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<th>A concentration phenomenon around a shrinking hole for solutions of mean field equations</th>
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<tr>
<td><strong>Author(s)</strong></td>
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<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 39(2) P.395-P.407</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>2002-06</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/7371">https://doi.org/10.18910/7371</a></td>
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<td><strong>DOI</strong></td>
<td>10.18910/7371</td>
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Osaka University
A CONCENTRATION PHENOMENON
AROUND A SHRINKING HOLE
FOR SOLUTIONS OF MEAN FIELD EQUATIONS

HIROSHI OHTSUKA

(Received June 26, 2000)

1. Introduction

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^2$. In this paper, we consider the following mean field equation in statistical mechanics of point vortices; see [6, 7, 15]:

$$\begin{align*}
-\Delta u &= \rho \frac{e^u}{\int_{\Omega} e^u} \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial \Omega.
\end{align*}$$

We note that the problem (P) for $\rho < 0$ is treated in [14]; see also [6, 7]. Analogous problems under Neumann boundary conditions are considered in relation to stationary problems of the Keller-Segel system of chemotaxis in [28]. Analogous problems on two-dimensional manifolds are also considered in relation to the prescribed Gauss curvature problem or Chern-Simons-Higgs gauge theory; see [12, 17, 26, 29] and references therein.

It should be also remarked that the following non-linear eigenvalue problem called the Gel’fand problem (see, for example, [3, 32]) also relates to our problem (P):

$$\begin{align*}
-\Delta u &= \lambda e^u \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial \Omega.
\end{align*}$$

Indeed, every solution of (G) corresponds to the solution of (P) for $\rho = \int_{\Omega} \lambda \exp u \, dx$.

(P) is the Euler-Lagrange equation of the following functional:

$$J_\rho(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \rho \log \int_{\Omega} e^u \quad \text{for} \quad u \in H_0^1(\Omega).$$

Caglioti et al. show the following facts on (P):
Facts 1.1 ([6]; see also [7]).

(1) From the Moser-Trudinger inequality [23],
\[ \inf_{u \in H^1_0(\Omega)} J_\rho(u) > -\infty \quad \text{for} \quad 0 < \rho \leq 8\pi. \]

Moreover, the problem (P) for \( 0 < \rho < 8\pi \) has a solution that minimizes \( J_\rho \).

(2) The disks admit no solution of (P) for every \( \rho \geq 8\pi \). More generally, let \( \Omega \) be a strictly star-shaped domain, that is, there exists a constant \( \alpha_0 > 0 \) such that \( (x \cdot \nu)(\int_{\partial \Omega} d\sigma)^{-1} \geq \alpha_0 \) on \( \partial \Omega \), where \( \nu \) is the exterior unit outer normal vector field to \( \partial \Omega \) and \( d\sigma \) is the arclength measure on \( \partial \Omega \). Then (P) admits no solutions if \( \rho \geq 4/\alpha_0 \) from the Pohožaev identity [27]. We note that \( \alpha_0 = 1/(2\pi) \) when \( \Omega \) is a disk.

(3) Each annulus admits the unique radial solution for every \( \rho \in \mathbb{R} \).

It should be remarked that parts of Fact 1.1 are already known as results on (G). Indeed, Bandle [3, Theorem 4.16] and Suzuki and Nagasaki [35, Lemma 3] obtained similar conclusions to Fact 1.1 (2) for (G) from the Pohožaev identity (see also [3, p. 201]). The existence of radial solutions of (G) on annuli was proved by Nagasaki and Suzuki [24] (see also [30, 32, 34]) and independently by Lin [19]. Their studies on the solutions are sufficient to obtain Fact 1.1 (3) for \( \rho > 0 \). We note that they also studied, in different ways, the existence of non-radial solutions of (G) on annuli. It should be also remarked that, in the course of the study of (G), Suzuki proved the unique existence of solutions of (P) when \( \Omega \) is simply connected and \( 0 < \rho < 8\pi \) [33] (see also [32, p. 263]).

