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AN ESTIMATE OF THE FIRST NON-ZERO EIGENVALUE OF THE LAPLACIAN BY THE RICCI CURVATURE ON EDGES OF GRAPHS

Taiki YAMADA

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Abstract

We define the distance between edges of graphs and study the coarse Ricci curvature on edges. We consider the Laplacian on edges based on the Jost-Horak's definition of the Laplacian on simplicial complexes. As one of our main results, we obtain an estimate of the first non-zero eigenvalue of the Laplacian by the Ricci curvature for a regular graph.

1. Introduction

The Ricci curvature is one of the most important concepts in Riemannian geometry. There are some definitions of the generalized Ricci curvature and Ollivier's coarse Ricci curvature is one of them (see [13], [14]). It is formulated by the Wasserstein distance on a metric space (X, d) with a random walk $m = \{m_x\}_{x\in X}$, where m_x is a probability measure on X. The coarse Ricci curvature is defined as, for two distinct points $x, y \in X$,

$$\kappa(x, y) := 1 - \frac{W(m_x, m_y)}{d(x, y)},$$

where $W(m_x, m_y)$ is the 1-Wasserstein distance between m_x and m_y. This definition was applied to graphs around 2010 and many researchers are focused on this. In 2010, Lin, Lu and Yau [11] defined the Ricci curvature on vertices of graphs by using the coarse Ricci curvature of the lazy random walk and they studied the Ricci curvature of product space of graphs and random graphs. In 2012, Jost and Liu [9] studied the relation between the Ricci curvature and the local clustering efficient. Recently, the Ricci curvature on graphs was applied to directed graphs [18] and internet topology [2] and so on. In this paper, we study the Ricci curvature on edges of graphs.

The study of the graph Laplacian has a long and prolific history. In 1847, Kirchhoff [10] was first defined the graph Laplacian on real valued functions. In the early 1970s, Fiedler [4] studied the relation between the first non-zero eigenvalue and the connectivity of a graph. After he was focused on the graph Laplacian, there has been many papers about the graph Laplacian and its spectrum. In 1935, Bottema [1] introduced the normalized the graph Laplacian and studied the relation between the normalized graph Laplacian and the transition probability operator on graphs. Moreover, in 2009, Ollivier [13] obtained an estimate of the first non-zero eigenvalue of the normalized graph Laplacian by a lower bound of the coarse

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Ricci curvature as follows.

Theorem 1.1 (Ollivier, [13])**.** *Let $\lambda_1^{\Delta_0}$ be the first non-zero eigenvalue of the normalized graph Laplace operator Δ_0. Suppose that on $(V(G), d, m)$, $\kappa(x, y) \geq \kappa^V$ for any $(x, y) \in E$ and for a positive real number κ^V. Then we have*

$$\lambda_1^{\Delta_0} \geq \kappa^V,$$

where $\kappa(x, y)$ is the Ricci curvature on x and y.

Theorem 1.1 is not optimal, so our motivation is obtaining the optimal estimate of the first non-zero eigenvalue of the graph Laplacian.

The graph Laplacian was generalized to simplicial complexes by Eckmann [3] as the discrete version of the Hodge theory. This graph Laplacian is called the *higher order combinatorial Laplacian*. In 2013, Jost and Horak [8] developed a general framework for Laplace operators defined by the combinatorial structure of a simplicial complex. Giving the concrete weight function defined by Definition 3.3, this Laplacian becomes each graph Laplacian defined before, and one of the most important properties is that the spectrum of this Laplacian on vertices coincides with that of this Laplacian on edges. In this paper, we arrange the weight function, and use this Laplacian. Considering the Ricci curvature on edges, we obtain the following theorem.

Theorem 1.2. *Let G be a d-regular graph and λ_1 the first non-zero eigenvalue of the Laplacian on vertices. Suppose that $\kappa(e, e') \geq \kappa^E$ for any $e, e' \in E$ and for a real number $\kappa^E > 0$. Then we have*

$$\lambda_1 \geq \kappa^E + \frac{2}{d} - 1, \tag{1.1}$$

where $\kappa(e, e')$ is the Ricci curvature on e and e', and the Laplacian here is defined by Jost-Horak (*see* Definition 3.4).

The estimate of Theorem 1.2 is optimal. In fact, any star graph satisfies equality of (1.1) (see Remark 4.5). We remark that for a graph G, we could consider the graph G^* over the vertex set $V(G^*) = E(G)$, however the spectrum of the normalized graph Laplacian on G does not coincide with that of the normalized graph Laplacian on G^* in general (see Remark 4.4), and the Laplacian on edges of G defined by Jost-Horak differs from the normalized graph Laplacian on G^*.

