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ON AUSLANDER-REITEN COMPONENTS AND SIMPLE MODULES FOR FINITE GROUP ALGEBRAS

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Introduction

Let G be a finite group, k a field of characteristic $p > 0$ and B a block of the group algebra kG. Let Θ be a connected component (AR-component for short) of the stable Auslander-Reiten quiver of B. Erdmann showed that if B is a wild block of kG, then the tree class of Θ is A_∞ [6]. In this note we investigate where simple modules lie in the Auslander-Reiten quiver of B. Let Λ be a symmetric algebra and M a simple Λ-module. Then the Auslander-Reiten sequence $\mathcal{A}(\Omega^{-1}M)$ terminating in $\Omega^{-1}M$ is of the form $0 \to \Omega M \to H_M \oplus P_M \to \Omega^{-1}M \to 0$, where Ω is the Heller operator, P_M is the projective cover of M and H_M is the heart $\mathrm{Rad}P_M/\mathrm{Soc}P_M$ of P_M (see [1, Proposition 4.11]), and sequences of this type will be called standard sequences. Therefore if the tree class of the AR-component Θ containing M is A_∞, then M lies at the end of Θ if and only if H_M is indecomposable.

In Section 1, we consider for general symmetric algebras what happens if some AR-component with tree class A_∞ contains a simple module not lying at the end of its AR-component. In Section 2 we give certain conditions which imply that all simple modules in B lie at the ends of AR-components.

The notation is almost standard. All the modules considered here are finite dimensional over k. Concerning some basic facts and terminologies used here, we refer to [2] and [5].

1. AR-components of symmetric algebras and simple modules

In the case of general symmetric algebras, Jost gave some conditions which imply that all simple modules contained in an AR-component with tree class A_∞ lie at the end of this component [7, Theorem 3.3]. Now we consider what happens if some simple module does not lie at the end of an AR-component with tree class A_∞. In this section, let Λ be a symmetric algebra and Θ an AR-component with tree class A_∞ of the stable Auslander-Reiten quiver of Λ, and suppose that Θ contains some simple Λ-module not lying at the end of Θ. Under this assumption Θ is of the form $\mathbf{Z}A_\infty$ or $(\mathbf{Z}/m)A_\infty$ (so called an m-tube), and we may assume that Θ or $\Omega\Theta$ contains some simple module S not lying at the end and that the wing $\mathcal{W}(S)$ spanned by S:

$$\begin{array}{ccccccccccccccc}
 &&&&&& S &&&&&&&& \\
 &&&&& {}^{\beta_1}\nearrow && \searrow_{\alpha_1} &&&&&&& \\
 &&&& \Omega^2X_1 &&&& X_1 &&&&&& \\
 &&& \nearrow && \searrow && \nearrow && \searrow_{\alpha_2} &&&&& \\
 && \Omega^4X_2 &&&& \Omega^2X_2 &&&& X_2 &&&& \\
 & \nearrow && \searrow && \nearrow && \searrow && \nearrow && \searrow_{\alpha_3} &&& \\
 && \cdot^{\cdot^{\cdot}} &&&&&&&&&& \ddots && \\
 & \nearrow & \searrow & \nearrow &&&&&&& \searrow & \nearrow & \searrow_{\alpha_{n-1}} && \\
 & \Omega^{2(n-1)}X_{n-1} && \Omega^{2(n-2)}X_{n-1} &&&&&&&& \Omega^2X_{n-1} && X_{n-1} & \\
 \nearrow & \searrow & \nearrow & \searrow &&&&&&& \nearrow & \searrow & \nearrow & \searrow_{\alpha_n} & \\
 \Omega^{2n}X_n && \Omega^{2(n-1)}X_n &&&&&&&&&& \Omega^2X_n && X_n
\end{array}$$

with $\Omega^{2i}X_n$ $(0 \leq i \leq n)$ lying at the end, satisfies the condition that

$(*)$ there are no projectives in $\mathcal{A}(\Omega^{2i}X_j)$ for $0 \leq i \leq j < n$.

Indeed, if this is not the case, then the AR-sequence $\mathcal{A}(\Omega^{2i}X_j)$ terminating in $\Omega^{2i}X_j$ is standard for some $1 \leq j \leq n-1$ and some $0 \leq i \leq j$ because standard ones are only those which involve projectives. Thus, $\Omega^{2i}X_j$ is isomorphic to $\Omega^{-1}S'$ for some simple module S', and S' does not lie at the end. Hence we start with S' instead of S, and therefore we finally get a wing with the above property $(*)$.

