (i), $\prod_p \frac{1}{1-\frac{a(p)}{p^{\sigma+it}}}$ $(\sigma > 1, t \in \mathbb{R})$ is convergent.

Let $\sigma \geq 1$ and $t \in \mathbb{R}$ again. First, from the proof of Claim 5, note that

$$\sum_{n\in\mathbb{N}_L} \frac{a(n)}{n^{\sigma+it}} = \Big(\prod_{j=1}^{L} \frac{1}{1-\frac{a(p_j)}{p_j^{\sigma+it}}}\Big) \prod_{j=1}^{L}\Big(1-\Big(\frac{a(p_j)}{p_j^{\sigma+it}}\Big)^{L+1}\Big),$$

$$\prod_{j=1}^{L}\Big(1-\Big(\frac{a(p_j)}{p_j^{\sigma+it}}\Big)^{L+1}\Big) = \exp\Big\{\sum_{j=1}^{L}\Big(-\Big(\frac{a(p_j)}{p_j^{\sigma+it}}\Big)^{L+1}\Big)\int_0^1 \frac{1}{1-(\frac{a(p_j)}{p_j^{\sigma+it}})^{L+1}s} ds\Big\},$$

$$\begin{aligned} &\left|\sum_{j=1}^{L}\Big(-\Big(\frac{a(p_j)}{p_j^{\sigma+it}}\Big)^{L+1}\Big)\int_0^1 \frac{1}{1-(\frac{a(p_j)}{p_j^{\sigma+it}})^{L+1}s} ds\right| \\ &\leq L\Big(\sup_p \frac{a(p)}{p}\Big)^{L+1} \frac{1}{1-(\sup_p \frac{a(p)}{p})^{L+1}} \to 0 \quad \text{as } L \to \infty. \end{aligned}$$

In the case where $\sigma > 1$,

$$\lim_{L\to\infty} \sum_{n\in\mathbb{N}_L} \frac{a(n)}{n^{\sigma+it}} = \prod_p \frac{1}{1-\frac{a(p)}{p^{\sigma+it}}}.$$

When $t = 0$, the monotone convergence theorem tells us that

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{\sigma}} = \sum_{n\in\mathbb{N}} \frac{a(n)}{n^{\sigma}} = \lim_{L\to\infty} \sum_{n\in\mathbb{N}_L} \frac{a(n)}{n^{\sigma}} < \infty.$$

Thus we have the absolute convergence of $\sum_{n=1}^{\infty} \frac{a(n)}{n^{\sigma+it}}$ and

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{\sigma+it}} = \prod_p \frac{1}{1-\frac{a(p)}{p^{\sigma+it}}}.$$

(iii) In the case where $\sigma < 1$,

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{\sigma}} \geq \sum_p \frac{a(p)}{p^{\sigma}} = \infty$$

by (i). In the case where $\sigma > 1$,

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{\sigma}} < \infty$$

by (ii).

From the proof of (ii), it follows that

$$\sum_{n=1}^{\infty}\frac{a(n)}{n} = \lim_{L\to\infty}\sum_{n\in\mathbb{N}_L}\frac{a(n)}{n} = \lim_{L\to\infty}\prod_{i=1}^{L}\frac{1}{1-\frac{a(p_i)}{p_i}} = \prod_p \frac{1}{1-\frac{a(p)}{p}},$$

$$\prod_p \frac{1}{1-\frac{a(p)}{p}} < \infty \underset{\text{iff}}{\Leftrightarrow} \sum_p \frac{a(p)}{p} < \infty.$$

Combining these, we have

$$\sum_{n=1}^{\infty}\frac{a(n)}{n} < \infty \underset{\text{iff}}{\Leftrightarrow} \sum_p \frac{a(p)}{p} < \infty.$$

□

Definition 6. (i) By virtue of Claim 10, for $s = \sigma + it$ $(\sigma \in (1,\infty), t \in \mathbb{R})$, we define

$$\eta(s;a) := \sum_{n=1}^{\infty}\frac{a(n)}{n^s} = \prod_p \frac{1}{1-\frac{a(p)}{p^s}}.$$

And, for $\sigma \in (1,\infty)$, we set

$$\eta_\sigma(t;a) := \frac{\eta(\sigma+it;a)}{\eta(\sigma;a)}, \quad t \in \mathbb{R}.$$

As before (cf. Definition 4), let $\mu_\sigma(dx;a)$ be a 1-dimensional probability measure corresponding to $\eta_\sigma(\cdot;a)$.

(ii) For a sequence $\{Y_p\}_p$ of independent geometric processes on some $(\Omega,\mathcal{F},P)$, we define

$$Z(\sigma;a) := \sum_p Y_p\Big(\frac{a(p)}{p^\sigma}\Big)\log p, \quad \sigma \in (1,\infty).$$

Then $(-Z(\sigma;a))_{1<\sigma<\infty}$ is a backwards Lévy process whose marginal distribution at σ is $\mu_\sigma(\cdot;a)$. But, as compared with $Z(\sigma;a)$ in Definition 5, there is the following difference: If $\sum_p \frac{a(p)}{p} < \infty$, then $Z(1+;a) < \infty$ a.s., and if $\sum_p \frac{a(p)}{p} = \infty$, then $Z(1+;a) = \infty$ a.s.

