



Title	On Auslander-Reiten components and simple modules for finite group algebras
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Citation	Osaka Journal of Mathematics. 1997, 34(3), p. 681-688
Version Type	VoR
URL	https://doi.org/10.18910/7367
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Kawata, S.
Osaka J. Math.
34 (1997), 681–688

ON AUSLANDER-REITEN COMPONENTS AND SIMPLE MODULES FOR FINITE GROUP ALGEBRAS

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(Received June 11, 1996)

Introduction

Let G be a finite group, k a field of characteristic $p > 0$ and B a block of the group algebra kG . Let Θ be a connected component (AR-component for short) of the stable Auslander-Reiten quiver of B . Erdmann showed that if B is a wild block of kG , then the tree class of Θ is A_∞ [6]. In this note we investigate where simple modules lie in the Auslander-Reiten quiver of B . Let Λ be a symmetric algebra and M a simple Λ -module. Then the Auslander-Reiten sequence $\mathcal{A}(\Omega^{-1}M)$ terminating in $\Omega^{-1}M$ is of the form $0 \rightarrow \Omega M \rightarrow H_M \oplus P_M \rightarrow \Omega^{-1}M \rightarrow 0$, where Ω is the Heller operator, P_M is the projective cover of M and H_M is the heart $\text{Rad}P_M/\text{Soc}P_M$ of P_M (see [1, Proposition 4.11]), and sequences of this type will be called standard sequences. Therefore if the tree class of the AR-component Θ containing M is A_∞ , then M lies at the end of Θ if and only if H_M is indecomposable.

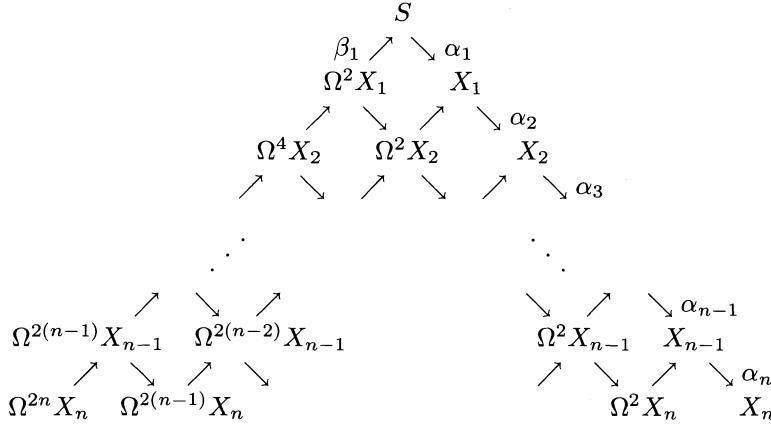
In Section 1, we consider for general symmetric algebras what happens if some AR-component with tree class A_∞ contains a simple module not lying at the end of its AR-component. In Section 2 we give certain conditions which imply that all simple modules in B lie at the ends of AR-components.

The notation is almost standard. All the modules considered here are finite dimensional over k . Concerning some basic facts and terminologies used here, we refer to [2] and [5].

1. AR-components of symmetric algebras and simple modules

In the case of general symmetric algebras, Jost gave some conditions which imply that all simple modules contained in an AR-component with tree class A_∞ lie at the end of this component [7, Theorem 3.3]. Now we consider what happens if some simple module does not lie at the end of an AR-component with tree class A_∞ . In this section, let Λ be a symmetric algebra and Θ an AR-component with tree class A_∞ of the stable Auslander-Reiten quiver of Λ , and suppose that Θ contains some simple Λ -module not lying at the end of Θ . Under this assumption Θ is of

the form $\mathbf{Z}A_\infty$ or $(\mathbf{Z}/m)A_\infty$ (so called an m -tube), and we may assume that Θ or $\Omega\Theta$ contains some simple module S not lying at the end and that the wing $\mathcal{W}(S)$ spanned by S :



with $\Omega^{2i}X_n$ ($0 \leq i \leq n$) lying at the end, satisfies the condition that

(*) there are no projectives in $\mathcal{A}(\Omega^{2i}X_j)$ for $0 \leq i \leq j < n$.