In the above situation, we shall see that the AR-sequences $\mathcal{A}(\Omega^{2i}X_n)$ terminating in $\Omega^{2i}X_n$ $(0 \leq i \leq n-1)$ are standard. Also in the case where Θ is an infinite m-tube, we shall see that $n+1 < m$, i.e., $X_n \ncong \Omega^{2i}X_n$ for $0 < i \leq n$.

First we recall the following easy result (see, e.g., the argument in [3, Section 3]), which will be used repeatedly.

Lemma 1.1. *Let $\mathcal{A}(U): 0 \to X \to Y \oplus Z \to U \to 0$ with Y and Z non-projective be an AR-sequence terminating in U. Assume that the irreducible map $\alpha: Y \to U$ is a monomorphism. Then the irreducible map $\alpha': X \to Z$ is also a monomorphism and* $\mathrm{Coker}\alpha \cong \mathrm{Coker}\alpha'$. *Dually, if the irreducible map $\alpha': X \to Z$ is an epimorphism, then the irreducible map $\alpha: Y \to U$ is also an epimorphism and* $\mathrm{Ker}\alpha \cong \mathrm{Ker}\alpha'$.

Now we give attention to the modules X_1, Ω^2X_1 and Ω^2X_2.

Lemma 1.2. *X_1, Ω^2X_1 and Ω^2X_2 are uniserial and their Loewy series are as follows for some simple Λ-modules T_1 and T_n :*

$$X_1 : \begin{pmatrix} T_1 \\ S \end{pmatrix}, \quad \Omega^2 X_1 : \begin{pmatrix} S \\ T_n \end{pmatrix}, \quad \Omega^2 X_2 : \begin{pmatrix} T_1 \\ S \\ T_n \end{pmatrix}.$$

Proof. Since S is simple, the irreducible map $\beta_1 : \Omega^2 X_1 \to S$ is an epimorphism and the irreducible map $\alpha_1 : S \to X_1$ is a monomorphism. By the property(*) and Lemma 1.1, it follows that $\mathcal{A}(X_n)$ and $\mathcal{A}(\Omega^{2(n-1)} X_n)$ are standard, i.e., $\Omega^{2(n-1)} X_n \cong \Omega^{-1} T_1$ and $X_n \cong \Omega^{-1} T_n$ for some simple Λ-modules T_1 and T_n. Also, Lemma 1.1 yields that $\mathrm{Coker}\alpha_1 \cong T_1$ and $\mathrm{Ker}\beta_1 \cong T_n$. □

Next we consider the modules X_i $(1 \leq i \leq n)$.

Lemma 1.3. *For the modules X_i and the irreducible maps $\alpha_i : X_{i-1} \to X_i$ $(1 \leq i \leq n)$, the following hold.*

(1) *The irreducible maps α_i are monomorphisms.*

(2) *$\Omega^{2(n-i)} X_n \cong \Omega^{-1} T_i$ for some simple Λ-module $T_i (1 \leq i \leq n)$.*

(3) *T_i appears in the head of X_i and the composition factors of X_i, from the head, are given by $\{T_i, T_{i-1}, \cdots, T_1, S\}$.*

(4) *The socle of X_i is isomorphic to S.*

Proof. In the case $i = 1$, the statements follow by Lemma 1.2. Assume that the statements hold for $X_j (1 \leq j \leq i-1)$. Note that the AR-sequences $\mathcal{A}(X_i)(1 \leq i \leq n-1)$ are not standard. We consider the following mesh:

$$\begin{tikzcd} & X_{i-1} \arrow[dr, "\alpha_i"] & \\ \Omega^2 X_i \arrow[ur] \arrow[dr, "\alpha"'] & & X_i \\ & \Omega^2 X_{i+1} \arrow[ur, "\beta"'] & \end{tikzcd}$$

(1) Assume contrary that α_i is an epimorphism. Since the socle of X_{i-1} is simple and isomorphic to S, the socle of $\mathrm{Ker}\alpha_i$ is isomorphic to S and S does not appear as a composition factor of X_i. Since the irreducible map $\beta : \Omega^2 X_{i+1} \to X_i$ is an epimorphism and $\mathrm{Ker}\beta \cong T_n$, S does not appear as a composition factor of $\Omega^2 X_{i+1}$. Now we see that $\Omega^2 X_1 = \begin{pmatrix} S \\ T_n \end{pmatrix} \subset \Omega^2 X_i$ by induction. However, since S lies in the head of $\Omega^2 X_1$, we have $\Omega^2 X_1 \subset \mathrm{Ker}\alpha$, where α is the irreducible map from $\Omega^2 X_i$ to $\Omega^2 X_{i+1}$, but this contradicts that $\mathrm{Ker}\alpha_i \cong \mathrm{Ker}\alpha$.

