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SOME REMARKS ON PL COLLAPSIBLE COVERS OF 2-DIMENSIONAL POLYHEDRA

Eugenio BORGHINI

Abstract

We analyze the topology and geometry of a polyhedron of dimension 2 according to the minimum size of a cover by PL collapsible polyhedra. We provide partial characterizations of the polyhedra of dimension 2 that can be decomposed as the union of two PL collapsible subpolyhedra in terms of their simple homotopy type and certain local properties.

1. Introduction

The Lusternik-Schnirelmann (L-S) category of a topological space X is the minimum cardinality of a cover of X by open sets which are contractible in the space. It is a classical homotopy invariant of a space, introduced in [12], which has became over the years an important tool in homotopy theory (see [5] for a good account on the subject). A natural upper bound for the L-S category of a space X is provided by its *geometric category* gcat(X), defined as the minimum number of open contractible sets that cover X. For a polyhedron P (i.e. the underlying topological space of some simplicial complex), the geometric category coincides with the minimum number of contractible subpolyhedra that cover P.

In this note we propose to study a variant of the geometric category in the context of compact connected polyhedra, which we call *PL geometric category* and denote it by plgcat. For this invariant, we replace the purely topological notion of "contractible" in the definition of geometric category by the more geometrically flavored notion of "PL collapsible" (refer to Section 2 for precise definitions). This point of view allows to exploit certain combinatorial properties of a space that admits triangulations while at the same time accounts for its inherent geometry and topology. We show in first place that the PL geometric category of a polyhedron of dimension n is bounded by $n + 1$, thus generalizing the corresponding result for geometric category (cf. [5, Proposition 3.2]). This implies that the PL geometric category of a non PL collapsible polyhedron of dimension 2 may only be 2 or 3. One of our main objectives is to understand the topological and geometrical properties that distinguish 2-dimensional polyhedra P with plgcat(P) = 2 from those with plgcat(P) = 3. In this direction, we find that the condition of having PL geometric category 2 is fairly restrictive. In particular, it determines the simple homotopy type of the polyhedron: by Proposition 2.10 below, such a polyhedron is simple homotopy equivalent to a wedge sum of spheres of dimension 1 and 2. Moreover, it is not difficult to verify that a contractible polyhedron P of dimension 2 with plgcat(P) = 2 satisfies the Andrews-Curtis conjecture [1], which states

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that a compact contractible 2-dimensional polyhedron 3-deforms to a point (see Remark 2.11). However, as observed in Section 2, the PL geometric category is not a (simple) homotopy invariant of a polyhedron. This leads us to study this invariant also from a local point of view. In this context we describe a special class of 2-dimensional polyhedra, which we call *inner-connected polyhedra*, defined by a property satisfied among others by all closed surfaces. We obtain a criterion which states that sufficiently "regular" inner-connected polyhedra cannot have PL geometric category 2 (see Theorem 2.18).

Section 3 is devoted to the computation of the PL geometric category of complexes associated to one-relator presentations. The main result of this section, Theorem 3.4, shows that it is possible to read off the PL geometric category of such a complex directly from the presentation.

In the recent works [6], [2] discrete versions of the L-S category and related invariants were introduced in the setting of finite simplicial complexes and finite topological spaces. These discrete versions depend mainly on the combinatorial structure of the involved spaces. Our notion relies more strongly on the topology and geometry of the underlying spaces.

2. PL geometric category

In this section we introduce the notion of *PL geometric category* of a polyhedron. We collect some necessary definitions first.

By a *polyhedron* we understand a topological space which admits triangulations, i.e. the underlying space of some simplicial complex. A subspace Q of a polyhedron P is a *subpolyhedron* if it is the underlying space of a subcomplex of some triangulation of P. We recall next the basic definitions of Whitehead's simple homotopy theory. Let K be a finite simplicial complex. A simplex σ of K is a *free face* of K if there is a unique simplex $\tau \in K$ containing σ. In that case, we say that there is an *elementary collapse* from K to $L = K \setminus \{\sigma, \tau\}$, denoted $K \searrow^e L$. More generally, K *collapses* to L, denoted by $K \searrow L$, if there is a sequence $K_1 = K, K_2, \ldots, K_r = L$ such that $K_i \searrow^e K_{i+1}$ for every i. We also say that L expands to K and denote $L \nearrow K$. The complex K is called *collapsible* if it collapses to a complex with only one vertex. A pair of simplicial complexes K and L are *simple homotopy equivalent* if there exists a finite sequence of complexes $K_1 = K, K_2, \ldots, K_r = L$ such that for every i either $K_i \nearrow^e K_{i+1}$ or $K_i \searrow^e K_{i+1}$. In that situation, we also say that there is an n-deformation from K to L if the dimension of complexes $K_1, \ldots K_r$ is at most n. A polyhedron P *PL collapses* to a subpolyhedron Q (and we still denote $P \searrow Q$) if there exist coherent triangulations K, L of P and Q respectively such that $K \searrow L$ (see [11, Ch.2]). A polyhedron P is called *PL collapsible* if it PL collapses to a point, (i.e. some simplicial complex that triangulates P collapses to a vertex).