Indeed, if this is not the case, then the AR-sequence $\mathcal{A}(\Omega^{2i}X_j)$ terminating in $\Omega^{2i}X_j$ is standard for some $1 \leq j \leq n-1$ and some $0 \leq i \leq j$ because standard ones are only those which involve projectives. Thus, $\Omega^{2i}X_j$ is isomorphic to $\Omega^{-1}S'$ for some simple module S' , and S' does not lie at the end. Hence we start with S' instead of S , and therefore we finally get a wing with the above property (*).

In the above situation, we shall see that the AR-sequences $\mathcal{A}(\Omega^{2i}X_n)$ terminating in $\Omega^{2i}X_n$ ($0 \leq i \leq n-1$) are standard. Also in the case where Θ is an infinite m -tube, we shall see that $n+1 < m$, i.e., $X_n \not\cong \Omega^{2i}X_n$ for $0 < i \leq n$.

First we recall the following easy result (see, e.g., the argument in [3, Section 3]), which will be used repeatedly.

Lemma 1.1. *Let $\mathcal{A}(U) : 0 \rightarrow X \rightarrow Y \oplus Z \rightarrow U \rightarrow 0$ with Y and Z non-projective be an AR-sequence terminating in U . Assume that the irreducible map $\alpha : Y \rightarrow U$ is a monomorphism. Then the irreducible map $\alpha' : X \rightarrow Z$ is also a monomorphism and $\text{Coker } \alpha \cong \text{Coker } \alpha'$. Dually, if the irreducible map $\alpha' : X \rightarrow Z$ is an epimorphism, then the irreducible map $\alpha : Y \rightarrow U$ is also an epimorphism and $\text{Ker } \alpha \cong \text{Ker } \alpha'$.*

Now we give attention to the modules X_1 , $\Omega^2 X_1$ and $\Omega^2 X_2$.

Lemma 1.2. *X_1 , $\Omega^2 X_1$ and $\Omega^2 X_2$ are uniserial and their Loewy series are as follows for some simple Λ -modules T_1 and T_n :*

$$X_1 : \begin{pmatrix} T_1 \\ S \end{pmatrix}, \quad \Omega^2 X_1 : \begin{pmatrix} S \\ T_n \end{pmatrix}, \quad \Omega^2 X_2 : \begin{pmatrix} T_1 \\ S \\ T_n \end{pmatrix}.$$

Proof. Since S is simple, the irreducible map $\beta_1 : \Omega^2 X_1 \rightarrow S$ is an epimorphism and the irreducible map $\alpha_1 : S \rightarrow X_1$ is a monomorphism. By the property(*) and Lemma 1.1, it follows that $\mathcal{A}(X_n)$ and $\mathcal{A}(\Omega^{2(n-1)} X_n)$ are standard, i.e., $\Omega^{2(n-1)} X_n \cong \Omega^{-1} T_1$ and $X_n \cong \Omega^{-1} T_n$ for some simple Λ -modules T_1 and T_n . Also, Lemma 1.1 yields that $\text{Coker} \alpha_1 \cong T_1$ and $\text{Ker} \beta_1 \cong T_n$. \square

Next we consider the modules X_i ($1 \leq i \leq n$).

