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# CLASSIFICATION OF ISOPARAMETRIC SUBMANIFOLDS ADMITTING A REFLECTIVE FOCAL SUBMANIFOLD IN SYMMETRIC SPACES OF NON-COMPACT TYPE

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**Abstract**

In this paper, we assume that all isoparametric submanifolds have flat section. The main purpose of this paper is to prove that, if a full irreducible complete isoparametric submanifold of codimension greater than one in a symmetric space of non-compact type admits a reflective focal submanifold and if it is of real analytic, then it is a principal orbit of a Hermann type action on the symmetric space. A hyperpolar action on a symmetric space of non-compact type admits a reflective singular orbit if and only if it is a Hermann type action. Hence is not extra the assumption that the isoparametric submanifold admits a reflective focal submanifold. Also, we prove that, if a full irreducible complete isoparametric submanifold of codimension greater than one in a symmetric space of non-compact type satisfies some additional conditions, then it is a principal orbit of the isotropy action of the symmetric space, where we need not impose that the submanifold is of real analytic. We use the building theory in the proof.

## 1. Introduction

In 1985, C. L. Terng ([35]) introduced the notion of an isoparametric submanifold (of general codimension) in a Euclidean space and, in 1995, C. L. Terng and G. Thorbergsson ([38]) introduced the notion of an *equifocal submanifold* in a (Riemannian) symmetric space as its generalized notion. This notion is defined as a compact submanifold with flat section, trivial normal holonomy group and parallel focal structure. Here ”with flat section” means that the images (which is called the normal umbrella) of the normal spaces of the submanifold by the normal exponential map are flat totally geodesic submanifolds and “the parallelity of the focal structure” means that, for any parallel normal vector field $\widetilde{v}$ of the submanifold, the focal radii along the normal geodesic $\gamma_{\widetilde{v}_x}$ with $\gamma'_{\widetilde{v}_x}(0) = \widetilde{v}_x$ are independent of the choice of $x$ (with considering the multiplicities), where $\gamma'_{\widetilde{v}_x}(0)$ is the velocity vector of $\gamma_{\widetilde{v}_x}$ at 0. Compact isoparametric hypersurfaces in a sphere or a hyperbolic space are equifocal. E. Heintze, X. Liu and C. Olmos ([11]) defined the notion of an *isoparametric submanifold* in a general complete Riemannian manifold as a (properly embedded) complete submanifold with section and trivial normal holonomy group whose sufficiently close parallel submanifolds are of constant mean curvature with respect to the radial direction. Here “with section“ means that the normal umbrellas of the submanifold are totally geodesic (the normal umbrellas are called *sections*).

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**Assumption.** In this paper, we assume that all isoparametric submanifolds have flat section, that is, the induced metric on the sections are flat.

For a compact submanifold in a symmetric space of compact type, they ([11]) proved that it is equifocal if and only if it is an isoparametric submanifold (with flat section). In 1989, C. L. Terng ([36]) introduced the notion of an isoparametric submanifold in a (separable) Hilbert space and initiated its research. In 1995, C. L. Terng and G. Thorbergsson ([38]) proved that the research of an equifocal submanifold in a symmetric space $G/K$ of compact type is reduced to that of an isoparametric submanifold in the Hilbert space $H^0([0,1],\mathfrak{g})$ through the composition of the parallel transport map $\phi : H^0([0,1],\mathfrak{g}) \to G$ for $G$ and the natural projection $\pi : G \to G/K$, where $\mathfrak{g}$ denotes the Lie algebra of $G$ and $H^0([0,1],\mathfrak{g})$ denotes the path space of all $L^2$-integrable curves(=paths) in $\mathfrak{g}$. Denote by $I(V)$ the group of all isometries of a (separable) Hilbert space $V$, where we note that $I(V)$ is not a Banach Lie group (see [9] and [13] (Appendix of [16] also)). Let $\widetilde{M}$ be a full irreducible complete isoparametric submanifold of codimension greater than one in $V$. Here "completeness" means "metric completeness", where we note that, for a Riemannian Hilbert manifold, if it is metrically complete, then it is also geodesically complete, but the converse does not necessarily hold (see [1]). In main theorems of [25], [26] and [27], we assumed that the submanifolds are metrically complete as anti-Kaehler Hilbert manifolds without mentioned. In Section 3, we shall state the definition of metric completeness of an anti-Kaehler Hilbert manifold. Throughout this paper, "completeness" means "metric completeness" and we shall write it without abbreviated. Set $H := \{F \in I(V) \mid F(\widetilde{M}) = \widetilde{M}\}$. In 1999, E. Heintze and X. Liu ([10]) proved that $\widetilde{M}$ is extrinsically homogeneous in the sense that $Hu = \widetilde{M}$ holds for any $u \in M$. This result is the infinite dimensional version of the extrinsic homogeneity theorem for a finite dimensional compact isoparametric submanifold in a Euclidean space by G. Thorbergsson ([39]). The extrinsic homogeneity theorem in [39] states that full irreducible compact isoparametric submanifolds of codimension greater than two in a Euclidean space are extrinsically homogeneous. G. Thorbergsson proved this statement by constructing the topological Tits building of spherical type associated to the isoparametric submanifold (in more general, he defined this topological Tits building for full irreducible isoparametric submanifolds of rank greater than one in a Euclidean space) and using it, where we note that, if the isoparametric submanifold is a principal orbit of the s-representation of an irreducible symmetric space $G/K$ of non-compact type and rank greater than one, then its associated topological Tits building coincides with the Tits building of the semi-simple Lie group $G$ (which is defined as the building having parabolic subgroups as vertices) as Tits building. Later, C. Olmos ([31]) proved this result by Thorbergsson in simpler method, that is, by constructing the normal homogeneous structure for the isoparametric submanifold and using the result in [32] (without use of the above topological Tits building), where the normal homogeneous structure means a certain kind of connection on the Whitney sum of the tangent bundle and the normal bundle of the submanifold. E. Heintze and X. Liu ([10]) proved the above extrinsic homogeneity theorem in the method similar to the proof in [31]. In 2002, by using the extrinsic homogeneity theorem of Heintze-Liu, U. Christ ([5]) proved that a full irreducible equifocal submanifold of codimension greater than one in a simply connected symmetric space of compact type is extrinsically homogeneous. However, there was a gap in his proof because the above group $H$ in the theorem of Heintze-Liu is not Banach Lie group but he interpreted it as a Banach Lie group. Let $I_b(V)$ be the subgroup of $I(V)$ generated by one-parameter transformation groups induced by the Killing vector fields defined entirely on $V$. It is easy to show that $I_b(V)$ is a Banach Lie group. Set $H_b := H \cap I_b(V)$, which is a Banach Lie subgroup of $I_b(V)$. Recently, C. Gorodski and E. Heintze ([7]) proved that $\widetilde{M}$ is extrinsically homogeneous in the sense that $H_b u = \widetilde{M}$ holds for any $u \in \widetilde{M}$. This improved extrinsic homogeneity theorem closed the gap in the proof of Christ. According to the extrinsic homogeneity theorem by Christ and Theorem 2.3 in [12], we can derive that, if $M$ is an irreducible equifocal(=isoparametric) submanifold of codimension greater than one in a simply connected symmetric space of compact type, then it is a principal orbit of a hyperpolar action. On the other hand, according to the classification of the hyperpolar actions by A. Kollross ([28]), all hyperpolar actions of cohomogeneity greater than one on the irreducible symmetric space of compact type are Hermann actions. Also, O. Goertsches and G. Thorbergsson ([8]) proved that principal orbits of Hermann actions are curvature-adapted, where "curvature-adaptedness" means that, for any unit normal vector $v$ of $M$, the normal Jacobi operator $R(v)$ preserves $T_xM$ ($x$ :the base point $v$) invariantly and that $R(v)$ commutes with the shape operator $A_v$, where $R$ is the curvature tensor of the ambient symmetric space and $R(v) := R(\cdot, v)v$. From these facts, we can derive the following fact:

*All complete equifocal(=isoparametric) submanifolds of codimension greater than one in simply connected irreducible symmetric spaces of compact type are principal orbits of Hermann actions and they are curvature-adapted.*

We can classify such equifocal submanifolds from this fact and the well-known classification of Hermann actions on simply connected irreducible symmetric spaces of compact type.

In 2000, by the discussion with G. Thorbergsson at Nagoya University (The 47-th Geometry Symposium), the author was very interesting in the following open problem:

*Is there a similar theory for equifocal submanifolds in simply connected non-compact symmetric spaces?*

This is one of seven open problems proposed in [38]. The author interpreted that this open problem means the following:

*Can we reduce the study of an equifocal submanifold in a simply connected non-compact symmetric space to the study of the lift of the submanifold to a Hilbert space through a Riemannian submersion (of the Hilbert space onto the symmetric space) or the study of the lift of some extended submanifold of the original submanifold (which is a submanifold in some extended symmetric space of the original symmetric space) to some pseudo-Hilbert space through a pseudo-Riemannian submersion (of the pseudo-Hilbert space onto the extended symmetric space)?*

Under this motivation, the author introduced the notion of a complex equifocal submanifold in a symmetric space of non-compact type and started its study. We shall explain why we introduced the notion of a complex equifocal submanifold in a symmetric space of non-compact type. When a non-compact submanifold $M$ in a symmetric space $G/K$ of non-compact type is deformed as its principal curvatures approach to zero, its focal set vanishes beyond the ideal boundary $(G/K)(\infty)$ of $G/K$. For example, when an open portion of a totally umbilic sphere (whose only principal curvature is greater than $\sqrt{-c}$) in a hyperbolic space of constant curvature $c(<0)$ is deformed as its principal curvatures approach to $\sqrt{-c}$ (remaining to be totally umbilic), its focal point approach to $(G/K)(\infty)$ and, when it furthermore is deformed as its principal curvatures approach to a positive value smaller than $\sqrt{-c}$ (remaining to be totally umbilic), the focal point vanishes beyond $(G/K)(\infty)$. According to these facts, we recognized that, for a non-compact submanifold in a symmetric space of non-compact type, the parallelity of the focal structure is not an essential condition. So, we ([15]) introduced the notion of a *complex focal radius* of the submanifold along the normal geodesic $\gamma_v$. See Section 2 about the definition of this notion. Furthermore, we ([15]) defined the notion of a *complex equifocal submanifold* as a (properly embedded) complete submanifold with flat section, trivial normal holonomy group and parallel complex focal structure, where we note that this submanifold should be called an equi-complex focal submanifold but we called it a complex equifocal submanifold for the simplicity. We proved that all isoparametric submanifolds (in the sense of [11]) are complex equifocal and that, conversely all curvature-adapted complex equifocal submanifolds are isoparametric (see Theorem 15 of [16]). Thus, for a complete submanifold in $G/K$, it is curvature-adapted complex equifocal if and only if it is curvature-adapted isoparametric. Hence, throughout this paper, we shall use the terminology "curvature-adapted isoparametric" more familiar than "curvature-adapted complex equifocal".

We consider the case where $M$ is of class $C^\omega$ (i.e., real analytic). Then we ([16]) defined the complexification $M^{\mathbb{C}}$ of $M$ as an anti-Kaehler submanifold in the anti-Kaehler symmetric space $G^{\mathbb{C}}/K^{\mathbb{C}}$. Here we note that $G^{\mathbb{C}}/K^{\mathbb{C}}$ is a space including both $G/K$ and its compact dual $G_\kappa/K$ as submanifolds transversal to each other and that it is interpreted as the complexification of both $G/K$ and $G_\kappa/K$. Also we note that the induced metric on $G/K$ coincides with the original metric of $G/K$ and that the induced metric on $G_\kappa/K$ is the $(-1)$-multiple of the metric of the symmetric space $G_\kappa/K$ of compact type. We ([16]) showed that $z$ is a complex focal radius of $M$ along the normal geodesic $\gamma_v$ if and only if $\gamma_v^{\mathbb{C}}(z)$ is a focal point of $M^{\mathbb{C}}$ along the complexified geodesic $\gamma_v^{\mathbb{C}}$. Here $\gamma_v^{\mathbb{C}}$ is defined by $\gamma_v^{\mathbb{C}}(z) := \gamma_{av+bJv}(1)$ $(z = a + b\sqrt{-1} \in \mathbb{C})$, where $J$ denotes the complex structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$ and $\gamma_{av+bJv}$ is the geodesic in $G^{\mathbb{C}}/K^{\mathbb{C}}$ with $\gamma'_{av+bJv}(0) = av + bJv$. Thus the complex focal radii of $M$ are the quantities indicating the positions of focal points of $M^{\mathbb{C}}$.

We ([16]) introduced the notion of an *anti-Kaehler isoparametric submanifold* in the infinite dimensional anti-Kaehler space and furthermore, defined the *parallel transport map* for $G^{\mathbb{C}}$ as an anti-Kaehler submersion of an infinite dimensional anti-Kaehler space (which is denoted by $H^0([0,1],\mathfrak{g}^{\mathbb{C}})$) consisting of certain kind of paths in the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of $G^{\mathbb{C}}$ onto $G^{\mathbb{C}}$. Denote by $\phi$ the parallel transport map for $G^{\mathbb{C}}$. We ([16]) proved that the research of a complex equifocal $C^\omega$-submanifold in a symmetric space $G/K$ of non-compact type is reduced to that of an anti-Kaehler isoparametric submanifold in the infinite dimensional anti-Kaehler space $H^0([0,1],\mathfrak{g}^{\mathbb{C}})$ by lifting the complexification of the original submanifold through the composition of the parallel transport map $\phi : H^0([0,1],\mathfrak{g}^{\mathbb{C}}) \to G^{\mathbb{C}}$ for $G^{\mathbb{C}}$ and the natural projection $\pi : G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$. More precisely, we showed that a $C^\omega$-submanifold in $G/K$ is complex equifocal if and only if the lift of its complexification to $H^0([0,1],\mathfrak{g}^{\mathbb{C}})$ is anti-Kaehler isoparametric.

We ([25]) proved that any full irreducible (metrically) complete anti-Kaehler isoparametric $C^\omega$-submanifold $\widetilde{M}$ with $J$-diagonalizable shape operators of codimension greater than one in an infinite dimensional anti-Kaehler space $V$ is extrinsically homogeneous in the sense that $Hu = \widetilde{M}$ holds for any $u \in \widetilde{M}$ as $H := \{F \in I_h(V) \,|\, F(\widetilde{M}) = \widetilde{M}\}$, where $I_h(V)$ denotes the group of all holomorphic isometries of $V$ and "with $J$-diagonalizable shape operators" means that the complexifications of the shape operators are diagonalized with respect to $J$-orthonormal bases. Note that we assumed that, in main theorem, an anti-Kaehler isoparametric submanifold is (metrically) complete without mentioned. Recently we ([26]) improved this extrinsic homogeneity theorem as follows.

**Fact 1.1.** *Let $\widetilde{M}$ be a full irreducible complete anti-Kaehler isoparametric $C^\omega$-submanifold with $J$-diagonalizable shape operators of codimension greater than one in an infinite dimensional anti-Kaehler space $V$. Then $\widetilde{M}$ is extrinsically homogeneous in the sense that $H_b u = \widetilde{M}$ holds for any $u \in \widetilde{M}$ as $H_b := \{F \in I_h^b(V) \,|\, F(\widetilde{M}) = \widetilde{M}\}$, where $I_h^b(V)$ denotes the subgroup of $I_h(V)$ generated by one-parameter transformation groups induced by holomorphic Killing vector fields defined entirely on $V$.*

Here we note that $I_h^b(V)$ is a Banach Lie group and $H_b$ is a Banach Lie subgroup of $I_h^b(V)$. Let $M$ be a complete curvature-adapted submanifold with flat section in a symmetric space $G/K$. If $G/K$ is of compact type or Euclidean type, then the following fact $(*_{\mathbb{R}})$ holds:

$(*_{\mathbb{R}})$ *For any unit normal vector $v$ of $M$, the nullity spaces for focal radii along the normal geodesic $\gamma_v$ span $T_xM \ominus (\mathrm{Ker}\, A_v \cap \mathrm{Ker}\, R(v))$.*

Here $T_xM \ominus (\mathrm{Ker}\, A_v \cap \mathrm{Ker}\, R(v))$ denotes $T_xM \cap (\mathrm{Ker}\, A_v \cap \mathrm{Ker}\, R(v))^\perp$. However, if $G/K$ is of non-compact type, then this fact $(*_{\mathbb{R}})$ does not necessarily hold. For example, in the case where $G/K$ is a hyperbolic space of constant curvature $c(< 0)$ and where $M$ is a hypersurface, $(*_{\mathbb{R}})$ holds if and only if all the absolute values of the principal curvatures of $M$ at each point are greater than $\sqrt{-c}$. So, in this paper, we consider the following condition:

$(*_{\mathbb{C}})$ *For any unit normal vector $v$ of $M$, the nullity spaces for complex focal radii along the normal geodesic $\gamma_v$ span $(T_xM)^{\mathbb{C}} \ominus (\mathrm{Ker}\, A_v \cap \mathrm{Ker}\, R(v))^{\mathbb{C}}$.*

This condition $(*_{\mathbb{C}})$ is the condition weaker than $(*_{\mathbb{R}})$. In the case where $G/K$ is of non-compact type, $(*_{\mathbb{C}})$ also does not necessarily hold. For example, in the case where $G/K$ is a hyperbolic space of constant curvature $c(< 0)$ and where $M$ is a hypersurface, $M$ satisfies $(*_{\mathbb{C}})$ if and only if all the principal curvatures of $M$ at each point of $M$ are not equal to $\pm\sqrt{-c}$.

In this paper, we first prove the following result.

**Theorem A.** *Let $M$ be a complete isoparametric submanifold in a symmetric space $G/K$ of non-compact type or compact type. If $M$ admits a reflective focal submanifold, then it is curvature-adapted.*

REMARK 1.1. In Theorem A, the condition of the existence of a reflective focal submanifold is indispensable. In fact, we have the following examples. Let $G = KAN$ be Iwasawa's decomposition of $G$. We can find many (complex) hyperpolar actions as subgroup actions of the solvable group $S := AN$. Since such hyperpolar actions admits no singular orbit, the principal orbits of the actions admit no focal submanifold. Among such actions, we can find ones whose principal orbits are not curvature-adapted (see [22]).

Next we prove the following extrinsic homogeneity theorem.

**Theorem B.** *Let $M$ be a full irreducible complete curvature-adapted isoparametric $C^\omega$-submanifold of codimension greater than one in a symmetric space $G/K$ of non-compact type. If $M$ satisfies the above condition $(*_{\mathbb{C}})$, then $M$ is extrinsically homogeneous.*

The proof of this theorem is performed by showing the extrinsic homogeneity of $M$ (see Theorem 7.1) by using Fact 1.1 through the anti-Kaehler submersion $\widetilde{\phi} := \pi \circ \phi : H^0([0,1], \mathfrak{g}^{\mathbb{C}}) \to G^{\mathbb{C}}/K^{\mathbb{C}}$ (see Figure 1).

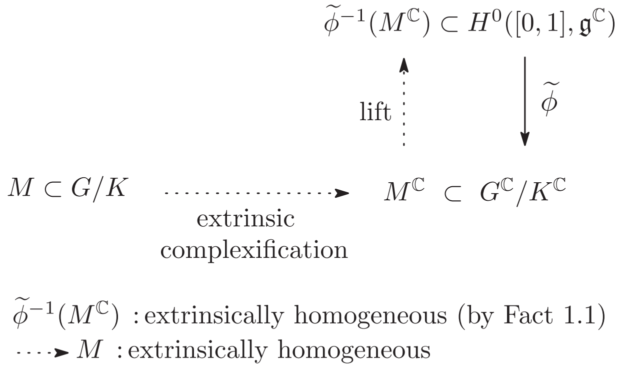



Fig. 1. The method of the proof of the extrinsic homogeneity

Let $G/K$ be a symmetric space of non-compact type and $H$ a closed subgroup of $G$. If there exists an involution $\sigma$ of $G$ with $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$, then we ([17]) called the $H$-action on $G/K$ an *action of Hermann type*, where $\mathrm{Fix}\,\sigma$ is the fixed point group of $\sigma$ and $(\mathrm{Fix}\,\sigma)_0$ is the identity component of $\mathrm{Fix}\,\sigma$. In [23], we called this kind of actions on semi-simple pseudo-Riemannian symmetric spaces (in more general) a *Hermann type action*. In this paper, we shall use this terminology. According to the result in [17], it follows that principal orbits of a Hermann type action are curvature-adapted complex equifocal (hence isoparametric) $C^\omega$-submanifolds and that they satisfy the condition $(*_{\mathbb{C}})$. Also, a Hermann type action admits a reflective singular orbit and hence the principal orbits of the action admit a reflective focal submanifold.

The main result of this paper is as follows.