Our interest is the behavior of $Z(\sigma;a)$ as $\sigma \searrow 1$. To see this, for a nonnegative, completely multiplicative arithmetical function $a(\cdot)$ satisfying (24) and (25), we further suppose the following:

$$\text{(26)} \qquad \begin{cases} \mathbb{R} \ni x \mapsto \displaystyle\sum_{p\le e^x}\frac{a(p)\log p}{p} \in [0,\infty) \text{ is regularly varying at } \infty \text{ with exponent} \\ \gamma \in [0,\infty). \end{cases}$$

First of all, note that $\gamma \le 1$ from Mertens' first theorem. For, let $L(\cdot)$[4] be a slowly varying function at ∞ defined as

$$\text{(27)} \qquad \sum_{p\le e^x}\frac{a(p)\log p}{p} = x^\gamma L(x),$$

then (24) and (6) tell us that

[4]In this paper, we call this $L(\cdot)$ a slowly varying part of a regularly varying function $x \mapsto \sum_{p\le e^x}\frac{a(p)\log p}{p}$.

$$x^{\gamma}L(x) \leq \sup_q a(q) \sum_{p\leq e^x} \frac{\log p}{p} = \sup_q a(q)(\log e^x + O(1)) = \sup_q a(q)(x + O(1)).$$

This implies

$$\limsup_{x\to\infty} x^{\gamma-1}L(x) \leq \sup_q a(q), \tag{28}$$

so that it must be that $\gamma \leq 1$.

We treat the following two cases:

Case 1 $\quad \gamma < 1$

or

$\gamma = 1$ and $\displaystyle\int^{\infty} \frac{L(x)}{x}dx < \infty$,

Case 2 $\quad \gamma = 1$ and $\displaystyle\int^{\infty} \frac{L(x)}{x}dx = \infty$.

Theorem 2. *In Case 1, $\sum_p \frac{a(p)}{p} < \infty$, and thus $Z(1+;a) < \infty$ a.s. In Case 2, $\sum_p \frac{a(p)}{p} = \infty$, and thus $Z(1+;a) = \infty$ a.s.*

Proof. First, by (10) and (27),

$$C(e^x) = x^{\gamma}L(x). \tag{29}$$

From (11), it follows that for $x \geq 2$,

$$\begin{aligned}
\sum_{p\leq x} \frac{a(p)}{p} &= \sum_{p\leq x} \frac{a(p)\log p}{p}\frac{1}{\log p} \\
&= \int_{(2-\varepsilon,x]} \frac{C(dt)}{\log t} \quad \text{[where } 0<\varepsilon<1\text{]} \\
&= \int_{(2-\varepsilon,x]} \left(d\left(\frac{C(t)}{\log t}\right) + C(t)\frac{1}{(\log t)^2}\frac{dt}{t}\right) \quad \text{[by integration by parts]} \\
&= \frac{C(x)}{\log x} + \int_2^x \frac{C(t)}{(\log t)^2}\frac{dt}{t} \quad \text{[since } C(t)=0 \ (^{\forall}t<2)\text{]} \\
&= \frac{C(x)}{\log x} + \int_{\log 2}^{\log x} \frac{C(e^s)}{s^2}ds \quad \text{[by the change of variables } \log t = s\text{]}.
\end{aligned}$$

By (29), this is rewritten as

$$\sum_{p\leq e^x} \frac{a(p)}{p} = \frac{L(x)}{x^{1-\gamma}} + \int_{\log 2}^{x} \frac{L(s)}{s^{2-\gamma}}ds.$$

In the case where $\gamma < 1$, $1-\gamma > 0$ and the slow variation of $L(\cdot)$ at ∞ yield that

$$\lim_{x\to\infty} \frac{L(x)}{x^{1-\gamma}} = 0,$$

$$\int^{\infty} \frac{L(s)}{s^{2-\gamma}}ds = \int^{\infty} \frac{L(s)}{s^{1+1-\gamma}}ds < \infty,$$

from which, it follows that

$$\sum_p \frac{a(p)}{p} < \infty.$$

In the case where $\gamma = 1$ and $\int^\infty \frac{L(x)}{x} dx < \infty$, (28) implies

$$\begin{aligned}\sum_p \frac{a(p)}{p} = \lim_{x\to\infty} \sum_{p\le e^x} \frac{a(p)}{p} &= \lim_{x\to\infty}\Big(L(x) + \int_{\log 2}^x \frac{L(s)}{s} ds\Big)\\ &\le \sup_q a(q) + \int_{\log 2}^\infty \frac{L(s)}{s} ds < \infty.\end{aligned}$$

