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# INTERSECTION NUMBER AND SOME METRICS ON TEICHMÜLLER SPACE

Zongliang SUN and Hui GUO

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**Abstract**

Let $T(X)$ be the Teichmüller space of a closed surface $X$ of genus $g \geq 2$, $C(X)$ be the space of geodesic currents on $X$, and $L : T(X) \to C(X)$ be the embedding introduced by Bonahon which maps a hyperbolic metric to its corresponding Liouville current. In this paper, we compare some quantitative relations and topological behaviors between the intersection number and the Teichmüller metric, the length spectrum metric and Thurston's asymmetric metrics on $T(X)$, respectively.

## 1. Introduction

We fix some notations first, for details see Section 2 and the references therein. Let $X$ be a closed surface of genus $g \geq 2$, $T(X)$ the Teichmüller space of $X$, and $C(X)$ the space of geodesic currents on $X$. Let $i : C(X) \times C(X) \to \mathbb{R}$ be the intersection number between geodesic currents, which is the generalization of geometric intersection number between (homotopy classes of) closed curves. It turns out that (see, e.g. [4, 5, 11]) much information about $C(X)$, such as the topology of $C(X)$, is governed by the intersection number. In [4], Bonahon established an embedding $L : T(X) \to C(X)$ of the Teichmüller space into $C(X)$, which maps a hyperbolic metric to the corresponding Liouville current. Moreover, Bonahon [4] was able to rebuild the Weil-Petersson metric and Thurston's compactification of the Teichmüller space with the aid of the intersection number between geodesic currents. In this paper, we make an attempt to study some aspects of the Teichmüller space by the intersection number, via Bonahon's embedding. We are mainly concerned with quantitative comparisons and topological behaviors of the intersection number and some metrics on $T(X)$.

There are many interesting metrics on $T(X)$, among them we have the Teichmüller metric $d_T$, the length spectrum metric $d_L$ and Thurston's asymmetric metrics $d_{P_i}$, $i = 1, 2$. These metrics have been studied by many authors [1, 2, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17]. As our first result, we describe some quantitative comparisons between the intersection number and these metrics as follows.

Throughout the paper, for notational convenience, we will frequently denote $L(\rho)$ by $L_\rho$ for $\rho \in T(X)$.

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**Theorem 1.1.** *Let $\sigma$ and $\rho$ be two points in $T(X)$, then*

$$\begin{aligned} i(L_\sigma, L_\rho) &\le \pi^2|\chi(X)|e^{d_{P_i}(\sigma,\rho)} \\ &\le \pi^2|\chi(X)|e^{d_L(\sigma,\rho)} \\ &\le \pi^2|\chi(X)|e^{d_T(\sigma,\rho)},\ i = 1, 2, \end{aligned}$$

*where $\chi(X)$ is the Euler characterization of X.*

Let $\{\rho_n\}_{n=0}^\infty$ be a sequence in $T(X)$. Theorem 1.1 implies that if $i(L_{\rho_n}, L_{\rho_0}) \to \infty$ $(n \to \infty)$, then each of the corresponding distance sequences (w.r.t. $d_T$, $d_L$ and $d_{P_i}$, $i = 1, 2$, respectively) will tend to infinity. Our second result implies that the converse to this is also true.

**Theorem 1.2.** *Let $\{\rho_n\}_{n=0}^\infty$ be a sequence in $T(X)$, then*

$$\begin{aligned} i(L_{\rho_n}, L_{\rho_0}) \to \infty &\Leftrightarrow d_T(\rho_n, \rho_0) \to \infty \\ &\Leftrightarrow d_L(\rho_n, \rho_0) \to \infty \\ &\Leftrightarrow d_{P_i}(\rho_n, \rho_0) \to \infty,\ i = 1, 2,\ n \to \infty. \end{aligned}$$

REMARK 1.3. In contrast to Theorem 1.1 and Theorem 1.2, we note that there do not exist inequalities which are inverse to those in Theorem 1.1. More precisely, none of $e^{d_T}$, $e^{d_L}$, $e^{d_{P_1}}$ and $e^{d_{P_2}}$ is bi-lipschitz to the intersection number. The reasoning of this observation is a little bit lengthy, we postpone the details to Section 5.