**Theorem C.** *Let $M$ be a full irreducible complete isoparametric $C^\omega$-submanifold of codimension greater than one in a symmetric space $G/K$ of non-compact type. If $M$ admits a reflective focal submanifold, then it is a principal orbit of a Hermann type action on $G/K$.*

From Theorem C and the list of Hermann type actions in [19], we can classify isoparmetric submanifolds as in Theorem C. See Section 9 in detail.

If $M$ is a principal orbit of the isotropy action of a symmetric space $G/K$ of non-compact type, then it satisfies the following condition:

$(*'_{\mathbb{R}})$ *For any unit normal vector $v$ of $M$, the nullity spaces for focal radii along the normal geodesic $\gamma_v$ span $T_xM$.*

By using the building theory, we prove that the following fact holds without the assumption of the real analyticity of the submanifold.

**Theorem D.** *Let $M$ be a full irreducible complete curvature-adapted isoparametric submanifold of codimension greater than two in an irreducible symmetric space $G/K$ of non-compact type. If $M$ satisfies the above condition $(*'_{\mathbb{R}})$, then $M$ is a principal orbit of the isotropy action of $G/K$.*

In Section 2-5, we shall recall the basic notions and facts. In Section 6, we shall prove Theorem A by using the basic facts stated in Section 5. In Section 7, we shall prove Theorem B by using Fact 1.1. In Section 8, we shall prove Theorems C (Main theorem) by using Theorems A and B. In Section 9, we shall classify isoparametric submanifolds as in Theorem C. In Section 10, we prove Theorem D.

## 2. Complex focal radius

In this section, we shall recall the notion of a complex focal radius and some facts related to it, which will be used in Sections 7 and 8. Let $M$ be a submanifold in a complete Riemannian manifold $N$, $\psi : T^\perp M \to M$ the normal bundle of $M$ and $\exp^\perp$ the normal exponential map of $M$. Denote by $\mathcal{V}$ the vertical distribution on $T^\perp M$ and $\mathcal{H}$ the horizontal distribution on $T^\perp M$ with respect to the normal connection of $M$. Let $v$ be a unit normal vector of $M$ at $x(\in M)$ and $r$ a real number. Denote by $\gamma_v$ the normal geodesic of $M$ of direction $v$ (i.e., $\gamma_v(s) = \exp^\perp(sv)$). If $\psi_*(\mathrm{Ker}\,\exp^\perp_{*rv}) \neq \{0\}$, then $\exp^\perp(rv)$ (resp. $r$) is called a *focal point* (resp. a *focal radius*) *of $M$ along $\gamma_v$*. For a focal radius $r$ of $M$ along $\gamma_v$, $\psi_*(\mathrm{Ker}\,\exp^\perp_{*rv})$ is called the *nullity space* for $r$ and its dimension is called the *multiplicity* of $r$. Denote by $\mathcal{FR}^{\mathbb{R}}_{M,v}$ the set of all focal radii of $M$ along $\gamma_v$. Set

$$\mathcal{F}^{\mathbb{R}}_{M,x} := \bigcup_{v\in T^\perp_x M \text{ s.t. } \|v\|=1} \{rv \,|\, r \in \mathcal{FR}^{\mathbb{R}}_{M,v}\},$$

which is called the *tangential focal set of $M$ at $x$*. Note that $\exp^\perp(\mathcal{F}^{\mathbb{R}}_{M,x})$ is the focal set of $M$ at $x$. If, for any $y \in M$, the normal umbrella $\Sigma_y := \exp^\perp(T^\perp_y M)$ is totally geodesic in $G/K$ and the induced metric on $\Sigma_y$ is flat, then $M$ is called a *submanifold with flat section*. Assume that $N$ is a symmetric space $G/K$ and that $M$ is a submanifold with flat section. Then we can show that, for any $rv \in T^\perp M$, $\mathrm{Ker}\,\exp^\perp_{*rv} \subset \mathcal{H}_{rv}$ holds and that

$$\exp^\perp_{*rv}(X^L_{rv}) = P_{\gamma_{rv}|_{[0,1]}}\left(\left(\cos(r\sqrt{R(v)}) - \frac{\sin(r\sqrt{R(v)})}{\sqrt{R(v)}} \circ A_v\right)(X)\right) \quad (X \in T_x M) \tag{2.1}$$

holds, where $X^L_{rv}$ is the horizontal lift of $X$ to $rv$, $P_{\gamma_{rv}|_{[0,1]}}$ is the parallel translation along the normal geodesic $\gamma_{rv}|_{[0,1]}$, $R(v)$ is the normal Jacobi operator $R(\bullet, v)v$ and $A$ is the shape tensor of $M$ (see Figure 2). Hence $\mathcal{FR}^{\mathbb{R}}_{M,v}$ coincide with the set of all zero points of the real-valued function

$$F_v(s) = \det\left(\cos(s\sqrt{R(v)}) - \frac{\sin(s\sqrt{R(v)})}{\sqrt{R(v)}} \circ A_v\right) \qquad (s \in \mathbb{R}).$$

In particular, in the case where $G/K$ is a Euclidean space, we have $F_v(s) = \det(\mathrm{id} - sA_v)$ (id : the identity transformation of $T_x M$). Hence $\mathcal{FR}^{\mathbb{R}}_{M,v}$ is equal to the set of all the inverse numbers of the eigenvalues of $A_v$ and the nullity space for $r \in \mathcal{FR}^{\mathbb{R}}_{M,v}$ is equal to $\mathrm{Ker}(A_v - \frac{1}{r}\mathrm{id})$.

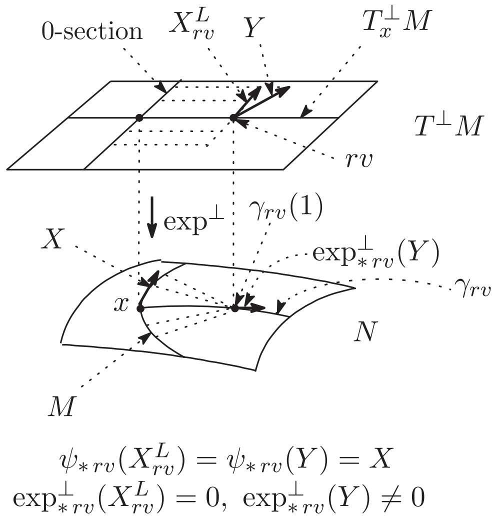



Fig. 2. Focal points of a submanifold with flat section

Therefore the nullity spaces for focal radii of $M$ along $\gamma_v$ span $T_xM \ominus \operatorname{Ker} A_v$. In the case where $G/K$ is a sphere of constant curvature $c(>0)$, we have

$$F_v(s) = \det\left(\cos(s\sqrt{c})\mathrm{id} - \frac{\sin(s\sqrt{c})}{\sqrt{c}}A_v\right).$$

Hence we have

$$\mathcal{FR}^{\mathbb{R}}_{M,v} = \left\{ \frac{1}{\sqrt{c}}\left(\arctan\frac{\sqrt{c}}{\lambda} + j\pi\right) \,\middle|\, \lambda : \text{the eigenvalue of } A_v, \quad j \in \mathbb{Z} \right\}$$

and the nullity space for $\frac{1}{\sqrt{c}}\left(\arctan\frac{\sqrt{c}}{\lambda} + j\pi\right)$ is equal to $\operatorname{Ker}(A_v - \lambda\,\mathrm{id})$, where we note that $\arctan\frac{\sqrt{c}}{\lambda}$ means $\frac{\pi}{2}$ when $\lambda = 0$. Therefore the nullity spaces for focal radii of $M$ along $\gamma_v$ span $T_xM$. Note that the focal set of $M$ at $x$ is given by

$$\mathcal{F}^{\mathbb{R}}_{M,x} = \bigcup_{v \in T^\perp_x M \text{ s.t. } \|v\|=1} \left\{ \gamma_v(r) \,\middle|\, r = \frac{1}{\sqrt{c}}\arctan\frac{\sqrt{c}}{\lambda} \text{ or } \frac{1}{\sqrt{c}}\left(\arctan\frac{\sqrt{c}}{\lambda};\pi\right) \right\}.$$

In the case where $G/K$ is a hyperbolic space of constant curvature $c(<0)$, we have

$$F_v(s) = \det\left(\cosh(s\sqrt{-c})\mathrm{id} - \frac{\sinh(s\sqrt{-c})}{\sqrt{-c}}A_v\right).$$

Hence we have

$$\mathcal{FR}^{\mathbb{R}}_{M,v} = \left\{ \frac{1}{\sqrt{-c}}\operatorname{arctanh}\frac{\sqrt{-c}}{\lambda} \,\middle|\, \lambda : \text{the eigenvalue of } A_v \text{ s.t. } |\lambda| > \sqrt{-c} \right\} \tag{2.2}$$

and the nullity space for $\frac{1}{\sqrt{-c}}\operatorname{arctanh}\frac{\sqrt{-c}}{\lambda}$ is equal to $\operatorname{Ker}(A_v - \lambda\,\mathrm{id})$. Therefore the nullity spaces for focal radii of $M$ along $\gamma_v$ span $T_xM$ if and only if all the absolute values of eigenvalues of $A_v$ is greater than $\sqrt{-c}$. As a non-compact submanifold $M$ with flat section in a symmetric space $G/K$ of non-compact type is deformed as its principal curvatures approach to zero, its focal set vanishes beyond the ideal boundary $(G/K)(\infty)$ of $G/K$. This fact follows from (2.2). According to this fact, we ([15]) considered that a focal radius of $M$ along the normal geodesic $\gamma_v$ should be defined in the complex number field $\mathbb{C}$. We ([15]) introduced the notion of a complex focal radius as the zero points of the complex-valued function $F_v^{\mathbb{C}}$ over $\mathbb{C}$ defined by

$$F_v^{\mathbb{C}}(z) := \det\left(\cos(z\sqrt{R(v)}^{\mathbb{C}}) - \frac{\sin(z\sqrt{R(v)}^{\mathbb{C}})}{\sqrt{R(v)}^{\mathbb{C}}} \circ A_v^{\mathbb{C}}\right) \qquad (z \in \mathbb{C}),$$

where $A_v^{\mathbb{C}}$ and $\sqrt{R(v)}^{\mathbb{C}}$ are the complexifications of $A_v$ and $\sqrt{R(v)}$, respectively. For a complex focal radius $z$ of $M$ along $\gamma_v$, $\mathrm{Ker}\left(\cos(z\sqrt{R(v)}^{\mathbb{C}}) - \frac{\sin(z\sqrt{R(v)}^{\mathbb{C}})}{\sqrt{R(v)}^{\mathbb{C}}} \circ A_v^{\mathbb{C}}\right)$ $(\subset (T_xM)^{\mathbb{C}})$ is called the *nullity space* for $z$ and its complex dimension is called the *multiplicity* of $z$. Denote by $\mathcal{FR}_{M,v}^{\mathbb{C}}$ the set of all complex focal radii of $M$ along $\gamma_v$. Set

$$\mathcal{F}_{M,x}^{\mathbb{C}} := \bigcup_{v\in T_x^{\perp}M \text{ s.t. } \|v\|=1} \{rv \mid r \in \mathcal{FR}_{M,v}^{\mathbb{C}}\} \quad (\subset (T_x^{\perp}M)^{\mathbb{C}}),$$

which is called the *tangential complex focal set of $M$ at $x$*. In the case where $G/K$ is a Euclidean space, we have $F_v^{\mathbb{C}}(z) = \det(\mathrm{id} - zA_v^{\mathbb{C}})$ (id : the identity transformation of $(T_xM)^{\mathbb{C}}$). Hence we have $\mathcal{FR}_{M,v}^{\mathbb{C}} = \mathcal{FR}_{M,v}^{\mathbb{R}}$ and the nullity space for $z \in \mathcal{FR}_{M,v}^{\mathbb{C}}$ is equal to $\mathrm{Ker}(A_v^{\mathbb{C}} - \frac{1}{z}\mathrm{id})$. Therefore the nullity spaces for complex focal radii of $M$ along $\gamma_v$ span $(T_xM)^{\mathbb{C}} \ominus \mathrm{Ker}\, A_v^{\mathbb{C}}$. Also, in the case where $G/K$ is a sphere of constant curvature $c(>0)$, we have

$$F_v^{\mathbb{C}}(z) = \det\left(\cos(z\sqrt{c})\mathrm{id} - \frac{\sin(z\sqrt{c})}{\sqrt{c}}A_v^{\mathbb{C}}\right).$$

Hence $\mathcal{FR}_{M,v}^{\mathbb{C}} = \mathcal{FR}_{M,v}^{\mathbb{R}}$ and the nullity space for $\frac{1}{\sqrt{c}}\left(\arctan\frac{\sqrt{c}}{\lambda} + j\pi\right)$ is equal to $\mathrm{Ker}(A_v^{\mathbb{C}} - \lambda\,\mathrm{id})$. Therefore the nullity spaces for complex focal radii of $M$ along $\gamma_v$ span $(T_xM)^{\mathbb{C}}$. Also, in the case where $G/K$ is a hyperbolic space of constant curvature $c(<0)$, we have

$$F_v^{\mathbb{C}}(z) = \det\left(\cos(\sqrt{-1}z\sqrt{-c})\mathrm{id} - \frac{\sin(\sqrt{-1}z\sqrt{-c})}{\sqrt{-1}\sqrt{-c}}A_v^{\mathbb{C}}\right),$$

where $\sqrt{-1}$ denotes the imaginary unit. Hence $\mathcal{FR}_{M,v}^{\mathbb{C}}$ is equal to

$$= \left\{\frac{1}{\sqrt{-c}}\left(\mathrm{arctanh}\frac{\sqrt{-c}}{\lambda} + j\pi\sqrt{-1}\right) \,\middle|\, \lambda : \text{the eigenvalue of } A_v \text{ s.t. } |\lambda| > \sqrt{-c}, \quad j \in \mathbb{Z}\right\} \cup$$
$$\left\{\frac{1}{\sqrt{-c}}\left(\mathrm{arctanh}\frac{\lambda}{\sqrt{-c}} + (j+\frac{1}{2})\pi\sqrt{-1}\right) \,\middle|\, \lambda : \text{the eigenvalue of } A_v \text{ s.t. } |\lambda| < \sqrt{-c}, \quad j \in \mathbb{Z}\right\},$$

the nullity space for $\frac{1}{\sqrt{-c}}\left(\mathrm{arctanh}\frac{\sqrt{-c}}{\lambda} + j\pi\sqrt{-1}\right)$ $(|\lambda| > \sqrt{-c})$ is equal to $\mathrm{Ker}(A_v^{\mathbb{C}} - \lambda\,\mathrm{id})$ and the nullity space for $\frac{1}{\sqrt{-c}}\left(\mathrm{arctanh}\frac{\lambda}{\sqrt{-c}} + (j+\frac{1}{2})\pi\sqrt{-1}\right)$ $(|\lambda| > \sqrt{-c})$ is equal to $\mathrm{Ker}(A_v^{\mathbb{C}} - \lambda\,\mathrm{id})$. Therefore the nullity spaces for complex focal radii of $M$ along $\gamma_v$ span $(T_xM)^{\mathbb{C}}$ if and only if all the eigenvalues of $A_v$ are not equal to $\pm\sqrt{-c}$.

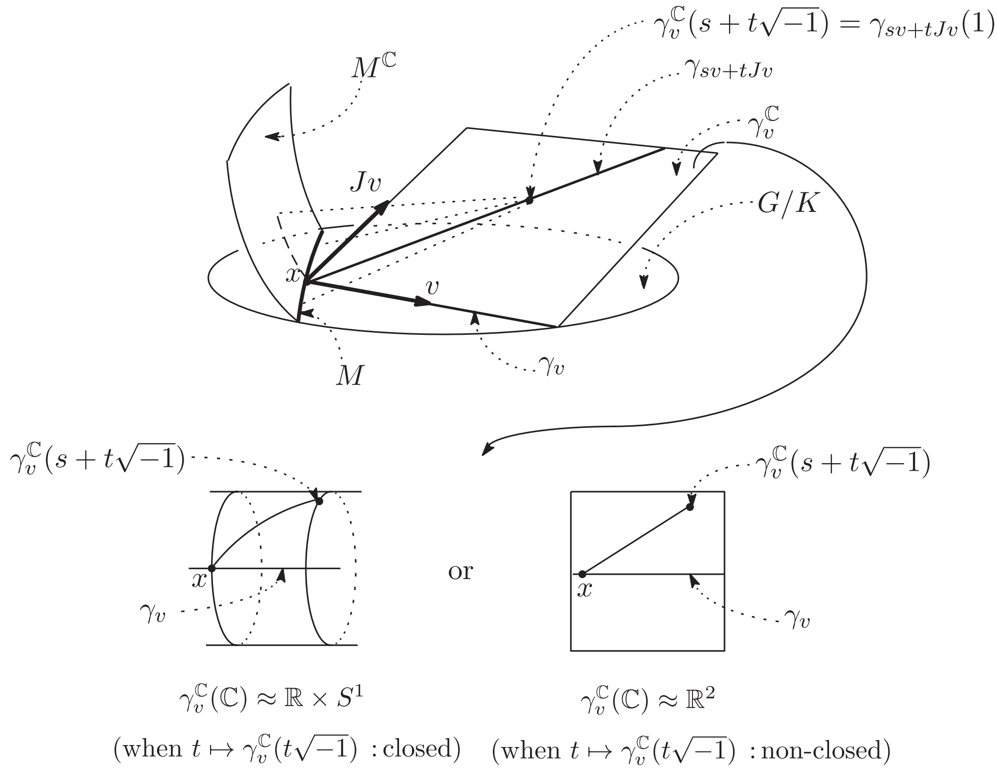
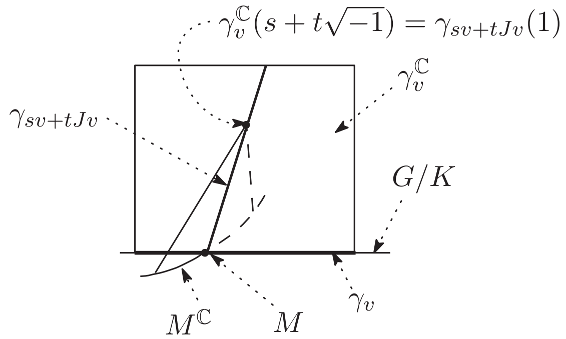



Fig. 3. The geometrical meaning of the complex focal radius




If we focus attention on $T_xM^{\mathbb{C}} \cap T_x\gamma(\mathbb{C}) = \{0\}$, then we should draw this figure.

Fig. 4. The geometrical meaning of the complex focal radius (continued)

Let $M$ be a $C^\omega$-submanifold with flat section in a symmetric space $G/K$. Then we can define the complete complexification $M^{\mathbb{C}}$ of $M$ as a (metrically) complete anti-Kaehler submanifold in the anti-Kaehler symmetric space $G^{\mathbb{C}}/K^{\mathbb{C}}$ associated with $G/K$ (see the proof of Theorem B in [20]). Denote by $J$ and $\widehat{R}$ the complex structure and the curvature tensor of $G^{\mathbb{C}}/K^{\mathbb{C}}$, respectively, and $\widehat{A}$ and $\widehat{\exp}^\perp$ the shape tensor and the normal exponential map of $M^{\mathbb{C}}$, respectively. Denote by $\widehat{\mathcal{H}}$ the horizontal distribution on the normal bundle $T^\perp(M^{\mathbb{C}})$ of $M^{\mathbb{C}}$ with respect to the normal connection of $M^{\mathbb{C}}$. Take $v \in T^\perp_xM (\subset T^\perp_x(M^{\mathbb{C}}))$ and $z = s + t\sqrt{-1} \in \mathbb{C}$ $(s, t \in \mathbb{R})$. Then we can show $\mathrm{Ker}\,\widehat{\exp}^\perp_{*sv+tJv} \subset \widehat{\mathcal{H}}_{sv+tJv}$ and

$$\widehat{\exp}^{\perp}_{*sv+tJv}(X^L_{sv+tJv}) = P_{\gamma_{sv+tJv}|_{[0,1]}}(Q_{v,z}(X)) \quad (X \in T_x(M^{\mathbb{C}})),$$

where $X^L_{s+tJv}$ is the horizontal lift of $X$ to $sv + tJv$, $P_{\gamma_{sv+tJv}}$ is the parallel translation along the normal geodesic $\gamma_{sv+tJv}$ of $M^{\mathbb{C}}$ and

$$Q_{v,z} := \cos\left(s\sqrt{\widehat{R}(v)} + t\left(J \circ \sqrt{\widehat{R}(v)}\right)\right) - \frac{\sin\left(s\sqrt{\widehat{R}(v)} + t\left(J \circ \sqrt{\widehat{R}(v)}\right)\right)}{\sqrt{\widehat{R}(v)}} \circ \widehat{A}_v$$

($\widehat{R}(v) := \widehat{R}(\bullet, v)v$). Hence $\widehat{\exp}^{\perp}(sv + tJv) = \gamma_{sv+tJv}(1) = \gamma^{\mathbb{C}}_v(s + t\sqrt{-1})$ is a focal point of $M^{\mathbb{C}}$ along the geodesic $\gamma_{sv+tJv}$ if and only if $z = s + t\sqrt{-1}$ is a zero point of the complex-valued function $\widehat{F}_v$ over $\mathbb{C}$ defined by $\widehat{F}_v(z) := \det Q_{v,z}$, where $Q_{v,z}$ is regarded as a $\mathbb{C}$-linear transformation of $T_x(M^{\mathbb{C}})$ regarded as a complex linear space by $J$. On the other hand, it is clear that the set of all zero points of $\widehat{F}_v$ is equal to that of $F^{\mathbb{C}}_v$. Therefore $z = s + t\sqrt{-1}$ is a complex focal radius of $M$ along $\gamma_v$ if and only if $\gamma^{\mathbb{C}}_v(s + t\sqrt{-1}) = \widehat{\exp}^{\perp}(sv + tJv)$ is a focal point of $M^{\mathbb{C}}$ along $\gamma_{sv+tJv}$ (see Figures 3 and 4). Hence we see that $\widehat{\exp}^{\perp}(\mathcal{F}^{\mathbb{C}}_{M,x})$ is the focal set of $M^{\mathbb{C}}$ ($\subset G^{\mathbb{C}}/K^{\mathbb{C}}$) at $x$, where we identify $(T_xM)^{\mathbb{C}}$ with $T_x(M^{\mathbb{C}})$. Thus we can grasp the geometrical meaning of the complex focal radius.