We note that, on general domains other than disks and annuli, it is not clear whether a solution of (P) for \( \rho \geq 8\pi \) exists. Caglioti et al. proved the existence of a minimizer of \( J_{8\pi}(\cdot) \), that is, a solution of (P) for \( \rho = 8\pi \) when \( \Omega \) is sufficiently thin by analyzing the dual functional to \( J_{8\pi}(\cdot) \) [6, p. 523]. In this case, supposing additionally that \( \Omega \) is strictly star-shaped and admits the unique solution of (P) for \( \rho = 8\pi \), they also proved the existence of a sequence \( \rho_\eta \to 8\pi + 0 \) such that (P) for \( \rho_\eta \) has at least two solutions [7, Theorem 7.1]. On the other hand, when \( \Omega \) is simply connected and satisfies some additional conditions, we know the existence of the Weston branch of large solutions \( (\lambda, u_\lambda) \) of (G) for sufficiently small \( \lambda \) [36], which blows up at one point in \( \Omega \) as \( \lambda \to 0 \). We note that Moseley [22] and subsequently Suzuki [31] (see also [32, Section 3.4]) reduced some sufficient conditions on \( \Omega \) to construct the branch. Suzuki and Nagasaki proved that the Weston branch satisfies
\[ \int_{\Omega} \lambda e^{u_\lambda} dx = 8\pi + C\lambda + o(\lambda) \quad \text{as} \quad \lambda \to 0, \]
where \( C \) is a constant determined by a conformal mapping \( B_1(0) \) onto \( \Omega \) [35, Appendix I] (see also [32, Proposition 4.36]). This formula indicates that, on the domains satisfying \( C > 0 \), the solutions of (P) for \( \rho > 8\pi \) and sufficiently close to \( 8\pi \)
exist. Moreover, Mizoguchi and Suzuki proved that the Weston branch and the trivial solution $(\lambda_n, u_n) = (0, 0)$ of (G) are connected under the additional conditions on $\Omega$ [21, Theorem 13]. This result indicates additionally the existence of solutions of (P) for $\rho = 8\pi$ as well as for $\rho > 8\pi$ and sufficiently close to $8\pi$ on the appropriate domains, an example of which is given in [21, pp. 207–208]. We note that this example is also thin in some sense. It should be remarked that Nagasaki and Suzuki [25] (see also [32, Section 3.3]) proved that, when a family of solutions $\{ (\lambda_n, u_n) \}$ of (G) on a general domain (not necessarily a simply connected one) satisfies $\lambda_n \to 0$ and $\int_\Omega \lambda_n \exp u_n \, dx \to \Sigma_0$ as $n \to \infty$, the limit $\Sigma_0$ must be $8\pi m$ for some $m \in \{ 0, \infty \} \cup \mathbb{N}$. They also proved that, when $m \in \mathbb{N}$, the solution $u_n$ of (G) blows up at distinct $m$ points in $\Omega$ as $n \to \infty$ and obtained several necessary conditions of the limiting function of $u_n$. We note that this result resembles the later results of Brezis and Merle [5] and Li and Shafrir [18], which we refer as Fact 2.5 in this paper.

Recently, Baraket and Pacard [4] considered the converse problem to this result of Nagasaki and Suzuki [25]. Baraket and Pacard gave, for each $m \in \mathbb{N}$, a sufficient condition of limiting functions that enables us to construct a one-parameter family of solutions $\{ (\lambda, u_\lambda) \}$ of (G) satisfying that $\int_\Omega \lambda \exp u_\lambda \, dx \to 8\pi m$ and $u_\lambda$ converges to such a limiting function as $\lambda \to 0$. This result also suggests the existence of solutions of (P) for $\rho = 8\pi m$ in the appropriate domains for each $m \in \mathbb{N}$.

Recently, Ding et al. proved the following fact by the minimax variational method:

**Fact 1.2** ([12]). On every smooth bounded domain whose complement contains a bounded region, that is, on every smooth bounded domain with a hole, the mean field equation (P) has a solution for all $\rho \in (8\pi, 16\pi)$.

The purpose of this paper is to investigate the behavior of this solution as the hole of the domain shrinks to a point. To simplify the presentation, assuming that $0 \in \Omega$, we study the behavior of solutions of (P) on $\Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon(0)}$ as $\varepsilon \to 0$, where $B_\varepsilon(0) = \{ x \in \mathbb{R}^2 : |x| < \varepsilon \}$. We refer (P) for $\Omega_\varepsilon$ as (P)$_\varepsilon$ and the functional $J_\rho(\cdot)$ on $H_0^1(\Omega_\varepsilon)$ for (P)$_\varepsilon$ as $J_\rho^\varepsilon(\cdot)$.

Here we recall the minimax method used in [12] for the case (P)$_\varepsilon$. Let $D_\rho^\varepsilon$ be a family of continuous functions $h : B_1(0) = \{ (r, \theta) : 0 \leq r < 1, \theta \in [0, 2\pi) \} \to H_0^1(\Omega_\varepsilon)$ satisfying

$$
\lim_{r \to 1} J_\rho^\varepsilon( (r, \theta) ) = -\infty 
$$

and

$$
\lim_{r \to 1} m_{\Omega_\varepsilon}( h(r, \cdot) ) \text{ is a continuous curve enclosing } B_\varepsilon(0),
$$
where

\[ m_{\Omega_e}(u) = \int_{\Omega_e} x \frac{e^{\varepsilon(x)}}{\int_{\Omega_e} e^{\varepsilon(x)}}. \]

Ding et al. proved that for every \( \rho \in (8\pi, 16\pi) \) the minimax value

\[ \alpha_{\rho}^\varepsilon = \inf_{h \in D_r} \sup_{u \in hB(0)} J_{\rho}^\varepsilon(u) \]

is achieved by a critical point of \( J_{\rho}^\varepsilon \) in \( H_0^1(\Omega_e) \), which is a solution of \( (P_\varepsilon) \).