Lemma 1.3. *For the modules X_i and the irreducible maps $\alpha_i : X_{i-1} \rightarrow X_i$ ($1 \leq i \leq n$), the following hold.*

- (1) *The irreducible maps α_i are monomorphisms.*
- (2) *$\Omega^{2(n-i)} X_n \cong \Omega^{-1} T_i$ for some simple Λ -module T_i ($1 \leq i \leq n$).*
- (3) *T_i appears in the head of X_i and the composition factors of X_i , from the head, are given by $\{T_i, T_{i-1}, \dots, T_1, S\}$.*
- (4) *The socle of X_i is isomorphic to S .*

Proof. In the case $i = 1$, the statements follow by Lemma 1.2. Assume that the statements hold for X_j ($1 \leq j \leq i-1$). Note that the AR-sequences $\mathcal{A}(X_i)$ ($1 \leq i \leq n-1$) are not standard. We consider the following mesh:

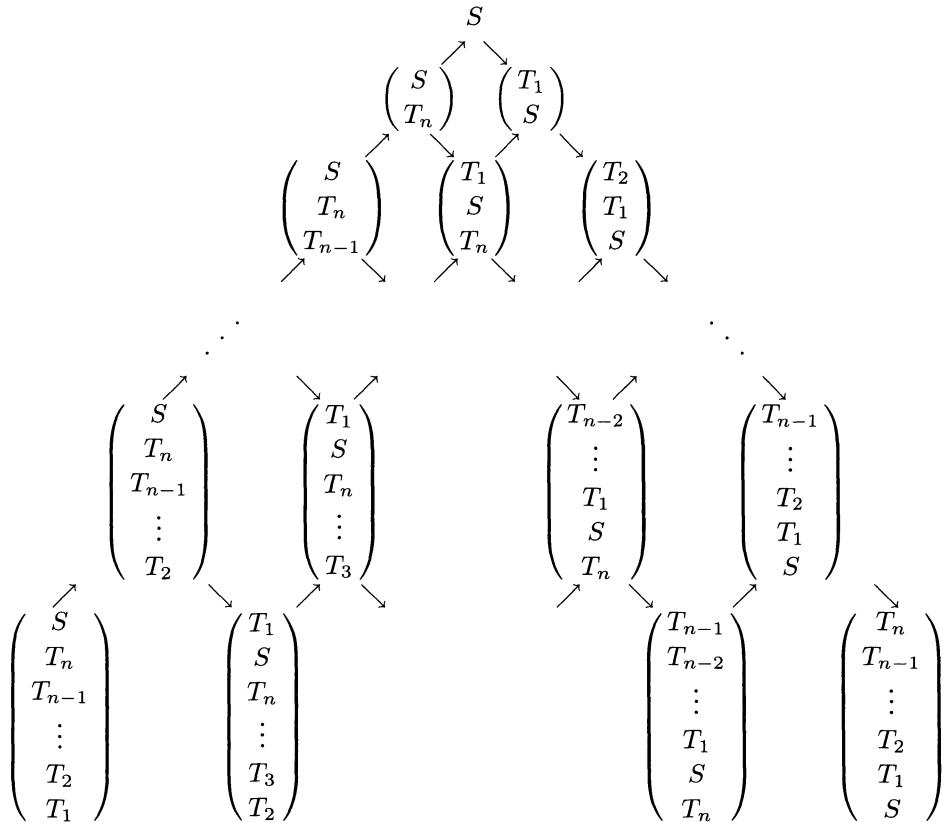
$$\begin{array}{ccc} & X_{i-1} & \\ \nearrow & & \searrow \alpha_i \\ \Omega^2 X_i & & X_i \\ \searrow \alpha & & \nearrow \beta \\ & \Omega^2 X_{i+1} & \end{array}$$

- (1) Assume contrary that α_i is an epimorphism. Since the socle of X_{i-1} is simple and isomorphic to S , the socle of $\text{Ker} \alpha_i$ is isomorphic to S and S does not appear as a composition factor of X_i . Since the irreducible map $\beta : \Omega^2 X_{i+1} \rightarrow X_i$ is an epimorphism and $\text{Ker} \beta \cong T_n$, S does not appear as a composition factor of $\Omega^2 X_{i+1}$. Now we see that $\Omega^2 X_1 = \begin{pmatrix} S \\ T_n \end{pmatrix} \subset \Omega^2 X_i$ by induction. However, since S lies in the head of $\Omega^2 X_1$, we have $\Omega^2 X_1 \subset \text{Ker} \alpha$, where α is the irreducible map from $\Omega^2 X_i$ to $\Omega^2 X_{i+1}$, but this contradicts that $\text{Ker} \alpha_i \cong \text{Ker} \alpha$.
- (2) Note that the statement (1) above, Lemma 1.1 and the property (*) imply that $\mathcal{A}(\Omega^{2(n-i)} X_n)$ is standard. Hence we have $\Omega^{2(n-i)} X_n \cong \Omega^{-1} T_i$ for some simple Λ -module T_i .