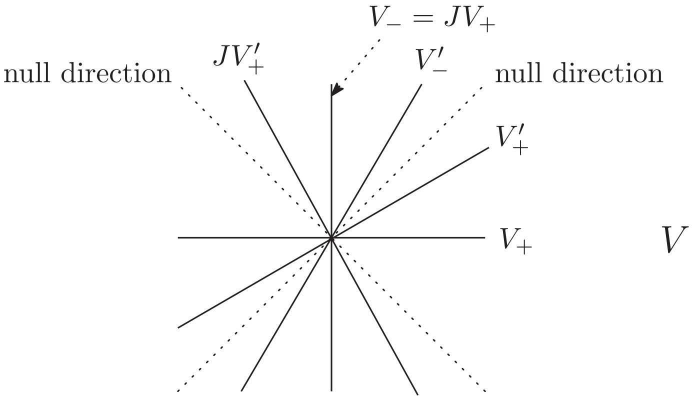
## 3. Anti-Kaehler isoparametric submanifold

In this section, we shall recall the notion of a proper anti-Kaehler isoparametric submanifold, which was introduced in [16]. Throughout this paper, we shall call this notion an *anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators* because this terminology seems to be more familiar than "proper anti-Kaehler isoparametric submanifold". Also, we shall state the definition of the (metrically) completeness of an anti-Kaehler Hilbert manifold. Furthermore we shall recall some facts related to this submanifold, which will be used in Sections 7 and 8.

First we shall recall the notion of an infinite dimensional anti-Kaehler space, where we shall define this notion more smartly than the definition in [16]. Let $V$ be an infinite dimensional topological real vector space, $J$ be a continuous linear operator of $V$ such that $J^2 = -\mathrm{id}$ and $\langle\ ,\ \rangle$ be a continuous non-degenerate symmetric bilinear form of $V$ such that $\langle JX, JY\rangle = -\langle X, Y\rangle$ holds for every $X, Y \in V$. It is easy to show that there uniquely exists an orthogonal time-space decomposition $V = V_- \oplus V_+$ compatible with $J$, that is, the decomposition such that $\langle\ ,\ \rangle|_{V_-\times V_-}$ is negative definite, $\langle\ ,\ \rangle|_{V_+\times V_+}$ is positive definite, $\langle\ ,\ \rangle|_{V_-\times V_+} = 0$ and that $JV_{\pm} = V_{\mp}$ (see Figure 5). Define an inner product $\langle\ ,\ \rangle^{\mathcal{I}}$ of $V$ by

$$\langle\ ,\ \rangle^{\mathcal{I}} := -\pi^*_{V_-}\langle\ ,\ \rangle + \pi^*_{V_+}\langle\ ,\ \rangle,$$

where $\pi_{V_{\pm}}$ denotes the projection of $V$ onto $V_{\pm}$. If $(V, \langle\ ,\ \rangle^{\mathcal{I}})$ is a Hilbert space and that the distance topology associated with $\langle\ ,\ \rangle^{\mathcal{I}}$ coincides with the original topology of $V$, then we ([16]) called $(V, \langle\ ,\ \rangle, J)$ the *infinite dimensional anti-Kaehler space*. Here we state that, from each (infinite dimensional separable) Hilbert space, an infinite dimensional anti-Kaehler space is constructed in natural manner. Let $(W, \langle\ ,\ \rangle_W)$ be a (infinite dimensional separable) Hilbert space and $V := W^{\mathbb{C}}$ be the complexification of $W$. Also, let $\langle\ ,\ \rangle^{\mathbb{C}}_W$ (: $V\times V \to \mathbb{C}$) be the complexification of $\langle\ ,\ \rangle_W$. We regard $V$ as a topological real vector space. Define a continuous ($\mathbb{R}$-)linear operator $J$ of $V$ by $Jv := \sqrt{-1}v$ ($v \in V$) and a continuous non-degenerate symmetric ($\mathbb{R}$-)bilinear form $\langle\ ,\ \rangle$ of $V$ by $\langle v_1, v_2\rangle := 2\mathrm{Re}(\langle\langle v_1, v_2\rangle\rangle)$ $(v_1, v_2 \in V)$, where $\mathrm{Re}(\cdot)$ is the real part of $(\cdot)$. Then $(V, \langle\ ,\ \rangle, J)$ is an infinite dimensional anti-Kaehler space and $V = \sqrt{-1}W \oplus W$ is the orthogonal time-space decomposition compatible with $J$.




$$\langle\ ,\ \rangle|_{V_-\times V_+} = 0,\ \langle\ ,\ \rangle^{\mathcal{I}}|_{V_-\times V_+} = 0$$
$$\langle\ ,\ \rangle|_{V'_-\times V'_+} = 0,\ \langle\ ,\ \rangle|_{V'_+\times JV'_+} \neq 0,\ \langle\ ,\ \rangle^{\mathcal{I}}|_{V'_+\times JV'_+} = 0$$

Fig. 5. The uniqueness of the orthogonal time-space decomposition compatible with $J$

Next we recall the notion of an anti-Kaeher Hilbert manifold, where we shall define this notion more smartly than the definition in [16]). Also, we define the (metrically) completeness of an anti-Kaehler Hilbert manifold. Let $N$ be a Hilbert manifold modelled on a (separable) Hilbert space $(V, \langle\ ,\ \rangle_V)$. Let $\langle\ ,\ \rangle$ be a (smooth) section of the $(0,2)$-tensor bundle $T^*M \otimes T^*M$ such that $\langle\ ,\ \rangle_x$ is a continuous non-degenerate symmetric bilinear form on $T_xM$ for each $x \in M$. Also, let $J$ be a (smooth) section of the $(1,1)$-tensor bundle $T^*M \otimes TM$ such that $J_x$ is a continuous linear operator of $T_xM$ for each $x \in M$, $J^2 = -\mathrm{id}$, $\nabla J = 0$ and that $\langle JX, JY\rangle = -\langle X, Y\rangle$ for every $X, Y \in TM$, where $\nabla$ denotes the Levi-Civita connection of $\langle\ ,\ \rangle$. For each $x \in M$, there uniquely exists an orthogonal time-space decomposition $T_xM = W^x_- \oplus W^x_+$ (with respect to $\langle\ ,\ \rangle_x$) compatible with $J_x$, that is, the decomposition such that $\langle\ ,\ \rangle_x|_{W^x_-\times W^x_-}$ is negative definite, $\langle\ ,\ \rangle_x|_{W^x_+\times W^x_+}$ is positive definite, $\langle\ ,\ \rangle_x|_{W^x_-\times W^x_+} = 0$ and that $J_xW^x_\pm = W^x_\mp$. Define an inner product $\langle\ ,\ \rangle^{\mathcal{I}}_x$ of $T_xM$ by

$$\langle\ ,\ \rangle^{\mathcal{I}}_x := -\pi^*_{W^x_-}\langle\ ,\ \rangle_x + \pi^*_{W^x_+}\langle\ ,\ \rangle_x,$$

where $\pi_{W^x_\pm}$ denotes the projection of $T_xM$ onto $W^x_\pm$. Let $\langle\ ,\ \rangle^{\mathcal{I}}$ be the section of $T^*M \otimes TM$ defined by assigning $\langle\ ,\ \rangle^{\mathcal{I}}_x$ to each $x \in M$. If $(T_xM, \langle\ ,\ \rangle^{\mathcal{I}}_x)$ is isometric to $(V, \langle\ ,\ \rangle_V)$ for each $x \in M$, then we ([16]) called $(M, \langle\ ,\ \rangle, J)$ an *anti-Kaehler Hilbert manifold*. If the Riemannian Hilbert manifold $(M, \langle\ ,\ \rangle^{\mathcal{I}})$ is complete, then we say that the anti-Kaehler Hilbert manifold $(M, \langle\ ,\ \rangle)$ is *(metrically) complete*. Note that the (metrically) completeness of a finite dimensional anti-Kaehler manifold also is defined similarly.

Let $f$ be an isometric immersion of an anti-Kaehler Hilbert manifold $(M, \langle\ ,\ \rangle, J)$ into an anti-Kaehler space $(V, \langle\ ,\ \rangle, \widetilde{J})$. If $\widetilde{J} \circ f_* = f_* \circ J$ holds, then we ([16]) called $M$ an *anti-Kaehler Hilbert submanifold in* $(V, \langle\ ,\ \rangle, \widetilde{J})$ *immersed by* $f$. If $M$ is of finite codimension and, for each $v \in T^\perp M$, the shape operator $A_v$ is a compact operator with respect to $f^*\langle\ ,\ \rangle^{\mathcal{I}}$, then we ([16]) called $M$ an *anti-Kaehler Fredholm submanifold*. Assume that $M$ is an embedded anti-Kaehler Fredholm submanifold in $V$, where $M \subset V$ and $f$ is the inclusion map of $M$ into $V$. For the simplicity, denote by the same symbol $J$ the complex structures of $M$ and $V$. Also, denote by $A$ the shape tensor of $M$. Fix a unit normal vector $v$ of $M$. If there exists $X(\neq 0) \in TM$ with $A_vX = aX + bJX$, then we call the complex number $a + b\sqrt{-1}$ a *$J$-eigenvalue of $A_v$* (or a *$J$-principal curvature of direction $v$*) and call $X$ a *$J$-eigenvector for $a + b\sqrt{-1}$*. Also, we call the space of all $J$-eigenvectors for $a + b\sqrt{-1}$ a *$J$-eigenspace for $a + b\sqrt{-1}$*. The $J$-eigenspaces are orthogonal to one another and they are $J$-invariant, respectively. We call the set of all $J$-eigenvalues of $A_v$ the *$J$-spectrum of $A_v$* and denote it by $\mathrm{Spec}_J A_v$. Since $M$ is an anti-Kaehler Fredholm submanifold, $\mathrm{Spec}_J A_v \setminus \{0\}$ is described as follows:

$$\mathrm{Spec}_J A_v \setminus \{0\} = \{\mu_i \,|\, i = 1, 2, \cdots\}$$

$$\left(\begin{array}{c} |\mu_i| > |\mu_{i+1}| \text{ or } \text{"}|\mu_i| = |\mu_{i+1}| \text{ \& } \mathrm{Re}\,\mu_i > \mathrm{Re}\,\mu_{i+1}\text{"} \\ \text{or } \text{"}|\mu_i| = |\mu_{i+1}| \text{ \& } \mathrm{Re}\,\mu_i = \mathrm{Re}\,\mu_{i+1} \text{ \& } \mathrm{Im}\,\mu_i = -\mathrm{Im}\,\mu_{i+1} > 0\text{"} \end{array}\right).$$

Also, the $J$-eigenspace for each $J$-eigenvalue of $A_v$ other than 0 is of finite dimension. We call the $J$-eigenvalue $\mu_i$ the *$i$-th $J$-principal curvature of direction $v$*. Assume that the normal holonomy group of $M$ is trivial. Fix a parallel normal vector field $\widetilde{v}$ of $M$. Assume that the number (which may be $\infty$) of distinct $J$-principal curvatures of direction $\widetilde{v}_x$ is independent of the choice of $x \in M$. Then we can define complex-valued functions $\widetilde{\mu}_i$ $(i = 1, 2, \cdots)$ on $M$ by assigning the $i$-th $J$-principal curvature of direction $\widetilde{v}_x$ to each $x \in M$. We call this function $\widetilde{\mu}_i$ the *$i$-th $J$-principal curvature function of direction $\widetilde{v}$*. The submanifold $M$ is called an *anti-Kaehler isoparametric submanifold* if it satisfies the following condition:

> For each parallel normal vector field $\widetilde{v}$ of $M$, the number of distinct $J$-principal curvatures of direction $\widetilde{v}_x$ is independent of the choice of $x \in M$, each $J$-principal curvature function of direction $\widetilde{v}$ is constant on $M$ and it has constant multiplicity.

Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal system of $T_xM$. If $\{e_i\}_{i=1}^{\infty} \cup \{Je_i\}_{i=1}^{\infty}$ is an orthonormal base of $T_xM$, then we call $\{e_i\}_{i=1}^{\infty}$ (rather than $\{e_i\}_{i=1}^{\infty} \cup \{Je_i\}_{i=1}^{\infty}$) a *$J$-orthonormal base*. If there exists a $J$-orthonormal base consisting of $J$-eigenvectors of $A_v$, then $A_v$ is said to *be diagonalized with respect to the $J$-orthonormal base*. If $M$ is anti-Kaehler isoparametric and, for each $v \in T^{\perp}M$, the shape operator $A_v$ is diagonalized with respect to a $J$-orthonormal base, then we ([16]) called $M$ a *proper anti-Kaehler isoparametric submanifold*. We named thus in similar to the terminology "*proper* isoparametric semi-Riemannian submanifold" used in [14]. Throughout this paper, we shall call this submanifold an *anti-Kaehler isoparametric submanifold with $J$-diagonalizable shape operators* because this terminology seems to be more familiar than "proper anti-Kaehler isoparametric submanifold". Assume that $M$ is an anti-Kaehler isoparametric submanifold with $J$-diagonalizable shape operators. Then, since the ambient space is flat and the normal holonomy group of $M$ is trivial, it follows from the Ricci equation that the shape operators $A_{v_1}$ and $A_{v_2}$ commute for arbitrary two unit normal vector $v_1$ and $v_2$ of $M$. Hence the shape operators $A_v$'s ($v \in T_x^{\perp}M$) are simultaneously diagonalized with respect to a $J$-orthonormal base. Let $\{E_i \,|\, i \in I\}$ be the family of distributions on $M$ such that, for each $x \in M$, $\{(E_i)_x \,|\, i \in I\}$ is the set of all common $J$-eigenspaces of

$A_v$'s ($v \in T^\perp_x M$). For each $x \in M$, we have $T_x M = \overline{\underset{i\in I}{\oplus}(E_i)_x}$, where $\overline{\underset{i\in I}{\oplus}(E_i)_x}$ denotes the closure of $\underset{i\in I}{\oplus}(E_i)_x$ with respect to $\langle\ ,\ \rangle^{\mathcal{I}}_x$. We regard $T^\perp_x M$ ($x \in M$) as a complex vector space by $J_x|_{T^\perp_x M}$ and denote the dual space of the complex vector space $T^\perp_x M$ by $(T^\perp_x M)^{*_{\mathbb{C}}}$. Also, denote by $(T^\perp M)^{*_{\mathbb{C}}}$ the complex vector bundle over $M$ having $(T^\perp_x M)^{*_{\mathbb{C}}}$ as the fibre over $x$. Let $\lambda_i$ ($i \in I$) be the section of $(T^\perp M)^{*_{\mathbb{C}}}$ such that $A_v = \mathrm{Re}(\lambda_i)_x(v)\mathrm{id} + \mathrm{Im}(\lambda_i)_x(v)J_x$ on $(E_i)_x$ for any $x \in M$ and any $v \in T^\perp_x M$. We call $\lambda_i$ ($i \in I$) *J-principal curvatures* of $M$ and $E_i$ ($i \in I$) *J-curvature distributions* of $M$. The distribution $E_i$ is integrable and each leaf of $E_i$ is a complex sphere. Each leaf of $E_i$ is called a *complex curvature sphere*. It is shown that there uniquely exists a normal vector field $n_i$ of $M$ with $\lambda_i(\cdot) = \langle n_i, \cdot\rangle - \sqrt{-1}\langle Jn_i, \cdot\rangle$ (see Lemma 5 of [16]). We call $n_i$ ($i \in I$) the *J-curvature normals* of $M$. Note that $n_i$ is parallel with respect to the complexification of the normal connection of $M$. Note that similarly are defined a (finite dimensional) proper anti-Kaehler isoparametric submanifold in a finite dimensional anti-Kaehler space, its $J$-principal curvatures, its $J$-curvature distributions and its $J$-curvature normals. Set $l^x_i := (\lambda_i)^{-1}_x(1)$. According to (i) of Theorem 2 in [16], the tangential focal set of $M$ at $x$ is equal to $\underset{i\in I}{\cup} l^x_i$. We call each $l^x_i$ a *complex focal hyperplane of M at x*. Let $\widetilde{v}$ be a parallel normal vector field of $M$. If $\widetilde{v}_x$ belongs to at least one $l_i$, then it is called a *focal normal vector field* of $M$. For a focal normal vector field $\widetilde{v}$, the focal map $f_{\widetilde{v}}$ is defined by $f_{\widetilde{v}}(x) := \exp^\perp(\widetilde{v}_x)$ ($x \in M$). The image $f_{\widetilde{v}}(M)$ is called a *focal submanifold* of $M$, where we denote by $F_{\widetilde{v}}$. For each $x \in F_{\widetilde{v}}$, the inverse image $f^{-1}_{\widetilde{v}}(x)$ is called a focal leaf of $M$. Denote by $T^x_i$ the complex reflection of order 2 with respect to $l^x_i$ (i.e., the rotation of angle $\pi$ having $l^x_i$ as the axis), which is an affine transformation of $T^\perp_x M$. Let $\mathcal{W}_x$ be the group generated by $T^x_i$'s ($i \in I$). According to Proposition 3.7 of [18], $\mathcal{W}_x$ is discrete. Furthermore, it follows from this fact that $\mathcal{W}_x$ is isomorphic an affine Weyl group. This group $\mathcal{W}_x$ is independent of the choice of $x \in M$ (up to group isomorphicness). Hence we simply denote it by $\mathcal{W}$. We call this group $\mathcal{W}$ the *complex Coxeter group associated with M*. According to Lemma 3.8 of [18], $W$ is decomposable (i.e., it is decomposed into a non-trivial product of two discrete complex reflection groups) if and only if there exist two $J$-invariant linear subspaces $P_1$ ($\neq \{0\}$) and $P_2$ ($\neq \{0\}$) of $T^\perp_x M$ such that $T^\perp_x M = P_1 \oplus P_2$ (orthogonal direct sum), $P_1 \cup P_2$ contains all $J$-curvature normals of $M$ at $x$ and that $P_i$ ($i = 1, 2$) contains at least one $J$- curvature normal of $M$ at $x$. Also, according to Theorem 1 of [18], $M$ is irreducible if and only if $\mathcal{W}$ is not decomposable.

## 4. Parallel transport map

Y. Maeda, S. Rosenberg and P. Tondeur ([29]) studied the minimality of the Gauge orbit in the space of the $H^0$-connections of the principal bundle $P$ having a compact semi-simple Lie group $G$ as the structure group over a compact Riemannian manifold $M$. Let $c^*P$ be the pull-back bundle of $P$ by a $C^\infty$-path $c : [0, 1] \to M$. The space of $H^0$-connections on $c^*P$ is identified with the (separable) Hilbert space $H^0([0, 1], \mathfrak{g})$ of the $H^0$-paths in the Lie algebra $\mathfrak{g}$ of $G$. The Hilbert Lie group $\Omega_e(G)(\subset H^1([0, 1], G))$ of $H^1$-loops at $e$ in $G$ acts on $H^0([0, 1], \mathfrak{g})$ as the subaction of the Gauge group on the space of connections, where $e$ is the identity element of $G$. The orbit map $\phi : H^0([0, 1], \mathfrak{g}) \to H^0([0, 1], \mathfrak{g})/\Omega_e(G) (= G)$ is called the *parallel transport map* for $G$. See [33], [36], [37], [38] and [34] about the study of the parallel transport map for a compact semi-simple Lie group. Now we shall consider the case where $[0,1]$ is replaced by the circle $S^1$ in the above definition. Then we shall explain that $\phi$ should be called the holonomy map for $G$. Let $\gamma : S^1 \to M$ be a $C^\infty$-loop. The space of $H^0$-connections on $\gamma^*P$ is identified with the (separable) Hilbert space $H^0(S^1, \mathfrak{g})$ of the $H^0$-loop in the Lie algebra $\mathfrak{g}$ of $G$. The Hilbert Lie group $\Omega_e(G)(\subset H^1(S^1, G))$ of $H^1$-loops at $e$ in $G$ acts on $H^0(S^1, \mathfrak{g})$ as the subaction of the Gauge group on the space of connections. We consider the orbit map $\phi : H^0(S^1, \mathfrak{g}) \to H^0(S^1, \mathfrak{g})/\Omega_e(G)(= G)$. Then, for each $C^\infty$-loop $\gamma : S^1 \to M$, $\phi(\gamma)$ is equal to the generator of the holonomy group (which is a cyclic subgroup of $G$) of the connection $\omega_\gamma$ of the trivial $G$-bundle $S^1 \times G \to S^1$ determined by $\gamma$. In this sense, $\phi$ should be called the holonomy map for $G$.