If a sequence $\{\rho_n\}_{n=0}^\infty$ leaves every compact subset in $T(X)$, we say it goes to infinity in $T(X)$. Note that this happens precisely when $d_T(\rho_n, \rho_0) \to \infty$ as $n \to \infty$. In particular, Theorem 1.2 provides equivalent characterizations for a sequence to go to infinity in $T(X)$, leading to the following direct corollary which was proved in [8, Theorem 2.25].

**Corollary 1.4.** *Let $\{\rho_n\}_{n=0}^\infty$ be a sequence in $T(X)$. As $n \to \infty$, if one of $d_T(\rho_n, \rho_0)$, $d_L(\rho_n, \rho_0)$ and $d_{P_i}(\rho_n, \rho_0)$ $(i = 1, 2)$ tends to infinity, then so do the others.*

Finally, as applications of Theorem 1.1, we make descriptions of various convergences in $T(X)$.

**Corollary 1.5.** *Let $\{\rho_n\}_{n=0}^\infty$ be a sequence in $T(X)$, then*

$$L_{\rho_n} \to L_{\rho_0} \Leftrightarrow d_{P_i}(\rho_n, \rho_0) \to 0,\ i = 1, 2,\ n \to \infty.$$

As a consequence of this corollary, we have the following

**Corollary 1.6.** *Let $\{\rho_n\}_{n=0}^\infty$ be a sequence in $T(X)$, then*

$$\begin{aligned} i(L_{\rho_n}, L_{\rho_0}) \to i(L_{\rho_0}, L_{\rho_0}) &\Leftrightarrow d_T(\rho_n, \rho_0) \to 0 \\ &\Leftrightarrow d_L(\rho_n, \rho_0) \to 0 \\ &\Leftrightarrow d_{P_i}(\rho_n, \rho_0) \to 0,\ i = 1, 2,\ n \to \infty. \end{aligned}$$

REMARK 1.7. Corollary 1.6 provides an approach to the topological equivalence of the Teichmüller metric, the length spectrum metric and Thurston's asymmetric metrics, which was studied in [2, 6, 12, 14], etc.

## 2. Preliminaries

**2.1. Metrics on Teichmüller space.** Let $X$ be a closed surface of genus $g \geq 2$. The Teichmüller space $T(X)$ is the space of equivalence classes $[S, f]$ of marked Riemann surfaces, where two marked Riemann surfaces $(S_1, f_1 : X \to S_1)$ and $(S_2, f_2 : X \to S_2)$ are equivalent if there exists a conformal mapping $h : S_1 \to S_2$ which is homotopic to $f_2 \circ f_1^{-1}$. By the uniformization theorem, $T(X)$ can also be viewed as the space of isotopy classes of hyperbolic metrics on $X$, where two hyperbolic metrics are isotopic if there exists an isometry between them which is isotopic to the identity.

The Teichmüller metric is defined by [1]

$$d_T([S_1, f_1], [S_2, f_2]) = \log\{\inf K(f)\},$$

where the infimum is taken over all $f : S_1 \to S_2$ in the homotopy class of $f_2 \circ f_1^{-1}$, and $K(f)$ is the maximal dilatation of $f$.

For a Riemann surface $S$, let $\gamma \subset S$ be a (homotopy class of an) essential closed curve and $l_S(\gamma)$ be the hyperbolic length of $\gamma$. Let $\Sigma_S$ be the set of homotopy classes of essential closed curves on $S$. The length spectrum metric $d_L$ is defined as [13, 14]

$$d_L([S_1, f_1], [S_2, f_2]) = \log \sup_{\gamma \in \Sigma_{S_1}} \{\frac{l_{S_2}(f(\gamma))}{l_{S_1}(\gamma)}, \frac{l_{S_1}(\gamma)}{l_{S_2}(f(\gamma))}\},$$

where $f = f_2 \circ f_1^{-1}$.