(3) This follows since $\text{Coker } \alpha_i \cong T_i$ by (2).
 (4) By the inductive hypothesis, we have $\text{Soc } X_{i-1} \cong S$. Since X_{i-1} is a maximal submodule of X_i and X_i is indecomposable, we have $\text{Soc } X_{i-1} = \text{Soc } X_i$.

□

Proposition 1.4. *Using the same notation as in Lemma 1.3, the wing $\mathcal{W}(S)$ spanned by S is as follows.*



In particular, all modules in $\mathcal{W}(S)$ are uniserial.

Proof. We continue to use the notation in Lemma 1.3. From Lemma 1.3(2) and the property (*), the irreducible maps $\Omega^{2s} X_i \rightarrow \Omega^{2s} X_{i+1}$ are monomorphisms and the irreducible maps $\Omega^{2(s+1)} X_{i+1} \rightarrow \Omega^{2s} X_i$ are epimorphisms for $1 \leq i \leq n-1$ and $0 \leq s \leq i$. Therefore X_i is a homomorphic image of $\Omega^{-1} T_i$ and the head of X_i is isomorphic to T_i . Thus X_i ($1 \leq i \leq n$) are uniserial. In particular X_{n-1} (=the heart of the projective cover of T_n) is uniserial and so is $\Omega^2 X_n (\cong \Omega T_n)$. Since

$\Omega^2 X_i$ ($1 \leq i \leq n$) are submodules of $\Omega^2 X_n$, they are uniserial. Using this argument repeatedly, we see that all modules in $\mathcal{W}(S)$ are uniserial. \square

From Lemma 1.3 and Proposition 1.4, we have the following immediately.

Theorem 1.5. *Let Λ be a symmetric algebra and Θ an AR-component of Λ with tree class A_∞ . Suppose that Θ contains some simple module not lying at the end of Θ . Then for some simple Λ -modules S, T_1, \dots, T_n the projective covers P_{T_i} of T_i ($1 \leq i \leq n$) are uniserial and the Loewy series are as follows :*

$$P_{T_1} : \begin{pmatrix} T_1 \\ S \\ T_n \\ T_{n-1} \\ \vdots \\ T_2 \\ T_1 \end{pmatrix}, \quad P_{T_2} : \begin{pmatrix} T_2 \\ T_1 \\ S \\ T_n \\ T_{n-1} \\ \vdots \\ T_3 \\ T_2 \end{pmatrix}, \quad \dots, \quad P_{T_i} : \begin{pmatrix} T_i \\ T_{i+1} \\ \vdots \\ T_2 \\ T_1 \\ S \\ T_n \\ T_{n-1} \\ \vdots \\ T_{i+1} \\ T_i \end{pmatrix}, \quad \dots, \quad P_{T_n} : \begin{pmatrix} T_n \\ T_{n-1} \\ \vdots \\ T_2 \\ T_1 \\ S \\ T_n \\ T_{n-1} \\ \vdots \\ T_2 \\ T_1 \\ S \\ T_n \end{pmatrix}.$$

In particular, the Cartan matrix for Λ looks like

$$\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & & \vdots \\ 1 & 1 & \ddots & \ddots & 1 \\ \vdots & \ddots & 2 & 1 & \\ 1 & 1 & \cdots & 1 & \\ 0 & & & & * \end{pmatrix}.$$