The polyhedra that we work with are assumed to be compact and connected. Likewise, the simplicial complexes are assumed to be finite and connected.

DEFINITION 2.1. Let P be a polyhedron. The *PL geometric category* plgcat(P) of P is the minimum number of PL collapsible subpolyhedra that cover P.

It is a well-known fact that the geometric category of a (compact, connected) polyhedron P of dimension n is at most $n + 1$. We will show an analogous result for PL geometric category, namely, that a polyhedron of dimension n is covered by at most $n + 1$ PL collapsible

subpolyhedra. The strategy for proving this, similarly as in the proof of the geometric category version, is to proceed by induction on the dimension of the polyhedron. However, for the inductive step to work in our context, a slight technical detour is needed. Specifically, we resort to the theory of strong homotopy types of [4].

DEFINITION 2.2 ([4]). Let K be a simplicial complex and $v \in K$ a vertex. We say that v is dominated by a vertex $v' \neq v$ if every maximal simplex that contains v also contains v'. If v is dominated by some vertex v', we say that there is an *elementary strong collapse* from K to $K \setminus v$ and denote $K \searrow\!\!\!\searrow^e K \setminus v$. In that situation we also say that there is an *elementary strong expansion* from $L = K \setminus v$ to K and denote it by $L^e \nearrow\!\!\!\nearrow K$. If there is a sequence of elementary strong collapses that starts in K and ends in L, we say that there is a *strong collapse* from K to L and denote $K \searrow\!\!\!\searrow L$. The inverse of a strong collapse is called a *strong expansion* and denoted by $L \nearrow\!\!\!\nearrow K$.

REMARK 2.3 ([4, Remark 2.4]). $K \searrow\!\!\!\searrow L$ implies that $K \searrow L$.

Recall that the *star* of a vertex v in a simplicial complex K is the subcomplex $\mathrm{st}_K(v) \subseteq K$ formed by the union of the simplices $\sigma \in K$ such that $\sigma \cup v \in K$. The *link* of v is the subcomplex $\mathrm{lk}_K(v) \subseteq \mathrm{st}_K(v)$ of the simplices that do not contain v. For a given simplex σ, its *boundary* $\dot\sigma$ is the subcomplex formed by the simplices τ strictly contained in σ.

REMARK 2.4. A vertex v in a simplicial complex K is dominated by v' if and only if the link $\mathrm{lk}_K(v)$ is a simplicial cone with apex v', i.e. $\mathrm{lk}_K(v) = v'M$ for certain subcomplex M.

Lemma 2.5. *Let σ_n be the standard n-simplex. Consider the subcomplex of the second barycentric subdivision of σ_n defined as $K_n := \sigma_n'' \setminus \mathrm{st}_{\sigma_n''}(\{v\})$, where v is the barycenter of σ_n. Then K_n strong collapses to $(\dot\sigma_n)''$.*