This paper is organized as follows. In the 2nd section, we define the Ricci curvature on edges of a graph and we prove some properties of the Ricci curvature. In the 3rd section, for a 1-dimensional abstract simplicial complex, we prove some properties of the Laplacian on edges defined in [8]. Moreover, we prove Theorem 1.2. In the 4th section, we treat some examples of graphs, and calculate the Ricci curvature and the eigenvalue of the Laplacian. In the 5th section, considering weighted graphs, and we generalize the results proved in the previous sections.

2. Properties of Ricci curvature on edges of graphs

Before we define the Ricci curvature on edges, we present some preliminaries.

In this paper, $G = (V, E)$ is an undirected connected simple finite graph, where V is the

set of the vertices and E the set of edges. That is,

(1) for any two vertices, there exists a path connecting them,
(2) there exists no loop and no multiple edge,
(3) the number of vertices and edges is finite.

For $x, y \in V$, we write (x, y) as an edge connecting x and y. We denote the set of vertices of G by $V(G)$ and the set of edges by $E(G)$.

DEFINITION 2.1. (1) Two distinct edges e and e' are *connected* if they have a common vertex.
(2) A *path* connecting two edges e and e' is a sequence of edges $\{e_i\}_{i=0}^n$ such that e_i and e_{i+1} are connected for $0 \le i \le n-1$ and $e_0 = e$, $e_n = e'$. We call n the *length* of the path.

The *distance* $d(e, e')$ between two edges $e, e' \in E$ is defined to be the length of a shortest path connecting e and e'.

For any $x \in V$, the *neighborhood* of x is defined as $\Gamma(x) := \{y \in V \mid (x, y) \in E\}$ and the *degree* of x, denoted by d_x^V, is the number of edges connecting v, i.e., $d_x^V = |\Gamma(x)|$. Similarly, we define the degree of an edge as following.

DEFINITION 2.2. For any $e \in E$, the *neighborhood* of e is defined as

$$\Gamma(e) := \{\bar{e} \in V \mid e \text{ and } \bar{e} \text{ are connected and } e \neq \bar{e}\}.$$

The *degree* of e, denoted by d_e, is the number of edges connecting e, i.e., $d_e = |\Gamma(e)|$. Moreover, G is a *d-regular graph* if every edge has the same degree d.

With these definitions, we define the Ricci curvature on edges.

DEFINITION 2.3. For any edge $e \in E$, we define a probability measure m_e on E by

$$m_e(\bar{e}) = \begin{cases} \dfrac{1}{d_e} & \text{if } \bar{e} \in \Gamma(e), \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

This defines a random walk $m = \{m_x\}_{x \in X}$.

DEFINITION 2.4. For two probability measures μ and ν on E, the 1-Wasserstein distance between μ and ν is written as

$$W(\mu, \nu) = \inf_A \sum_{\bar{e}, \bar{e}' \in E} A(\bar{e}, \bar{e}')\, d(\bar{e}, \bar{e}'),$$

where $A : E \times E \to [0, 1]$ runs over all maps satisfying

$$\begin{cases} \sum_{\bar{e}' \in E} A(\bar{e}, \bar{e}') = \mu(\bar{e}), \\ \sum_{\bar{e} \in E} A(\bar{e}, \bar{e}') = \nu(\bar{e}'). \end{cases}$$

Such a map A is called a *coupling* between μ and ν.

REMARK 2.5. There exists a coupling A that attains the Wasserstein distance (see [7], [16] and [17]).

One of the most important properties of the Wasserstein distance is the Kantorovich-Rubinstein duality as stated as follows.

Proposition 2.6 (Kantorovich, Rubinstein)**.** *For two probability measures μ and ν on E, the 1-Wasserstein distance between μ and ν is written as*

$$W(\mu,\nu) = \sup_{f:1-Lip} \sum_{\bar{e}\in E} f(\bar{e})\,(m_e(\bar{e}) - m_{e'}(\bar{e})),$$

where the supremum is taken over all functions on G that satisfy $|f(e) - f(e')| \le d(e,e')$ for any $e, e' \in E$.

DEFINITION 2.7. For any two distinct edges $e, e' \in E$, the *Ricci curvature* of e and e' is defined as

$$\kappa(e,e') = 1 - \frac{W(m_e, m_{e'})}{d(e,e')}.$$

If we apply some properties of the Ricci curvature on vertices proved in [9] and [11] to the Ricci curvature on edges, then we obtain the following results.