In the following, we assume each element of \( H_0^1(\Omega_e) \) to be an element of \( H_0^1(\Omega) \) by extending it by 0 on \( B_\varepsilon(0) \). Our result is stated as follows:

**Main Theorem.** Fix \( \rho \in (8\pi, 16\pi) \), a sequence \( \varepsilon_n \downarrow 0 \) as \( n \to \infty \), and a solution \( u_n \) of \( (P_{\varepsilon_n}) \) that attains the minimax value \( \alpha_{\rho}^{\varepsilon_n} \). Then,

\[ \frac{e^{\varepsilon_n}}{\int_{\Omega} e^{\varepsilon_n}} \to \delta_0 \text{ weakly } * \text{ in } M(\Omega) \text{ as } n \to \infty, \]

where \( M(\Omega) = C(\Omega)^* \) denotes the space of signed Radon measures over the compact space \( \Omega \) and \( \delta_0 \) denotes the Dirac measure supported at the origin \( 0 \in \Omega \).

We note that Lewandowski [16] obtained a concentration phenomenon similar to our Main Theorem in the following higher dimensional problem with the critical Sobolev exponent:

\[
\begin{align*}
(P') \quad -\Delta u &= u^{(N+2)/(N-2)} \quad \text{in } \Omega \subset \mathbb{R}^N \quad \text{for } N \geq 5, \\
\quad u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Assuming that \( \Omega \) is a smooth bounded star-shaped domain and \( 0 \in \Omega \), Lewandowski considered \( (P') \) also on the domain \( \Omega_e = \Omega \setminus \overline{B_\varepsilon(0)} \). We note that Coron [9] proved that \( \Omega_e \) admits a solution of \( (P') \) for sufficiently small \( \varepsilon \); see also [2] for the more general existence result for \( (P') \). Let \( u_\varepsilon \) be a solution of \( (P') \) on \( \Omega_e \) satisfying the appropriate conditions. Then Lewandowski proved that \( |\nabla u_\varepsilon|^2 \to (S_N)^{N/2}\delta_0 \) as \( \varepsilon \to 0 \), where \( S_N \) is the best constant in the Sobolev inequality, that is, \( S_N = \inf \{ \int_{\mathbb{R}^N} |\nabla u|^2 : u \in H^1(\mathbb{R}^N), \|u\|_{L^{2N}/N-2(\mathbb{R}^N)} = 1 \} \).

In contrast to our results, Lewandowski proved more on the behavior of \( u_\varepsilon \) as \( \varepsilon \to 0 \). Indeed, let \( u_\varepsilon(x) \) be a blow-up around an appropriate point \( a_\varepsilon \in \mathbb{R}^N \), that is, \( u_\varepsilon(x) = t_\varepsilon^{(N-2)/2} u_\varepsilon(t_\varepsilon(x + a_\varepsilon)) \) for appropriately chosen \( t_\varepsilon \in (0, 1) \) satisfying \( t_\varepsilon \to 0 \). Then \( u_\varepsilon(x) \) converges to a solution of \( (P') \) for \( \Omega = \mathbb{R}^N \) in an appropriate topology.

Thus, also for our problem, it is natural to ask more precise behavior of \( u_n \) itself. It seems interesting to study the behavior of \( u_n \) by the blow-up analysis for \( (P) \) developed by Li and Shafrir [18], though the author now thinks that it seems difficult.
2. Proof of Main Theorem

The key of the proof of Main Theorem is the following estimate on the minimax value $\alpha_\rho^\varepsilon$:

**Proposition 2.1.** For every $\rho \in (8\pi, 16\pi)$,

$$\alpha_\rho^\varepsilon \longrightarrow -\infty \quad \text{as} \quad \varepsilon \longrightarrow 0.$$  

Assuming this proposition, which we prove in Section 3, we prove Main Theorem in this section.

Set

$$\mu_n(x) = \frac{e^{\mu_n(x)}}{\int_{\Omega} e^{\mu_n(x)} dx}.$$  

We regard $\{\mu_n\}$ as a bounded set in $M(\Omega)$. Thus, choosing a subsequence if necessary, we may assume that

$$\mu_n \longrightarrow \mu_\infty \text{ weakly * in } M(\Omega) \text{ as } n \longrightarrow \infty$$

for some $\mu_\infty \in M(\Omega)$. In the rest of this section, we prove $\mu_\infty$ is always $\delta_0$, which implies that $\mu_n \longrightarrow \delta_0$ without choosing a subsequence, that is, we obtain Main Theorem.