Proof. We view the simplices of the second subdivision of σ_n as chains of simplices of σ_n' ordered by inclusion. That is, we think of a vertex of σ_n'' as a 0-chain $\{\tau\}$ where τ is some simplex of σ_n', an edge of σ_n'' as a 1-chain $\{\tau \subseteq \eta\}$ for some simplices τ, η of σ_n' and so on. Let $w \in \mathrm{lk}_{\sigma_n''}(\{v\})$ be a vertex. This means that there is a 1-simplex in σ_n'' with vertices $\{v\}$ and w, so that there is a chain of inclusion of simplices of σ_n' $\{\{v\} \subseteq w\}$. Suppose $w = \{e\}$, where $e = \{v, a\}$ is a 1-simplex of σ_n'. Then, any maximal simplex of σ_n'' containing w either contains $\{v\}$ or $\{a\}$. Since $\{v\} \notin K_n$, this shows that w is dominated by $\{a\}$ in K_n. Consider now the complex $\tilde{K}_n$ obtained from K_n by removing all the vertices of the form $\{e\}$ for e a 1-simplex of σ_n' containing v. Take a vertex $u \in \mathrm{lk}_{\sigma_n''}(\{v\}) \cap \tilde{K}_n$. Suppose that $u = \{\tau\}$ for some 2-simplex $\tau = \{v, a, b\}$ of σ_n'. Since $\tilde{K}_n$ does not contain vertices of the form $\{\{v, x\}\}$, nor the vertex $\{v\}$, any maximal simplex of $\tilde{K}_n$ containing u also contains $\{a, b\}$. Hence u is dominated by $\{a, b\}$ in $\tilde{K}_n$. By removing the vertices of $\mathrm{lk}_{\sigma_n''}(\{v\})$ in non-decreasing order of the dimension of the simplex of σ_n' that they represent, we see that $K_n \searrow\!\!\!\searrow (\dot\sigma_n)''$.   □

Lemma 2.6. *Let K, L be simplicial complexes such that $L \nearrow\!\!\!\nearrow K$. If L can be covered by n strong collapsible subcomplexes, so does K.*

Proof. Let $\{L_1, \ldots, L_n\}$ be a cover of L by n strong collapsible subcomplexes and assume that there is an elementary strong expansion from L to K, say $L = K \setminus v$ for a certain $v \in K$.

Let $v' \in K$ be a vertex that dominates v, so that $\mathrm{lk}_K(v) = v'M$ for some subcomplex M of L. For each $1 \leq i \leq n$, define the subcomplex K_i of K as

$$K_i = \begin{cases} L_i \cup v(v'M \cap L_i) \text{ if } v'M \cap L_i \neq \emptyset, \\ L_i \text{ otherwise.} \end{cases}$$

If $v'M \cap L_i$ is nonempty, then $v \in K_i$ and is clearly dominated by v' because $\mathrm{lk}_{K_i}(v) = v'(M \cap L_i)$. In any case, K_i strong collapses to L_i and is therefore strong collapsible. This shows that K is covered by n strong collapsible subcomplexes. The conclusion follows by induction on the number of elementary strong expansions from L to K.   □

We are now able to prove that the PL geometric category of a polyhedron of dimension n is bounded from above by $n + 1$. We will prove the following slightly stronger result.

Proposition 2.7. *Let K be a complex of dimension n. Then, the second barycentric subdivision K'' of K can be covered by $n + 1$ strong collapsible subcomplexes.*

Proof. Proceed by induction on n, the dimension of K. When $n = 1$, K is a simplicial graph. We show first that in this case K' admits a cover by two strong collapsible subcomplexes. In order to produce the strong collapsible cover, let T be a spanning tree of the graph K and note that each edge $e \in K \setminus T$ becomes the union of two edges in K', say $e = e_1 \cup e_2$. Consider the following subcomplexes of K':

$$K_1 = T' \cup \bigcup_{e \in K \setminus T} e_1 \,,\, K_2 = T' \cup \bigcup_{e \in K \setminus T} e_2.$$

As K_1, K_2 both strong collapse to T', they are strong collapsible and they clearly cover K'. Since their barycentric subdivisions are also strong collapsible, the base case is complete.

Let now K be a simplicial complex of dimension n. By inductive hypothesis, the second barycentric subdivision of the $(n-1)$-skeleton $\left(K^{(n-1)}\right)''$ of K can be covered by n strong collapsible subcomplexes $K_1, \ldots, K_n$. Let $v_1, \ldots, v_r$ be the barycenters of the maximal simplices of K. By Lemma 2.5, we see that $\left(K^{(n-1)}\right)'' \nearrow\nearrow K'' \setminus \bigcup_{i=1}^{r} \mathrm{st}_{K''}(\{v_i\})$ and so Lemma 2.6 implies that this last complex is covered by n strong collapsible subcomplexes. Since K'' is connected and $\mathrm{st}_{K''}(\{v_i\})$ is strong collapsible for every i, we can include their union in a strong collapsible subcomplex of K''.   □

As a consequence, in dimension 1 the PL geometric category only distinguishes trees (contractible graphs) from the rest of graphs. The first non trivial case is the class of 2-dimensional polyhedra. Since PL collapsible polyhedra are relatively well understood, the interest is centered in understanding the difference between polyhedra of PL geometric category 2 from those of PL geometric category 3. Our first step in this direction concerns the simple homotopy type of a polyhedron P of dimension 2 with $\mathrm{plgcat}(P) = 2$. By a result of C.T.C. Wall [13], a polyhedron P with $\mathrm{gcat}(P) = 2$ has the homotopy type of a finite wedge sum of spheres of dimension 1 and 2. We show that a polyhedron P with $\mathrm{plgcat}(P) = 2$ 3-deforms to the suspension of a graph.