Proposition 2.8. *If $\kappa(e,e') \ge \kappa_0$ for any edges e and e' with $d(e,e') = 1$, then $\kappa(e,e') \ge \kappa_0$ for any pair of edges (e,e').*

Since Proposition 2.8 is proved in the same way as Lemma 2.3 in [11], the proof is omitted.

Theorem 2.9. *For any edges e and e' with $d(e,e') = 1$, we have*

$$\begin{aligned}\kappa(e,e') &\ge -2\left(1 - \frac{1}{d_e} - \frac{1}{d_{e'}}\right)_+ \\ &= \begin{cases} -2 + \dfrac{2}{d_e} + \dfrac{2}{d_{e'}}, & \text{if } d_e > 1 \text{ and } d_{e'} > 1 \\ 0, & \textit{otherwise}. \end{cases}\end{aligned}$$

Theorem 2.10. *For any edges $e \in E$ and $e' \in \Gamma(e)$, we have*

$$\kappa(e,e') \le \frac{|\,\Gamma(e) \cup \Gamma(e')\,|}{\max\{d_e, d_{e'}\}}.$$

Since Theorem 2.9 and Theorem 2.10 are proved in the same way as Proposition 2 and Theorem 5 in [9], these proofs are omitted.

3. Properties of the Laplacian on edges in the case $\dim G = 1$

First we review the definition of the cohomology. Let K be an abstract simplicial complex and $S_i(K)$ the set of all i-faces of K. We say that a face F is *oriented* if we choose an ordering on its vertices and write $[F]$.

The i-th chain group $C_i(K,\mathbb{R})$ of K with coefficients in $\mathbb{R}$ is a vector space over the

real field $\mathbb{R}$ with basis $B_i(K,\mathbb{R}) = \{[F] \mid F \in S_i(K)\}$, and the i-th cochain group $C^i(K,\mathbb{R})$ is defined as the dual of the i-th chain group.

DEFINITION 3.1. For the cochain groups, the *simplicial coboundary maps* $\delta_i : C^i(K,\mathbb{R}) \to C^{i+1}(K,\mathbb{R})$ are defined by

$$(\delta_i f)([v_0,\cdots,v_{i+1}]) = \sum_{j=0}^{i+1}(-1)^j f([v_0,\cdots,\hat{v_j},\cdots,v_{i+1}])$$

for $f \in C^i(K,\mathbb{R})$, where $\hat{v_j}$ means that the vertex v_j has been omitted.

Note that the one-dimensional vector space $C^{-1}(K,\mathbb{R})$ is generated by a function f_{-1} with $f_{-1}(\emptyset) = 1$. The δ_i are the connecting maps in the *augmented cochain complex* of K with coefficients in $\mathbb{R}$, i.e, the sequence of the vector spaces with the linear transformations

$$\cdots \xleftarrow{\delta_{i+1}} C^{i+1}(K,\mathbb{R}) \xleftarrow{\delta_i} C^i(K,\mathbb{R}) \xleftarrow{\delta_{i-1}} \cdots \xleftarrow{\delta_1} C^1(K,\mathbb{R}) \xleftarrow{\delta_0} C^0(K,\mathbb{R}) \xleftarrow{\delta_{-1}} C^{-1}(K,\mathbb{R}) \leftarrow 0$$

It is easy to show that $\delta_i\delta_{i-1} = 0$, thus the image of δ_{i-1} is contained in the kernel of δ_i. We define the reduced cohomology group by

$$\tilde{H}^i(K,\mathbb{R}) := \ker\delta_i / \operatorname{im}\delta_{i-1}.$$

In order to define the Laplacian, we define the boundary of the oriented face and the inner product.

DEFINITION 3.2. Let $F' = \{v_0,\cdots,v_{i+1}\}$ be an $(i+1)$-face of K and $F = \left\{v_0,\cdots,\hat{v_j},\cdots,v_{i+1}\right\}$ an i-face of F'. The *boundary of the oriented face* $[F']$ is

$$\partial[F'] = \sum_j (-1)^j [v_0,\cdots,\hat{v_j},\cdots,v_{i+1}],$$

and the sign of $[F]$ in the boundary of $[F']$ is denoted by $\operatorname{sgn}([F],\partial[F'])$ and is equal to $(-1)^j$.

DEFINITION 3.3. The *inner product* on the space $C^i(K,\mathbb{R})$ is

$$(f,g)_{C^i} = \sum_{F\in S_i(K)} w(F)\, f([F])\, g([F]) \tag{3.1}$$

for $f, g \in C^i(K,\mathbb{R})$, where $w : \bigcup_{i=0} S_i(K) \to \mathbb{R}^+$ is a function with $w(\emptyset) = 0$. We call w the weight function on K.