We ([16]) defined the notion of the parallel transport map for the complexification $G^{\mathbb{C}}$ of a semi-simple Lie group $G$. In this section, we recall this notion and some facts related to this notion, which will be used in Sections 7 and 8. Let $K$ be a maximal compact subgroup of $G$, $\mathfrak{g}$ (resp. $\mathfrak{k}$) the Lie algebra of $G$ (resp. $K$) and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$. Also, let $\langle\ ,\ \rangle$ be the $\mathrm{Ad}_G(G)$-invariant non-degenerate inner product of $\mathfrak{g}$. The Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is an orthogonal time-space decomposition of $\mathfrak{g}$ with respect to $\langle\ ,\ \rangle$, that is, $\langle\ ,\ \rangle|_{\mathfrak{k}\times\mathfrak{k}}$ is negative definite, $\langle\ ,\ \rangle|_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite and $\langle\ ,\ \rangle|_{\mathfrak{k}\times\mathfrak{p}}$ vanishes. Set $\langle\ ,\ \rangle^A := 2\mathrm{Re}\langle\ ,\ \rangle^{\mathbb{C}}$, where $\langle\ ,\ \rangle^{\mathbb{C}}$ is the complexification of $\langle\ ,\ \rangle$ (which is a $\mathbb{C}$-bilinear form of $\mathfrak{g}^{\mathbb{C}}$). The $\mathbb{R}$-bilinear form $\langle\ ,\ \rangle^A$ on $\mathfrak{g}^{\mathbb{C}}$ regarded as a real Lie algebra induces a bi-invariant pseudo-Riemannian metric on $G^{\mathbb{C}}$ and furthermore a $G^{\mathbb{C}}$-invariant anti-Kaehler metric on $G^{\mathbb{C}}/K^{\mathbb{C}}$. It is clear that $\mathfrak{g}^{\mathbb{C}} = (\mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}) \oplus (\sqrt{-1}\mathfrak{k} \oplus \mathfrak{p})$ is an orthogonal time-space decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\langle\ ,\ \rangle^A$. For the simplicity, set $\mathfrak{g}^{\mathbb{C}}_- := \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}$ and $\mathfrak{g}^{\mathbb{C}}_+ := \sqrt{-1}\mathfrak{k} \oplus \mathfrak{p}$. Note that $\mathfrak{g}^{\mathbb{C}}_-$ is the compact real form of $\mathfrak{g}^{\mathbb{C}}$. Set $\langle\ ,\ \rangle^{\mathcal{I}} := -\pi^*_{\mathfrak{g}^{\mathbb{C}}_-}\langle\ ,\ \rangle^A + \pi^*_{\mathfrak{g}^{\mathbb{C}}_+}\langle\ ,\ \rangle^A$, where $\pi_{\mathfrak{g}^{\mathbb{C}}_-}$ (resp. $\pi_{\mathfrak{g}^{\mathbb{C}}_+}$) is the projection of $\mathfrak{g}^{\mathbb{C}}$ onto $\mathfrak{g}^{\mathbb{C}}_-$ (resp. $\mathfrak{g}^{\mathbb{C}}_+$). Let $H^0([0,1], \mathfrak{g}^{\mathbb{C}})$ be the space of all $L^2$-integrable paths $u : [0,1] \to \mathfrak{g}^{\mathbb{C}}$ with respect to $\langle\ ,\ \rangle^{\mathcal{I}}$ and $H^0([0,1], \mathfrak{g}^{\mathbb{C}}_-)$ (resp. $H^0([0,1], \mathfrak{g}^{\mathbb{C}}_+)$) the space of all $L^2$-integrable paths $u : [0,1] \to \mathfrak{g}^{\mathbb{C}}_-$ (resp. $u : [0,1] \to \mathfrak{g}^{\mathbb{C}}_+$) with respect to $-\langle\ ,\ \rangle^A|_{\mathfrak{g}^{\mathbb{C}}_-\times\mathfrak{g}^{\mathbb{C}}_-}$ (resp. $\langle\ ,\ \rangle^A|_{\mathfrak{g}^{\mathbb{C}}_+\times\mathfrak{g}^{\mathbb{C}}_+}$). Clearly we have $H^0([0,1], \mathfrak{g}^{\mathbb{C}}) = H^0([0,1], \mathfrak{g}^{\mathbb{C}}_-) \oplus H^0([0,1], \mathfrak{g}^{\mathbb{C}}_+)$. Define a non-degenerate inner product $\langle\ ,\ \rangle^A_0$ of $H^0([0,1], \mathfrak{g}^{\mathbb{C}})$ by $\langle u, v\rangle^A_0 := \int_0^1 \langle u(t), v(t)\rangle^A dt$. It is easy to show that the decomposition $H^0([0,1], \mathfrak{g}^{\mathbb{C}}) = H^0([0,1], \mathfrak{g}^{\mathbb{C}}_-) \oplus H^0([0,1], \mathfrak{g}^{\mathbb{C}}_+)$ is an orthogonal time-space decomposition with respect to $\langle\ ,\ \rangle^A_0$. For the simplicity, set $H^{0,\mathbb{C}}_\varepsilon := H^0([0,1], \mathfrak{g}^{\mathbb{C}}_\varepsilon)$ ($\varepsilon = -$ or $+$) and $\langle\ ,\ \rangle^{\mathcal{I}}_0 := -\pi^*_{H^{0,\mathbb{C}}_-}\langle\ ,\ \rangle^A_0 + \pi^*_{H^{0,\mathbb{C}}_+}\langle\ ,\ \rangle^A_0$, where $\pi_{H^{0,\mathbb{C}}_-}$ (resp. $\pi_{H^{0,\mathbb{C}}_+}$) is the projection of $H^0([0,1], \mathfrak{g}^{\mathbb{C}})$ onto $H^{0,\mathbb{C}}_-$ (resp. $H^{0,\mathbb{C}}_+$). It is clear that $\langle u, v\rangle^{\mathcal{I}}_0 = \int_0^1 \langle u(t), v(t)\rangle^{\mathcal{I}} dt$ ($u, v \in H^0([0,1], \mathfrak{g}^{\mathbb{C}})$). Hence $(H^0([0,1], \mathfrak{g}^{\mathbb{C}}), \langle\ ,\ \rangle^{\mathcal{I}}_0)$ is a Hilbert space, that is, $(H^0([0,1], \mathfrak{g}^{\mathbb{C}}), \langle\ ,\ \rangle^A_0)$ is a pseudo-Hilbert space in the sense of [15]. Let $J$ be the endomorphism of $\mathfrak{g}^{\mathbb{C}}$ defined by $JX = \sqrt{-1}X$ ($X \in \mathfrak{g}^{\mathbb{C}}$). Denote by the same symbol $J$ the bi-invariant almost complex structure of $G^{\mathbb{C}}$ induced from $J$. Define the endomorphism $\widetilde{J}$ of $H^0([0,1], \mathfrak{g}^{\mathbb{C}})$ by $\widetilde{J}u = \sqrt{-1}u$ ($u \in H^0([0,1], \mathfrak{g}^{\mathbb{C}})$). From $\widetilde{J}H^{0,\mathbb{C}}_+ = H^{0,\mathbb{C}}_-$, $\widetilde{J}H^{0,\mathbb{C}}_- = H^{0,\mathbb{C}}_+$ and $\langle \widetilde{J}u, \widetilde{J}v\rangle^A_0 = -\langle u, v\rangle^A_0$ ($u, v \in H^0([0,1], \mathfrak{g}^{\mathbb{C}})$), the space $(H^0([0,1], \mathfrak{g}^{\mathbb{C}}), \langle\ ,\ \rangle^A_0, \widetilde{J})$ is an anti-Kaehler space. Let $H^1([0,1], \mathfrak{g}^{\mathbb{C}})$ be a pseudo-Hilbert subspace of $H^0([0,1], \mathfrak{g}^{\mathbb{C}})$ consisting of all absolutely continuous paths $u : [0,1] \to \mathfrak{g}^{\mathbb{C}}$ such that the weak derivative $u'$ of $u$ is squared integrable (with respect to $\langle\ ,\ \rangle^{\mathcal{I}}$). Also, let $H^1([0,1], G^{\mathbb{C}})$ be the Hilbert Lie group of all absolutely continuous paths $g : [0,1] \to G^{\mathbb{C}}$ such that the weak derivative $g'$ of $g$ is squared integrable (with respect to $\langle\ ,\ \rangle^{\mathcal{I}}$), that is, $g_*^{-1}g' \in H^0([0,1], \mathfrak{g}^{\mathbb{C}})$. Define a map $\phi : H^0([0,1], \mathfrak{g}^{\mathbb{C}}) \to G^{\mathbb{C}}$ by $\phi(u) := g_u(1)$ ($u \in H^0([0,1], \mathfrak{g}^{\mathbb{C}})$), where $g_u$ is the element of $H^1([0,1],G^{\mathbb{C}})$ with $g_u(0) = e$ and $g_{u*}^{-1}g'_u = u$. This map is called the *parallel transport map for* $G^{\mathbb{C}}$. This map is an anti-Kaehler submersion. Set $P(G^{\mathbb{C}}, e\times G^{\mathbb{C}}) := \{g \in H^1([0,1],G^{\mathbb{C}})\,|\,g(0) = e\}$ and $\Omega_e(G^{\mathbb{C}}) := \{g \in H^1([0,1],G^{\mathbb{C}})\,|\,g(0) = g(1) = e\}$. The group $H^1([0,1],G^{\mathbb{C}})$ acts on $H^0([0,1],\mathfrak{g}^{\mathbb{C}})$ as the action of the gauge transformation group on the space of connections, that is,

$$g * u := \mathrm{Ad}_{G^{\mathbb{C}}}(g)u - g'g_*^{-1} \qquad (g \in H^1([0,1],G^{\mathbb{C}}),\ u \in H^0([0,1],\mathfrak{g}^{\mathbb{C}})).$$

It is shown that the following facts hold:

(i) The above action of $H^1([0,1],G^{\mathbb{C}})$ on $H^0([0,1],\mathfrak{g}^{\mathbb{C}})$ is isometric,
(ii) The above action of $P(G^{\mathbb{C}}, e\times G^{\mathbb{C}})$ in $H^0([0,1],\mathfrak{g}^{\mathbb{C}})$ is transitive and free,
(iii) $\phi(g*u) = (L_{g(0)}\circ R_{g(1)}^{-1})(\phi(u))$ for $g \in H^1([0,1],G^{\mathbb{C}})$ and $u \in H^0([0,1],\mathfrak{g}^{\mathbb{C}})$,
(iv) $\phi : H^0([0,1],\mathfrak{g}^{\mathbb{C}}) \to G^{\mathbb{C}}$ is regarded as a $\Omega_e(G^{\mathbb{C}})$-bundle.
(v) If $\phi(u) = (L_{x_0}\circ R_{x_1}^{-1})(\phi(v))$ $(u, v \in H^0([0,1],\mathfrak{g}^{\mathbb{C}}),\ x_0, x_1 \in G^{\mathbb{C}})$, then there exists $g \in H^1([0,1],G^{\mathbb{C}})$ such that $g(0) = x_0,\ g(1) = x_1$ and $u = g * v$. In particular, it follows that any $u \in H^0([0,1],\mathfrak{g}^{\mathbb{C}})$ is described as $u = g * \hat{0}$ in terms of some $g \in P(G^{\mathbb{C}}, G^{\mathbb{C}}\times e)$.

## 5. Partial tubes

In this section, we recall some facts for partial tubes in a symmetric space, which will use to prove Theorem A in the next section. For a submanifold $F$ in a symmetric space $G/K$ of non-positive (or non-negative) curvature, M. Br*ü*ck ([3]) introduced a certain kind of partial tube with flat section including the normal holonomy tube, where $F$ is assumed to admit the $\varepsilon$-tube for a sufficiently small positive number $\varepsilon$. This notion is defined as follows. Let $\varepsilon_\gamma := \inf\{|r|\,|\,r$ : focal radius of $M$ along $\gamma\}$, where $\gamma$ is a unit speed normal geodesic of $F$. Set

$$\varepsilon_F := \inf\{\varepsilon_\gamma\,|\,\gamma : \text{unit speed normal geodesic of } F\}.$$

Assume that $\varepsilon_F > 0$. Fix $x_0 \in F$. Let $\mathfrak{C}_{x_0} := \{c : [0,1] \to F$ : a piecewise smooth path with $c(0) = x_0\}$, $\Phi^0_{x_0}$ be the restricted normal holonomy group of $F$ at $x_0$ and $\mathfrak{L}_{x_0}$ be the Lie subalgebra of $\mathfrak{so}(T^\perp_{x_0}F)$ generated by $\{P_c^{-1}\circ \mathrm{pr}_{T^\perp_{c(1)}F}\circ R_{c(1)}(P_cv_1, P_cv_2)\circ P_c\,|\,v_1, v_2 \in T^\perp_{x_0}M,\ c \in \mathfrak{C}_{x_0}\}$, where $R$ denotes the curvature tensor of $G/K$ and $P_c$ is the parallel transport along $c$ with respect to the normal connection $\nabla^\perp$ of $F$ and $\mathrm{pr}_{T^\perp_{c(1)}F}$ is the orthogonal projection onto $T^\perp_{c(1)}F$. Also, let $\widehat{\mathfrak{L}}_{x_0}$ be the Lie algebra generated by $\mathfrak{L}_{x_0}$ and $\mathrm{Lie}\,\Phi^0_{x_0}$. Let $L_{x_0} := \exp\,\mathfrak{L}_{x_0}$ and $\widehat{L}_{x_0} := \exp\,\widehat{\mathfrak{L}}_{x_0}$, where exp is the exponential map of $GL(T^\perp_{x_0}F)$. Note that $L_{x_0}$ and $\widehat{L}_{x_0}$ are Lie subgroups of $SO(T^\perp_{x_0}F)$. For $v_0 \in T^\perp_{x_0}F$, define a subbundle $B_{v_0}(F)$ of the normal bundle $T^\perp F$ of $F$ by

$$B_{v_0}(F) := \{P_c(gv_0)\ |\ g \in \widehat{L}_{x_0},\ c \in \mathfrak{C}_{x_0}\}$$

and $\widetilde{B}_{v_0}(F) := \exp^\perp(B_{v_0}(F))$, where $\exp^\perp$ denotes the normal exponential map of $F$. For each vector $v_0$ with $\|v_0\| < \varepsilon_F$, $\widetilde{B}_{v_0}(F)$ is an immersed submanifold, that is, a partial tube over $F$ whose fibre over $x_0$ is $\exp^\perp(\widehat{L}_{x_0}v_0)$. M. Brück proved the following fact.

**Theorem 5.1** ([3])**.** *Let $M$ be an equifocal submanifold in a symmetric space of non-positive (or non-negative) curvature having a focal submanifold $F$. If the sections of $M$ are* *properly embedded, then $M$ is equal to the partial tube $\widetilde{B}_{v_0}(F)$, where $v_0$ is an element of $T^\perp F$ with $\exp^\perp(v_0) \in M$, and each fibre of $\widetilde{B}_{v_0}(F)(= M)$ is the image by $\exp^\perp$ of a principal orbit of an orthogonal representation on the normal space of $F$ which is equivalent to the direct sum representation of some s-representations and a trivial representation.*

According to the proof of this theorem in [3], we can derive the following fact.

**Theorem 5.2.** *Let $M$ be an isoparametric submanifold in a symmetric space of non-compact type having a focal submanifold $F$. If the sections of $M$ are properly embedded, then $M$ is equal to the partial tube $\widetilde{B}_{v_0}(F)$, where $v_0$ is an element of $T^\perp F$ with $\exp^\perp(v_0) \in M$, and each fibre of $\widetilde{B}_{v_0}(F)(= M)$ is the image by $\exp^\perp$ of a principal orbit of an orthogonal representation on the normal space of $F$ which is equivalent to the direct sum representation of some s-representations and a trivial representation.*

We recall the notion of a (general) partial tube. Let $F$ be a submanifold in a Riemannian manifold $N$, $T^\perp F$ be the normal bundle of $F$, $\exp^\perp$ be the normal exponential map of $F$, $\nabla^\perp$ be the normal connection of $F$ and $\bar{A}$ be the shape tensor of $F$. Let $t(F)$ be a submanifold of $T^\perp F$ which is given as the sum of some normal holonomy subbundles of $T^\perp F$ and $\widetilde{t}(F) := \exp^\perp(t(F))$. If $\exp^\perp|_{t(F)}$ is an immersion, then $\widetilde{t}(F)$ is called a *partial tube* over $F$. Denote by $A$ the shape tensor of $\widetilde{t}(F)$ and $\mathcal{V}$ (resp. $\mathcal{H}$) the vertical distribution (resp. the horizontal distribution) on $T^\perp F$, where "horizontal distribution" means that it is horizontal with respect to $\nabla^\perp$. Denote by $\widetilde{X}_\xi$ the horizontal lift of $X \in T_xF$ to $\xi \in \widetilde{t}(F)_x$. Denote by $A^x$ the shape tensor of the fibre $\widetilde{t}(F)_x := \exp^\perp(t(F) \cap T^\perp_x F)$ of $\widetilde{t}(F)$ over $x(\in F)$ in the normal umbrella $\Sigma_x := \exp^\perp(T^\perp_x F)$. For a $C^\omega$-function $\Psi$ and a linear transformation $Q$, we define a linear transformation $\Psi(Q)$ by

$$\Psi(Q) := \sum_{k=0}^{\infty} \frac{\Psi^{(k)}(0)}{k!} Q^k.$$

According to the proof of Proposition 3.1 and Corollary 3.2 in [17], we have the following facts for $A$.

**Proposition 5.3.** *Assume that $N$ is a symmetric space $G/K$ of compact type or non-compact type and that $F$ is a submanifold with section, where "with section" means that the normal umbrella $\Sigma_y := \exp^\perp(T^\perp_y F)$ at each $y \in F$ is totally geodesic in $G/K$ ($\Sigma_y$ is then called the section of $F$ through $y$). Let $v \in t(F)_x := t(F) \cap T^\perp_x F$ and $w \in T^\perp_v \widetilde{t}(F)$. Then the following statements* (i) *and* (ii) *hold:*

(i) *For $X \in \mathcal{V}_v$, we have $A_w X = A^x_w X$;*

(ii) *Set $\bar{w} := (P_{\gamma_v|_{[0,1]}})^{-1}(w)$, where $\gamma_v$ is the geodesic in $G/K$ with $\gamma_v'(0) = v$ and $P_{\gamma_v|_{[0,1]}}$ is the parallel transport map along $\gamma_v|_{[0,1]}$. Assume that the sectional curvature for the* 2*-plane* $\mathrm{Span}\{v, \bar{w}\}$ *is equal to zero. For $Y \in \mathcal{H}_v$, we have*

$$A_w \widetilde{Y}_v = P_{\gamma_v|_{[0,1]}} \Bigg( -(\mathrm{ad}(\bar{w}) \circ \sinh(\mathrm{ad}(v)))(Y) + \frac{\sinh(\mathrm{ad}(v))}{\mathrm{ad}(v)} (\bar{A}_{\bar{w}} Y) + \left( \left( \frac{\cosh(\mathrm{ad}(v)) - \mathrm{id}}{\mathrm{ad}(v)} - \frac{\sinh(\mathrm{ad}(v)) - \mathrm{ad}(v)}{\mathrm{ad}(v)^2} \right) \circ \mathrm{ad}(\bar{w}) \right) (\bar{A}_v Y) \Bigg),$$

*where* ad *denotes the adjoint representation of the Lie algebra $\mathfrak{g}$ of $G$, and* $\mathrm{ad}(v)^2$ *and* $\mathrm{ad}(v) \circ \mathrm{ad}(\bar{w})$ *are regarded as a linear transformations of* $\mathfrak{p} := \mathrm{Ker}(\theta_{*e} + \mathrm{id})(\approx T_{eK}(G/K))$ ($\theta$ :

*the Cartan involution of $G$ with* $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$*). In particular, if $F$ is reflective, $R(v)Y = b_1^2 Y$ and $R(\bar{w})Y = b_2^2 Y$, then we have*

$$A_w \widetilde{Y}_v = -b_2 \tanh b_1 \widetilde{Y}_v,$$

*where $R$ denotes the curvature tensor of $G/K$, $R(\bullet)$ denotes the normal Jacobi operator for $(\bullet)$ and $b_i$ ($i = 1, 2$) are real numbers (resp. purely imaginary numbers) when $G/K$ is of non-compact type (resp. of compact type).*

## 6. Proof of Theorem A

In this section, we shall prove Theorem A. Let $M$ be as in Theorem A and $F$ be a reflective focal submanifold of $M$. Denote by $A$ the shape tensor of $M$ and $R$ the curvature tensor of $G/K$. Without loss of generality, we may assume that $o := eK \in F$. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of $G$ (resp. $K$) and $\theta$ be an Cartan involution of $G$ with $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$, where $\mathrm{Fix}\,\theta$ denotes the fixed point set of $\theta$ and $(\mathrm{Fix}\,\theta)_0$ denotes the identity component of $\mathrm{Fix}\,\theta$. Denote by the same symbol $\theta$ the involution (i.e., $\theta_{*e}$) of $\mathfrak{g}$ induced form $\theta$ and set $\mathfrak{p} := \mathrm{Ker}(\theta + \mathrm{id}_{\mathfrak{g}})$, which is identified with the tangent space $T_o(G/K)$. Denote by Exp the exponential map of $G/K$ at $o$.