Lemma 2.8. *Let K be a simplicial complex of dimension 2 which is covered by collapsible subcomplexes K_1, K_2. Then there is a 3-deformation from K to the suspension $\Sigma(K_1 \cap K_2)$ of $K_1 \cap K_2$.*

Proof. Cone off K_1, K_2 with vertices v_1, v_2. This gives an expansion $K \nearrow v_1 K_1 \cup v_2 K_2$. Collapse every new simplex based on a simplex contained in K_1 or K_2 but not in both. Hence, $K \nearrow v_1 K_1 \cup v_2 K_2 \searrow v_1(K_1 \cap K_2) \cup v_2(K_1 \cap K_2)$, which is the desired 3-deformation. $\square$

Lemma 2.9. *Let K be a collapsible simplicial complex of dimension* 2 *and L a subcomplex of K. If* $\dim L = 2$, *L collapses to a graph, i.e. a complex of dimension 1.*

Proof. Choose an ordering $\sigma_1, \sigma_2, \ldots, \sigma_r$ of the 2-simplices of K that induces a valid sequence of collapses. It is clear then that the first 2-simplex of L appearing in that list must have a free face in L and hence L collapses to a subcomplex with one fewer 2-simplex. By induction on the number of 2-simplices of L, it follows that L collapses to a graph. $\square$

Proposition 2.10. *Let P be a polyhedron of dimension* 2 *such that* $\mathrm{plgcat}(P) = 2$. *Then P 3-deforms to the suspension of a graph.*

Proof. Take a triangulation K of P covered by collapsible subcomplexes K_1, K_2. By Lemma 2.8, K 3-deforms to $\Sigma(K_1 \cap K_2) = v_1(K_1 \cap K_2) \cup v_2(K_1 \cap K_2)$ and by Lemma 2.9 $K_1 \cap K_2$ collapses to a 1-dimensional subcomplex G. It follows that $v_i(K_1 \cap K_2) \searrow v_i G$ for $i = 1, 2$, and hence K 3-deforms to the suspension of G. $\square$

REMARK 2.11. As a consequence of Proposition 2.10, the Andrews-Curtis conjecture is satisfied by contractible polyhedra which admit a cover by two PL collapsible subpolyhedra. Indeed, let P be a contractible polyhedron covered by collapsible subpolyhedra P_1, P_2. From the Mayer Vietoris sequence, the intersection $P_1 \cap P_2$ has trivial homology and by Lemma 2.9, $P_1 \cap P_2$ collapses to a tree. By Proposition 2.10, P 3-deforms to a point.

As it was to be expected, the property of having PL geometric category 2 is not a (simple) homotopy invariant of a polyhedron. To illustrate this point, we invoke the classical example used by Fox [7] to show that the geometric category is not a homotopy invariant. Let P_1 be the wedge sum of S^2 and two circles and let P_2 be the space obtained from S^2 by identifying three distinct points. Notice that P_1 and P_2 are simply homotopy equivalent (in fact, there is a 3-deformation from P_1 to P_2). By splitting every sphere in P_1 in two, we see that P_1 admits a cover by two PL collapsible subpolyhedra and hence $\mathrm{plgcat}(P_1) = 2$. On the other hand, since P_2 does not admit covers by two contractible subpolyhedra by [7, § 39], $\mathrm{plgcat}(P_2) = 3$.

Thus, the global simple homotopy type is not enough to characterize 2-dimensional polyhedra of PL geometric category 2. A study of a more local nature is required. In this context a special class of polyhedra of dimension 2, which we proceed to describe, appears naturally.

DEFINITION 2.12. Let K be a simplicial complex of dimension 2. We say that an edge of K is *inner* if it is a face of exactly two 2-simplices of K.

Recall that a simplicial complex K of dimension n is *homogeneous* or *pure* if all of its maximal simplices have dimension n.