For the innner product on $C^i(K,\mathbb{R})$, the adjoint $\delta_i^* : C^{i+1}(K,\mathbb{R}) \to C^i(K,\mathbb{R})$ of the coboundary operator δ_i is defined by

$$(\delta_i f_1, f_2)_{C^{i+1}} = (f_1, \delta_i^* f_2)_{C^i},$$

for every $f_1 \in C^i(K,\mathbb{R})$ and $f_2 \in C^{i+1}(K,\mathbb{R})$. The adjoint operator δ_i^* is expressed as

$$(\delta_i^* f)(F) = \frac{1}{w(F)} \sum_{F'\in S_{i+1}(K):F\in\partial F'} \operatorname{sgn}([F],\partial[F'])f([F'])$$

for $f \in C^{i+1}(K, \mathbb{R})$.

DEFINITION 3.4 (HORAK, JOST, [8]). We define the Laplace operators on $C^i(K, \mathbb{R})$ as follows.

(1) The *i-dimensional combinatorial up Laplace operator* or *simply i-up Laplace operator* is defined by

$$\mathcal{L}_i^{up}(K) := \delta_i^* \delta_i.$$

(2) The *i-dimensional combinatorial down Laplace operator* or *simply i-down Laplace operator* is defined by

$$\mathcal{L}_i^{down}(K) := \delta_{i-1} \delta_{i-1}^*.$$

(3) The *i-dimensional combinatorial Laplace operator* or simply *i*-Laplace operator is defined by

$$\mathcal{L}_i(K) := \mathcal{L}_i^{up}(K) + \mathcal{L}_i^{down}(K) = \delta_i^* \delta_i + \delta_{i-1} \delta_{i-1}^*.$$

REMARK 3.5. Since their operators are all self-adjoint and non-negative, the eigenvalues are real and non-negative. In addition, since $\delta_i \delta_{i-1} = 0$ and $\delta_{i-1}^* \delta_i^* = 0$, we have

$$\operatorname{im} \mathcal{L}_i^{down} \subset \ker \mathcal{L}_i^{up},$$
$$\operatorname{im} \mathcal{L}_i^{up} \subset \ker \mathcal{L}_i^{down}.$$

Thus, λ is a non-zero eigenvalue of $\mathcal{L}_i$ if and only if it is a non-zero eigenvalue of $\mathcal{L}_i^{up}$ or $\mathcal{L}_i^{down}$.

We represent the Laplace operators by the matrix form. Let D_i be the matrix corresponding to the operator δ_i, D_i^T its transpose and W_i the diagonal matrix representing their scalar product on C^i, then the operators $\mathcal{L}_i^{up}(K)$ and $\mathcal{L}_i^{down}(K)$ are expressed as

$$\mathcal{L}_i^{up}(K) = W_i^{-1} D_i^T W_{i+1} D_i$$

and

$$\mathcal{L}_i^{down}(K) = D_{i-1} W_{i-1}^{-1} D_{i-1}^T W_i.$$

Since a graph is a 1-dimensional abstract simplicial complex and we have $W_{-1} = w(\emptyset) = 0$, we obtain $\mathcal{L}_0^{down}(G) = 0$ and $\mathcal{L}_1^{up}(G) = 0$. As the result, it is sufficient to consider the following Laplace operator:

$$\mathcal{L}_0(G) = \mathcal{L}_0^{up}(G) = W_0^{-1} D_0^T W_1 D_0$$

and

$$\mathcal{L}_1(G) = \mathcal{L}_1^{down}(G) = D_0 W_0^{-1} D_0^T W_1.$$

By the matrix form, λ is a non-zero eigenvalue of $\mathcal{L}_0(G)$ if and only if it is a non-zero eigenvalue of $\mathcal{L}_1(G)$.

REMARK 3.6. If W_0 and W_1 are the identity matrix, then the 0-Laplace operator $\mathcal{L}_0(G)$ corresponds to the graph Laplacian, so the operator does not depend on the orientation of

faces. On the other hand, the 1-Laplace operator $\mathcal{L}_1(G)$ depends on the orientation of faces. However, the 1-Laplacian has the same spectrum as the 0-Laplacian, except for the multiplicity of the eigenvalue 0, so the spectrum of $\mathcal{L}_1(G)$ does not depend on the orientation of faces.