Proof. By Lemma 1.3(2), the AR-sequences $\mathcal{A}(\Omega^{2(n-i)} X_n)$ are standard and $\Omega^{2(n-i)} X_n \cong \Omega^{-1} T_i$ for some simple Λ -modules T_i ($1 \leq i \leq n$). Also by Proposition 1.4, the hearts $\Omega^{2(n-i)} X_{n-1}$ of P_{T_i} are uniserial and their Loewy series, from the head, are given by $T_{i-1}, T_{i-2}, \dots, T_1, S, T_n, T_{n-1}, \dots, T_{i+1}$. We claim that $T_i \not\cong T_j$ if $i \neq j$. Indeed, since S appears only in the $(i+1)$ th head of P_{T_i} , we have $P_{T_i} \not\cong P_{T_j}$ if $i \neq j$. \square

REMARK 1.6. Under the same notation as in Proposition 1.4, suppose that Θ is an infinite m -tube. Then it follows that $n+1 < m$ since $\Omega^{-1} T_i \not\cong \Omega^{-1} T_j$ if $i \neq j$.

2. AR-components of group algebras and simple modules

In this section, we show that under certain conditions all simple modules in a wild block B of the group algebra kG lie at the ends of the AR-components.

Theorem 2.1. *Let B be a wild block of kG . Suppose that G has a non-trivial normal p -subgroup and k is algebraically closed. Then all simple modules in B lie at the ends of the AR-components.*

Proof. Let \mathcal{Q} be a non-trivial normal p -subgroup of G . Assume contrary that some simple module in B does not lie at the end. Then for some simple modules S, T_1, \dots, T_n , the projective covers P_{T_i} of T_i ($1 \leq i \leq n$) are uniserial and the Loewy series are as in Theorem 1.5. In particular the Cartan integers $c_{T_i T_i} = 2$.

CLAIM 1. $n = 1$, i.e., for some simple modules S and T , the Loewy series of the projective cover P_T of T is given by T, S, T .

Proof of the Claim 1. From the result of Tsushima [10, Lemma 3], T_i are projective as $k(G/\mathcal{Q})$ -modules, i.e., $\text{vx}(T_i) = \mathcal{Q}$ and the trivial $k\mathcal{Q}$ -module $k_{\mathcal{Q}}$ is a source of T_i . Now assume contrary that $n \geq 2$. Since $T_1 \cong \Omega^2 T_2$, it follows that $k_{\mathcal{Q}} \cong \Omega^2 k_{\mathcal{Q}}$ and \mathcal{Q} is cyclic. However, by the result of Erdmann [4, Theorem] T_i belong to a block with a cyclic defect group, a contradiction.

CLAIM 2. We have $p = 2$ and \mathcal{Q} is the Klein four group.

Proof of the Claim 2. Since $T \downarrow_{\mathcal{Q}}$ and $S \downarrow_{\mathcal{Q}}$ are direct sums of copies of $k_{\mathcal{Q}}$, the length of Loewy series of $P_T \downarrow_{\mathcal{Q}}$ is at most 3. Hence \mathcal{Q} is the Klein four group by the result of Okuyama [9].

Let H_S be the heart of the projective cover P_S of S and Θ the AR-component containing ΩS . Then $H_S \cong T \oplus X$ for some indecomposable non-projective module X . We consider the wing spanned by X :

$$\begin{array}{ccccc}
 & & X & & \\
 & \nearrow & & \searrow & \\
 \Omega S & \rightarrow & P_S & \rightarrow & \Omega^{-1} S \\
 \nearrow & \searrow & \nearrow & \searrow & \\
 \Omega^2 T & & T & & \Omega^{-2} T.
 \end{array}$$

CLAIM 3. $\text{vx}(X) \not\cong \mathcal{Q}$.