Definition 2.13. Let K be a homogeneous 2-dimensional simplicial complex. We say that K is *inner-connected* if any pair of 2-simplices σ, τ of K is connected by a sequence of 2-simplices $\sigma = \eta_1, \eta_2, \ldots, \eta_r = \tau$ such that $\eta_i \cap \eta_{i+1}$ is an inner edge of K for each $1 \le i < r$. We call such a sequence an *inner sequence*. We say that a polyhedron P is inner-connected if one (=all) of its triangulations is inner-connected.

Recall that K is strongly connected if it is homogeneous and for every pair of 2-simplices σ, τ of K there is a sequence of 2-simplices of K $\sigma_1 = \sigma, \sigma_2, \ldots, \sigma_n = \tau$ such that $\sigma_i \cap \sigma_{i+1}$ is an edge of K for each i. Obviously, an inner-connected complex of dimension 2 is strongly connected. A strongly connected complex of dimension 2 is a *pseudosurface* if each of its edges is a face of at most two 2-simplices .

Example 2.14. Surfaces or more generally pseudosurfaces are inner-connected. The presentation complex associated to finite one-relator presentation in which every generator appears at least once in the relator is also inner-connected.

Consider a 2-dimensional polyhedron P which is the union of two collapsible subpolyhedra P_1, P_2. We know by Lemma 2.9 that the intersection $P_1 \cap P_2$ collapses to a graph. The main reason why inner-connected polyhedra are useful as a technical tool is the following: if P is assumed to be inner-connected, it is possible to deform P_1 and P_2 so that $P_1 \cap P_2$ *is a graph.*

Lemma 2.15. *Let K be an inner-connected and non collapsible simplicial complex of dimension 2. Suppose that K is the union of collapsible subcomplexes K_1, K_2. Then there exist collapsible subcomplexes L_1, L_2 such that $K = L_1 \cup L_2$ and $L_1 \cap L_2$ is 1-dimensional.*

Proof. Suppose $K_1 \cap K_2$ has at least one 2-simplex η. Since $K_1 \cap K_2$ is a proper subcomplex of K, we can find a 2-simplex not in $K_1 \cap K_2$ and an inner sequence joining it to η. Then there are 2-simplices σ, τ together with an inner edge $e = \sigma \cap \tau$ such that $\tau \in K_1 \cap K_2$ but $\sigma \notin K_1 \cap K_2$. Without loss of generality, suppose $\sigma \in K_1$. Then e is a free face of the complex K_2, which implies that we can remove τ from K_2. That is, the complexes K_1 and $\tilde{K}_2 = K_2 \setminus \tau$ form again a collapsible cover of K and $K_1 \cap \tilde{K}_2$ has one fewer 2-simplex than $K_1 \cap K_2$. It follows by induction that it is possible to find collapsible subcomplexes L_1, L_2 that cover K and intersect in a graph.   □

Even if a polyhedron P admits covers by two PL collapsible subpolyhedra intersecting in a graph, the possible structure of these intersection graphs is constrained by the local topology of P. More concretely, we show that the topology of small neighborhoods around a point that is a leaf of an intersection graph satisfies certain condition.

Definition 2.16 ([9]). Let K be a simplicial complex. A vertex v of K is a *bridge* if $K \setminus v$ has more connected components than K. We say that v is *splittable* if the link $\mathrm{lk}_K(v)$ has bridges. Note that it makes sense to say that a point in a polyhedron is splittable because this property depends only on the homeomorphism type of a small closed neighborhood around the point and not on a specific triangulation of the space.

The statement and proof of the following lemma are based on results from [3, 9].

Lemma 2.17. *Let K be a homogeneous complex of dimension 2 which admits a collapsible cover of size two. Suppose additionally that the link of every non splittable vertex of K is connected. Then, there exist collapsible subcomplexes L_1, L_2 that cover K and such that every leaf of the 1-skeleton $(L_1 \cap L_2)^{(1)}$ of $L_1 \cap L_2$ is a splittable vertex of K.*