REMARK 3.7. If W_1 is the identity matrix and if $w(v_i)$ is the degree of v_i, then the obtained operator is called the *normalized combinatorial Laplace operator* and is denoted by Δ_0.

In this paper, we want to obtain an estimate of the first non-zero eigenvalue of the 1-Laplacian by the Ollivier's Ricci curvature. In order to do so, we assume that W_0 is an identity matrix and $w(e_i) = 1/d_{e_i}$, and the obtained operator is denoted by $\mathcal{L}'_1$.
Before we obtain an estimate of the eigenvalues of the Laplacian, we decompose the Laplacian into the diagonal component and the other part

$$\mathcal{L}'_1 = \mathrm{Diag}(\mathcal{L}'_1) + L^{down},$$

where

$$(L^{down} f)(e) = \sum_{e' \in E(G): v = e \cap e'} \mathrm{sgn}([v], \partial[e])\, \mathrm{sgn}([v], \partial[e'])\, m_e(e') f(e') \frac{d_e}{d_{e'}}.$$

Proof of Theorem 1.2. By the definition of the Laplacian, λ is a non-zero eigenvalue of $\mathcal{L}'_0(G)$ if and only if it is a non-zero eigenvalue of $\mathcal{L}'_1(G)$. So we consider the eigenvalue of $\mathcal{L}'_1(G)$. Let f be an eigenfunction with respect to λ_1. Fix any two edges $e = (x, y), e' = (y, z) \in E(G)$ with $d(e, e') = 1$.

Case 1. If e and e' are not contained in any triangle, then we orient edges $\bar{e} \in \Gamma(e) \cup \Gamma(e')$ as follows (see Fig.1).

(1) If $\bar{e} = e$, or if $\bar{e}$ and e are connected by x, then we orient $\bar{e}$ with $\mathrm{sgn}([x], \partial[\bar{e}]) = -1$.
(2) If $\bar{e}$ and e are connected by y, or if $\bar{e}$ and e' are connected by y, then we orient $\bar{e}$ with $\mathrm{sgn}([y], \partial[\bar{e}]) = 1$.
(3) If $\bar{e} = e'$, or if $\bar{e}$ and e are connected by z, then we orient $\bar{e}$ with $\mathrm{sgn}([z], \partial[\bar{e}]) = -1$.

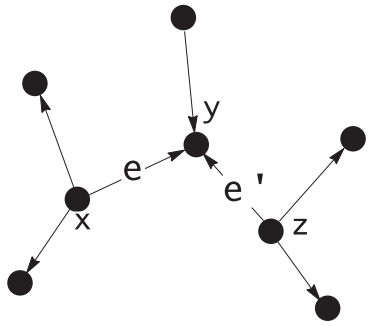

Fig. 1. Orientation of $\Gamma(e) \cup \Gamma(e')$

Then we obtain

$$\begin{cases} (L^{down}f)(e) = \sum_{\bar{e}\in\Gamma(e)} m_e(\bar{e})f(\bar{e}) = \lambda_1 f(e) - \dfrac{2}{d}f(e), \\ (L^{down}f)(e') = \sum_{\bar{e'}\in\Gamma(e')} m_{e'}(\bar{e'})f(\bar{e'}) = \lambda_1 f(e') - \dfrac{2}{d}f(e'). \end{cases} \tag{3.2}$$

If f is a constant function, then $\lambda_1 = 1 + 2/d$. In this case, there is nothing to prove. In fact, by the definition of the Ricci curvature, we see $\kappa \leq 1$. So, we have

$$\kappa + \frac{2}{d} - 1 \leq \frac{2}{d} < \lambda_1.$$

Thus we assume that f is not a constant function and by scaling f if necessarily,

$$\sup_{e,e'\in E: d(e,e')=1} |f(e) - f(e')| = 1. \tag{3.3}$$

By Proposition 2.6 and (3.2), we have

$$\begin{aligned} d(e,e')(1-\kappa) &\geq W(m_e, m_{e'}) \\ &\geq \sum_{\bar{e}\in\Gamma(e)\cup\Gamma(e')} f(\bar{e})\,(m_e(\bar{e}) - m_{e'}(\bar{e})) \\ &= \sum_{\bar{e}\in\Gamma(e)} m_e(\bar{e})f(\bar{e}) - \sum_{\bar{e'}\in\Gamma(e')} m_{e'}(\bar{e'})f(\bar{e'}) \\ &\geq \lambda_1 f(e) - \frac{2}{d}f(e) - \lambda_1 f(e') + \frac{2}{d}f(e') \\ &= -\left(\frac{2}{d} - \lambda_1\right)(f(e) - f(e')). \end{aligned}$$