(2) Note that the statement (1) above, Lemma 1.1 and the property (*) imply that $\mathcal{A}(\Omega^{2(n-i)} X_n)$ is standard. Hence we have $\Omega^{2(n-i)} X_n \cong \Omega^{-1} T_i$ for some simple Λ-module T_i.

(3) This follows since $\mathrm{Coker}\alpha_i \cong T_i$ by (2).

(4) By the inductive hypothesis, we have $\mathrm{Soc}X_{i-1} \cong S$. Since X_{i-1} is a maximal submodule of X_i and X_i is indecomposable, we have $\mathrm{Soc}X_{i-1} = \mathrm{Soc}X_i$. $\square$

Proposition 1.4. *Using the same notation as in Lemma* 1.3, *the wing $\mathcal{W}(S)$ spanned by S is as follows.*

$$
\begin{tikzcd}[column sep=tiny, row sep=small]
 & & & & S \arrow[dr] & & & & \\
 & & & \begin{pmatrix} S \\ T_n \end{pmatrix} \arrow[ur] \arrow[dr] & & \begin{pmatrix} T_1 \\ S \end{pmatrix} \arrow[dr] & & & \\
 & & \begin{pmatrix} S \\ T_n \\ T_{n-1} \end{pmatrix} \arrow[ur] \arrow[dr] & & \begin{pmatrix} T_1 \\ S \\ T_n \end{pmatrix} \arrow[ur] \arrow[dr] & & \begin{pmatrix} T_2 \\ T_1 \\ S \end{pmatrix} \arrow[dr] & & \\
 & \iddots \arrow[ur] & & \arrow[ur] & & \arrow[ur] & & \ddots & \\
 & \begin{pmatrix} S \\ T_n \\ T_{n-1} \\ \vdots \\ T_2 \end{pmatrix} \arrow[dr] & & \begin{pmatrix} T_1 \\ S \\ T_n \\ \vdots \\ T_3 \end{pmatrix} \arrow[dr] & & \begin{pmatrix} T_{n-2} \\ \vdots \\ T_1 \\ S \\ T_n \end{pmatrix} \arrow[dr] & & \begin{pmatrix} T_{n-1} \\ \vdots \\ T_2 \\ T_1 \\ S \end{pmatrix} \arrow[dr] & \\
\begin{pmatrix} S \\ T_n \\ T_{n-1} \\ \vdots \\ T_2 \\ T_1 \end{pmatrix} \arrow[ur] & & \begin{pmatrix} T_1 \\ S \\ T_n \\ \vdots \\ T_3 \\ T_2 \end{pmatrix} \arrow[ur] & & \arrow[ur] & & \begin{pmatrix} T_{n-1} \\ T_{n-2} \\ \vdots \\ T_1 \\ S \\ T_n \end{pmatrix} \arrow[ur] & & \begin{pmatrix} T_n \\ T_{n-1} \\ \vdots \\ T_2 \\ T_1 \\ S \end{pmatrix}
\end{tikzcd}
$$

In particular, all modules in $\mathcal{W}(S)$ are uniserial.

Proof. We continue to use the notation in Lemma 1.3. From Lemma 1.3(2) and the property $(*)$, the irreducible maps $\Omega^{2s}X_i \to \Omega^{2s}X_{i+1}$ are monomorphisms and the irreducible maps $\Omega^{2(s+1)}X_{i+1} \to \Omega^{2s}X_i$ are epimorphisms for $1 \le i \le n-1$ and $0 \le s \le i$. Therefore X_i is a homomorphic image of $\Omega^{-1}T_i$ and the head of X_i is isomorphic to T_i. Thus X_i $(1 \le i \le n)$ are uniserial. In particular X_{n-1}(=the heart of the projective cover of T_n) is uniserial and so is $\Omega^2 X_n (\cong \Omega T_n)$. Since

$\Omega^2 X_i$ $(1 \leq i \leq n)$ are submodules of $\Omega^2 X_n$, they are uniserial. Using this argument repeatedly, we see that all modules in $\mathcal{W}(S)$ are uniserial. $\square$

From Lemma 1.3 and Proposition 1.4, we have the following immediately.