Proof. Let K_1 and K_2 be subcomplexes of K that form a collapsible cover of K. Take $\eta = vw \in (K_1 \cap K_2)^{(1)}$ an edge such that w is a leaf, i.e. $\mathrm{lk}_{(K_1 \cap K_2)^{(1)}}(w) = v$, but not a splittable vertex. Suppose in first place that η is not maximal in either of the subcomplexes K_1, K_2, so that there exist vertices $v_i \in K_i$ with $vwv_i \in K_i$ for $i = 1, 2$. As w is not a splittable vertex, we can find a path joining v_1 and v_2 in $\mathrm{lk}_K(w) \setminus v$. But then there must be at least another edge in $\mathrm{lk}_{K_1 \cap K_2}(w)$ contradicting the hypothesis that η is a leaf of $(K_1 \cap K_2)^{(1)}$. Suppose now η is maximal in K_1 and take $\tau = v_2\eta$ a 2-simplex of K_2 containing η (we can find one by homogeneity of K). We show that in this case K_1 collapses to $K_1 \setminus w$. If it was not the case, there should be another edge $\eta' \in K_1$ hanging from w. By the homogeneity of K, η' is the face of some 2-simplex $\sigma = v_1\eta'$ which per force is in K_1 but not in K_2. Since by hypothesis w is not splittable and has connected link, there is a path in $\mathrm{lk}_K(w) \setminus v$ joining v_1 to v_2 and so w cannot be a leaf of $(K_1 \cap K_2)^{(1)}$, a contradiction. By performing the collapses that correspond to edges in the second case, we may assume the the leaves of $(K_1 \cap K_2)^{(1)}$ are splittable vertices. $\square$

Consider again a 2-dimensional polyhedron P covered by collapsible subpolyhedra P_1, P_2. A straightforward computation using the (reduced) Mayer-Vietoris long sequence reveals that

$$\tilde{H}_0(P_1 \cap P_2) \equiv H_1(P), \quad H_1(P_1 \cap P_2) \equiv H_2(P),$$

where the homology groups are taken with coefficients in $\mathbb{Z}$. From Proposition 2.10, we know that $H_1(P)$ and $H_2(P)$ are finitely generated free abelian groups. Suppose that $\mathrm{rk}\, H_2(P) < \mathrm{rk}\, H_1(P)$. Since by Lemma 2.9 the polyhedron $P_1 \cap P_2$ collapses to a graph, at least two connected components of $P_1 \cap P_2$ are collapsible (because at least two of them are acyclic). When these components are graphs and P is homogeneous (for example, this is the case if P is inner-connected), by Lemma 2.17 its leaves should be located in splittable vertices or vertices with non connected links. Thus, P should have at least two such vertices. The conclusion reached in this paragraph is roughly that an inner-connected polyhedron which is regular both in a local and a global sense does not admit PL collapsible covers of size two.

Theorem 2.18. *Let P be an inner-connected polyhedron of dimension 2 such that $H_2(P) \equiv 0$ or $\mathrm{rk}\, H_2(P) < \mathrm{rk}\, H_1(P)$. Suppose additionally that P is not PL collapsible, has at most one splittable vertex and that the link of every non splittable vertex is connected. Then $\mathrm{plgcat}(P) = 3$.*

Proof. The case $\mathrm{rk}\, H_2(P) < \mathrm{rk}\, H_1(P)$ was already treated in the paragraph above. Suppose then $H_2(P) \equiv H_1(P) \equiv 0$ and that P is the union of PL collapsible subpolyhedra P_1, P_2 that intersect in a graph. Hence, $P_1 \cap P_2$ is a tree and since we may assume by Lemma 2.17 that its leaves are located in splittable vertices, $P_1 \cap P_2$ should be a point. It follows that P is a wedge sum of PL collapsible polyhedra, which contradicts the hypothesis that P be inner-connected. $\square$

Example 2.19. The dunce hat D is an inner-connected contractible polyhedron with only one splittable vertex and such that every other vertex has connected link. Hence, by Theorem 2.18 no triangulation of D admits a cover by two collapsible subcomplexes. In fact, we can say a little more. The dunce hat D can be viewed as the presentation complex associated to the one-relator presentation $\langle\, a\,|\,aaa^{-1}\,\rangle$ (see the first paragraph of Section 3). More generally, by Theorem 2.18 none of the presentation complexes associated to a presentation of the form $\langle\, a\,|\,a^n a^{-(n-1)}\,\rangle$ ($n \geq 2$) admits a cover by two PL collapsible subpolyhedra.

We remark that this example generalizes a result from [8], where the authors show that a specific triangulation of the dunce hat cannot be written as the union of two collapsible subcomplexes.