In the case where $\gamma = 1$ and $\int^\infty \frac{L(x)}{x} dx = \infty$,

$$\sum_p \frac{a(p)}{p} = \lim_{x\to\infty} \sum_{p\le e^x} \frac{a(p)}{p} \ge \lim_{x\to\infty} \int_{\log 2}^x \frac{L(s)}{s} ds = \int_{\log 2}^\infty \frac{L(s)}{s} ds = \infty.$$

□

Claim 11. *In Case 2,* $-\dfrac{\eta'(\sigma; a)}{\eta(\sigma; a)} \sim \dfrac{1}{\sigma-1} L\Big(\dfrac{1}{\sigma-1}\Big)$ *as* $\sigma \searrow 1$.

Proof. By 1° in the proof of Claim 6,

$$-\frac{\eta'(\sigma; a)}{\eta(\sigma; a)} = \sum_p \frac{a(p)}{p^\sigma} \log p + \sum_p \frac{a(p)^2}{p^{2\sigma}} \frac{\log p}{1 - \frac{a(p)}{p^\sigma}}.$$

Since the second term is convergent as $\sigma \searrow 1$, we may investigate the asymptotics of the first term as $\sigma \searrow 1$.

By (11) and (29),

$$\begin{aligned}(30)\quad \sum_p \frac{a(p)}{p^\sigma} \log p &= \sum_p \frac{a(p)\log p}{p} \frac{1}{p^{\sigma-1}}\\ &= \int_{(2-\varepsilon,\infty)} \frac{C(dx)}{x^{\sigma-1}} \quad [\text{where } 0 < \varepsilon < 1]\\ &= \int_{(2-\varepsilon,\infty)} \Big(d\Big(\frac{C(x)}{x^{\sigma-1}}\Big) - C(x)(1-\sigma)x^{-\sigma} dx\Big) \quad [\text{by integration by parts}]\\ &= (\sigma-1)\int_1^\infty \frac{C(x)}{x^\sigma} dx \quad \left[\begin{array}{l}\text{since } C(t) = 0\ (^\forall t < 2) \text{ and by (29),}\\ \qquad \frac{C(x)}{x^{\sigma-1}} = \frac{C(e^{\log x})}{x^{\sigma-1}} = \frac{\log x\, L(\log x)}{x^{\sigma-1}} \to 0 \text{ as } x \to \infty\end{array}\right]\\ &= (\sigma-1)\int_0^\infty e^{-(\sigma-1)s} C(e^s) ds \quad [\text{by the change of variables } \log x = s]\\ &= (\sigma-1)\int_0^\infty e^{-(\sigma-1)s} d\Big(\int_0^s C(e^x) dx\Big).\end{aligned}$$

Here, since, by (29), $x \mapsto C(e^x)$ is regularly varying at ∞ with exponent 1, Feller [4, Chapter VIII, Theorem 1] tells us that

$$\frac{sC(e^s)}{\int_0^s C(e^x) dx} \to 2 \quad \text{as } s \to \infty,$$

and thus

$$\int_0^s C(e^x)dx \sim \frac{1}{2}sC(e^s) = \frac{1}{2}s^2L(s) \quad \text{as } s \to \infty.$$

This, by Feller [4, Chapter XIII, Theorem 2], implies

$$\int_0^\infty e^{-(\sigma-1)s}d\Big(\int_0^s C(e^x)dx\Big) \sim (\sigma-1)^{-2}L\Big(\frac{1}{\sigma-1}\Big) \quad \text{as } \sigma \searrow 1. \tag{31}$$

Therefore we have

$$\sum_p \frac{a(p)}{p^\sigma}\log p \sim \frac{1}{\sigma-1}L\Big(\frac{1}{\sigma-1}\Big) \quad \text{as } \sigma \searrow 1,$$

which is the assertion of the claim. □

We divide Case 2 into three cases:

Case 2.1 $\lim_{x\to\infty} L(x) = \tau \in (0,\infty)$,

Case 2.2 $\lim_{x\to\infty} L(x) = 0$,

Case 2.3 Neither Case 2.1 nor Case 2.2.