We prove $\mu_\infty = \delta_0$ by the following three steps:

**STEP 1.** $\mu_\infty = \delta_{x_\infty}$ for some $x_\infty \in \Omega$.

**STEP 2.** $x_\infty \notin \partial \Omega$.

**STEP 3.** $x_\infty \notin \Omega \setminus \{0\}$, that is, $x_\infty = 0$.

We start from recalling the improved Moser-Trudinger inequality:

**Fact 2.2** ([12, Lemma 2.2]; see also [1, Théorème 4] and [8, Theorem 2.1]). Let $S_1$ and $S_2$ be two subsets of $\Omega$ satisfying $\text{dist}(S_1, S_2) \geq \delta_0 > 0$ and let $\gamma_0$ be a number satisfying $\gamma_0 \in (0, 1/2)$. Then for any $\varepsilon > 0$, there exists a constant $c = c(\varepsilon, \delta_0, \gamma_0) > 0$ such that

$$\int_{\Omega} e^u \leq c \exp \left\{ \frac{1}{32\pi - \varepsilon} \int_{\Omega} |\nabla u|^2 + c \right\}$$

holds for all $u \in H^1_{0}(\Omega)$ satisfying

$$\frac{\int_{S_1} e^u}{\int_{\Omega} e^u} \geq \gamma_0 \quad \text{and} \quad \frac{\int_{S_1} e^u}{\int_{\Omega} e^u} \geq \gamma_0.$$ 

From Fact 2.2, we obtain the following lemma:
**Lemma 2.3.** Suppose that a sequence \( \{u_n\} \subset H_0^1(\Omega) \) satisfies 
\[
J_\rho(u_n) \to -\infty \quad \text{and} \quad m_{\Omega^1}(u_n) \left(= \int_{\Omega} x e^{u_n} \right) \to x_\infty \quad \text{as} \quad n \to \infty
\]
for some \( \rho \in (8\pi, 16\pi) \) and some \( x_\infty \in \mathbb{R}^2 \). Then \( x_\infty \in \bar{\Omega} \) and
\[
\mu_n \to \delta_{x_\infty} \quad \text{weakly} \star \quad \text{in} \quad M(\Omega).
\]

Although we are able to prove this lemma easily by similar argument to the proof of [12, Lemma 2.3], we give a proof of Lemma 2.3 in Appendix for convenience.

Proof of Step 1. It is obvious that \( J_\rho(u_n) \leq J_\rho^{\varepsilon_n}(u_n) \) because \( u_n \equiv 0 \) in \( B_{\varepsilon_n}(0) \). Thus \( J_\rho(u_n)(\leq J_\rho^{\varepsilon_n}(u_n) = O_\rho^{\varepsilon_n}) \to -\infty \) as \( n \to \infty \) from Proposition 2.1. On the other hand, there exists a subsequence of \( m_{\Omega^1}(u_n) \) that converges because \( \Omega \) is bounded. Using Lemma 2.3, we obtain the conclusion because \( \mu_n \to \mu_{x_\infty} \).

To make the next step, we recall the following fact from [10]:

**Fact 2.4** ([10, p. 51 (8’)]; see also [20, p. 628].) Let \( N \subset \mathbb{R}^2 \) be a neighborhood of \( \partial \Omega \) (not \( \partial \Omega^1 \)) and set \( \omega_0 = \Omega \cap N \). Then, there exist positive constants \( \varepsilon, \gamma \), and \( C \) depending on \( \partial \Omega \) and \( \omega_0 \) satisfying the following properties: \( \omega = \{ x \in \omega_0 : \text{dist}(x, \partial \Omega) < \varepsilon \} \) is a subset of \( \omega_0 \) and, for all \( x \in \omega \), there exists a measurable set \( I_x \) such that
\begin{enumerate}
  
  1. \( \text{meas}(I_x) \geq \gamma \),
  2. \( I_x \subset \{ y \in \omega_0 : \text{dist}(y, \partial \Omega) \geq \varepsilon / 2 \} \),
  3. \( u(x) \leq C u(\xi) \) for all \( \xi \in I_x \),
\end{enumerate}
where \( u \) is any \( C^2(\omega_0) \) function satisfying
\[
-\Delta u = f(u) \quad \text{and} \quad u > 0 \quad \text{in} \quad \omega_0 \cap \Omega \subset \Omega,
\]
\[
u = 0 \quad \text{on} \quad \omega_0 \cap \partial \Omega (= \partial \Omega)
\]
for some locally Lipschitz function \( f : \mathbb{R} \to \mathbb{R} \).