Example 2.20. The standard Bing's house with two rooms admits a PL collapsible cover of size two (to see this, split the complex in two halves, each one containing the walls which support the vertical tunnels). Note that unlike the dunce hat, the Bing's house with two rooms is not inner-connected. Moreover, as a consequence of the proof of Theorem 2.18 it is impossible to cover this polyhedron by two PL collapsible subpolyhedra intersecting in a graph.

3. The geometry of one-relator presentations

We use the results of the previous section to provide a complete characterization of one-relator presentation complexes that admit a PL collapsible cover of size two.

Recall that associated to a finite presentation $\mathcal{P} = \langle\, X\,|\,R\,\rangle$ there is a topological model built as follows. Let $K = \vee_{x \in X} S^1_x$ be a wedge sum of 1-spheres indexed by X. Every word $r \in R$ spells out a combinatorial loop on the space K based on the wedge point, which is used to attach a 2-cell on K. The resulting 2-dimensional CW-complex is called the *presentation complex* of $\mathcal{P}$ and is denoted by $K_{\mathcal{P}}$. Since the attaching maps are combinatorial, the presentation complex $K_{\mathcal{P}}$ is a polyhedron (see [10, Chapter 2] for more details). When the set R consists of only one word r the presentation $\langle\, X\,|\,r\,\rangle$ is called a *one-relator presentation*.

In what follows, we will assume that the one-relator presentation complexes are homogeneous, that is, every generator appears in the relator. There is no loss of generality in this assumption. Indeed, if it was not the case, the associated complex $K_{\mathcal{P}}$ would decompose as a wedge sum of a bouquet of 1-spheres and a homogeneous one-relator complex $K_{\mathcal{Q}}$. It is easy to see then that to compute $\mathrm{plgcat}(K_{\mathcal{P}})$, it is enough to compute $\mathrm{plgcat}(K_{\mathcal{Q}})$.

Lemma 3.1. *Let K be a connected simplicial complex and let $L_1, \ldots, L_n$ be disjoint collapsible subcomplexes of K. Then there exists a collapsible subcomplex of K containing $\bigcup_{i=1}^{n} L_i$.*

Proof. Since K is connected and L_1 is disjoint with $\bigcup_{i=2}^{n} L_i$, there exists a simple path p in the 1-skeleton of K joining a vertex of L_1 with a vertex of some L_i ($i \neq 1$) with no edges in $\bigcup_{i=1}^{n} L_i$. Consider the subcomplex M of K defined as $M := L_1 \cup p \cup L_i$. Since L_1 and L_i are collapsible, they collapse to any of its vertices and hence M collapses to p which is in turn collapsible. The result now follows from induction. $\square$

Proposition 3.2. *Let $\mathcal{P} = \langle x_1, \ldots, x_k \,|\, r \rangle$ be a finite one-relator presentation and suppose that r admits an algebraic collapse, that is, there is a generator x which occurs only once in r with exponent ± 1. Then $K_{\mathcal{P}}$ admits a cover by two PL collapsible subpolyhedra, that is, $\mathrm{plgcat}(K_{\mathcal{P}}) \leq 2$.*

Proof. We may assume that $x = x_1$ and $r = x^{\pm 1} a_1 \ldots a_{m-1}$, where each a_i is equal to some $x_j^{\pm 1}$, $j \neq 1$. Picture the complex $K_{\mathcal{P}}$ as a disk with the boundary subdivided in m edges labeled in counterclockwise order according to r. Subdivide the edge labeled x in $2(m-1) + 1$ edges and subdivide the rest of the edges in three edges. Join the $2i$-th edge of the subdivided x to the central edge of (the edge labeled as) a_i by a 2-dimensional strip inside the disk in such a way that the strips are pairwise disjoint (see Figure 1).
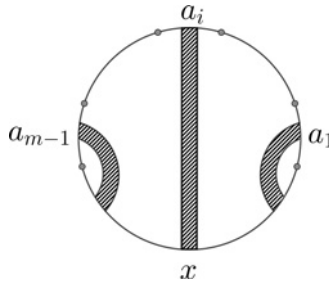

Fig. 1. The strips (shaded) PL collapse to a tree through the edge which intersects the edge labeled x.