Since Case 2.3 is hard to deal with, this case is excluded from our consideration. Case 2.1 is $C(e^x) = x(\tau + o(1))$ as $x \to \infty$, i.e.,

$$\sum_{p\le x} \frac{a(p)\log p}{p} = (\log x)(\tau + o(1)) \quad \text{as } x \to \infty.$$

This is just the condition (4), so the answer to Case 2.1 is given from Corollary 1 in the following way:

$$\text{the distribution of } (\sigma-1)Z(\sigma;a) \to \mathbf{1}_{(0,\infty)}(x)\frac{1}{\Gamma(\tau)}x^{\tau-1}e^{-x}dx \quad \text{as } \sigma \searrow 1.$$

Theorem 3. *In Case 2.2,* $\lim_{\sigma\searrow 1} E[e^{-\lambda(\sigma-1)Z(\sigma;a)}] = 1$ *($^\forall\lambda > 0$). Thus*

$$(\sigma-1)Z(\sigma;a) \to 0 \quad \textit{i.p.} \quad \textit{as } \sigma \searrow 1.$$

Proof. Fix $\lambda > 0$. First, by (14),

$$\begin{aligned}
E[e^{-\lambda(\sigma-1)Z(\sigma;a)}] &= \prod_p E\Big[e^{-\lambda(\sigma-1)(\log p)Y_p\left(\frac{a(p)}{p^\sigma}\right)}\Big] \\
&= \prod_p \frac{1-\frac{a(p)}{p^\sigma}}{1-\frac{a(p)}{p^\sigma}\frac{1}{p^{\lambda(\sigma-1)}}} \\
&= \prod_p \exp\Big\{-\frac{a(p)}{p^\sigma} - \frac{a(p)^2}{p^{2\sigma}}\int_0^1 \frac{s}{1-\frac{a(p)}{p^\sigma}s}ds \\
&\qquad + \frac{a(p)}{p^\sigma}\frac{1}{p^{\lambda(\sigma-1)}} + \frac{a(p)^2}{p^{2\sigma}}\frac{1}{p^{2\lambda(\sigma-1)}}\int_0^1 \frac{s}{1-\frac{a(p)}{p^\sigma}\frac{s}{p^{\lambda(\sigma-1)}}}ds\Big\} \\
&= \exp\Big\{-\sum_p \frac{a(p)}{p^\sigma}\Big(1-\frac{1}{p^{\lambda(\sigma-1)}}\Big) - \sum_p \frac{a(p)^2}{p^{2\sigma}}\int_0^1 \frac{s}{1-\frac{a(p)}{p^\sigma}s}ds
\end{aligned}$$

$$+\sum_p \frac{a(p)^2}{p^{2\sigma}} \frac{1}{p^{2\lambda(\sigma-1)}} \int_0^1 \frac{s}{1-\frac{a(p)}{p^\sigma}\frac{s}{p^{\lambda(\sigma-1)}}} ds\Big\}.$$

Since, as $\sigma \searrow 1$,

$$\sum_p \frac{a(p)^2}{p^{2\sigma}} \int_0^1 \frac{s}{1-\frac{a(p)}{p^\sigma}s} ds \to \sum_p \frac{a(p)^2}{p^2} \int_0^1 \frac{s}{1-\frac{a(p)}{p}s} ds,$$

$$\sum_p \frac{a(p)^2}{p^{2\sigma}} \frac{1}{p^{2\lambda(\sigma-1)}} \int_0^1 \frac{s}{1-\frac{a(p)}{p^\sigma}\frac{s}{p^{\lambda(\sigma-1)}}} ds \to \sum_p \frac{a(p)^2}{p^2} \int_0^1 \frac{s}{1-\frac{a(p)}{p}s} ds,$$

it suffices to show that

$$\sum_p \frac{a(p)}{p^\sigma}\Big(1-\frac{1}{p^{\lambda(\sigma-1)}}\Big) \to 0. \tag{32}$$

Rewriting $1-\frac{1}{p^{\lambda(\sigma-1)}}$ as

$$1-\frac{1}{p^{\lambda(\sigma-1)}} = \lambda(\sigma-1)\int_0^1 \frac{1}{p^{\lambda(\sigma-1)t}} \log p\, dt$$

and then using (30) yield that

$$\begin{aligned}\sum_p \frac{a(p)}{p^\sigma}\Big(1-\frac{1}{p^{\lambda(\sigma-1)}}\Big) \\ = \lambda(\sigma-1)\int_0^1 (\sigma-1)(1+\lambda t)dt \int_0^\infty e^{-(\sigma-1)(1+\lambda t)s} d\Big(\int_0^s C(e^x)dx\Big).\end{aligned} \tag{33}$$

Here, by (31),

$$\kappa\int_0^\infty e^{-\kappa s} d\Big(\int_0^s C(e^x)dx\Big) \sim \frac{1}{\kappa}L\Big(\frac{1}{\kappa}\Big) \quad \text{as } \kappa \searrow 0,$$

and thus, for $0 < {}^\forall\varepsilon < 1$,

$${}^\exists\delta > 0 \text{ s.t. } 0<\kappa<\delta \Rightarrow 1-\varepsilon < \frac{\kappa\int_0^\infty e^{-\kappa s} d(\int_0^s C(e^x)dx)}{\frac{1}{\kappa}L(\frac{1}{\kappa})} < 1+\varepsilon.$$