Theorem 1.5. *Let Λ be a symmetric algebra and Θ an AR-component of Λ with tree class A_∞. Suppose that Θ contains some simple module not lying at the end of Θ. Then for some simple Λ-modules $S, T_1, \cdots, T_n$ the projective covers P_{T_i} of T_i $(1 \leq i \leq n)$ are uniserial and the Loewy series are as follows:*

$$P_{T_1} : \begin{pmatrix} T_1 \\ S \\ T_n \\ T_{n-1} \\ \vdots \\ \\ \vdots \\ T_2 \\ T_1 \end{pmatrix}, \quad P_{T_2} : \begin{pmatrix} T_2 \\ T_1 \\ S \\ T_n \\ T_{n-1} \\ \vdots \\ \\ \vdots \\ T_3 \\ T_2 \end{pmatrix}, \quad \cdots, \quad P_{T_i} : \begin{pmatrix} T_i \\ T_{i+1} \\ \vdots \\ T_2 \\ T_1 \\ S \\ T_n \\ T_{n-1} \\ \vdots \\ T_{i+1} \\ T_i \end{pmatrix}, \quad \cdots, \quad P_{T_n} : \begin{pmatrix} T_n \\ T_{n-1} \\ \vdots \\ \\ \vdots \\ T_2 \\ T_1 \\ S \\ T_n \end{pmatrix}.$$

In particular, the Cartan matrix for Λ looks like $\left(\begin{array}{cccccc|c} 2 & 1 & 1 & \cdots & 1 & & \\ 1 & 2 & 1 & & \vdots & & \\ 1 & 1 & \ddots & \ddots & 1 & & 0 \\ \vdots & & \ddots & 2 & 1 & & \\ 1 & 1 & \cdots & 1 & & & \\ & & 0 & & & & * \end{array}\right)$.

Proof. By Lemma 1.3(2), the AR-sequences $\mathcal{A}(\Omega^{2(n-i)} X_n)$ are standard and $\Omega^{2(n-i)} X_n \cong \Omega^{-1} T_i$ for some simple Λ-modules $T_i (1 \leq i \leq n)$. Also by Proposition 1.4, the hearts $\Omega^{2(n-i)} X_{n-1}$ of P_{T_i} are uniserial and their Loewy series, from the head, are given by $T_{i-1}, T_{i-2}, \cdots, T_1, S, T_n, T_{n-1}, \cdots, T_{i+1}$. We claim that $T_i \not\cong T_j$ if $i \neq j$. Indeed, since S appears only in the $(i+1)$th head of P_{T_i}, we have $P_{T_i} \not\cong P_{T_j}$ if $i \neq j$. $\square$

REMARK 1.6. Under the same notation as in Proposition 1.4, suppose that Θ is an infinite m-tube. Then it follows that $n+1 < m$ since $\Omega^{-1} T_i \not\cong \Omega^{-1} T_j$ if $i \neq j$.

2. AR-components of group algebras and simple modules

In this section, we show that under certain conditions all simple modules in a wild block B of the group algebra kG lie at the ends of the AR-components.

Theorem 2.1. *Let B be a wild block of kG. Suppose that G has a non-trivial normal p-subgroup and k is algebraically closed. Then all simple modules in B lie at the ends of the AR-components.*

Proof. Let Q be a non-trivial normal p-subgroup of G. Assume contrary that some simple module in B does not lie at the end. Then for some simple modules $S, T_1, \cdots, T_n$, the projective covers P_{T_i} of T_i $(1 \leq i \leq n)$ are uniserial and the Loewy series are as in Theorem 1.5. In particular the Cartan integers $c_{T_iT_i} = 2$.

CLAIM 1. $n = 1$, i.e., for some simple modules S and T, the Loewy series of the projective cover P_T of T is given by T, S, T.

Proof of the Claim 1. From the result of Tsushima [10, Lemma 3], T_i are projective as $k(G/Q)$-modules, i.e., $\mathrm{vx}(T_i) = Q$ and the trivial kQ-module k_Q is a source of T_i. Now assume contrary that $n \geq 2$. Since $T_1 \cong \Omega^2 T_2$, it follows that $k_Q \cong \Omega^2 k_Q$ and Q is cyclic. However, by the result of Erdmann [4, Theorem] T_i belong to a block with a cyclic defect group, a contradiction.