Proof of Theorem A. Take $Z_0 \in \mathfrak{p}$ with $\mathrm{Exp}\,Z_0 \in M$. Set $x_0 := \mathrm{Exp}\,Z_0$, $\mathfrak{t} := T_oF$, $\mathfrak{t}^\perp := T_o^\perp F$ and $\mathfrak{b} := (\exp Z_0)_{*o}^{-1}(T_{x_0}^\perp M)$. Since $F$ is reflective, both $\mathfrak{t}$ and $\mathfrak{t}^\perp$ are Lie triple systems. Also it is clear that $\mathfrak{b}$ is a maximal abelian subspace of $\mathfrak{t}^\perp$ containing $Z_0$. Take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ including $\mathfrak{b}$. Let $\triangle$ be a (restricted) root system with respect to $\mathfrak{a}$ and set $\triangle_{\mathfrak{b}} := \{\alpha|_{\mathfrak{b}} \mid \alpha \in \triangle\}$. Let $(\triangle_{\mathfrak{b}})_+$ be the positive root system under a lexicographic ordering of $\mathfrak{b}^*$, $\mathfrak{p}_\beta$ be the root space for $\beta \in (\triangle_{\mathfrak{b}})_+$. Set $(\triangle_{\mathfrak{b}})_+^V := \{\beta \in (\triangle_{\mathfrak{b}})_+ \mid \mathfrak{p}_\beta \cap \mathfrak{t}^\perp \neq \{0\}\}$ and $(\triangle_{\mathfrak{b}})_+^H := \{\beta \in (\triangle_{\mathfrak{b}})_+ \mid \mathfrak{p}_\beta \cap \mathfrak{t} \neq \{0\}\}$. Since $\mathfrak{t}$ and $\mathfrak{t}^\perp$ are $\mathrm{ad}(\mathfrak{b})$-invariant, we have

$$\mathfrak{t}^\perp = \mathfrak{b} \oplus \left( \bigoplus_{\beta \in (\triangle_{\mathfrak{b}})_+^V} (\mathfrak{p}_\beta \cap \mathfrak{t}^\perp) \right)$$

and

$$\mathfrak{t} = \mathfrak{z}_{\mathfrak{t}}(\mathfrak{b}) \oplus \left( \bigoplus_{\beta \in (\triangle_{\mathfrak{b}})_+^H} (\mathfrak{p}_\beta \cap \mathfrak{t}) \right),$$

where $\mathfrak{z}_{\mathfrak{t}}(\mathfrak{b})$ denotes the centralizer of $\mathfrak{b}$ in $\mathfrak{t}$. For the convenience, we denote the centralizer $\mathfrak{z}_{\mathfrak{p}}(\mathfrak{b})$ of $\mathfrak{b}$ in $\mathfrak{p}$ by $\mathfrak{p}_0$. It is clear that $\mathfrak{z}_{\mathfrak{t}}(\mathfrak{b}) = \mathfrak{p}_0 \cap \mathfrak{t}$. Let $\widetilde{B}_{Z_0}(F)$ be the partial tube over $F$ through $x_0$ stated in the previous section. According to Theorem 5.2, $M = \widetilde{B}_{Z_0}(F)$ holds and each fibre of this tube is the image by the normal exponential map of a principal orbit of an orthogonal representation on the normal space of $F$ given as the direct sum representation of some s-representations and a trivial representation, which implies that each fibre of this tube is a principal orbit of the isotropy action of the symmetric space $\mathrm{Exp}(\mathfrak{t}^\perp)$. Take any $v \in T_{\mathrm{Exp}\,Z_0}^\perp M$. Then we have

$$R(v)|_{(\exp Z_0)_{*o}(\mathfrak{p}_\beta)} = -\beta(v)^2\,\mathrm{id} \quad (\beta \in (\triangle_{\mathfrak{b}})_+ \cup \{0\}).$$

According to (ii) of Proposition 5.3, we can derive that the horizontal lift (which is denoted by $(\mathfrak{p}_\beta \cap \mathfrak{t})_{Z_0}^L$ of $\mathfrak{p}_\beta \cap \mathfrak{t}$ ($\beta \in (\triangle_{\mathfrak{b}})_+^H \cup \{0\}$) to $\mathrm{Exp}\,Z_0$ is included by an eigenspace of $A_v$.

According to (i) of Proposition 5.3 and the fact that each fibre of $M = \widetilde{B}_{Z_0}(F)$ is the image by the normal exponential map of a principal orbit of an orthogonal representation on the normal space of $F$ given as the direct sum representation of some s-representations and a trivial representation, we can derive that $(\exp Z_0)_*(\mathfrak{p}_\beta \cap \mathfrak{t}^\perp)$ $(\beta \in (\triangle_\mathfrak{b})^V_+)$ is included by an eigenspace of $A_v$. Also we have

$$T_{\mathrm{Exp}\, Z_0} M = \left( \underset{\beta \in (\triangle_\mathfrak{b})^H_+ \cup \{0\}}{\oplus} (\mathfrak{p}_\beta \cap \mathfrak{t})^L_{Z_0} \right) \oplus \left( \underset{\beta \in (\triangle_\mathfrak{b})^V_+}{\oplus} (\exp Z_0)_{*o}(\mathfrak{p}_\beta \cap \mathfrak{t}^\perp) \right).$$

From the above facts, it follows that this decomposition is the common eigenspace decomposition of $A_v$ and $R(v)$. Hence $A_v$ and $R(v)$ commute. It is clear that the same fact holds at other points of $M$. Hence $M$ is curvature-adapted. □

## 7. Proof of Theorem B

In this section, we shall prove Theorem B. Let $M$ be as in Theorem B and $M^{\mathbb{C}}$ be the complete extrinsic complexification of $M$. See the proof of Theorem B in [20] about the construction of the complete extrinsic complexification of $M$. Let $\pi$ be the natural projection of $G^{\mathbb{C}}$ onto $G^{\mathbb{C}}/K^{\mathbb{C}}$ and $\phi : H^0([0,1], \mathfrak{g}^{\mathbb{C}}) \to G^{\mathbb{C}}$ the parallel transport map for $G^{\mathbb{C}}$. Set $\widehat{M}^{\mathbb{C}} := \pi^{-1}(M^{\mathbb{C}})$ and $\widetilde{M}^{\mathbb{C}} := (\pi \circ \phi)^{-1}(M^{\mathbb{C}})$. Without loss of generality, we may assume that $K^{\mathbb{C}}$ is connected and that $G^{\mathbb{C}}$ is simply connected. Hence $\widetilde{M}^{\mathbb{C}}$ is connected. Denote by $A$ the shape tensor of $M$ and $R$ the curvature tensor of $G/K$. First we shall show the following fact.

**Proposition 7.1.** *The lifted submanifold $\widetilde{M}^{\mathbb{C}}$ is a full irreducible complete anti-Kaehler isoparametric submanifold with $J$-diagonalizable shape operators.*

Proof. Fix $x \in M$ and a unit normal vector $v$ of $M$ at $x$. Denote by $\mathrm{Spec}\, A_v$ and $\mathrm{Spec}\, R(v)$ the spectrum of $A_v$ and $R(v)$, respectively. For each $(\lambda, \mu) \in (\mathrm{Spec}\, A_v) \times (\mathrm{Spec}\, R(v))$, set

$$D_{\lambda\mu} := \mathrm{Ker}(A_v - \lambda\, \mathrm{id}) \cap \mathrm{Ker}(R(v) - \mu\, \mathrm{id}).$$

Also, set

$$\mathcal{S} := \{(\lambda, \mu) \in \mathrm{Spec}\, A_v \times \mathrm{Spec}\, R(v) \,|\, D_{\lambda\mu} \neq \{0\}\}, \quad \mathcal{S}_+ := \{(\lambda, \mu) \in \mathcal{S},\, |\, |\lambda| > \sqrt{-\mu}\}$$
$$\text{and} \quad \mathcal{S}_- := \{(\lambda, \mu) \in \mathcal{S} \,|\, |\lambda| < \sqrt{-\mu}\}.$$

Since $M$ is curvature-adapted, $T_x M = \underset{(\lambda,\mu)\in\mathcal{S}}{\oplus} D_{\lambda\mu}$ holds. For the simplicity, set

$$Q(z) := \cos\left(z\sqrt{R(v)^{\mathbb{C}}}\right) - \frac{\sin\left(z\sqrt{R(v)^{\mathbb{C}}}\right)}{\sqrt{R(v)^{\mathbb{C}}}} \circ A_v^{\mathbb{C}}.$$

Clearly we have

$$Q(z)|_{D_{\lambda\mu}} = \left( \cos(\sqrt{-1} z \sqrt{-\mu}) - \frac{\lambda \sin(\sqrt{-1} z \sqrt{-\mu})}{\sqrt{-1}\sqrt{-\mu}} \right) \mathrm{id}.$$

Hence, if $(\lambda, \mu) \in \mathcal{S}_+$ and $\mu \neq 0$, then $\frac{1}{\sqrt{-\mu}}\left( \mathrm{arctanh} \frac{\sqrt{-\mu}}{\lambda} + k\pi\sqrt{-1} \right)$ $(k \in \mathbb{Z})$ are complex focal radii along $\gamma_v$ including $D_{\lambda\mu}$ as its nullity space. Also, if $(\lambda, \mu) \in \mathcal{S}_-$ and $\mu \neq 0$, then $\frac{1}{\sqrt{-\mu}}\left( \mathrm{arctanh} \frac{\lambda}{\sqrt{-\mu}} + (k + \frac{1}{2})\pi\sqrt{-1} \right)$ $(k \in \mathbb{Z})$ are complex focal radii along $\gamma_v$ including $D_{\lambda\mu}$ as its nullity space. Also, if $\lambda \in \operatorname{Spec} A_v \setminus \{0\}$ satisfying $(\lambda, 0) \in \mathcal{S}$, then $\frac{1}{\lambda}$ is a focal radii along $\gamma_v$ including $D_{\lambda 0}$ as its nullity space. Also, if $|\lambda| = \sqrt{-\mu}$, then there exists no complex focal radius along $\gamma_v$ including $D_{\lambda\mu}$ as its nullity space. Hence, since $M$ satisfies the condition $(*_{\mathbb{C}})$, there exists no $(\lambda, \mu) \in \mathcal{S}$ satisfying $|\lambda| = \sqrt{-\mu} \neq 0$. Thus we have

$$T_x M = D_{00} \oplus \left( \bigoplus_{(\lambda,\mu)\in\mathcal{S}_+\cup\mathcal{S}_-} D_{\lambda\mu} \right). \tag{7.1}$$

Denote by $\widetilde{A}$ the shape tensor of $\widetilde{M}^{\mathbb{C}}$. Fix $u \in (\pi \circ \phi)^{-1}(x)$. Let $v_u^L$ be the horizontal lift of $v$ to $u$. Then it follows from the above facts and Proposition 4 of [16] that

$$\begin{aligned}\operatorname{Spec} \widetilde{A}_{v_u^L} = & \{\lambda \,|\, \lambda \in \operatorname{Spec} A_v \text{ s.t. } (\lambda, 0) \in \mathcal{S}_+\} \\ & \bigcup \left\{ \frac{\sqrt{-\mu}}{\operatorname{arctanh}\frac{\sqrt{-\mu}}{\lambda} + \pi k\sqrt{-1}} \,\middle|\, (\lambda,\mu) \in \mathcal{S}_+ \text{ s.t. } \mu \neq 0, \quad k \in \mathbb{Z} \right\} \\ & \bigcup \left\{ \frac{\sqrt{-\mu}}{\operatorname{arctanh}\frac{\lambda}{\sqrt{-\mu}} + (k+\frac{1}{2})\pi\sqrt{-1}} \,\middle|\, (\lambda,\mu) \in \mathcal{S}_-, \quad k \in \mathbb{Z} \right\}.\end{aligned}$$

For the simplicity, set

$$\Lambda^+_{\lambda,\mu,k} := \frac{\sqrt{-\mu}}{\operatorname{arctanh}\frac{\sqrt{-\mu}}{\lambda} + \pi k\sqrt{-1}} \quad ((\lambda,\mu) \in \mathcal{S}_+ \text{ s.t. } \mu \neq 0,\ k \in \mathbb{Z})$$

and

$$\Lambda^-_{\lambda,\mu,k} := \frac{\sqrt{-\mu}}{\operatorname{arctanh}\frac{\lambda}{\sqrt{-\mu}} + (k+\frac{1}{2})\pi\sqrt{-1}} \quad ((\lambda,\mu) \in \mathcal{S}_-,\ k \in \mathbb{Z}).$$

Also, we set

$$\begin{aligned}\widetilde{D}_\lambda &:= \operatorname{Ker}\left(\widetilde{A}_{v_u^L} - \lambda \operatorname{id}\right) \quad (\lambda \in \operatorname{Spec} A_v \text{ s.t. } (\lambda, 0) \in \mathcal{S}_+), \\ \widetilde{D}_{\Lambda^+_{\lambda,\mu,k}} &:= \operatorname{Ker}\left(\widetilde{A}_{v_u^L} - \Lambda^+_{\lambda,\mu,k} \operatorname{id}\right) \quad ((\lambda,\mu) \in \mathcal{S}_+ \text{ s.t. } \mu \neq 0, \quad k \in \mathbb{Z})\end{aligned}$$

and

$$\widetilde{D}_{\Lambda^-_{\lambda,\mu,k}} := \operatorname{Ker}\left(\widetilde{A}_{v_u^L} - \Lambda^-_{\lambda,\mu,k} \operatorname{id}\right) \quad ((\lambda,\mu) \in \mathcal{S}_-,\ k \in \mathbb{Z}).$$

Furthermore, by using (7.1) and Lemma 9 of [16] (see Lemma 7.3 of [15] also), we can derive that $T_u\widetilde{M}^{\mathbb{C}}$ is equal to

$$\overline{\left( \bigoplus_{\lambda\in\operatorname{Spec} A_v \text{ s.t. } (\lambda,0)\in\mathcal{S}_+} \widetilde{D}_\lambda \right) \oplus \left( \bigoplus_{(\lambda,\mu)\in\mathcal{S}_+ \text{ s.t. } \mu\neq 0} \bigoplus_{k\in\mathbb{Z}} \widetilde{D}_{\Lambda^+_{\lambda,\mu,k}} \right) \oplus \left( \bigoplus_{(\lambda,\mu)\in\mathcal{S}_-} \bigoplus_{k\in\mathbb{Z}} \widetilde{D}_{\Lambda^-_{\lambda,\mu,k}} \right)}.$$

This implies that $\widetilde{A}_{v_u^L}$ is diagonalized with respect to a $J$-orthonormal base of $T_u\widetilde{M}^{\mathbb{C}}$. Therefore it follows that $\widetilde{M}^{\mathbb{C}}$ is an anti-Kaehler isoparametric submanifold with $J$-diagonalizable shape operators from the arbitrariness of $x, v$ and $u$. Since $M$ is irreducible, it follows from Theorem 2 of [18] that the complex Coxeter group associated with $M$ is not decomposable, where we note that the complex Coxeter groups associated with $M$ is equal to the complex Coxeter groups associated with $\widetilde{M}^{\mathbb{C}}$ (see Introduction of [18]). Hence, it follows from Theorem 1 of [18] that $\widetilde{M}^{\mathbb{C}}$ is irreducible. Also, since $M$ is full, it is shown that the $J$-curvature normals of $\widetilde{M}^{\mathbb{C}}$ span the normal space of $\widetilde{M}^{\mathbb{C}}$ at each point of $\widetilde{M}^{\mathbb{C}}$ (see the discussion in the proof of Theorem 2 of [18]). Furthermore, it follows from this fact that $\widetilde{M}^{\mathbb{C}}$ is full (see the discussion in the proof of Theorem 1 of [18]). The completeness of $\widetilde{M}^{\mathbb{C}}$ follows from the completeness of $M^{\mathbb{C}}$ and the fact that the fibres of $\pi\circ\phi$ are isometric to the complete anti-Kaehler Hilbert manifold $P(G^{\mathbb{C}},\{e\}\times K^{\mathbb{C}})$, where $P(G^{\mathbb{C}},\{e\}\times K^{\mathbb{C}})$ denotes the Hilbert Lie group $\{g\in H^1([0,1],G^{\mathbb{C}})\,|\,(g(0),g(1))\in\{e\}\times K^{\mathbb{C}}\}$ equipped with the natural complete anti-Kaehler metric. This completes the proof. □

Remark 7.1. According to this proposition, $M$ is proper complex equifocal in the sense of [17].

Without loss of generality, we may assume $\hat{0}\in\widetilde{M}^{\mathbb{C}}$ and hence $e\in\widehat{M}^{\mathbb{C}}$. For the simplicity, set $V:=H^0([0,1],\mathfrak{g}^{\mathbb{C}})$, $\langle\ ,\ \rangle:=\langle\ ,\ \rangle_0^A$ and $\langle\ ,\ \rangle^{\mathcal{I}}:=\langle\ ,\ \rangle_0^{\mathcal{I}}$. Also, denote by $\|\cdot\|$ the norm associated with $\langle\ ,\ \rangle^{\mathcal{I}}$. Let $\mathcal{K}^h$ be the Lie algebra of all holomorphic Killing vector fields defined entirely on $V$ and $\mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$ the Lie subalgebra of $\mathcal{K}^h$ consisting of elements of $\mathcal{K}^h$ which are tangent to $\widetilde{M}^{\mathbb{C}}$ along $\widetilde{M}^{\mathbb{C}}$. Also, denote by $\mathfrak{o}_{AK}(V)$ be the Lie algebra of all continuous skew-symmetric complex linear maps from $V$ to oneself. Any $X\in\mathcal{K}^h$ is described as $X_u=Au+b$ $(u\in V)$ for some $A\in\mathfrak{o}_{AK}(V)$ and some $b\in V$. Hence $\mathcal{K}^h$ is identified with $\mathfrak{o}_{AK}(V)\ltimes V$. Give $\mathfrak{o}_{AK}(V)$ the operator norm (which we denote by $\|\cdot\|_{\rm op}$) associated with $\langle\ ,\ \rangle^{\mathcal{I}}$ and $\mathcal{K}^h$ the product norm of this norm $\|\cdot\|_{\rm op}$ of $\mathfrak{o}_{AK}(V)$ and the norm $\|\cdot\|$ of $V$. Then the space $\mathcal{K}^h$ is a Banach Lie algebra with respect to this norm. Let $I_h(V)$ be the group of all holomorphic isometries of $V$ and $I_h^b(V)$ be the subgroup of $I_h(V)$ generated by one-parameter transformation groups induced by elements of $\mathcal{K}^h$. Since $\mathcal{K}^h$ is a Banach Lie algebra, $I_h^b(V)$ is a Banach Lie group. Note that, for a general holomorphic isometry $f$ of $V$, $\left.\frac{d}{dt}\right|_{t=0}(f_t)_*$ is defined on a dense linear subspace of $V$ but it is not necessarily defined entirely on $V$ (see Example in Appendix of [26]). It is clear that $\mathcal{K}^h$ is the Banach Lie algebra of this Banach Lie group $I_h^b(V)$. Let $H_b$ be the closed Banach Lie subgroup of $I_h^b(V)$ of all elements of $I_h^b(V)$ preserving $\widetilde{M}^{\mathbb{C}}$ invariantly. From Fact 1.1 stated in Introduction and Proposition 7.1, we can derive the following extrinsic homogeneity theorem.