Thurston's asymmetric metrics $d_{P_1}$ and $d_{P_2}$ are defined as (see [17])

$$d_{P_1}([S_1, f_1],\ [S_2, f_2]) = \log \sup_{\gamma \in \Sigma_{S_1}} \{\frac{l_{S_2}(f(\gamma))}{l_{S_1}(\gamma)}\}$$

and

$$d_{P_2}([S_1, f_1],\ [S_2, f_2]) = \log \sup_{\gamma \in \Sigma_{S_1}} \{\frac{l_{S_1}(\gamma)}{l_{S_2}(f(\gamma))}\},$$

where $f = f_2 \circ f_1^{-1}$. Both $d_{P_1}$ and $d_{P_2}$ satisfy the positive definiteness and triangle inequality in the definition of a metric [17]. However, as observed by Thurston [17], they are not symmetric. In [7], $d_{P_i}$ $(i = 1, 2)$ is also called Thurston's pseudometric.

The classical Schwarz lemma implies that a conformal mapping preserves hyperbolic lengths. The length spectrum metric and Thurston's asymmetric metrics reveal the extent to which the converse to the Schwarz lemma holds.

The following distortion result of Sorvali and Wolpert is well-known.

**Lemma 2.1** ([13, 18])**.** *Let $f : S_1 \to S_2$ be a $K$-quasiconformal mapping between hyperbolic Riemann surfaces, then*

$$\frac{l_{S_2}(f(\gamma))}{l_{S_1}(\gamma)} \leq K(f)$$

*holds for all closed curves $\gamma \subset S_1$.*

From this lemma and the definitions, one gets directly a comparison of the above metrics.

**Lemma 2.2.**

$$d_{P_i} \le d_L \le d_T, \ i = 1, 2$$

**2.2. Geodesic current, intersection number and Bonahon's embedding.** Following [4], we introduce the concept of geodesic currents. Let $S$ be a Riemann surface, and $p : \tilde{S} \to S$ be the universal covering of $S$ with the induced hyperbolic metric on $\tilde{S}$ (thus making $\tilde{S}$ isometric to the Poincaré upper half plane $\mathbb{H}$). Let $S^1_\infty$ be the circle at infinity of $\tilde{S}$, and denote by $\mathcal{G}(\tilde{S})$ the space of un-oriented geodesics on $\tilde{S}$. Then $\mathcal{G}(\tilde{S})$ is homeomorphic to the open Möbius band $(S^1_\infty \times S^1_\infty - \triangle)/\sim$, where $\triangle \subset S^1_\infty \times S^1_\infty$ is the diagonal and $(\theta_1, \theta_2) \sim (\theta_2, \theta_1)$. The fundamental group $\pi_1(S)$ acts isometrically on $\tilde{S}$ and hence on $\mathcal{G}(\tilde{S})$. A geodesic current on $S$ is defined to be a positive Borel measure on $\mathcal{G}(\tilde{S})$ which is $\pi_1(S)$-invariant and locally finite.

The space $\mathcal{G}(\mathbb{H})$ of geodesics on $\mathbb{H}$ has a canonical measure which is known as the Liouville measure. For a hyperbolic metric $\rho \in T(X)$ which is represented by a Riemann surface $S$, its Liouville current $L_\rho$ is defined by considering the isometry from $\tilde{S}$ to $\mathbb{H}$ and the corresponding pull-back of the Liouville measure. Bonahon [4, Propositions 14 and 15] showed that

$$i(\alpha, L_\rho) = l_\rho(\alpha) \tag{2.1}$$

and

$$i(L_\rho, L_\rho) = \pi^2 |\chi(X)| \tag{2.2}$$

hold for every closed curve $\alpha$ and $\rho \in T(X)$.