We note that Fact 2.4 is proved by the moving plane method established in [13].

Proof of Step 2. Fix a neighborhood \( N \subset \mathbb{R}^2 \) of \( \partial \Omega \) such that \( \Omega_{\varepsilon_n} \cap N \) is independent of \( n \). Applying Fact 2.4 to this \( N \), we obtain \( \omega \subset \bar{\Omega} \) satisfying the several properties stated in Fact 2.4. We prove below that \( \sup_n \| u_n \|_{L^\infty(\omega)} < \infty \), which prevents \( x_\infty \in \partial \Omega \) since \( \int_{\Omega} e^{u_n} \to \infty \) as \( n \to \infty \) from Proposition 2.1 and Fact 1.1 (1).

Let \( \omega_1 = \bigcup_{x \in \omega} I_x \subset \Omega \). Then we obtain that
\[
0 \leq u_n(x) \leq \frac{C}{\gamma} \int_{I_x} u_n(y)dy \leq \frac{C}{\gamma} \| u_n \|_{L^1(\omega_1)} \leq \frac{C}{\gamma} \| u_n \|_{L^1(\Omega)} \quad \text{for every} \quad x \in \omega,
\]
that is,
\[ \sup_n \| u_n \|_{L^\infty(\Omega)} \leq C \sup_n \| u_n \|_{L^1(\Omega)}. \]

It is rather standard to estimate \( \sup_n \| u_n \|_{L^1(\Omega)} \). Indeed, let
\[ \psi_n(x) = \int_{\Omega_n} \left( \frac{1}{2\pi} \log |x - y|^{-1} - \frac{1}{2\pi} \log[\text{diam}(\Omega)]^{-1} \right) \rho \mu_n(y) dy. \]

It is obvious that
\[ -\Delta \psi_n = \rho \mu_n = \rho \int_{\Omega_n} e^{u_n} \quad \text{in} \quad \Omega_n, \]
\[ \psi_n \geq 0 \quad \text{on} \quad \partial \Omega_n. \]

We note that \( \psi_n - u_n \) is harmonic in \( \Omega_n \) and non-negative on \( \partial \Omega_n \). Applying the maximum principle of harmonic functions to \( \psi_n - u_n \), we obtain \( \psi_n - u_n \geq 0 \), that is, \( \psi_n \geq u_n \) in \( \Omega_n \). Using the Young inequality for convolutions, we obtain
\[ \| u_n \|_{L^1(\Omega)} = \| u_n \|_{L^1(\Omega_n)} \leq \| \psi_n \|_{L^1(\Omega_n)} \leq \frac{\rho}{2\pi} \| \log |\cdot|^{-1} \|_{L^1(\Omega)} \cdot \| \mu_n \|_{L^1(\Omega_n)} + C' \]
\[ \leq C'' < \infty \]
for some constants \( C' \) and \( C'' \) independent of \( n \) because \( \| \mu_n \|_{L^1(\Omega_n)} \equiv 1 \).

To make the final step, we recall the results of [5] and [18] concerning the solutions of \( -\Delta u = V(x) \exp u \). Combining their results, we obtain the following fact:

**Fact 2.5** ([5, Theorem 3] and [18, Theorem]). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) and let \( \{ u_n \} \subset C(\Omega) \) be a sequence of solutions of
\[ -\Delta w = \rho e^w \quad \text{in} \quad \mathcal{D}'(\Omega) \]
for some \( \rho > 0 \) such that \( \sup_n \int_{\Omega} e^{u_n} < \infty \). Then, there exists a subsequence \( \{ u_{n_k} \} \) satisfying one of the following alternatives:

1. \( \{ u_{n_k} \} \) is bounded in \( L^\infty_{\text{loc}}(\Omega) \).
2. \( u_{n_k} \to -\infty \) uniformly on compact subsets of \( \Omega \).
3. there exists a finite non-empty blow-up set \( S = \{ a_1, \ldots, a_m \} \subset \Omega \) such that, for any \( i = 1, \ldots, m \), there exists \( \{ x_{n_k} \} \subset \Omega \) satisfying \( x_{n_k} \to a_i \), \( u_{n_k}(x_{n_k}) \to \infty \), and \( u_{n_k}(x) \to -\infty \) uniformly on compact subsets of \( \Omega \setminus S \). Moreover, \( \rho \exp(u_{n_k}(\cdot)) \to \sum_{i=1}^{m} 8\pi m_i \delta_{a_i} \) weakly in the sense of measures on \( \Omega \), where \( m_i \) is a positive integer for all \( i = 1, \ldots, m \).
It should be remarked that, prior to [5] and [18], an analogous result to Facts 2.5 for the asymptotic behavior of the solutions of (G) as \( \lambda \to 0 \) exists [25], which we mentioned in Section 1.