Since, for $1 < \sigma < 1 + \frac{\delta}{1+\lambda}$,

$$0 < (\sigma-1)(1+\lambda t) \le (\sigma-1)(1+\lambda) < \delta \quad (0 \le {}^\forall t \le 1),$$

it follows that

$$\frac{(\sigma-1)(1+\lambda t)\int_0^\infty e^{-(\sigma-1)(1+\lambda t)s} d(\int_0^s C(e^x)dx)}{\frac{1}{(\sigma-1)(1+\lambda t)}L(\frac{1}{(\sigma-1)(1+\lambda t)})} < 1+\varepsilon \quad (0 \le {}^\forall t \le 1).$$

Using this estimate in (33), we have

$$0 \le \sum_p \frac{a(p)}{p^\sigma}\Big(1-\frac{1}{p^{\lambda(\sigma-1)}}\Big) \le (1+\varepsilon)\int_0^1 \frac{\lambda}{1+\lambda t} L\Big(\frac{1}{(\sigma-1)(1+\lambda t)}\Big)dt.$$

Finally, noting that by $\lim_{x\to\infty} L(x) = 0$ and the bounded convergence theorem,

$$\lim_{\sigma\searrow 1}\int_0^1 \frac{\lambda}{1+\lambda t}L\Big(\frac{1}{(\sigma-1)(1+\lambda t)}\Big)dt = 0,$$

we obtain (32). □

To investigate the behavior of $Z(\sigma;a)$ as $\sigma \searrow 1$ in Case 2.2 in more detail, we suppose the following:

(34) $u \mapsto L(e^u)$ is regularly varying at ∞ with exponent δ.

Then note that $-1 \le \delta \le 0$. For, let $l(\cdot)$ be its slowly varying part, then

$$\begin{aligned}\infty &= \int^{\infty} \frac{L(x)}{x}dx \quad \text{[cf. Case 2]}\\ &= \int^{\infty} L(e^u)du \quad \text{[by the change of variables } \log x = u]\\ &= \int^{\infty} u^{\delta} l(u)du,\end{aligned}$$

$$0 = \lim_{x\to\infty} L(x) \;\; \text{[cf. Case 2.2]} = \lim_{u\to\infty} L(e^u) = \lim_{u\to\infty} u^{\delta} l(u).$$

These convergences imply neither $\delta < -1$ nor $\delta > 0$, i.e., $-1 \le \delta \le 0$.

We divide Case 2.2 into the following cases:

Case 2.2.1 $\quad -1 < \delta \le 0$,

Case 2.2.2 $\quad \delta = -1$,

Case 2.2.2.1 $\quad \lim_{u\to\infty} l(u) = \infty$,

Case 2.2.2.2 $\quad \lim_{u\to\infty} l(u) = \kappa \in (0,\infty)$,

Case 2.2.2.3 $\quad \lim_{u\to\infty} l(u) = 0$.

Theorem 4. (i) *In Case 2.2.1 or Case 2.2.2.1,* $\lim_{\sigma\searrow 1} E[e^{-\lambda(\sigma-1)^{\Delta}Z(\sigma;a)}] = 0$ *(*$^{\forall}\lambda > 0,\ 0 \le {}^{\forall}\Delta < 1$*). Thus*

$$(\sigma-1)^{\Delta}Z(\sigma;a) \to \infty \quad \textit{i.p.} \quad \textit{as } \sigma \searrow 1, \quad 0 \le {}^{\forall}\Delta < 1.$$

(ii) *In Case 2.2.2.2,* $\lim_{\sigma\searrow 1} E[e^{-\lambda(\sigma-1)^{\Delta}Z(\sigma;a)}] = \Delta^{\kappa}$ *(*$^{\forall}\lambda > 0,\ 0 < {}^{\forall}\Delta \le 1$*). Thus,* $(\sigma-1)^{\Delta}Z(\sigma;a)$ *being regarded as a* $[0,\infty]$*-valued random variable,*

$$\begin{aligned}&\text{the distribution of } (\sigma-1)^{\Delta}Z(\sigma;a)\\ &\to \Delta^{\kappa}\delta_0 + (1-\Delta^{\kappa})\delta_{\infty} \quad \textit{as } \sigma \searrow 1, \quad 0 < {}^{\forall}\Delta \le 1.\end{aligned}$$

(iii) *In Case 2.2.2.3,* $\lim_{\sigma\searrow 1} E[e^{-\lambda(\sigma-1)^{\Delta}Z(\sigma;a)}] = 1$ *(*$^{\forall}\lambda > 0,\ 0 < {}^{\forall}\Delta \le 1$*). Thus*

$$(\sigma-1)^{\Delta}Z(\sigma;a) \to 0 \quad \textit{i.p.} \quad \textit{as } \sigma \searrow 1, \quad 0 < {}^{\forall}\Delta \le 1.$$