The notion of geometric intersection number between (homotopy classes of) two essential closed curves plays an important role in Teichmüller theory and related topics. It extends [4] naturally to the notion of intersection number between two geodesic currents. Actually, if we let $\mathcal{C}(X)$ be the space of homotopy classes of closed curves and $C(X)$ be the space of geodesic currents on $X$ endowed with the weak* topology, then

**Theorem 2.3** ([3, 4])**.** *Both $\mathcal{C}(X)$ and $T(X)$ embed into $C(X)$.*

The embedding of $\mathcal{C}(X)$ to $C(X)$ is given by considering the Dirac measure, and the embedding of $T(X)$ to $C(X)$ is given by the mapping $L : T(X) \to C(X)$ which maps a hyperbolic metric $\rho$ to its Liouville current $L_\rho$.

**Theorem 2.4** ([3, 4])**.** *There is a continuous, symmetric, bilinear extension of the intersection number from $i : \mathcal{C}(X) \times \mathcal{C}(X) \to \mathbb{R}$ to $i : C(X) \times C(X) \to \mathbb{R}$.*

Recall that a measure has full support if its support is the entire space, or equivalently if every non-empty open subset has positive measure. As is well-known, it follows from the definition that Liouville currents for hyperbolic metrics have full support. A geodesic current $\mu \in C(X)$ binds [4, 10] if every geodesic on the universal covering $\tilde{X}$ intersects transversely a geodesic in the support of $\mu$, or equivalently if $i(\mu, \nu) > 0$ for every $\nu \in C(X), \nu \neq 0$. Therefore, for any $\rho \in T(X)$, the Liouville current $L_\rho$ binds.

In the sequel, we will not distinguish a closed curve $\gamma \in \mathcal{C}(X)$ from its image in $C(X)$, and we will use the same symbol $\gamma$ to denote each of them. We will also abuse the notations $L(\rho)$ and $L_\rho$ for $\rho \in T(X)$.

By virtue of the intersection number, Otal [11] give a criterion on separation of points in $C(X)$.

**Theorem 2.5** ([11]). *Let $\mu_1, \mu_2 \in C(X)$. Then $\mu_1 = \mu_2$ if and only if $i(\mu_1, \gamma) = i(\mu_2, \gamma)$ for every $\gamma \in C(X)$.*

The following interesting result of Thurston simplifies Otal's criterion (Theorem 2.5) on separation of points in $T(X)$ and also in $L(T(X))$.

**Theorem 2.6** ([4]). *Let $\sigma$ and $\rho$ be two points in $T(X)$. Then*

$$i(L_\sigma, L_\sigma) \leq i(L_\sigma, L_\rho).$$

*Furthermore, equality holds if and only if $\sigma$ and $\rho$ represent the same point in $T(X)$.*

As a corollary to this theorem, we have

**Corollary 2.7.** *Let $\{\rho_n\}_{n=0}^\infty$ be a sequence in $T(X)$. Then $L_{\rho_n} \to L_{\rho_0}$ if and only if $i(L_{\rho_n}, L_{\rho_0}) \to i(L_{\rho_0}, L_{\rho_0})$, $n \to \infty$.*

To see this corollary, for one thing, if $L_{\rho_n} \to L_{\rho_0}$, then from the continuity of the intersection number, $i(L_{\rho_n}, L_{\rho_0}) \to i(L_{\rho_0}, L_{\rho_0})$, $n \to \infty$. For the other, suppose $i(L_{\rho_n}, L_{\rho_0}) \to i(L_{\rho_0}, L_{\rho_0})$, $n \to \infty$. By a result of Bonahon [4, Proposition 4], since $L_{\rho_0}$ binds, these $\rho_n$ lie in a compact subset of $T(X)$. Let $\lambda_0 \in T(X)$ be the limit point of $\rho_n$. Then $i(L_{\lambda_0}, L_{\rho_0}) = i(L_{\rho_0}, L_{\rho_0})$. But Theorem 2.6 implies that the only possibility is $\lambda_0 = \rho_0$. Hence $\rho_n \to \rho_0$ and $L_{\rho_n} \to L_{\rho_0}$, $n \to \infty$.