Proof of Step 3. Suppose \( x_\infty \in \Omega_e \setminus \{0\} \). Then, we are able to take \( R > 0 \) such that \( \Omega_e \supset B_R(x_\infty) \) for sufficient large \( n \). Let \( w_n(x) = u_n(x) - \log \int_{\Omega_e} u_n(x) dx \). Then this \( \{w_n(x)\} \) satisfies the assumptions of Fact 2.5 on the bounded domain \( B_R(x_\infty) \). Since \( \rho \exp(u_n) = \rho \mu_n \to \rho \delta_{x_\infty} \), only the alternative (3) is able to occur with \( S = \{x_\infty\} \) and \( \rho \) must be \( 8\pi m \) for some positive integer \( m \). Nevertheless, \( \rho \in (8\pi, 16\pi) \) from the hypothesis. This is a contradiction and we obtain \( x_\infty = 0 \).

\[ \Box \]

3. Estimate of the minimax value

To prove Proposition 2.1, it is enough to construct \( h_\varepsilon \in D^\rho_{B_0} \) such that

\[ \sup_{u \in H_e(B_0)} J^\varepsilon(u) \to -\infty \quad \text{as} \quad \varepsilon \to 0, \]

Fix \( s_0 > 0 \) and set

\[ u_s(t) = \begin{cases} 4 \log \frac{s_0}{s} & 0 \leq t \leq s, \\ 4 \log \frac{s_0}{t} & s \leq t \leq s_0, \\ 0 & s_0 \leq t. \end{cases} \]

We use \( u_s(t) \) to construct \( h_\varepsilon \). It is obvious that \( u_{s,p}(x) = u_s(|x - p|) \in H_0^1(\Omega_e) \subset H_0^1(\Omega) \) if \( B_{s_0}(p) \subset \Omega \). Moreover, we are able to obtain the following estimates:

**Proposition 3.1.** Suppose \( B_{s_0}(p) \subset \Omega_e \). Then we obtain

\[ \int_{\Omega_e} |\nabla u_{s,p}|^2 = \int_{B_{s_0}(0) \setminus B_0} |\nabla (u_s(|x|))|^2 = 32\pi \log \frac{s_0}{s}, \]

\[ \int_{\Omega_e} e^{\mu_{s,p}} \geq \int_{B_{s_0}(0) \setminus B_0} e^{u_s(|x|)} = \frac{1}{s^2} \pi s_0^4 \left[ 1 - \left( \frac{s}{s_0} \right)^2 \right], \]

for every \( 0 < s < s_0 \). Especially, we have

\[ \frac{e^{\mu_{s,p}}}{\int_{\Omega_e} e^{\mu_{s,p}}} \to \delta_p \quad \text{weakly} \ast \quad \text{in} \quad M(\tilde{\Omega}) \quad \text{as} \quad s \to 0, \]

\[ J^\varepsilon(u_{s,p}) \leq -2(\rho - 8\pi) \log \frac{1}{s} + O(1) \to -\infty \quad \text{as} \quad s \to 0, \]

where \( O(1) \) is independent of \( \varepsilon \) and \( p \).

Since we obtain Proposition 3.1 by elementary calculations, we omit the proof.
We are able to take positive numbers $R$ and $s_0 \leq R$ such that $B_{4R}(0) \setminus B_{2R}(0) \subset \Omega_\varepsilon$ for sufficiently small $\varepsilon$ and $B_{4R_0}(0) \subset \Omega$. Take $s = s(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that $\varepsilon \leq s \leq s_0$, which we specify later. We define

$$h_\varepsilon^0(r, \theta)(x) := \begin{cases} u_{\iota, \theta}(x) & 0 \leq r \leq \frac{1}{2}, \\ u_{2(1-r)\iota, \theta}(x) & \frac{1}{2} \leq r < 1, \end{cases}$$

where $p(r, \theta) = (r \cos \theta, r \sin \theta) \in \mathbb{R}^2$. From (3.4) and (3.5), it is easy to see that $h_\varepsilon^0(r, \theta)(\cdot)$ satisfies (1.1) and (1.2), though $h_\varepsilon^0(r, \theta)(\cdot) \notin H^1_0(\Omega_\varepsilon)$ if $r$ is small, that is, $h_\varepsilon^0(\cdot) \notin D^\varepsilon_\rho$ yet.