Proof. Fix $\lambda > 0$ and $0 < \Delta < 1$. First, from the proof of Theorem 3, it is seen that as $\sigma \searrow 1$,

$$E[e^{-\lambda(\sigma-1)^{\Delta}Z(\sigma;a)}] \sim \exp\Big\{-\sum_p \frac{a(p)}{p^{\sigma}}\Big(1-\frac{1}{p^{\lambda(\sigma-1)^{\Delta}}}\Big)\Big\},$$

$$
\begin{aligned}
&\sum_p \frac{a(p)}{p^\sigma}\Big(1-\frac{1}{p^{\lambda(\sigma-1)^\Delta}}\Big)\\
&=\lambda(\sigma-1)^\Delta\int_0^1(\sigma-1+\lambda(\sigma-1)^\Delta t)dt\int_0^\infty e^{-(\sigma-1+\lambda(\sigma-1)^\Delta t)s}d\Big(\int_0^s C(e^x)dx\Big)\\
&\sim\lambda(\sigma-1)^\Delta\int_0^1\frac{1}{\sigma-1+\lambda(\sigma-1)^\Delta t}L\Big(\frac{1}{\sigma-1+\lambda(\sigma-1)^\Delta t}\Big)dt\\
&=\int_{\log\frac{1}{\sigma-1+\lambda(\sigma-1)^\Delta}}^{\log\frac{1}{\sigma-1}}L(e^u)du\quad\Big[\text{by the change of variables }\log\tfrac{1}{\sigma-1+\lambda(\sigma-1)^\Delta t}=u\Big]\\
&=\int_{\log\frac{1}{\sigma-1+\lambda(\sigma-1)^\Delta}}^{\log\frac{1}{\sigma-1}}u^\delta l(u)du.
\end{aligned}
$$

In the case where $-1<\delta\le 0$, we take $\varepsilon>0$ such that $-1<\delta-\varepsilon<\delta\le 0$. Since $u^{-\varepsilon}<l(u)<u^{\varepsilon}$ $(u\gg 1)$ by the slow variation of $l(\cdot)$ at ∞, and $\delta-\varepsilon+1>0$, it follows that

$$
\begin{aligned}
\int_{\log\frac{1}{\sigma-1+\lambda(\sigma-1)^\Delta}}^{\log\frac{1}{\sigma-1}}u^\delta l(u)du &\ge \int_{\log\frac{1}{\sigma-1+\lambda(\sigma-1)^\Delta}}^{\log\frac{1}{\sigma-1}}u^{\delta-\varepsilon}du\\
&=\frac{1}{\delta-\varepsilon+1}\Big(\log\frac{1}{\sigma-1}\Big)^{\delta-\varepsilon+1}\bigg(1-\bigg(\Delta+\frac{\log\frac{1}{\lambda+(\sigma-1)^{1-\Delta}}}{\log\frac{1}{\sigma-1}}\bigg)^{\delta-\varepsilon+1}\bigg)\\
&\to\infty\quad\text{as }\sigma\searrow 1.
\end{aligned}
$$

Thus

$$\lim_{\sigma\searrow 1}E[e^{-\lambda(\sigma-1)^\Delta Z(\sigma;a)}]=0.$$

In the case where $\delta=-1$,

$$
\begin{aligned}
\int_{\log\frac{1}{\sigma-1+\lambda(\sigma-1)^\Delta}}^{\log\frac{1}{\sigma-1}}u^{-1}l(u)du &= \int_{\log\log\frac{1}{\sigma-1+\lambda(\sigma-1)^\Delta}}^{\log\log\frac{1}{\sigma-1}}l(e^v)dv\quad[\text{by the change of variables }\log u=v]\\
&=\int_{\alpha(\sigma)}^{\beta(\sigma)}f(v)dv\quad\left[\begin{array}{l}\text{where, for simplicity}\\ \qquad f(v):=l(e^v),\\ \qquad \alpha(\sigma):=\log\log\frac{1}{\sigma-1+\lambda(\sigma-1)^\Delta},\\ \qquad \beta(\sigma):=\log\log\frac{1}{\sigma-1}\end{array}\right]\\
&=\int_0^{\beta(\sigma)-\alpha(\sigma)}\frac{f(w+\alpha(\sigma))}{f(\alpha(\sigma))}dwf(\alpha(\sigma)).
\end{aligned}
$$

Here, note that as $\sigma\searrow 1$,

$$
\begin{aligned}
&\beta(\sigma)-\alpha(\sigma)=\log\frac{1}{\Delta+\frac{\log\frac{1}{\lambda+(\sigma-1)^{1-\Delta}}}{\log\frac{1}{\sigma-1}}}\to\log\frac{1}{\Delta},\\
&\alpha(\sigma)\to\infty,\\
&\frac{f(w+\alpha(\sigma))}{f(\alpha(\sigma))}=\frac{l(e^we^{\alpha(\sigma)})}{l(e^{\alpha(\sigma)})}\underset{\mathrm{c}}{\rightrightarrows}1\quad[\text{by the slow variation of }l(\cdot)\text{ at }\infty].
\end{aligned}
$$