## 3. Proof of Theorem 1.1

Proof. From Lemma 2.2, we only need to show

$$i(L_\sigma, L_\rho) \leq \pi^2 |\chi(X)| e^{d_{P_2}(\sigma,\rho)},$$

while the case for $d_{P_1}$ follows from the symmetry of the intersection number.

By (2.1) and the definition of $d_{P_2}$,

$$\frac{i(L_\sigma, \gamma)}{i(L_\rho, \gamma)} \leq e^{d_{P_2}(\sigma,\rho)}$$

holds for any closed curve $\gamma$. Together with the continuity and linearity of the intersection number, it follows from the density (see, e.g. [4]) of weighted closed curves in $C(X)$ that

$$\frac{i(L_\sigma, \mu)}{i(L_\rho, \mu)} \leq e^{d_{P_2}(\sigma,\rho)}$$

holds for any $\mu \in C(X)$. In particular, the above inequality holds for $\mu = L_\rho$. The proof follows in view of (2.2). □

We record the following corollary for later use.

**Corollary 3.1.** *Let $\{\rho_n\}_{n=0}^\infty$ be a sequence in $T(X)$. If $i(L_{\rho_n}, L_{\rho_0}) \to \infty$, then each of the sequences $\{d_T(\rho_n, \rho_0)\}_{n=0}^\infty$, $\{d_L(\rho_n, \rho_0)\}_{n=0}^\infty$ and $\{d_{P_i}(\rho_n, \rho_0)\}_{n=0}^\infty$ $(i = 1, 2)$ will tend to infinity as $n \to \infty$.*

## 4. Proof of Theorem 1.2

To prove Theorem 1.2, we need the following lemma which is converse to Corollary 3.1.

**Lemma 4.1.** *Let* $\{\rho_n\}_{n=0}^{\infty}$ *be a sequence in* $T(X)$*. If one of* $d_T(\rho_n,\rho_0)$*,* $d_L(\rho_n,\rho_0)$ *and* $d_{P_i}(\rho_n,\rho_0)$ $(i=1,2)$ *tends to infinity as* $n\to\infty$*, then* $i(L_{\rho_n},L_{\rho_0})$ *tends to infinity as* $n\to\infty$*.*

Proof. By Lemma 2.2, it suffices to prove the implication that $d_T(\rho_n,\rho_0)\to\infty$ implies $i(L_{\rho_n},L_{\rho_0})\to\infty$, $n\to\infty$. If $d_T(\rho_n,\rho_0)\to\infty$, then $\rho_n$ leaves every compact subset in $T(X)$. Hence under the embedding $L: T(X)\to C(X)$, $L_{\rho_n}$ leaves every compact subset in $C(X)$. We claim that $i(L_{\rho_n},L_{\rho_0})\to\infty$ as $n\to\infty$. To see this, assume the contrary that $i(L_{\rho_n},L_{\rho_0})\leq M$ for some constant $M$, for all $n$. Then all the $L_{\rho_n}$ are contained in the subset

$$\mathcal{A}=\{\mu\in C(X): i(\mu,L_{\rho_0})\leq M\}. \tag{4.1}$$

Because $L_{\rho_0}$ binds, from (4.1) and a result of Bonahon [4, Proposition 4], $\mathcal{A}$ is a compact subset in $C(X)$. Consequently, all the points $\rho_n$ lie in a compact subset in $T(X)$, a contradiction. □

Proof of Theorem 1.2. The proof is a combination of Corollary 3.1 and Lemma 4.1. □

## 5. Explanation to Remark 1.3

In this section, we give the promised explanation to Remark 1.3 in the introduction. Recall that Remark 1.3 asserts none of $e^{d_T}$, $e^{d_L}$, $e^{d_{P_1}}$ and $e^{d_{P_2}}$ is bi-lipschitz to the intersection number.