We introduce the following logarithmic cut-off function, which is also used in [11]:

$$\eta_\varepsilon(t) := \begin{cases} 0, & 0 \leq t \leq \varepsilon \\ -\frac{2 \log(t/\varepsilon)}{\log \varepsilon}, & \varepsilon \leq t \leq \sqrt{\varepsilon}, \\ 1, & \sqrt{\varepsilon} \leq t. \end{cases}$$

Let

$$h_\varepsilon(r, \theta)(x) := \eta_\varepsilon(|x|)h_\varepsilon^0(r, \theta)(x).$$

This $h_\varepsilon$ obviously belongs to $D^\varepsilon_\rho$ and we are able to prove the following fact:

**Proposition 3.2.** For every $\delta > 0$, if we take sufficiently small positive number $\sigma < 1/2$ and set $s = \varepsilon^\sigma (\geq \sqrt{\varepsilon} \geq \varepsilon)$, we obtain

$$\sup_{(r, \theta) \in B_1(0)} J^\varepsilon_\rho(h_\varepsilon(r, \theta)(x)) \leq -2\sigma \{ \rho - (1 + \delta)8\pi \} \log \frac{1}{\varepsilon} + O(1) \quad \text{as} \quad \varepsilon \to 0.$$

**Proof.** We note that $h_\varepsilon(r, \theta)(x) \equiv h_\varepsilon^0(r, \theta)(x)$ if $1/2 \leq r < 1$. From (3.5), we obtain that

$$J^\varepsilon_\rho(h_\varepsilon(r, \theta)) = J^\varepsilon_\rho(u_{2(1-r)\iota, \theta}(x)) \leq -2\rho - 8\pi \log \frac{1}{2(1-r)\iota} + O(1)$$

$$\leq -2\rho - 8\pi \log \frac{1}{\sqrt{s}} + O(1) \quad \text{as} \quad s \to 0 \quad \text{if} \quad \frac{1}{2} \leq r < 1. \quad (3.6)$$

For every $r \leq 1/2$ and every $\delta > 0$, we obtain

$$\int_{\Omega_\varepsilon} |\nabla h_\varepsilon(r, \theta)|^2$$

$$\leq \left( 1 + \frac{\delta}{2} \right) \int_{\Omega_\varepsilon} |\nabla h_\varepsilon^0(r, \theta)|^2 + C(\delta) \left( \sup_{x \in \Omega_\varepsilon} |h_\varepsilon^0(r, \theta)(x)| \right)^2 \int_{\Omega_\varepsilon} |\nabla (\eta_\varepsilon(|x|))|^2, \quad (3.7)$$
where $C(\delta)$ is a constant depending only on $\delta$. We note that $h_0^0(r, \theta)(x)$ is a translation of $u_0(|x|)$ if $0 \leq r \leq 1/2$ and $\text{supp} h_0^0(1/2, \theta) = B_{S_0}(p(2R, \theta)) \subset \Omega_{\varepsilon}$. Thus we obtain from (3.2) that

$$
\int_{\Omega_{\varepsilon}} |\nabla h_0^0(r, \theta)|^2 \leq 
\int_{\Omega_{\varepsilon}} \left| (\nabla h_0^0) \left( \frac{1}{2}, \theta \right) \right|^2
= \int_{\Omega_{\varepsilon}} |\nabla u_{5, p(2R, \theta)}|^2 = 32\pi \log \frac{1}{s} + O(1) \quad \text{as} \quad s \to 0.
$$

It is easy to see that

$$
\sup_{x \in \Omega_{\varepsilon}} |h_0^0(r, \theta)(x)| = \sup_{r} |u_{5}(r)| = 4 \log \frac{S_0}{s}
$$

and

$$
\int_{\Omega_{\varepsilon}} |\nabla (\eta_0(|x|))|^2 = \frac{4\pi}{\log(1/\varepsilon)}.
$$

Combining (3.7–10) and choosing $s = \varepsilon^\sigma$ for sufficiently small $\sigma \in (0, 1/2)$, we obtain

$$
\int_{\Omega_{\varepsilon}} |\nabla h_0^0(r, \theta)|^2 \leq 32\pi \left( 1 + \frac{\delta}{2} \right) \log \frac{1}{s} + C(\delta)' \frac{\log s}{\log \varepsilon} \frac{1}{s} + O(1),
$$

(3.11)

$$
\leq 32\pi \sigma (1 + \delta) \log \frac{1}{\varepsilon} + O(1) \quad \text{as} \quad \varepsilon \to 0,
$$

where $C(\delta)'$ is a constant independent of $\varepsilon$.