From these, it follows that

$$\int_0^{\beta(\sigma)-\alpha(\sigma)} \frac{f(w+\alpha(\sigma))}{f(\alpha(\sigma))}dw \to \log\frac{1}{\Delta} \quad \text{as } \sigma \searrow 1,$$

so that

$$\lim_{\sigma\searrow 1}\int_{\log\frac{1}{\sigma-1+\lambda(\sigma-1)^{\Delta}}}^{\log\frac{1}{\sigma-1}} u^{-1}l(u)du = \begin{cases} \infty & \text{if } \lim_{u\to\infty} l(u)=\infty, \\ \kappa\log\dfrac{1}{\Delta} = -\log\Delta^{\kappa} & \text{if } \lim_{u\to\infty} l(u)=\kappa\in(0,\infty), \\ 0 & \text{if } \lim_{u\to\infty} l(u)=0. \end{cases}$$

Thus

$$\lim_{\sigma\searrow 1} E\big[e^{-\lambda(\sigma-1)^{\Delta}Z(\sigma;a)}\big] = \begin{cases} 0 & \text{if } \lim_{u\to\infty} l(u)=\infty, \\ \Delta^{\kappa} & \text{if } \lim_{u\to\infty} l(u)=\kappa\in(0,\infty), \\ 1 & \text{if } \lim_{u\to\infty} l(u)=0. \end{cases}$$

□

Before closing this paper, we give some examples of $a(\cdot)$. For this, we need the following lemma:

Lemma 2. *Let $t_0>0$, and $f:(t_0,\infty)\to(0,\infty)$ be of class C^1 and ultimately non-increasing, i.e., $^{\exists}t_1>t_0$ s.t. $f'\le 0$ on $[t_1,\infty)$. Then, for $q_0 := \min\{p\text{:prime}; t_0<p\}$ and $0<\varepsilon<(q_0-t_0)\wedge(q_0-1)$,*

$$\sum_{t_0<p\le x}\frac{\log p}{p}f(p) = \int_{q_0-\varepsilon}^{x}\frac{f(t)}{t}dt + O(1) \quad \text{as } x\to\infty.$$

Proof. For simplicity, we set

$$M(x) = \sum_{p\le x}\frac{\log p}{p}, \quad x\in\mathbb{R}.$$

Clearly $M(\cdot)$ is non-decreasing, right-continuous, $M(x)=0$ ($^{\forall}x<2$), and

$$M(dx) = \sum_{p}\frac{\log p}{p}\delta_p(dx).$$

Note that $\sup_{x\ge 1}|M(x)-\log x|<\infty$ by Mertens' first theorem (6). By integration by parts,

$$\begin{aligned}
\sum_{t_0<p\le x}\frac{\log p}{p}f(p) &= \sum_{q_0-\varepsilon<p\le x}\frac{\log p}{p}f(p) \\
&= \int_{(q_0-\varepsilon,x]} f(t)M(dt) \\
&= \int_{(q_0-\varepsilon,x]} (d(f(t)M(t)) - M(t)f'(t)dt) \\
&= f(x)(\log x+\eta(x)) - f(q_0-\varepsilon)(\log(q_0-\varepsilon)+\eta(q_0-\varepsilon)) \\
&\quad - \int_{q_0-\varepsilon}^{x}(\log t+\eta(t))f'(t)dt \quad [\text{where } \eta(x) := M(x)-\log x]
\end{aligned}$$

$$= \int_{q_0-\varepsilon}^{x} \frac{f(t)}{t}dt + f(x)\eta(x) - f(q_0-\varepsilon)\eta(q_0-\varepsilon) - \int_{q_0-\varepsilon}^{x} \eta(t)f'(t)dt$$
$$= \int_{q_0-\varepsilon}^{x} \frac{f(t)}{t}dt + O(1) - \int_{q_0-\varepsilon}^{x} \eta(t)f'(t)dt \quad \text{as } x\to\infty.$$

Here, since $-f' \geq 0$ on $[t_1,\infty)$,

$$\begin{aligned}\left|-\int_{q_0-\varepsilon}^{x} \eta(t)f'(t)dt\right| &= \left|-\int_{q_0-\varepsilon}^{t_1\vee 1} \eta(t)f'(t)dt + \int_{t_1\vee 1}^{x} \eta(t)(-f'(t))dt\right| \\ &\leq \left|\int_{q_0-\varepsilon}^{t_1\vee 1} \eta(t)f'(t)dt\right| + \int_{t_1\vee 1}^{x} |\eta(t)|(-f'(t))dt \\ &\leq \left|\int_{q_0-\varepsilon}^{t_1\vee 1} \eta(t)f'(t)dt\right| + \Big(\sup_{t\geq 1}|\eta(t)|\Big)(-f(x)+f(t_1\vee 1)) \\ &= O(1) \quad \text{as } x\to\infty.\end{aligned}$$