By the symmetry of e and e', we obtain

$$|f(e) - f(e')| \leq \frac{1-\kappa}{|2/d - \lambda_1|}.$$

Then, the above inequality holds for any edge, which together with (3.3), implies

$$\lambda_1 \geq \kappa + \frac{2}{d} - 1.$$

Case 2. If e and e' are contained in a triangle, then we put $e_0 := (x, z)$. We orient the edge e_0 with $\mathrm{sgn}([x], \partial[e_0]) = 1$, and orient other edges $\bar{e} \in (\Gamma(e) \cup \Gamma(e')) \setminus \{e_0\}$ in the same way as Case 1 (see Fig.2).

Then we obtain

$$\begin{cases} (L^{down}f)(e) = \sum_{\bar{e}\in\Gamma(e)} m_e(\bar{e})f(\bar{e}) = \lambda_1 f(e) - \dfrac{2}{d}f(e), \\ (L^{down}f)(e') = \sum_{\bar{e'}\in\Gamma(e')\setminus\{e_0\}} m_{e'}(\bar{e'})f(\bar{e'}) - \dfrac{1}{d}f(e_0) = \lambda_1 f(e') - \dfrac{2}{d}f(e'), \end{cases} \tag{3.4}$$

If f is a constant function, then $\lambda_1 = 0$. This is a contradiction. Thus f is not a constant function. By scaling f if necessarily, we assume (3.3). We focus on e_0. By (3.4), we have

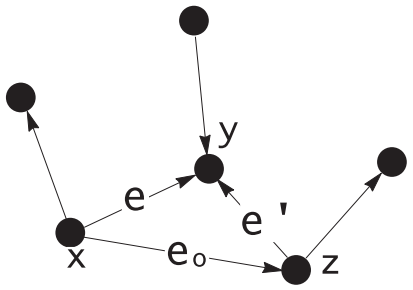

Fig. 2. Orientation of $\Gamma(e) \cup \Gamma(e')$

$$\begin{aligned}
\lambda_1 f(e_0) &= \frac{2}{d} f(e_0) + \frac{1}{d} f(e) + \sum_{\bar{e} \in \Gamma(e) \setminus (\Gamma(e') \cup \{e'\})} m_e(\bar{e}) f(\bar{e}) - \frac{1}{d} f(e') - \sum_{\bar{e}' \in \Gamma(e') \setminus (\Gamma(e) \cup \{e\})} m_{e'}(\bar{e}') f(\bar{e}') \\
&= \frac{2}{d} f(e_0) + \frac{2}{d} f(e) + \sum_{\bar{e} \in \Gamma(e) \setminus \Gamma(e')} m_e(\bar{e}) f(\bar{e}) - \frac{2}{d} f(e') - \sum_{\bar{e}' \in \Gamma(e') \setminus \Gamma(e)} m_{e'}(\bar{e}') f(\bar{e}') \\
&= \frac{2}{d} f(e_0) + \lambda_1 f(e) - \lambda_1 f(e_0) - \frac{2}{d} f(e_0) \\
&= \lambda_1 (f(e) - f(e')).
\end{aligned}$$

Thus we obtain

$$f(e_0) = f(e) - f(e'). \tag{3.5}$$

By Proposition 2.6 and (3.5), we have

$$\begin{aligned}
d(e, e')(1 - \kappa) &\geq W(m_e, m_{e'}) \\
&\geq \sum_{\bar{e} \in (\Gamma(e) \cup \Gamma(e'))} f(\bar{e})(m_e(\bar{e}) - m_{e'}(\bar{e})) \\
&= \sum_{\bar{e} \in \Gamma(e) \setminus \{e_0\}} m_e(\bar{e}) f(\bar{e}) - \sum_{\bar{e}' \in \Gamma(e') \setminus \{e_0\}} m_{e'}(\bar{e}') f(\bar{e}') \\
&= \lambda_1 f(e) - \frac{2}{d} f(e) - \frac{1}{d} f(e_0) - \lambda_1 f(e') + \frac{2}{d} f(e') - \frac{1}{d} f(e_0) \\
&\geq \lambda_1 f(e) - \frac{2}{d} f(e) - \lambda_1 f(e') + \frac{2}{d} f(e') - \frac{2}{d} f(e_0) \\
&= -\left(\frac{2}{d} - \lambda_1 \right) (f(e) - f(e')) - \frac{2}{d} (f(e) - f(e')).
\end{aligned}$$

By the symmetry of e and e' and $d(e, e') \geq 1$, we obtain

$$|4/d - \lambda_1| \, |f(e) - f(e')| \quad \leq \quad 1 - \kappa$$

Then, the above inequality holds for any edge, which together with (3.3), implies

$$\lambda_1 \geq \kappa + \frac{4}{d} - 1.$$

This completes the proof. □

4. Example

In this section, we treat some examples of graphs and calculate the eigenvalues of $\mathcal{L}'_1$ and the Ricci curvature.