**Lemma 7.2.** *We have* $\widetilde{M}^{\mathbb{C}}=H_b\cdot\hat{0}$.

Denote by $\rho$ the homomorphism from $H^1([0,1],G^{\mathbb{C}})$ to $I_h(V)$ defined by assigning $g*\cdot$ to each $g\in H^1([0,1],G^{\mathbb{C}})$ (i.e., $\rho(g)(u):=g*u$ $(g\in H^1([0,1],G^{\mathbb{C}}),\ u\in V)$), where $g*u$ is as stated in Section 4.

**Lemma 7.3.** *The group* $\rho(H^1([0,1],G^{\mathbb{C}}))$ *is a closed subgroup of* $I_h^b(V)$.

Proof. Take an arbitrary $v\in H^1([0,1],\mathfrak{g}^{\mathbb{C}})$ and set $\psi_s:=\rho(\exp\circ sv)$, where exp is the exponential map of $G^{\mathbb{C}}$. Note that $\exp\circ sv$ is equal to the image of $sv\in H^1([0,1],\mathfrak{g}^{\mathbb{C}})$ by the exponential map of $H^1([0,1],G^{\mathbb{C}})$. The group $\{\psi_s\,|\,s\in\mathbb{R}\}$ is a one-parameter transformation group consisting of holomorphic isometries of $V$. The holomorphic Killing vector field $X$ associated with $\{\psi_s\,|\,s\in\mathbb{R}\}$ is given by

$$X_u=\left.\frac{d}{ds}\right|_{s=0}\psi_s(u)=\left.\frac{d}{ds}\right|_{s=0}(\exp\circ sv)*u={\rm ad}(v)(u)-v'.$$

Set $I_c:=\{t\in[0,1]\,|\,\max{\rm Spec}(-{\rm ad}(v(t))^2)\geq c\}$ and $c_0:=\min\{c\,|\,I_c$ is of measure zero in

[0, 1]}, where ad is the adjoint operator of $\mathfrak{g}^{\mathbb{C}}$. Then we have

$$\|\mathrm{ad}(v)u\|^2 = \int_0^1 \langle \mathrm{ad}(v(t))u(t), \mathrm{ad}(v(t))u(t)\rangle^{\mathcal{I}}\, dt$$
$$= -\int_0^1 \langle \mathrm{ad}(v(t))^2 u(t), u(t)\rangle^{\mathcal{I}}\, dt \le c_0\|u\|^2,$$

where $\langle\ ,\ \rangle^{\mathcal{I}}$ is the inner product of $\mathfrak{g}^{\mathbb{C}}$ stated in Section 2. Thus $\mathrm{ad}(v)$ is bounded. Hence we have $X \in \mathcal{K}^h$, that is, $\rho(\exp\circ v) \in I_h^b(V)$. Therefore, it follows from the arbitrariness of $v$ that $\rho(H^1([0,1], G^{\mathbb{C}}))$ is a subgroup of $I_h^b(V)$. The closedness of $\rho(H^1([0,1], G^{\mathbb{C}}))$ is trivial. □

In the proof of Theorem B, it is key to show the following fact.

**Proposition 7.4.** *The above group $H_b$ is a subgroup of $\rho(H^1([0,1], G^{\mathbb{C}}))$.*

To prove this proposition, we prepare some lemmas. For $X \in \mathcal{K}^h$, we define a map $F_X : \Omega_e(G^{\mathbb{C}}) \to \mathfrak{g}^{\mathbb{C}}$ by $F_X(g) := \phi_{*\hat{0}}((\rho(g)_*X)_{\hat{0}})$. For the simplicity, denote by Ad the adjoint operator $\mathrm{Ad}_{G^{\mathbb{C}}}$ of $G^{\mathbb{C}}$. For this map $F_X$, we have the following fact.

**Lemma 7.4.1.** (i) *For $g \in \Omega_e(G^{\mathbb{C}})$, $F_X(g) = \int_0^1 \mathrm{Ad}(g)(X_{\rho(g^{-1})(\hat{0})})dt$.*
(ii) *If $X \in \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$, then the image of $F_X$ is included by $T_e\widehat{M}^{\mathbb{C}}$.*

Proof. Let $\{\psi_s\}_{s\in\mathbb{R}}$ be the one-parameter transformation group associated with $X$. For each $g \in \Omega_e(G^{\mathbb{C}})$, we have

$$(\rho(g)_*X)_{\hat{0}} = \left.\frac{d}{ds}\right|_{s=0} \rho(g)(\psi_s(g^{-1} * \hat{0}))$$
$$= \left.\frac{d}{ds}\right|_{s=0} (\mathrm{Ad}(g)(\psi_s(\rho(g^{-1})(\hat{0}))) - g' g_*^{-1}) = \mathrm{Ad}(g)(X_{\rho(g^{-1})(\hat{0})}).$$

Also we have $\phi_{*\hat{0}}(u) = \int_0^1 u(t)dt$ ($u \in T_{\hat{0}}V(=V)$) (see Lemma 6 of [16]). Hence we obtain the relation in (i). Since $g \in \Omega_e(G^{\mathbb{C}})$, it maps each fibre of $\phi$ to oneself. Hence, if $X \in \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$, then $\rho(g)_*X \in \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$. In particular, we have $(\rho(g)_*X)_{\hat{0}} \in T_{\hat{0}}\widetilde{M}^{\mathbb{C}}$. Therefore we obtain $F_X(g) \in \phi_{*\hat{0}}(T_{\hat{0}}\widetilde{M}^{\mathbb{C}}) = T_e\widehat{M}^{\mathbb{C}}$. □

For $v \in H^1([0,1], \mathfrak{g}^{\mathbb{C}})$, we define a vector field $X^v$ on $V$ by $(X^v)_u := [v, u] - v'$ ($u \in V$). Let $\{\exp\circ sv \mid s \in \mathbb{R}\}$ be the one-parameter subgroup of $H^1([0,1], G^{\mathbb{C}})$ associated with $v$. Then the holomorphic Killing vector field associated with the one-parameter transformation group $\{\rho(\exp\circ sv) \mid s \in \mathbb{R}\}$ of $V$ is equal to $X^v$. Furthermore, we can show $X^v \in \mathcal{K}^h_b$ by the discussion in the proof of Lemma 7.4.1. For $X^v$, we have the following fact.

**Lemma 7.4.2.** *The map $F_{X^v}$ is a constant map.*

Proof. Take elements $g_1$ and $g_2$ of $\Omega_e(G^{\mathbb{C}})$. Since $\rho(g_i)$ maps each fibre of $\phi$ to oneself by the fact (iii) for $\phi$ (stated in Section 2), we have $\phi\circ\rho(g_i) = \phi$ ($i = 1, 2$) and hence

$$F_{X^v}(g_i) = \phi_{*\hat{0}}((\rho(g_i)_*(X^v))_{\hat{0}}) = \phi_{*\rho(g_i^{-1})(\hat{0})}((X^v)_{\rho(g_i^{-1})(\hat{0})}) \quad (i = 1, 2). \tag{7.2}$$

Since $\rho(\exp\circ sv)$ maps the fibres of $\phi$ to them by the fact (iii) for $\phi$ and $\phi(\rho(g_1^{-1})(\hat{0})) = \phi(\rho(g_2^{-1})(\hat{0}))$, we have $\phi(\rho(\exp sv)(\rho(g_1^{-1})(\hat{0}))) = \phi(\rho(\exp sv)(\rho(g_2^{-1})(\hat{0})))$ and hence

$\phi_{*\rho(g_1^{-1})(\hat{0})}(X^v_{\rho(g_1^{-1})(\hat{0})}) = \phi_{*\rho(g_2^{-1})(\hat{0})}(X^v_{\rho(g_2^{-1})(\hat{0})})$. From this relation and (7.2), we obtain $F_{X^v}(g_1) = F_{X^v}(g_2)$. Therefore it follows from the arbitrariness of $g_1$ and $g_2$ that $F_{X^v}$ is a constant map. □

For each $u \in V$, denote by $\widetilde{u}$ the element $t \mapsto \int_0^t u(t)dt$ $(0 \le t \le 1)$ of $H^1([0,1], \mathfrak{g}^{\mathbb{C}})$. Also we have the following fact for $F_X$.

**Lemma 7.4.3.** (i) *The map $X \mapsto F_X$ is linear.*

(ii) $F_X(g_1 g_2) = F_{\rho(g_2)_* X}(g_1)$ *$(g_1, g_2 \in \Omega_e(G^{\mathbb{C}}))$.*

(iii) $(dF_X)_g \circ (dR_g)_{\hat{e}} = (dF_{\rho(g)_* X})_{\hat{e}}$ *$(g \in \Omega_e(G^{\mathbb{C}}))$.*

(iv) *If $X_u = Au + b$ $(u \in V)$ for some linear transformation $A$ of $V$ and some $b \in V$, then we have $(dF_X)_{\hat{e}}(u) = \int_0^1 (A + \mathrm{ad}(\widetilde{b}))u'dt$ $(u \in \Omega_0(\mathfrak{g}^{\mathbb{C}}))$, where* ad *is the adjoint representation of $\mathfrak{g}^{\mathbb{C}}$ and $\Omega_0(\mathfrak{g}^{\mathbb{C}}) := \{u \in H^1([0,1], \mathfrak{g}^{\mathbb{C}}) \,|\, u(0) = u(1) = 0\}$.*

(v) *If $X, \overline{X} \in \mathcal{K}^h$ and if $\overline{X} - X = X^v$ for some $v \in H^1([0,1], \mathfrak{g}^{\mathbb{C}})$, then $F_{\overline{X}} - F_X$ is a constant map.*

Proof. The statements (i) ∼ (iii) are trivial. The statement (iv) is shown by imitating the proof of Proposition 2.3 of [5]. The statement (v) follows from Lemma 7.4.2 and (i) directly. □

By imitating the proof of Theorem 2.2 of [5], we can show the following fact in terms of Lemmas 7.3.1∼7.3.3.

**Lemma 7.4.4.** *Let $X$ be an element of $\mathcal{K}^h$ given by $X_u := [v, u] - b$ $(u \in V)$ for some $v, b \in V$. If $X \in \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$, then we have $v \in H^1([0,1], \mathfrak{g}^{\mathbb{C}})$ and $b = v'$ (i.e., $X = X^v$).*

Proof. Set $\overline{X} := X - X^{\widetilde{b}}$ and $w := v - \widetilde{b}$. First we consider the case where $G^{\mathbb{C}}$ is simple. From $\overline{X} = \mathrm{ad}(w)$, we have

$$(\rho(g)_*\overline{X})_u = \rho(g)_*(\overline{X}_{\rho(g^{-1})(u)}) = \mathrm{Ad}(g)([w, \rho(g^{-1})(u)]) = [\mathrm{Ad}(g)w,\ u - g * \hat{0}] \quad (u \in V).$$

From this relation and (i) of Lemma 7.4.1, we have

(7.3)

$$\begin{aligned}
(dF_{\rho(g)_*\overline{X}})_{\hat{e}}(u) &= \left.\frac{d}{ds}\right|_{s=0} F_{\rho(g)_*\overline{X}}(\exp\ su) \\
&= \left.\frac{d}{ds}\right|_{s=0} \int_0^1 \mathrm{Ad}(\exp\ su)((\rho(g)_*\overline{X})_{\rho(\exp(-su))(\hat{0})})dt \\
&= \int_0^1 \left([u,\ (\rho(g)_*\overline{X})_{\hat{0}}] + \left.\frac{d}{ds}\right|_{s=0} (\rho(g)_*\overline{X})_{\rho(\exp(-su))(\hat{0})}\right) dt \\
&= \int_0^1 \left(-[u,\ [\mathrm{Ad}(g)w, g * \hat{0}]] + \left.\frac{d}{ds}\right|_{s=0} [\mathrm{Ad}(g)w, \rho(\exp(-su))(\hat{0}) - g * \hat{0}]\right) dt \\
&= \int_0^1 \left([u,\ [\mathrm{Ad}(g)w, g' g_*^{-1}]] - [\mathrm{Ad}(g)w,\ \left.\frac{d}{ds}\right|_{s=0} ((\exp(-su))' \exp(-su)_*^{-1})]\right) dt \\
&= \int_0^1 \left([u,\ [\mathrm{Ad}(g)w, g' g_*^{-1}]] + [\mathrm{Ad}(g)w,\ u']\right) dt \\
&= [u(t),\ [\widetilde{\mathrm{Ad}(g)w, g' g_*^{-1}}](t)]\Big|_{t=1} - [u(t),\ [\widetilde{\mathrm{Ad}(g)w, g' g_*^{-1}}](t)]\Big|_{t=0}
\end{aligned}$$

$$
- \int_0^1 [u', [\widetilde{\mathrm{Ad}(g)w, g'g_*^{-1}}]]dt + \int_0^1 [\mathrm{Ad}(g)w, u']dt
$$
$$
= \int_0^1 [[\widetilde{\mathrm{Ad}(g)w, g'g_*^{-1}}] + \mathrm{Ad}(g)w, \ u']dt
$$

for $u \in T_{\hat{e}}(\Omega_e(G^{\mathbb{C}}))(= \Omega_0(\mathfrak{g}^{\mathbb{C}}))$, where each of the notation $'$ means the derivative with respect to $t$, $\hat{e}$ is the constant path at the identity element $e$ of $G^{\mathbb{C}}$ and $\Omega_0(\mathfrak{g}^{\mathbb{C}}) := \{u \in H^1([0,1],\mathfrak{g}^{\mathbb{C}}) \,|\, u(0) = u(1) = 0\}$. According to (ii) of Lemma 7.4.1, we have $\mathrm{Im}\, F_X \subset T_e\widehat{M}^{\mathbb{C}}$ and hence $\dim_{\mathbb{C}}(\mathrm{Span}_{\mathbb{C}}\mathrm{Im}\, F_X) \le \dim_{\mathbb{C}} T_e\widehat{M}^{\mathbb{C}} \le \dim_{\mathbb{C}}\mathfrak{g}^{\mathbb{C}} - 2$, where $\mathrm{Span}_{\mathbb{C}}(\cdot)$ means the complex linear span of $(\cdot)$ and $\dim_{\mathbb{C}}(\cdot)$ means the complex dimension of $(\cdot)$. Since $F_{\overline{X}} - F_X$ is a constant map by (v) of Lemma 7.4.3, we have $\dim_{\mathbb{C}}(\mathrm{Span}_{\mathbb{C}}\mathrm{Im}\, F_{\overline{X}}) \le \dim_{\mathbb{C}}\mathfrak{g}^{\mathbb{C}} - 1$, that is, $\dim_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}} \ominus \mathrm{Span}_{\mathbb{C}}\mathrm{Im}\, F_{\overline{X}}) \ge 1$. Take $Y(\neq 0) \in \mathfrak{g}^{\mathbb{C}} \ominus \mathrm{Span}_{\mathbb{C}}\, \mathrm{Im}\, F_{\overline{X}}$. Also, take $g \in \Omega_e(G^{\mathbb{C}})$ and $u \in T_{\hat{e}}(\Omega_e(G^{\mathbb{C}}))$. By using (iii) of Lemma 7.4.3 and (7.3), we have

$$
\begin{aligned}
\langle (dF_{\overline{X}})_g((dR_g)_{\hat{e}}(u)), Y\rangle^A &= \langle (dF_{\rho(g)_*\overline{X}})_{\hat{e}}(u), Y\rangle^A \\
&= \langle \int_0^1 [[\widetilde{\mathrm{Ad}(g)w, g'g_*^{-1}}] + \mathrm{Ad}(g)w, \ u']dt, \ Y\rangle^A \\
&= \int_0^1 \langle [[\widetilde{\mathrm{Ad}(g)w, g'g_*^{-1}}] + \mathrm{Ad}(g)w, \ u'], \ Y\rangle^A \, dt \\
&= -\int_0^1 \langle u', \ [[\widetilde{\mathrm{Ad}(g)w, g'g_*^{-1}}] + \mathrm{Ad}(g)w, \ Y]\rangle^A \, dt \\
&= -\langle u', \ [[\widetilde{\mathrm{Ad}(g)w, g'g_*^{-1}}] + \mathrm{Ad}(g)w, \ Y]\rangle,
\end{aligned}
$$

where $\langle \ \ \rangle^A$ is the non-degenerate symmetric bilinear form of $\mathfrak{g}^{\mathbb{C}}$ stated in Section 4. For the simplicity, we set $\eta := [\widetilde{\mathrm{Ad}(g)w, g'g_*^{-1}}] + \mathrm{Ad}(g)w$. On the other hand, from $(dF_{\overline{X}})_g((dR_g)_{\hat{e}}(u)) \in \mathrm{Span}_{\mathbb{C}}\mathrm{Im}\, F_{\overline{X}}$, we have $\langle (dF_{\overline{X}})_g((dR_g)_{\hat{e}}(u)), Y\rangle^A = 0$. Hence we have $\langle u', [\eta, Y]\rangle = 0$. The space $\Omega_0(\mathfrak{g}^{\mathbb{C}})$ is identified with the vertical space (which is denoted by $\mathcal{V}_{\hat{0}}$) at $\hat{0}$ of $\phi$ under the correspondence $u \mapsto u'$ ($u \in \Omega_0(\mathfrak{g}^{\mathbb{C}})$), where we note that $\phi_{*\hat{0}}(u') = \int_0^1 u'(t)\, dt = 0$ by Lemma 6 of [16] (hence $u' \in \mathcal{V}_{\hat{0}}$). Hence, from the arbitrariness of $u$, it follows that $[\eta, Y]$ belongs to the horizontal space (which is denoted by $\mathcal{H}_{\hat{0}}$) at $\hat{0}$ of $\phi$. Since $G^{\mathbb{C}}$ has no center, there exists $Z \in \mathfrak{g}^{\mathbb{C}}$ with $[Y, Z] \neq 0$. Set $W := [Y, Z]$. By using Lemma 6 of [16], we can show that $\mathcal{H}_{\hat{0}}$ is equal to the set of all constant paths in $\mathfrak{g}^{\mathbb{C}}$. Hence it follows from $[\eta, Y] \in \mathcal{H}_{\hat{0}}$ that $[\eta, Y]$ is a constant path. Furthermore it follows from $\langle \eta, W\rangle^A = \langle [\eta, Y], Z\rangle^A$ that $\langle \eta, W\rangle^A$ is constant, that is,

$$
\langle [\widetilde{\mathrm{Ad}(g)w, g'g_*^{-1}}], W\rangle^A + \langle \mathrm{Ad}(g)w, W\rangle^A = \mathrm{const.} \tag{7.4}
$$

Since $\mathfrak{g}^{\mathbb{C}}$ has no center, there exists $\overline{W} \in \mathfrak{g}^{\mathbb{C}}$ with $[W, \overline{W}] \neq 0$. Since $G^{\mathbb{C}}$ is simple, $\mathrm{Ad}(G^{\mathbb{C}})[W, \overline{W}]$ is full in $\mathfrak{g}^{\mathbb{C}}$. Hence there exist $h_1, \cdots, h_{2m} \in G^{\mathbb{C}}$ such that $(\mathrm{Ad}(h_1)[W, \overline{W}], \cdots, \mathrm{Ad}(h_{2m})[W, \overline{W}])$ is a base of $\mathfrak{g}^{\mathbb{C}}$ (regarded as a real vector space), where $m := \dim_{\mathbb{C}} G^{\mathbb{C}}$. For a sufficiently small $\varepsilon > 0$, we take $g_i \in \Omega_e(G^{\mathbb{C}})$ with $g_i|_{(\varepsilon, 1-\varepsilon)} = h_i$ ($i = 1, \cdots, 2m$). Since $g_i$ ($i = 1, \cdots, 2m$) are constant over $[\varepsilon, 1-\varepsilon]$, it follows form (7.4) ($g = g_i$-case) that $\langle w, \mathrm{Ad}(h_i^{-1})W\rangle^A$ ($i = 1, \cdots, 2m$) are constant over $[\varepsilon, 1-\varepsilon]$. Hence $w$ is constant over $[\varepsilon, 1-\varepsilon]$. Hence it follows from the arbitrariness of $\varepsilon$ that $w$ is constant over $[0,1]$. That is, we obtain $b = v'$ and hence $v \in H^1([0,1], \mathfrak{g}^{\mathbb{C}})$.