This, together with the preceding, implies the assertion of the lemma. □

Letting $f(t) = \frac{1}{(\log t)^b}$ or $\frac{1}{(\log\log t)^c}$ or $\frac{1}{(\log\log t)^c(\log\log\log t)^d}$ or $\frac{1}{(\log\log\log t)^d}$ in Lemma 2 yields the following example, whose details are omitted:

EXAMPLE 4. (i) For $b>0$,

$$\sum_{p\leq x}\frac{\log p}{p}\frac{1}{(\log p)^b} = \begin{cases} O(1) & \text{if } b>1, \\ \log\log x + O(1) & \text{if } b=1, \\ \frac{1}{1-b}(\log x)^{1-b} + O(1) & \text{if } 0<b<1 \end{cases} \qquad \text{as } x\to\infty.$$

Thus

$$\sum_{p}\frac{\log p}{p}\frac{1}{(\log p)^b} \begin{cases} <\infty & \text{if } b>1, \\ =\infty & \text{if } 0<b\leq 1. \end{cases}$$

In the latter case, $x \mapsto \sum_{p\leq e^x}\frac{\log p}{p}\frac{1}{(\log p)^b}$ is regularly varying at ∞ with exponent $1-b\in[0,1)$.
(ii) For $c>0$,

$$\sum_{e<p\leq x}\frac{\log p}{p}\frac{1}{(\log\log p)^c} \sim \frac{\log x}{(\log\log x)^c} \quad \text{as } x\to\infty.$$

Thus $\sum_{p>e}\frac{\log p}{p}\frac{1}{(\log\log p)^c} = \infty$, and $x\mapsto \sum_{e<p\leq e^x}\frac{\log p}{p}\frac{1}{(\log\log p)^c}$ is regularly varying at ∞ with exponent 1 and its slowly varying part $L(x)\sim\frac{1}{(\log x)^c}$ as $x\to\infty$, so that

$$\lim_{x\to\infty}L(x)=0, \qquad \int^{\infty}\frac{L(x)}{x}dx \begin{cases} <\infty & \text{if } c>1, \\ =\infty & \text{if } 0<c\leq 1. \end{cases}$$

In the latter case, $u\mapsto L(e^u)$ is regularly varying at ∞ with exponent $-c\in[-1,0)$.
(iii) For $c>0$ and $d\in\mathbb{R}$,

$$\sum_{e^e<p\leq x}\frac{\log p}{p}\frac{1}{(\log\log p)^c(\log\log\log p)^d} \sim \frac{\log x}{(\log\log x)^c(\log\log\log x)^d} \quad \text{as } x\to\infty.$$

Thus $\sum_{p>e^e} \frac{\log p}{p} \frac{1}{(\log\log p)^c (\log\log\log)^d} = \infty$, and $x \mapsto \sum_{e^e<p\le e^x} \frac{\log p}{p} \frac{1}{(\log\log p)^c (\log\log\log p)^d}$ is regularly varying at ∞ with exponent 1 and its slowly varying part $L(x) \sim \frac{1}{(\log x)^c (\log\log x)^d}$ as $x \to \infty$, so that

$$\lim_{x\to\infty} L(x) = 0, \qquad \int^{\infty} \frac{L(x)}{x} dx \begin{cases} < \infty & \text{if } c > 1 \text{ or } c = 1 \text{ and } d > 1, \\ = \infty & \text{if } 0 < c < 1 \text{ or } c = 1 \text{ and } d \le 1. \end{cases}$$

In the latter case, $u \mapsto L(e^u)$ is regularly varying at ∞ with exponent $-c \in [-1, 0)$ and its slowly varying part $l(u) \sim \frac{1}{(\log u)^d}$ as $u \to \infty$, so that in the case where $c = 1$ and $d \le 1$,

$$\lim_{u\to\infty} l(u) = \begin{cases} \infty & \text{if } d < 0, \\ 1 & \text{if } d = 0, \\ 0 & \text{if } 0 < d \le 1. \end{cases}$$

(iv) For $d > 0$,

$$\sum_{e^e<p\le x} \frac{\log p}{p} \frac{1}{(\log\log\log p)^d} \sim \frac{\log x}{(\log\log\log x)^d} \quad \text{as } x \to \infty.$$

Thus $\sum_{p>e^e} \frac{\log p}{p} \frac{1}{(\log\log\log)^d} = \infty$, and $x \mapsto \sum_{e^e<p\le e^x} \frac{\log p}{p} \frac{1}{(\log\log\log p)^d}$ is regularly varying at ∞ with exponent 1 and its slowly varying part $L(x) \sim \frac{1}{(\log\log x)^d}$ as $x \to \infty$, so that

$$\lim_{x\to\infty} L(x) = 0, \qquad \int^{\infty} \frac{L(x)}{x} dx = \infty,$$

$u \mapsto L(e^u)$ is regularly varying at ∞ with exponent 0 (= slowly varying at ∞).

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