On the other hand, we obtain from (3.3) that

$$
\int_{\Omega_{\varepsilon}} e^{h_0^0(r, \theta)} \geq \int_{\Omega_{\varepsilon}} e^{h_0^0(0, \theta)} \geq \int_{B_{S_0}(0) \setminus B_{S_0}(0)} e^{h_0^0(|x|)}
\geq \frac{1}{S^2 \pi S_0^4} \left[ 1 - \left( \frac{S}{S_0} \right)^2 \right]
= \frac{1}{\varepsilon^{2\sigma} \pi S_0^4} \left[ 1 - \left( \frac{\varepsilon^\sigma}{S_0} \right)^2 \right].
$$

Combining (3.11–12), we obtain

$$
J_{\rho}^\varepsilon(h_0^0(r, \theta)) \leq -2\sigma \left( \rho - (1 + \delta)8\pi \right) \log \frac{1}{\varepsilon} + O(1) \quad \text{as} \quad \varepsilon \to 0 \quad \text{if} \quad 0 \leq r \leq \frac{1}{2}.
$$

(3.13)

Thus we obtain the conclusion from (3.6) and (3.13). \qed

Proof of Proposition 2.1. As we assumed that $\rho > 8\pi$, we are able to take a sufficiently small $\delta > 0$ such that $\rho - (1 + \delta)8\pi > 0$. Then $h_0^\varepsilon$ satisfies required property (3.1). \qed
Appendix. Proof of Lemma 2.3

It is enough to see that, for every sufficiently small $r > 0$, there exists $x_{r,\mathcal{H}} \in \bar{\Omega}$ such that

$$\int_{\bar{\Omega} \cap B_r(x_{r,\mathcal{H}})} \mu_n \geq 1 - r$$

if $n$ is sufficiently large.

(2.1) is equivalent to the inequality

$$\frac{\rho - 16\pi + (\varepsilon/2)}{32\pi - \varepsilon} \int_{\Omega} |\nabla u|^2 + J_\rho(u) \geq -\rho \log C - \rho C.$$ 

Since we assumed that $\rho < 16\pi$, we are able to take a sufficiently small $\varepsilon$ such that $\rho - 16\pi + (\varepsilon/2) < 0$. Then (A.2) with this $\varepsilon$ does not hold for $u_n$ with sufficiently large $n$ because $J_\rho(u_n) \to -\infty$. Accordingly, for every $\delta_0 > 0$, every two subsets $S_1$ and $S_2$ of $\bar{\Omega}$ satisfying $\text{dist}(S_1, S_2) \geq \delta_0 > 0$, and every $\gamma_0 \in (0, 1/2)$, we obtain

$$\min \left( \frac{\int_{S_1} e^{\mu_n}}{\int_{\Omega} e^{\mu_n}}, \frac{\int_{S_2} e^{\mu_n}}{\int_{\Omega} e^{\mu_n}} \right) = \min \left( \int_{S_1} \mu_n, \int_{S_2} \mu_n \right) < \gamma_0$$

if $n$ is sufficiently large.

Let $Q_n(r)$ be the concentration function of $\mu_n$, that is,

$$Q_n(r) = \sup_{x \in \bar{\Omega}} \int_{\Omega \cap B_r(x)} \mu_n.$$ 

For every $r > 0$, take $x_{r,\mathcal{H}} \in \bar{\Omega}$ such that $\int_{\bar{\Omega} \cap B_r(x_{r,\mathcal{H}})} \mu_n = Q_n(r/2)$. Applying (A.3) for $\delta_0 = r/2$, $S_1 = \Omega \cap B_{r/2}(x_{r,\mathcal{H}})$, and $S_2 = \bar{\Omega} \setminus B_r(x_{r,\mathcal{H}})$, we obtain that, for every $\gamma_0 \in (0, 1/2),

$$\min \left( \int_{S_1} \mu_n, \int_{S_2} \mu_n \right) = \min \left( Q_n \left( \frac{r}{2} \right), 1 - \int_{\bar{\Omega} \setminus B_r(x_{r,\mathcal{H}})} \mu_n \right) < \gamma_0$$

if $n$ is sufficiently large.

Since $\int_{\bar{\Omega}} \mu_n \equiv 1$, it is easy to see that there exists a constant $C$ independent of $n$ such that

$$Q_n(r) \geq Cr^2 \quad \text{for every} \quad 0 < r \leq \text{diam}(\Omega).$$

Taking sufficiently small $\gamma_0$ such that $Q_n(r/2) \geq \gamma_0 > 0$, we obtain (A.1) from (A.4). □

Acknowledgement. I would like to thank Professor Atsushi Inoue and Professor Takashi Suzuki for their useful suggestions.
References


[25] K. Nagasaki and T. Suzuki: *Asymptotic analysis for two-dimensional elliptic eigenvalue prob-
lems with exponentially dominated nonlinearities, Asymptot. Anal. 3 (1990), 173–188.


[27] S.I. Pohožaev: Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. 6 (1965), 1408–1411.


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