Both the subpolyhedron P_1 formed by the union of these strips and its complement P_2 consist of a disjoint union of PL collapsible subpolyhedra of $K_{\mathcal{P}}$. Hence by Lemma 3.1, P_1 and P_2 may be included in PL collapsible polyhedra Q_1 and Q_2 that cover $K_{\mathcal{P}}$. □

The presentation complex of a (homogeneous) one-relator presentation is an inner-connected polyhedron and each of its points has a connected link, except possibly the wedge point. Moreover, if such a complex admits no algebraic collapses only the wedge point may be splittable. It is a consequence of Theorem 2.18 that most such complexes do not admit PL collapsible covers of size two.

Proposition 3.3. *Let $\mathcal{P} = \langle x_1, \ldots, x_k \,|\, r \rangle$ be a finite one-relator presentation such that r does not admit algebraic collapses. Then $\mathrm{plgcat}(K_{\mathcal{P}}) = 2$ if and only $\mathcal{P}$ is of the form $\langle x \,|\, (xx^{-1})^{\pm 1} \rangle$.*

Proof. By cellular homology, the group $H_2(K_{\mathcal{P}})$ is free abelian of rank at most 1. Moreover, by a straightforward Euler characteristic computation we know that

$$\mathrm{rk}\, H_2(K_{\mathcal{P}}) - \mathrm{rk}\, H_1(K_{\mathcal{P}}) = 1 - k.$$

Hence, if $\mathcal{P}$ has $k > 1$ generators, we have $\mathrm{rk}\, H_1(K_{\mathcal{P}}) < \mathrm{rk}\, H_2(K_{\mathcal{P}})$ and the conclusion follows from Theorem 2.18. The case $H_2(P) \equiv 0$ is also covered by Theorem 2.18.

It remains then to analyze the case of one-relator presentations with one generator and non trivial second homology group. Those are exactly the presentations of the form $\langle x \,|\, r \rangle$, where r is a word on letters x, x^{-1} with total exponent 0. Suppose that a triangulation of $K_{\mathcal{P}}$ admits a cover by collapsible subcomplexes K_1, K_2. We may assume that $K_1 \cap K_2$ is

a graph with at most one leaf. Since $H_0(K_1 \cap K_2) \equiv \mathbb{Z}^2$ and $H_1(K_1 \cap K_2) \equiv \mathbb{Z}$, one of the connected components of $K_1 \cap K_2$ is acyclic and therefore consists of only one point. For this to be possible, the link of the wedge point must have more than one connected component. By drawing the Whitehead graph of r (see [10, Ch.6]), we see that this is the case only for presentations of the form $\langle x \,|\, (xx^{-1})^{\pm n} \rangle$, $n \in \mathbb{N}$. Call C the other connected component of $K_1 \cap K_2$. Since it is a connected graph with one cycle and no leaves, C is homeomorphic to S^1. Moreover, it is not difficult to show that the intersection of this component with the loop x may by assumed to be 0-dimensional, that is, a finite set of points. Also, notice that the intersection $C \cap x$ is not empty. Indeed, suppose that the loop x is entirely contained in K_1 (the argument for K_2 is identical). Since the homology class determined by x is a generator of the first homology group $H_1(K_{\mathcal{P}})$, K_1 does not have trivial H_1, a contradiction. Let then v be a point in $C \cap x$ and let a, b the edges of the subdivision of x that contain v. Since the intersection of C with the loop x is 0-dimensional, we may assume that $a \in K_1 \setminus K_2$ and $b \in K_2 \setminus K_1$. The edges a, b are faces of $2n$ 2-simplices in any triangulation of $K_{\mathcal{P}}$. Furthermore, the (open) star of v is homeomorphic to a union of $2n$ half euclidean planes with the x axis identified. It follows that vertex v has valency $2n$ in the graph C. This is impossible unless $n = 1$. Finally, observe that the complex associated to a presentation of the form $\langle x \,|\, (xx^{-1})^{\pm 1} \rangle$ is homeomorphic to a 2-sphere with its poles identified and so admits a cover by two PL collapsible subpolyhedra. $\square$

As a corollary to Propositions 3.2 and 3.3 we obtain the announced characterization, which shows that the property of admitting a PL collapsible cover of size two is very restrictive for this class.

Theorem 3.4. *Let* $\mathcal{P} = \langle x_1, \dots, x_k \,|\, r \rangle$ *be a finite one-relator presentation. Then* $K_{\mathcal{P}}$ *can covered by two PL collapsible subpolyhedra if and only if* r *admits an algebraic collapse or* $\mathcal{P}$ *is of the form* $\langle x \,|\, (xx^{-1})^{\pm 1} \rangle$.

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