In [6], Li showed the existence of two sequences $\{\sigma_n\}_{n=1}^{\infty}$ and $\{\tau_n\}_{n=1}^{\infty}$ in $T(X)$, such that as $n\to\infty$,

$$d_L(\sigma_n,\tau_n)\to 0, \text{ while } d_T(\sigma_n,\tau_n)\to\infty.$$

In view of Theorem 1.1, this implies $e^{d_T}$ is not bi-lipschitz to the intersection number.

Liu [7] proved that Thurston's asymmetric metrics $d_{P_1}$ and $d_{P_2}$ are not bi-lipschitz to each other, and that the length spectrum metric is not bi-lipschitz to Thurston's asymmetric metrics. Following the proof in [7] with very slight modification, it can be easily shown that $d_{P_1}$ is not $(1,k)$-quasi-isometric to $d_{P_2}$ for any constant $k$, where two metrics $d_1$ and $d_2$ are $(l,k)$-quasi-isometric if

$$l\,d_2(\cdot,\cdot)-k\leq d_1(\cdot,\cdot)\leq l\,d_2(\cdot,\cdot)+k.$$

Suppose $e^{d_{P_1}}$ is bi-lipschitz to the intersection number. It then follows from Theorem 1.1 that there exists some $k$ such that

$$d_{P_1}(\sigma,\rho)\leq d_{P_2}(\sigma,\rho)+k$$

holds for $\sigma,\rho\in T(X)$. Since by the definitions $d_{P_1}(\sigma,\rho)=d_{P_2}(\rho,\sigma)$, we deduce from the above inequality that $d_{P_1}$ and $d_{P_2}$ are $(1,k)$-quasi-isometric, a contradiction. This shows $e^{d_{P_1}}$ is not bi-lipschitz to the intersection number. By the symmetry of the intersection number and the equality $d_{P_1}(\sigma,\rho)=d_{P_2}(\rho,\sigma)$, it follows that $e^{d_{P_2}}$ is not bi-lipschitz to the intersection number.

Finally, suppose $e^{d_L}$ is bi-lipschitz to the intersection number. Again from Theorem 1.1, we infer that $e^{d_{P_1}}$ is bi-lipschitz to the intersection number, a contradiction.

## 6. Proofs of Corollary 1.5 and Corollary 1.6

Proof of Corollary 1.5. We will show

$$d_{P_1}(\rho_n,\rho_0)\to 0 \Leftrightarrow L_{\rho_n}\to L_{\rho_0},$$

while

$$d_{P_2}(\rho_n,\rho_0)\to 0 \Leftrightarrow L_{\rho_n}\to L_{\rho_0}$$

can be proved similarly.

First, suppose

$$d_{P_1}(\rho_n,\rho_0)\to 0,\ n\to\infty.$$

By (2.2), Theorem 2.6 and Theorem 1.1, we have

$$\pi^2|\chi(X)|\le i(L_{\rho_n},L_{\rho_0})\le \pi^2|\chi(X)|e^{d_{P_1}(\rho_n,\rho_0)}.$$

Thus

$$\lim_{n\to\infty} i(L_{\rho_n},L_{\rho_0}) = i(L_{\rho_0},L_{\rho_0}).$$

We conclude from Corollary 2.7 that

$$L_{\rho_n}\to L_{\rho_0},\ n\to\infty.$$

Conversely, suppose

$$L_{\rho_n}\to L_{\rho_0},\ n\to\infty.$$

From the embedding $L : T(X)\to C(X)$ it follows that

$$d_T(\rho_n,\rho_0)\to 0,\ n\to\infty.$$

By Lemma 2.2, this implies

$$d_{P_1}(\rho_n,\rho_0)\to 0,\ n\to\infty.$$

□

Proof of Corollary 1.6. From Corollary 2.7 and the embedding $L : T(X)\to C(X)$, we know

$$i(L_{\rho_n},L_{\rho_0})\to i(L_{\rho_0},L_{\rho_0}) \Leftrightarrow d_T(\rho_n,\rho_0)\to 0,\ n\to\infty.$$