CLAIM 2. We have $p = 2$ and Q is the Klein four group.

Proof of the Claim 2. Since $T\downarrow_Q$ and $S\downarrow_Q$ are direct sums of copies of k_Q, the length of Loewy series of $P_T\downarrow_Q$ is at most 3. Hence Q is the Klein four group by the result of Okuyama [9].

Let H_S be the heart of the projective cover P_S of S and Θ the AR-component containing ΩS. Then $H_S \cong T \oplus X$ for some indecomposable non-projective module X. We consider the wing spanned by X :

$$\begin{tikzcd}[column sep=small]
 & & X \arrow[dr] & & \\
 & \Omega S \arrow[r] \arrow[ur] \arrow[dr] & P_S \arrow[r] & \Omega^{-1} S \arrow[dr] & \\
\Omega^2 T \arrow[ur] & & T \arrow[ur] & & \Omega^{-2} T.
\end{tikzcd}$$

CLAIM 3. $\mathrm{vx}(X) \gneqq Q$.

Proof of the Claim 3. Assume contrary that $\mathrm{vx}(X) = Q$. Note that the AR-component containing S is not a tube, since $k_Q | S\downarrow_Q$ and k_Q is not a periodic module. Since $\mathrm{vx}(T) = Q$, from the result of Okuyama and Uno [8, Theorem], all the indecomposable modules in Θ have the same vertex Q. Since Θ is of the form $\mathbf{Z}A_\infty$, the class of the Q-sources of the indecomposable modules in Θ consists of infinitely many Ω^2-orbits. However this would be impossible because non-periodic indecomposable kQ-modules are the syzygies of the trivial module k_Q only (see, e.g., [2]).

Now we consider the following two cases.

CASE 1. $\mathrm{vx}(\Omega^{-1}S) \gneqq Q$. The AR-sequence $\mathcal{A}(\Omega^{-1}S)\downarrow_Q$ restricted to Q splits [2, Proposition 4.12.10]. However, $\Omega^{-1}S\downarrow_Q$ (resp. $\Omega S\downarrow_Q$) is a direct sum of copies of $\Omega^{-1}k_Q$ (resp. Ωk_Q) but $T\downarrow_Q$ is a direct sum of copies of k_Q, a contradiction.

CASE 2. $\mathrm{vx}(\Omega^{-1}S) = Q$, $\mathrm{vx}(X) \gneqq Q$. The AR-sequence $\mathcal{A}(\Omega^{-1}S)\downarrow_Q$ restricted to Q is a direct sum of split sequences and m copies of AR-sequence $\mathcal{A}(\Omega^{-1}k_Q)$ for some m. Since $S\downarrow_Q\cong(\dim S)k_Q$ and $\Omega^{-1}S\downarrow_Q\cong(\dim S)\Omega k_Q$, we have $m \leq \dim S$. On the other hand, since $\dim(\mathrm{Soc}(P_S\downarrow_Q)) \geq \dim S$ and $(\dim S)kQ|P_S\downarrow_Q$, we have $m \geq \dim S$. Therefore $m = \dim S$. This means that the AR-sequence $\mathcal{A}(\Omega^{-1}S)\downarrow_Q$ restricted to Q is a direct sum of $(\dim S)$ copies of the AR-sequence $\mathcal{A}(\Omega^{-1}k_Q)$ and $X\downarrow_Q$ is a direct sum of copies of k_Q. Since $\mathrm{vx}(X) \gneqq Q$, the AR-sequence $\mathcal{A}(X)\downarrow_Q$ restricted to Q splits. However Ωk_Q is a direct summand of $\Omega S\downarrow_Q$, which is a direct summand of the middle term of $\mathcal{A}(X)\downarrow_Q$, a contradiction. □

Corollary 2.2. *Let B a wild block of kG. Suppose that G is p-solvable and k is algebraically closed. Then all simple modules in B lie at the ends of the AR-components.*

Proof. Assume that some simple module does not lie at the end. Then by Theorem 1.5 and the result of Tsushima [10, Theorem], there is a finite group H with normal p-subgroup such that B and kH are Morita equivalent. However by Theorem 2.1 all simple kH-modules lie at the ends of the AR-components, a contradiction. □

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