Proof of the Claim 3. Assume contrary that $\text{vx}(X) = \mathcal{Q}$. Note that the AR-component containing S is not a tube, since $k_{\mathcal{Q}}|S \downarrow_{\mathcal{Q}}$ and $k_{\mathcal{Q}}$ is not a periodic

module. Since $\text{vx}(T) = \mathcal{Q}$, from the result of Okuyama and Uno [8, Theorem], all the indecomposable modules in Θ have the same vertex \mathcal{Q} . Since Θ is of the form $\mathbf{Z}A_\infty$, the class of the \mathcal{Q} -sources of the indecomposable modules in Θ consists of infinitely many Ω^2 -orbits. However this would be impossible because non-periodic indecomposable $k\mathcal{Q}$ -modules are the syzygies of the trivial module $k_{\mathcal{Q}}$ only (see, e.g., [2]).

Now we consider the following two cases.

CASE 1. $\text{vx}(\Omega^{-1}S) \not\geq \mathcal{Q}$. The AR-sequence $\mathcal{A}(\Omega^{-1}S) \downarrow_{\mathcal{Q}}$ restricted to \mathcal{Q} splits [2, Proposition 4.12.10]. However, $\Omega^{-1}S \downarrow_{\mathcal{Q}}$ (resp. $\Omega S \downarrow_{\mathcal{Q}}$) is a direct sum of copies of $\Omega^{-1}k_{\mathcal{Q}}$ (resp. $\Omega k_{\mathcal{Q}}$) but $T \downarrow_{\mathcal{Q}}$ is a direct sum of copies of $k_{\mathcal{Q}}$, a contradiction.

CASE 2. $\text{vx}(\Omega^{-1}S) = \mathcal{Q}$, $\text{vx}(X) \not\geq \mathcal{Q}$. The AR-sequence $\mathcal{A}(\Omega^{-1}S) \downarrow_{\mathcal{Q}}$ restricted to \mathcal{Q} is a direct sum of split sequences and m copies of AR-sequence $\mathcal{A}(\Omega^{-1}k_{\mathcal{Q}})$ for some m . Since $S \downarrow_{\mathcal{Q}} \cong (\dim S)k_{\mathcal{Q}}$ and $\Omega^{-1}S \downarrow_{\mathcal{Q}} \cong (\dim S)\Omega k_{\mathcal{Q}}$, we have $m \leq \dim S$. On the other hand, since $\dim(\text{Soc}(P_S \downarrow_{\mathcal{Q}})) \geq \dim S$ and $(\dim S)k_{\mathcal{Q}}|P_S \downarrow_{\mathcal{Q}}$, we have $m \geq \dim S$. Therefore $m = \dim S$. This means that the AR-sequence $\mathcal{A}(\Omega^{-1}S) \downarrow_{\mathcal{Q}}$ restricted to \mathcal{Q} is a direct sum of $(\dim S)$ copies of the AR-sequence $\mathcal{A}(\Omega^{-1}k_{\mathcal{Q}})$ and $X \downarrow_{\mathcal{Q}}$ is a direct sum of copies of $k_{\mathcal{Q}}$. Since $\text{vx}(X) \not\geq \mathcal{Q}$, the AR-sequence $\mathcal{A}(X) \downarrow_{\mathcal{Q}}$ restricted to \mathcal{Q} splits. However $\Omega k_{\mathcal{Q}}$ is a direct summand of $\Omega S \downarrow_{\mathcal{Q}}$, which is a direct summand of the middle term of $\mathcal{A}(X) \downarrow_{\mathcal{Q}}$, a contradiction. \square

Corollary 2.2. *Let B a wild block of kG . Suppose that G is p -solvable and k is algebraically closed. Then all simple modules in B lie at the ends of the AR-components.*

Proof. Assume that some simple module does not lie at the end. Then by Theorem 1.5 and the result of Tsushima [10, Theorem], there is a finite group H with normal p -subgroup such that B and kH are Morita equivalent. However by Theorem 2.1 all simple kH -modules lie at the ends of the AR-components, a contradiction. \square

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