EXAMPLE 4.1 (COMPLETE GRAPH K_n). Let K_n denote the complete graph with n vertices. While the Ricci curvatures on vertices of K_n are equal to $(n-2)/(n-1)$, the Ricci curvature on edges of K_n is equal to $1/2$. On the other hand, the first non-zero eigenvalue of $\mathcal{L}'_1(K_n)$ is equal to $n/2(n-2)$.

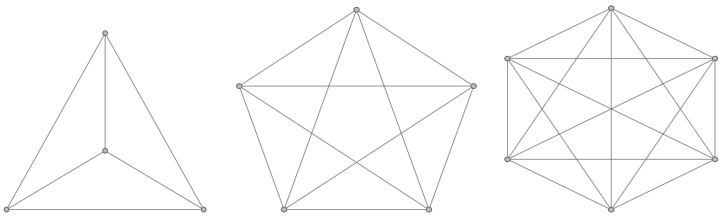

Fig. 3. Complete graph

EXAMPLE 4.2 (CYCLE C_n). The Ricci curvatures on vertices and those on edges of the cycle C_n with length $n \geq 4$ are all zero. On the other hand, the eigenvalues of $\mathcal{L}'_1(C_4)$ are 0, 1 and 2, and the eigenvalues of $\mathcal{L}'_1(C_5)$ are 0 and $(5 \pm \sqrt{5})/4$.

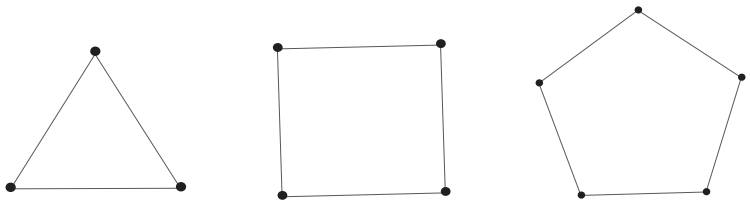

Fig. 4. Cycle

EXAMPLE 4.3 (COMPLETE BIPARTITE GRAPH $K_{n,m}$). While the Ricci curvatures on vertices of any complete bipartite graph $K_{n,m}$ are all zero, the Ricci curvatures on edges $e = (x, y), e' = (y, z) \in E(K_{n,m})$ is

$$\frac{d_y^V - 2}{m + n - 2}.$$

If $n = 1$, then $K_{1,m}$ is called the *star graph*, and the Ricci curvatures on edges of $K_{1,m}$ are $(m-2)/(m-1)$.

REMARK 4.4. For a graph G, we define the graph G^* over the vertex set $V(G^*) = E(G)$ such that $e, e' \in V(G^*)$ are connected if e and e' have a common vertex of G. Then, $K^*_{1,m}$ is

the complete graph K_{m+1}. The first non-zero eigenvalue of $\Delta_0(K_{1,m})$ is 1, and the first non-zero eigenvalue of $\Delta_0(K_{m+1})$ is $m+1/m$. So, the spectrum of the normalized graph Laplacian does not always coincide with each other.

REMARK 4.5. The first non-zero eigenvalue λ_1 of $\mathcal{L}'_1(K_{1,m})$ is equal to $1/(m-1)$. So, if $m \geq 3$, then λ_1 is smaller than the Ricci curvatures. Since the degree of any edge is $m-1$, we have

$$\kappa + \frac{2}{d} - 1 = \frac{1}{m-1} = \lambda_1.$$

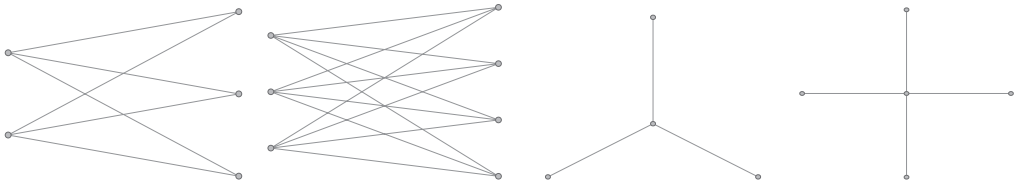

Fig. 5. Complete bipartite graph and star graph

EXAMPLE 4.6 (TREE T). While the Ricci curvature on vertices $x, y \in V(T)$ is $-2(1-1/d_x^V - 1/d_y^V)$, the Ricci curvature on edges $e = (x,y), e' = (y,z) \in E(T)$ is

$$\kappa(e,e') = \frac{d_y^V}{\min\{d_e, d_{e'}\}} + \frac{2d_y^V - 2}{\max\{d_e, d_{e'}\}} - 2.$$

If T is a d-regular tree, then the Ricci curvature on edges is $(-d+2)/(2d-2)$.