By Corollary 1.5 and Corollary 2.7,

$$i(L_{\rho_n},L_{\rho_0})\to i(L_{\rho_0},L_{\rho_0}) \Leftrightarrow d_{P_i}(\rho_n,\rho_0)\to 0,\ i=1,2,\ n\to\infty.$$

In view of Lemma 2.2, we conclude

$$i(L_{\rho_n}, L_{\rho_0}) \to i(L_{\rho_0}, L_{\rho_0}) \Leftrightarrow d_L(\rho_n, \rho_0) \to 0,\ n \to \infty.$$

□

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## References

[1] L.V. Ahlfors: Lectures on quasiconformal mappings: Second edition, with additional chapters by C.J. Earle, I. Kra, M. Shishikura and J.H. Hubbard, Providence, Rhode Island, USA: American Mathematical Society, 2006.

[2] D. Alessandrini, L. Liu, A. Papadopoulos, W. Su and Z. Sun: *On Fenchel-Nielsen coordinates on Teichmüller spaces of surfaces of infinite type*, Ann. Acad. Sci. Fenn. Math. A **36** (2011), 621–659.

[3] F. Bonahon: *Bouts des variétés hyperboliques de dimension 3*, Ann. of Math. **124** (1986), 71–158.

[4] F. Bonahon: *The geometry of Teichmüller space via geodesic currents*, Invent. Math. **92** (1988), 139–162.

[5] M. Duchin, C.J. Leininger and K. Rafi: *Length spectra and degeneration of flat metrics*, Invent. Math. **182** (2010), 231–277.

[6] Z. Li: *Length spectrums of Riemann surfaces and the Teichmüller metric*, Bull. London Math. Soc. **35** (2003), 247–254.

[7] L. Liu: *Asymmetry of Thurston's pseudometric*, Acta Sci. Natur. Univ. **39** (2000), 6–9.

[8] L. Liu and A. Papadopoulos: *Some metrics on Teichmüller spaces of surfaces of infinite type*, Trans. Amer. Math. Soc. **363** (2011), 4109–4134.

[9] L. Liu, Z. Sun and H. Wei: *Topological equivalence of metrics in Teichmüller space*, Ann. Acad. Sci. Fenn. Math. **33** (2008), 159–170.

[10] C. McMullen: *Teichmüller theory notes*, 2005, available at: http://www.math.harvard.edu/˜ctm/home/text/class/harvard/275/05/html/home/course/course.pdf

[11] J.-P. Otal: *Le spectre marqué des longueurs des surfaces à courbure négative*, Ann. of Math. **131** (1990), 151–162.

[12] A. Papadopoulos and G. Théret: *On the topology defined by Thurston's asymmetric metric*, Math. Proc. Cambridge Philos. Soc. **142** (2007), 487–496.

[13] T. Sorvali: *The boundary mapping induced by an isomorphism of covering groups*, Ann. Acad. Sci. Fenn. Math. **526** (1972), 1–31.

[14] T. Sorvali: *On Teichmüller space of tori*, Ann. Acad. Sci. Fenn. Math. **1** (1975), 7–11.

[15] Z. Sun: *On quadratic differential metrics on a closed Riemann surface*, Tokyo J. Math. **21** (2014), 221–234.

[16] Z. Sun and H. Guo: *Energy, Hopf differential, and metrics on Teichmüller space*, Bull. Malays. Math. Sci. Soc. **41** (2018), 1223–1231.

[17] W.P. Thurston: *Minimal stretch maps between hyperbolic surfaces*, arXiv: math/9801039v1, 1998.

[18] S. Wolpert: *The length spectra as moduli for compact Riemann surfaces*, Ann. of Math. (2) **109** (1979), 323–351.

Zongliang Sun
Department of Mathematics
Shenzhen University
Shenzhen 518060
P. R. China

Hui Guo
Department of Mathematics
Shenzhen University
Shenzhen 518060
P. R. China
e-mail: hguo@szu.edu.cn