Next we consider the case where $G^{\mathbb{C}}$ is not simple. Let $G^{\mathbb{C}} = G_1^{\mathbb{C}} \times \cdots \times G_k^{\mathbb{C}}$ be the irreducible decomposition of $G^{\mathbb{C}}$ and $\mathfrak{g}_i^{\mathbb{C}}$ be the Lie algebra of $G_i^{\mathbb{C}}$ $(i=1,\cdots,k)$. Let $\mathfrak{g}_{\overline{X}}^{\mathbb{C}}$ be the maximal ideal of $\mathfrak{g}^{\mathbb{C}}$ such that the orthogonal projection of $w=v-\widetilde{b}$ onto the ideal is a constant path, where we note that any ideal of $\mathfrak{g}^{\mathbb{C}}$ is equal to the direct sum of some $\mathfrak{g}_i^{\mathbb{C}}$'s and hence it is a non-degenerate subspace with respect to $\langle\ ,\ \rangle^A$. Now we shall show

$$(\mathfrak{g}_{\overline{X}}^{\mathbb{C}})^{\perp}\subset T_e\widehat{M}^{\mathbb{C}}, \tag{7.5}$$

where $(\mathfrak{g}_{\overline{X}}^{\mathbb{C}})^{\perp}$ is the orthogonal complement of $\mathfrak{g}_{\overline{X}}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$ with respect to $\langle\ ,\ \rangle^A$. Let $V_i:=H^0([0,1],\mathfrak{g}_i^{\mathbb{C}})$ $(i=1,\cdots,k)$. It is clear that $V=V_1\oplus\cdots\oplus V_k$ (orthogonal direct sum). The holomorphic Killing vector field $\overline{X}$ is described as $\overline{X}=\overline{X}_1^L+\cdots+\overline{X}_k^L$ in terms of some holomorphic Killing vector field $\overline{X}_i$ on $V_i$ $(i=1,\cdots,k)$, where $\overline{X}_i^L$ is the holomorphic Killing vector field on $V$ defined by $(\overline{X}_i^L)_u=(\overline{X}_i)_{u_i}$ $(u=(u_1,\cdots,u_k)\in V)$. For $g=(g_1,\cdots,g_k)\in\Omega_e(G^{\mathbb{C}})(=\Omega_e(G_1^{\mathbb{C}})\times\cdots\times\Omega_e(G_k^{\mathbb{C}}))$, we have $\mathrm{Ad}(g)(\overline{X}_{\rho(g^{-1})(\hat{0})})=\sum_{i=1}^{k}\mathrm{Ad}_i(g_i)((\overline{X}_i)_{\rho_i(g_i^{-1})(\hat{0})})$, where $\mathrm{Ad}_i$ denotes the adjoint representation of $G_i^{\mathbb{C}}$ and $\rho_i$ denotes the homomorphism from $H^1([0,1],G_i^{\mathbb{C}})$ to $I_h(V_i)$ defined in similar to $\rho$. Hence, from (i) of Lemma 7.4.1, we have $F_{\overline{X}}(g)=\sum_{i=1}^{k}F^i_{\overline{X}_i}(g_i)$, where $F^i_{\overline{X}_i}$ is the map from $\Omega_e(G_i^{\mathbb{C}})$ to $\mathfrak{g}_i^{\mathbb{C}}$ defined in similar to $F_{\overline{X}}$. Therefore we obtain $\mathrm{Span}_{\mathbb{C}}\mathrm{Im}\,F_{\overline{X}}=\bigoplus_{i=1}^{k}\mathrm{Span}_{\mathbb{C}}\mathrm{Im}\,F^i_{\overline{X}_i}$. Let $v=\sum_{i=1}^{k}v_i$ and $\widetilde{b}=\sum_{i=1}^{k}\widetilde{b}_i$, where $v_i,\widetilde{b}_i\in V_i$ $(i=1,\cdots,k)$. Since $\mathfrak{g}_{\overline{X}}^{\mathbb{C}}$ is an ideal of $\mathfrak{g}^{\mathbb{C}}$, it is described as $\mathfrak{g}_{\overline{X}}^{\mathbb{C}}=\bigoplus_{i\in I}\mathfrak{g}_i^{\mathbb{C}}$ $(I\subset\{1,\cdots,k\})$. Since $v_i-\widetilde{b}_i$ $(i\in I)$ are constant paths by the definition of $\mathfrak{g}_{\overline{X}}^{\mathbb{C}}$, $\mathrm{Ad}_i(g_i)[v_i-\widetilde{b}_i,\rho(g_i^{-1})(\hat{0})]$ $(i\in I)$ are loops and hence

$$F^i_{\overline{X}_i}(g_i)=\int_0^1\mathrm{Ad}_i(g_i)[v_i-\widetilde{b}_i,\rho(g_i^{-1})(\hat{0})]dt=0\qquad(i\in I).$$

Hence we have

$$\mathrm{Span}_{\mathbb{C}}\mathrm{Im}\,F_{\overline{X}}\subset(\mathfrak{g}_{\overline{X}}^{\mathbb{C}})^{\perp}(=\bigoplus_{i\notin I}\mathfrak{g}_i^{\mathbb{C}}).$$

Also we can show $\mathrm{Span}_{\mathbb{C}}\mathrm{Im}\,F^i_{\overline{X}_i}=\mathfrak{g}_i^{\mathbb{C}}$ $(i\notin I)$. Therefore we obtain

$$\mathrm{Span}_{\mathbb{C}}\mathrm{Im}\,F_{\overline{X}}=(\mathfrak{g}_{\overline{X}}^{\mathbb{C}})^{\perp}. \tag{7.6}$$

Also, since $F_{\overline{X}}-F_X$ is a constant map by (v) of Lemma 7.4.3 and $0\in\mathrm{Im}\,F_{\overline{X}}$, we have

$$\mathrm{Span}_{\mathbb{C}}\mathrm{Im}\,F_{\overline{X}}\subset\mathrm{Span}_{\mathbb{C}}\mathrm{Im}\,F_X. \tag{7.7}$$

From (7.6), (7.7) and (ii) of Lemma 7.4.1, we obtain $(\mathfrak{g}_{\overline{X}}^{\mathbb{C}})^{\perp}\subset T_e\widehat{M}^{\mathbb{C}}$. Next we shall show that $(R_g)_*((\mathfrak{g}_{\overline{X}}^{\mathbb{C}})^{\perp})\subset T_g\widehat{M}^{\mathbb{C}}$ for any $g\in\widehat{M}^{\mathbb{C}}$. Fix $g\in\widehat{M}^{\mathbb{C}}$. Define $\widehat{g}\in H^1([0,1],G^{\mathbb{C}})$ with $\widehat{g}(0)=e$ and $\widehat{g}(1)=g$ by $\widehat{g}(t):=\exp tY$ for some $Y\in\mathfrak{g}^{\mathbb{C}}$. Since $\phi\circ\rho(\widehat{g})=R_g^{-1}\circ\phi$, we have $\phi^{-1}(R_g^{-1}(\widehat{M}^{\mathbb{C}}))=\rho(\widehat{g})(\widetilde{M}^{\mathbb{C}})$. Also we have $\rho(\widehat{g})_*X\in\mathcal{K}^h_{\rho(\widehat{g})(\widetilde{M}^{\mathbb{C}})}$. Hence, by imitating the above discussion, we can show

$$(\mathfrak{g}^{\mathbb{C}}_{\overline{\rho(\widehat{g})_*X}})^{\perp}\subset T_eR_g^{-1}(\widehat{M}^{\mathbb{C}})=(R_g)_*^{-1}(T_g\widehat{M}^{\mathbb{C}}). \tag{7.8}$$

Also, we have

(7.9) $(\rho(\widehat{g})_*X)_u = \rho(\hat{g})_*(X_{\rho(\hat{g})^{-1}(u)}) = [\mathrm{Ad}(\widehat{g})v, u] - [\mathrm{Ad}(\widehat{g})v, \rho(\widehat{g})(\hat{0})] - \mathrm{Ad}(\widehat{g})b.$

Set $\overline{v} := \mathrm{Ad}(\widehat{g})v$ and $\overline{b} := [\mathrm{Ad}(\widehat{g})v, \rho(\widehat{g})(\hat{0})] + \mathrm{Ad}(\widehat{g})b$. Denote by $\mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}_{\overline{X}}}$ the orthogonal projection of $\mathfrak{g}^{\mathbb{C}}$ onto $\mathfrak{g}^{\mathbb{C}}_{\overline{X}}$. Since $\mathrm{Ad}(\widehat{g})$ preserves each $\mathfrak{g}^{\mathbb{C}}_i$ invariantly, it preserves $\mathfrak{g}^{\mathbb{C}}_{\overline{X}}$ and $(\mathfrak{g}^{\mathbb{C}}_{\overline{X}})^{\perp}$ invariantly, respectively. Hence we have $\mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}_{\overline{X}}} \circ \mathrm{Ad}(\widehat{g}) = \mathrm{Ad}(\widehat{g}) \circ \mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}_{\overline{X}}}$ and $\mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}_{\overline{X}}} \circ \mathrm{ad}(Y) = \mathrm{ad}(Y) \circ \mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}_{\overline{X}}}$. Also, we have $\rho(\widehat{g})(\hat{0}) = -Y = -\mathrm{Ad}(\widehat{g})Y$. By using these facts and noticing that $\mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}_{\overline{X}}}(v - \widetilde{b})$ is a constant path, we have

$$\begin{aligned}
\frac{d}{dt}\mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}_{\overline{X}}}\left(\overline{v} - \widetilde{\overline{b}}\right) &= \frac{d}{dt}\mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}_{\overline{X}}}\left(\mathrm{Ad}(\widehat{g})(v - \widetilde{b}) + \mathrm{Ad}(\widehat{g})\widetilde{b} - \widetilde{[\mathrm{Ad}(\widehat{g})v, \rho(\widehat{g})(\hat{0})]} - \widetilde{\mathrm{Ad}(\widehat{g})b}\right)\\
&= \mathrm{Ad}(\widehat{g})[Y, \mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}_{\overline{X}}}(v - \widetilde{b})] + \mathrm{Ad}(\widehat{g})[Y, \mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}_{\overline{X}}}(\widetilde{b})]\\
&\quad + \mathrm{Ad}(\widehat{g})\mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}_{\overline{X}}}(b) + \mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}_{\overline{X}}}[\mathrm{Ad}(\widehat{g})v, Y] - \mathrm{Ad}(\widehat{g})\mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}_{\overline{X}}}(b)\\
&= (\mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}_{\overline{X}}} \circ \mathrm{Ad}(\widehat{g}))\left([Y, v - \widetilde{b}] + [Y, \widetilde{b}] + [v, Y]\right) = 0.
\end{aligned}$$

Thus $\mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}_{\overline{X}}}(\overline{v} - \widetilde{\overline{b}})$ is a constant path. This fact together with (7.9) implies $\mathfrak{g}^{\mathbb{C}}_{\overline{X}} \subset \mathfrak{g}^{\mathbb{C}}_{\overline{\rho(\widehat{g})_*X}}$. By exchanging the roles of $X$ and $\rho(\widehat{g})_*X$, we have $\mathfrak{g}^{\mathbb{C}}_{\overline{\rho(\widehat{g})_*X}} \subset \mathfrak{g}^{\mathbb{C}}_{\overline{X}}$. Thus we obtain $\mathfrak{g}^{\mathbb{C}}_{\overline{X}} = \mathfrak{g}^{\mathbb{C}}_{\overline{\rho(\widehat{g})_*X}}$. Therefore the relation $(R_g)_*(\mathfrak{g}^{\mathbb{C}}_{\overline{X}})^{\perp} \subset T_g\widehat{M}^{\mathbb{C}}$ follows from (7.8). Since this relation holds for any $g \in \widehat{M}^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}}_{\overline{X}}$ is an ideal of $\mathfrak{g}^{\mathbb{C}}$, we have $\widehat{M}^{\mathbb{C}} = \widehat{M}^{\mathbb{C}\prime} \times G^{\mathbb{C}\perp}_{\overline{X}} \subset G^{\mathbb{C}}_{\overline{X}} \times G^{\mathbb{C}\perp}_{\overline{X}} (= G^{\mathbb{C}})$ for some submanifold $\widehat{M}^{\mathbb{C}\prime}$ in $G^{\mathbb{C}}_{\overline{X}}$, where $G^{\mathbb{C}}_{\overline{X}} := \exp(\mathfrak{g}^{\mathbb{C}}_{\overline{X}})$ and $G^{\mathbb{C}\perp}_{\overline{X}} := \exp((\mathfrak{g}^{\mathbb{C}}_{\overline{X}})^{\perp})$. Since $\widehat{M}^{\mathbb{C}}$ is irreducible and $\dim \widehat{M}^{\mathbb{C}} < \dim G^{\mathbb{C}}$, we have $(\mathfrak{g}^{\mathbb{C}}_{\overline{X}})^{\perp} = \{0\}$, that is, $\mathfrak{g}^{\mathbb{C}}_{\overline{X}} = \mathfrak{g}^{\mathbb{C}}$. This implies that $v - \widetilde{b}$ is a constant path. Therefore we obtain $b = v'$ and hence $v \in H^1([0,1], \mathfrak{g}^{\mathbb{C}})$. □

Also we have the following fact.

**Lemma 7.4.5.** *The set $\mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$ is closed in $\mathcal{K}^h$.*

Proof. Denote by $\overline{\mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}}$ the closure of $\mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$ in $\mathcal{K}^h$. Take $X \in \overline{\mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}}$. Then there exists a sequence $\{X_n\}_{n=1}^{\infty}$ in $\mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$ with $\lim_{n\to\infty} X_n = X$ (in $\mathcal{K}^h$). Let $(X_n)_u = A_n u + b_n$ ($A_n \in \mathfrak{o}_{AK}(V),\ b_n \in V$) and $X_u = Au + b$ ($A \in \mathfrak{o}_{AK}(V),\ b \in V$). From $\lim_{n\to\infty} X_n = X$ (in $\mathcal{K}^h$), we have $\lim_{n\to\infty} A_n = A$ (in $\mathfrak{o}_{AK}(V)$) and hence $\lim_{n\to\infty} A_n u = Au$ ($u \in V$). Also, we have $\lim_{n\to\infty} b_n = b$. Hence we have $\lim_{n\to\infty}(X_n)_u = X_u$ ($u \in V$). For each $u \in \widetilde{M}^{\mathbb{C}}$, denote by $\mathrm{pr}^{\perp}_u$ the orthogonal projection of $V$ onto $T^{\perp}_u\widetilde{M}^{\mathbb{C}}$. Since $\dim T^{\perp}_u\widetilde{M}^{\mathbb{C}} < \infty$, $\mathrm{pr}^{\perp}_u$ is a compact operator. Hence, since $\mathrm{pr}^{\perp}_u((X_n)_u) = 0$ for all $n$, we obtain $\mathrm{pr}^{\perp}_u(X_u) = 0$ and hence $X \in \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$. Therefore we obtain $\overline{\mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}} = \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$. □

Take $v \in V$ and $X \in \mathcal{K}^h$. Also, define $g_n \in H^1([0,1], G^{\mathbb{C}})$ ($n \in \mathbf{N}$) by $g_n(t) := \exp(n\widetilde{v}(t))$ and a vector field $X^v_n$ ($n \in \mathbf{N}$) by $X^v_n := \frac{1}{n}\rho(g_n)_*X$. Since $\rho(g_n) \in I^b_h(V)$ by Lemma 7.3, we have $X^v_n \in \mathcal{K}^h$. Let $X_u = Au + b$ ($A \in \mathfrak{o}_{AK}(V),\ b \in V$), where $u \in V$, and $(X^v_n)_u = A^v_n u + b^v_n$ ($A^v_n$ : a skew-symmetric complex linear map from the domain of $X^v_n$ to $V$, $b^v_n \in V$), where $u$ is an arbitrary point of the domain of $X^v_n$. Then we have

$$(X^v_n)_u = \frac{1}{n}\mathrm{Ad}(g_n)(X_{\rho(g_n^{-1})(u)}) = \frac{1}{n}\mathrm{Ad}(g_n)(A\rho(g_n^{-1})(u) + b)$$

$$= \frac{1}{n}(\mathrm{Ad}(g_n) \circ A \circ \mathrm{Ad}(g_n^{-1}))(u) + \frac{1}{n}\mathrm{Ad}(g_n)(A\rho(g_n^{-1})(\hat{0}) + b)$$

and hence

$$A_n^v = \frac{1}{n}\mathrm{Ad}(g_n) \circ A \circ \mathrm{Ad}(g_n^{-1}) \quad \text{and} \quad b_n^v = \frac{1}{n}\mathrm{Ad}(g_n)A(\rho(g_n^{-1})(\hat{0}) + b). \tag{7.10}$$

From the first relation in (7.10), we have $A_n^v \in \mathfrak{o}_{AK}(V)$ and hence $X_n^v \in \mathcal{K}^h$.

For $\{X_n^v\}_{n=1}^\infty$, we have the following fact.

**Lemma 7.4.6.** *If $X \in \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$ and $v$ is an element of $H_-^{0,\mathbb{C}}$ with*

$$\exp\left(n\int_0^1 v(t)\,dt\right) = e \quad (n \in \mathbb{N}),$$

*then there exists a subsequence of $\{X_n^v\}_{n=1}^\infty$ converging to the zero vector field.*

Proof. Take $u \in V$. Let $u = u_- + u_+$ ($u_- \in H_-^{0,\mathbb{C}}$, $u_+ \in H_+^{0,\mathbb{C}}$). Then we have

$$(\mathrm{Ad}(g_n)u_\varepsilon)(t) = \mathrm{Ad}(\exp(n\widetilde{v}(t)))u_\varepsilon(t) = \exp(\mathrm{ad}(n\widetilde{v}(t)))u_\varepsilon(t) \in \mathfrak{g}_\varepsilon^{\mathbb{C}} \;\; (\varepsilon = - \text{ or } +)$$

for each $t \in [0,1]$ because $\widetilde{v}(t) \in \mathfrak{g}_-^{\mathbb{C}}$ $(0 \le t \le 1)$ by the assumption and $[\mathfrak{g}_-^{\mathbb{C}}, \mathfrak{g}_\varepsilon^{\mathbb{C}}] \subset \mathfrak{g}_\varepsilon^{\mathbb{C}}$ ($\varepsilon = -$ or $+$). Hence we have

$$\begin{aligned}\langle \mathrm{Ad}(g_n)u, \mathrm{Ad}(g_n)u\rangle^{\mathcal{I}} &= -\langle \mathrm{Ad}(g_n)u_- \mathrm{Ad}(g_n)u_-\rangle + \langle \mathrm{Ad}(g_n)u_+, \mathrm{Ad}(g_n)u_+\rangle \\ &= -\langle u_-, u_-\rangle + \langle u_+, u_+\rangle = \langle u, u\rangle^{\mathcal{I}}.\end{aligned}$$

Therefore, by using (7.10), we can show $\|A_n^v\|_{\mathrm{op}} = \frac{1}{n}\|A\|_{\mathrm{op}} \to 0$ $(n \to \infty)$. Also, since $v \in \mathfrak{g}_-^{\mathbb{C}}$ and $G_-^{\mathbb{C}}$ is a compact Lie group, we have

$$\|\rho(g_n^{-1})(\hat{0})\| = \| - (g_n^{-1})'(g_n^{-1})_*^{-1}\| = \|(g_n^{-1})'\| = \|\exp_*(nv)\| \le n\|v\|.$$

and hence

$$\|b_n^v\| \le \frac{1}{n}\left(\|A\rho(g_n^{-1})(\hat{0})\| + \|b\|\right) \le \|A\|_{\mathrm{op}} \cdot \|v\| + \frac{1}{n}\|b\| \to \|A\|_{\mathrm{op}} \cdot \|v\| \;\; (n \to \infty).$$

Since the sequence $\{X_n^v \,|\, n \in \mathbf{N}\}$ in $\mathcal{K}^h$ is bounded, there exists its convergent subsequence $\{X_{n_j}^v\}_{j=1}^\infty$. Set $X_\infty^v := \lim_{j\to\infty} X_{n_j}^v$. From $\lim_{n\to\infty} A_n^v = 0$, $X_\infty^v$ is a parallel Killing vector field on $V$. From $\exp\left(n\int_0^1 v(t)\,dt\right) = e$, we have $g_n \in \Omega_e(G^{\mathbb{C}})$ and hence $\rho(g_n)(\widetilde{M}^{\mathbb{C}}) = \widetilde{M}^{\mathbb{C}}$. This fact together with $X \in \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$ deduces $X_n^v \in \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$. Also, from $\|A_n^v\|_{\mathrm{op}} = \frac{1}{n}\|A\|_{\mathrm{op}} < \infty$, we have $X_n^v \in \mathcal{K}^h$. Hence we have $X_n^v \in \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$. Therefore we have $X_\infty^v \in \overline{\mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}}$. Furthermore, from Lemma 7.4.5, we have $X_\infty^v \in \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$. Thus, since $X_\infty^v$ is parallel and $X_\infty^v \in \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$, it follows from Lemma 7.4.4 that $X_\infty^v = 0$. This completes the proof. □

On the other hand, we have the following fact.

**Lemma 7.4.7.** *Let $X$ be an element of $\mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$ given by $X_u = Au + b$ ($u \in V$) for some $A \in \mathfrak{o}_{AK}(V)$ and some $b \in V$, $Y$ an element of $\mathfrak{g}_-^{\mathbb{C}}$ and $f$ an element of $H^0([0,1],\mathbb{C})(= H^0([0,1],\mathbb{R}^2))$ satisfying $\int_0^1 f(t)dt = 0$ or $f =$ const. Then we have $A(fY) = [Y, w]$ for some*

$w \in V$.