5. Relation between a weight function and a weighted graph

The i-Laplacian depends on the weight function w. On the other hand, the Ricci curvature depends on the Wasserstein distance. If we figure out a relation between the weight function and the Wasserstein distance, then we obtain an estimate of the non-zero eigenvalues of i-Laplacian in more general setting. So, we consider a weighted graph G that has the weight corresponding the weight function. Let G be a weighted undirected graph, and we generalize definitions of the distance, the degree, and a probability measure.

DEFINITION 5.1. For two edges e_0 and e_n, let $\{e_i\}_{i=0}^n$ be the shortest path connecting e_0 and e_n. The *distance* between e_0 and e_n is defined by

$$d(e_0, e_n) = \sum_{j=1}^{n} w(v_j),$$

where v_j is a vertex such that e_{j-1} and e_j are connected by v_j.

DEFINITION 5.2. For any edge e, the *degree* of e is defined by

$$d_e = \sum_{\bar{e} \in \Gamma(e)} w(\bar{e}).$$

DEFINITION 5.3. For any edge e, a probability measure m_e is defined by

$$m_e(\bar{e}) = \begin{cases} \dfrac{w(\bar{e})}{d_e} & \text{if } \bar{e} \in \Gamma(e), \\ 0 & \text{otherwise.} \end{cases}$$

By this definition, the operator L^{down} is expressed as

$$\begin{aligned} (L^{down} f)(e) &= \sum_{\bar{e}\in\Gamma(e):e\cap\bar{e}=v} \frac{w(\bar{e})}{w(v)} \mathrm{sgn}([v], \partial[e])\mathrm{sgn}([v], \partial[\bar{e}]) f(\bar{e}) \\ &= \sum_{\bar{e}\in\Gamma(e):e\cap\bar{e}=v} m_e(\bar{e})\mathrm{sgn}([v], \partial[e])\mathrm{sgn}([v], \partial[\bar{e}]) \frac{d_e}{w(v)} f(\bar{e}). \end{aligned}$$

We obtain lower and upper bounds of the Ricci curvature on a weighted graph.

Theorem 5.4. *For any pair of neighboring edges e and e', we have*

$$\kappa(e, e') \geq -2\left(1 - \frac{w(e')}{d_e} - \frac{w(e)}{d_{e'}}\right)_+$$

Theorem 5.5. *We assume that the weight function on V is a constant, say w_0. For any pair of neighboring edges e and e', we have*

$$\kappa(e, e') \leq \frac{w_\cap}{\max\{d_e, d_{e'}\}},$$

where $w_\cap := \sum_{\bar{e}\in\Gamma(e)\cap\Gamma(e')} w(\bar{e})$.

REMARK 5.6. Theorems 5.4 and 5.5 are proved in the same way as those of Theorems 2.9 and 2.10. If $w(v) = 1$, $w(e) = 1/d_e$, and $|\Gamma(e)| = d_e$ for any vertex v and any edge e, these results coincide with Theorems 2.9 and 2.10.

Hereafter, we assume that the weight function on E is a constant, say w_1. We generalize the estimate of the first non-zero eigenvalue of $\mathcal{L}_1^{down}$ as follows.

Theorem 5.7. *Let λ_1 be the first non-zero eigenvalue of $\mathcal{L}_1^{down}$. Suppose that $\kappa(e, e') \geq \kappa$ for any $e, e' \in E$ and for a real number $\kappa > 0$. Then, we have*

$$\lambda_1 \geq \{d(\kappa - 1) + 2\} \frac{w_1}{w_0}.$$

REMARK 5.8. Theorem 5.7 is proved in the same way as that of Theorem 1.2. If $w_0 = 1$, $\Gamma = d$ and $w_1 = 1/d$, the result coincides Theorem 1.2.

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Research Institute for Humanity and Nature
Kyoto 603–8047
Japan
e-mail: t.yamada@chikyu.ac.jp