Proof. Set $v := fY$. Define $\widetilde{f} \in H^1([0,1],\mathbb{C})$ by $\widetilde{f}(t) := \int_0^t f(t)dt$ $(0 \le t \le 1)$. Let $A(fY)(t) = u_1(t) + u_2(t)$ ($u_1(t) \in \mathrm{Ker}\,\mathrm{ad}(Y)$ and $u_2(t) \in \mathrm{Im}\,\mathrm{ad}(Y)$), and $u_i(t) = u_i^-(t) + u_i^+(t)$ ($u_i^-(t) \in \mathfrak{g}_-^{\mathbb{C}}$, $u_i^+(t) \in \mathfrak{g}_+^{\mathbb{C}}$) $(i = 1,2)$ and $b(t) = b^-(t) + b^+(t)$ ($b^-(t) \in \mathfrak{g}_-^{\mathbb{C}}$, $b^+(t) \in \mathfrak{g}_+^{\mathbb{C}}$). Let $g_n(t) := \exp(n\widetilde{v}(t)) = \exp(n\widetilde{f}(t)Y)$. From (7.10) and $\mathrm{Ad}(g_n)|_{\mathrm{Ker}\,\mathrm{ad}(Y)} = \mathrm{id}$, we have

$$\begin{aligned} b_n^v &= \frac{1}{n}\mathrm{Ad}(g_n)(A\rho(g_n^{-1})(\hat{0}) + b) = \mathrm{Ad}(g_n)\left(A(fY) + \frac{b}{n}\right) \\ &= u_1 + \mathrm{Ad}(g_n)(u_2 + \frac{b}{n}). \end{aligned}$$

Since $\mathrm{Ad}(g_n)$ preserves $\mathfrak{g}_-^{\mathbb{C}}$ and $\mathfrak{g}_+^{\mathbb{C}}$ invariantly, respectively, and $\mathrm{Ad}(g_n)|_{\mathrm{Ker}\,\mathrm{ad}(Y)} = \mathrm{id}$, we have

$$\begin{aligned} \langle b_n^v, u_1\rangle^{\mathcal{I}} &= \langle u_1, u_1\rangle^{\mathcal{I}} + \langle \mathrm{Ad}(g_n)(u_2 + \frac{b}{n}), \mathrm{Ad}(g_n)u_1\rangle^{\mathcal{I}} \\ &= \langle u_1, u_1\rangle^{\mathcal{I}} + \frac{1}{n}\langle b, u_1\rangle^{\mathcal{I}} \to \langle u_1, u_1\rangle^{\mathcal{I}} \quad (n \to \infty). \end{aligned}$$

First we consider the case where "$\int_0^1 f(t)dt = 0$" or "$f = \mathrm{const}$ and $Y$ is the initial vector of a closed geodesic in $G_-^{\mathbb{C}}$ of period $f$". Then we have $\exp\left(n\int_0^1 v(t)dt\right) = e$ $(n \in \mathbb{N})$. Also we have $v \in H_-^{0,\mathbb{C}}$ because of $Y \in \mathfrak{g}_-^{\mathbb{C}}$. Hence, according to Lemma 7.4.6, there exists a subsequence $\{X_{n_i}^v\}_{i=1}^\infty$ of $\{X_n\}_{n=1}^\infty$ converging to the zero vector field. Clearly we have $\lim_{i\to\infty} b_{n_i}^v = 0$ and hence $u_1 = 0$. Thus we see that $A(fY)(t) \in \mathrm{Im}\,\mathrm{ad}(Y)$ holds for all $t \in [0,1]$. That is, we have $A(fY) = [Y, w]$ for some $w \in V$. Next we consider the case where $f = \mathrm{const}$ and $Y$ is the initial vector of a closed geodesic in $G_-^{\mathbb{C}}$ (not necessarily of period $f$). Let $a$ be the period of the closed geodesic. Since $aY$ is the initial vector of a closed geodesic in $G_-^{\mathbb{C}}$ of period one, it follows from the above discussion that $A(aY) = [Y, \bar{w}]$ holds for some $\bar{w} \in V$. Hence we have

$$A(fY) = \frac{f}{a}A(aY) = \frac{f}{a}[Y, \bar{w}] = \left[Y, \frac{f}{a}\bar{w}\right].$$

Next we consider the case where $f = \mathrm{const}$ and $Y$ is the initial vector of non-closed geodesic in $G_-^{\mathbb{C}}$. Set

$$B := \{Z \mid Z : \text{ the initial vector of a closed geodesic in } G_-^{\mathbb{C}}\}.$$

Since $\mathfrak{g}_-^{\mathbb{C}}$ is the compact real of $\mathfrak{g}^{\mathbb{C}}$, $B$ is dense in $\mathfrak{g}_-^{\mathbb{C}}$. Take a sequence $\{Z_i\}_{i=1}^\infty$ in $B$ with $\lim_{i\to\infty} Z_i = fY$. As showed in the above, there exists $w_i \in V$ with $A(Z_i) = [Z_i, w_i]$ for each $i$. We can show that the sequence $\{w_i\}_{i=1}^\infty$ is a convergent sequence and that

$$A(fY) = \lim_{i\to\infty}[Z_i, w_i] = [Y, f\lim_{i\to\infty} w_i].$$

This completes the proof. □

Since $w$ in this lemma depends on $X$, $f$ and $Y$, we denote it by $w_{X,f,Y}$. According to Lemma 2.10 of [5], we have the following fact.

**Lemma 7.4.8.** *Let $B$ be a map from $\mathfrak{g}_-^{\mathbb{C}}$ to oneself defined by $B(Y) = [\mu(Y), Y]$ ($Y \in \mathfrak{g}_-^{\mathbb{C}}$) in terms of a map $\mu : \mathfrak{g}_-^{\mathbb{C}} \to \mathfrak{g}_-^{\mathbb{C}}$. If $B$ is linear, then $\mu$ is a constant map.*

By using Lemmas 7.3.7 and 7.3.8, we can show the following fact.

**Lemma 7.4.9.** *Fix* $X \in \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$ *and* $f \in H^0([0,1],\mathbb{C})$ *satisfying* $\int_0^1 f(t)dt = 0$ *or* $f = \text{const}$. *Then* $w_{X,f,Y}$ *is independent of the choice of* $Y \in \mathfrak{g}_-^{\mathbb{C}}$.

Proof. For the simplicity, set $w_Y := w_{X,f,Y}$. Define a linear map $B_1^t : \mathfrak{g}_-^{\mathbb{C}} \to \mathfrak{g}_-^{\mathbb{C}}$ by $B_1^t(Y) := A(fY)(t)_{\mathfrak{g}_-^{\mathbb{C}}}$ ($Y \in \mathfrak{g}_-^{\mathbb{C}}$) and a linear map $B_2^t : \mathfrak{g}_-^{\mathbb{C}} \to \mathfrak{g}_-^{\mathbb{C}}$ by $B_2^t(Y) := \sqrt{-1}(A(fY)(t)_{\mathfrak{g}_+^{\mathbb{C}}})$ ($Y \in \mathfrak{g}_-^{\mathbb{C}}$), where $(\cdot)_{\mathfrak{g}_\varepsilon^{\mathbb{C}}}$ ($\varepsilon = -$ or $+$) is the $\mathfrak{g}_\varepsilon^{\mathbb{C}}$-component of $(\cdot)$. Since $A(fY) = [Y, w_Y]$, we have $B_1^t(Y) = [Y, w_Y(t)_{\mathfrak{g}_-^{\mathbb{C}}}]$ and $B_2^t(Y) = [Y, \sqrt{-1}w_Y(t)_{\mathfrak{g}_+^{\mathbb{C}}}]$, it follows from Lemma 7.4.8 that, for each $t \in [0,1]$, $w_Y(t)_{\mathfrak{g}_-^{\mathbb{C}}}$ and $w_Y(t)_{\mathfrak{g}_+^{\mathbb{C}}}$ are independent of the choice of $Y \in \mathfrak{g}_-^{\mathbb{C}}$. Hence $w_Y$ is independent of the choice of $Y \in \mathfrak{g}_-^{\mathbb{C}}$. □

According to this lemma, $w_{X,f,Y}$ is independent of the choice of $Y \in \mathfrak{g}_-^{\mathbb{C}}$, we denote it by $w_{X,f}$. Define $\psi_n \in H^0([0,1],\mathbb{C})$ by $\psi_n(t) = \exp(2n\pi\sqrt{-1}t)$ ($0 \le t \le 1$), where $n \in \mathbb{Z}$.

**Lemma 7.4.10.** *For each* $X \in \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$ *and each* $f \in H^0([0,1],\mathbb{C})$ *satisfying* $\int_0^1 f(t)dt = 0$ *or* $f = \text{const}$, *we have* $w_{X,f} = f w_{X,1}$, *where the subscript* 1 *in* $w_{X,1}$ *means* $1 \in H^0([0,1],\mathbb{C})$.

Proof. Let $\langle\ ,\ \rangle^{\mathbb{C}}$ be the complexification of the $\mathrm{Ad}(G)$-invariant non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ of $\mathfrak{g}$ inducing the metric of $G/K$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\sqrt{-1}\mathfrak{p}$ and $\mathfrak{g}_-^{\mathbb{C}} = \mathfrak{z}_{\mathfrak{g}_-^{\mathbb{C}}}(\mathfrak{a}) + \sum_{\alpha\in\triangle}(\mathfrak{g}_-^{\mathbb{C}})_\alpha$ the root space decomposition of $\mathfrak{g}_-^{\mathbb{C}}$ with respect to $\mathfrak{a}$, where $\mathfrak{z}_{\mathfrak{g}_-^{\mathbb{C}}}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}_-^{\mathbb{C}}$ and $\triangle := \{\alpha \in \mathfrak{a}^* \,|\, (\mathfrak{g}_-^{\mathbb{C}})_\alpha \neq \{0\}\}$ ($(\mathfrak{g}_-^{\mathbb{C}})_\alpha := \{Z \in \mathfrak{g}_-^{\mathbb{C}} \,|\, \mathrm{ad}(a)Z = \sqrt{-1}\alpha(a)Z\ (\forall a \in \mathfrak{a})\}$). For any $\alpha \in \triangle$ and any $n \in \mathbf{N} \cup \{0\}$, define $H_\alpha \in \mathfrak{a}$ by $\langle H_\alpha, \cdot\rangle = \alpha(\cdot)$ and $c_{\alpha,n} := \frac{2n\pi\sqrt{-1}}{\alpha(H_\alpha)}$. Define $g_{\alpha,n} \in H^1([0,1],G^{\mathbb{C}})$ by $g_{\alpha,n}(t) := \exp(tc_{\alpha,n}H_\alpha)$ ($0 \le t \le 1$). It is clear that $g_{\alpha,n} \in \Omega_e(G^{\mathbb{C}})$. Let $\overline{X}_{\alpha,n} := \rho(g_{\alpha,n})_*^{-1}X$. Since $\rho(g_{\alpha,n})(\widetilde{M}^{\mathbb{C}}) = \widetilde{M}^{\mathbb{C}}$, $\overline{X}_{\alpha,n}$ is tangent to $\widetilde{M}^{\mathbb{C}}$ along $\widetilde{M}^{\mathbb{C}}$. Also, we can show $\overline{X}_{\alpha,n} \in \mathcal{K}^h$. Hence we have $\overline{X}_{\alpha,n} \in \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$. Let $(\overline{X}_{\alpha,n})_u = \overline{A}_{\alpha,n}u + \overline{b}_{\alpha,n}$ ($\overline{A}_{\alpha,n} \in \mathfrak{o}_{AK}(V)$, $\overline{b}_{\alpha,n} \in V$). We can show that $\overline{A}_{\alpha,n} = \mathrm{Ad}(g_{\alpha,n})^{-1} \circ A \circ \mathrm{Ad}(g_{\alpha,n})$ in similar to the first relation in (7.10). Take any $Y_0 \in \mathfrak{z}_{\mathfrak{g}_-^{\mathbb{C}}}(\mathfrak{a})$ and any $Y_\alpha \in (\mathfrak{g}_-^{\mathbb{C}})_\alpha$. Then, from $\mathrm{Ad}(g_{\alpha,n})Y_0 = Y_0$, we have

$$\begin{aligned}[\mathrm{Ad}(g_{\alpha,n})w_{\overline{X}_{\alpha,n},1}, Y_0] &= [\mathrm{Ad}(g_{\alpha,n})w_{\overline{X}_{\alpha,n},1}, \mathrm{Ad}(g_{\alpha,n})Y_0] \\ &= -\mathrm{Ad}(g_{\alpha,n})(\overline{A}_{\alpha,n}Y_0) = -A(\mathrm{Ad}(g_{\alpha,n})Y_0) = -AY_0 = [w_{X,1}, Y_0].\end{aligned}$$

It follows from the arbitrariness of $Y_0 (\in \mathfrak{z}_{\mathfrak{g}_-^{\mathbb{C}}}(\mathfrak{a}))$ that

$$\mathrm{Im}(\mathrm{Ad}(g_{\alpha,n})w_{\overline{X}_{\alpha,n},1} - w_{X,1}) \subset \mathfrak{a}. \tag{7.11}$$

Also, from $\mathrm{Ad}(g_{\alpha,n})Y_\alpha = \psi_n Y_\alpha$, we have

$$\begin{aligned}[\mathrm{Ad}(g_{\alpha,n})w_{\overline{X}_{\alpha,n},1}, Y_\alpha] &= \psi_{-n}[\mathrm{Ad}(g_{\alpha,n})w_{\overline{X}_{\alpha,n},1}, \mathrm{Ad}(g_{\alpha,n})Y_\alpha] \\ &= -\psi_{-n}\mathrm{Ad}(g_{\alpha,n})(\overline{A}_{\alpha,n}Y_\alpha) = -\psi_{-n}A(\mathrm{Ad}(g_{\alpha,n})Y_\alpha) \\ &= -\psi_{-n}A(\psi_n Y_\alpha) = \psi_{-n}[w_{X,\psi_n}, Y_\alpha]\end{aligned}$$

and hence

$$[\mathrm{Ad}(g_{\alpha,n})w_{\overline{X}_{\alpha,n},1} - \psi_{-n}w_{X,\psi_n}, Y_\alpha] = 0.$$

It follows from the arbitrariness of $Y_\alpha (\in (\mathfrak{g}_-^{\mathbb{C}})_\alpha)$ that

$$\mathrm{Im}\left(\mathrm{Ad}(g_{\alpha,n})w_{\overline{X}_{\alpha,n},1}-\psi_{-n}w_{X,\psi_n}\right)\subset\mathfrak{z}_{\mathfrak{g}_-^{\mathbb{C}}}((\mathfrak{g}_-^{\mathbb{C}})_\alpha).$$

This together with (7.11) implies

$$\mathrm{Im}\left(\psi_n w_{X,1}-w_{X,\psi_n}\right)\subset\mathfrak{a}\oplus\mathfrak{z}_{\mathfrak{g}_-^{\mathbb{C}}}((\mathfrak{g}_-^{\mathbb{C}})_\alpha).$$

From the arbitrariness of $\alpha$, we obtain

$$\mathrm{Im}\left(\psi_n w_{X,1}-w_{X,\psi_n}\right)\subset\mathfrak{a}\oplus\left(\bigcap_{\alpha\in\triangle}\mathfrak{z}_{\mathfrak{g}_-^{\mathbb{C}}}((\mathfrak{g}_-^{\mathbb{C}})_\alpha)\right)=\mathfrak{a}.$$

Take another maximal abelian subspace $\mathfrak{a}'$ of $\sqrt{-1}\mathfrak{p}$ with $\mathfrak{a}'\cap\mathfrak{a}=\{0\}$. Similarly we can show

$$\mathrm{Im}\left(\psi_n w_{X,1}-w_{X,\psi_n}\right)\subset\mathfrak{a}'$$

and hence

$$w_{X,\psi_n}=\psi_n w_{X,1}. \tag{7.12}$$

Take any $f\in H^0([0,1],\mathbb{C})$ satisfying $\int_0^1 f(t)dt=0$ or $f=\mathrm{const}$. Let $f=\sum_{n=-\infty}^{\infty}c_n\psi_n$ be the Fourier's expansion of $f$, where $c_n$ is constant for each $n$. Then, since $A$ is continuous and linear, we have

$$A(fY)=\sum_{n=-\infty}^{\infty}c_nA(\psi_nY)\quad(Y\in\mathfrak{g}_-^{\mathbb{C}}). \tag{7.13}$$

From (7.12) and (7.13), we obtain

$$[Y,w_{X,f}]=A(fY)=\sum_{n=-\infty}^{\infty}c_n[Y,w_{X,\psi_n}]=[Y,fw_{X,1}]\quad(Y\in\mathfrak{g}_-^{\mathbb{C}}).$$

Thus $w_{X,f}-fw_{X,1}$ belongs to the center of $\mathfrak{g}_-^{\mathbb{C}}$. Therefore, since $\mathfrak{g}_-^{\mathbb{C}}$ has no center, we obtain $w_{X,f}=fw_{X,1}$. □

From Lemmas 7.3.7 and 7.3.10, we have the following fact.

**Lemma 7.4.11.** *Let $X$ be an element of $\mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$ given by $X_u=Au+b$ $(u\in V)$ for some $A\in\mathfrak{o}_{AK}(V)$ and $b\in V$. Then we have $A=\mathrm{ad}(v)$ for some $v\in V$.*

Proof. Take any $u\in V$ and a base $\{e_1,\cdots,e_m\}$ of $\mathfrak{g}_-^{\mathbb{C}}$. Let $u=\sum_{i=1}^{m}u_ie_i$ and $u_i=\sum_{n=-\infty}^{\infty}c_{i,n}\psi_n$ be the Fourier expansion of $u_i$. Then, since $A$ is continuous and linear, we have $Au=\sum_{n=-\infty}^{\infty}\sum_{i=1}^{m}c_{i,n}A(\psi_ne_i)$. According to Lemmas 7.3.7 and 7.3.10, we have $A(fY)=[w_{X,1},fY]$ for any $Y\in\mathfrak{g}_-^{\mathbb{C}}$ and any $f\in H^0([0,1],\mathbb{C})$ satisfying $\int_0^1 f(t)dt=0$ or $f=\mathrm{const}$. Hence we have

$$Au=\sum_{n=-\infty}^{\infty}\sum_{i=1}^{m}c_{i,n}[w_{X,1},\psi_ne_i]=[w_{X,1},u].$$

Thus we obtain $A=\mathrm{ad}(w_{X,1})$. □

By using Lemmas 7.3.4 and 7.3.11, we shall prove Proposition 7.4.

Proof of Proposition 7.4. Take any $X \in \mathrm{Lie}\, H_b$. Since $\mathrm{Lie}\, H_b \subset \mathcal{K}^h_{\widetilde{M}^{\mathbb{C}}}$, it follows from Lemmas 7.3.4 and 7.3.11 that $X = X^v$ for some $v \in V$. Since $X^v$ is the holomorphic Killing vector field associated with an one-parameter subgroup $\{\rho(\exp \circ sv) \mid s \in \mathbb{R}\}$ of $\rho(H^1([0,1], G^{\mathbb{C}}))$, we have $X \in \mathrm{Lie}\,\rho(H^1([0,1], G^{\mathbb{C}}))$. Hence we obtain $\mathrm{Lie}\, H_b \subset \mathrm{Lie}\,\rho(H^1([0,1], G^{\mathbb{C}})$, that is, $H_b \subset \rho(H^1([0,1], G^{\mathbb{C}}))$. □

By using Proposition 7.4, we shall prove Theorem B.

Proof of Theorem B. Since $H_b$ is a subgroup of $\rho(H^1([0,1], G^{\mathbb{C}}))$ by Proposition 7.4, we have $H_b = \rho(Q)$ for some subgroup $Q$ of $H^1([0,1], G^{\mathbb{C}})$. Let $Q'$ be a closed connected subgroup of $G^{\mathbb{C}} \times G^{\mathbb{C}}$ generated by $\{(h(0), h(1)) \mid h \in Q\}$. Since $\phi \circ \rho(h) = (L_{h(0)} \circ R^{-1}_{h(1)}) \circ \phi$ for each $h \in H$, we have $\widehat{M}^{\mathbb{C}} = Q' \cdot e$, where $e$ is the identity element of $G^{\mathbb{C}}$. Here we note that $G^{\mathbb{C}} \times G^{\mathbb{C}}$ acts on $G^{\mathbb{C}}$ by $(g_1, g_2) \cdot g := (L_{g_1} \circ R^{-1}_{g_2})(g)$ $(g_1, g_2, g \in G^{\mathbb{C}})$. Set $\widehat{M} := \pi_{\mathbb{R}}^{-1}(M)$, where $\pi_{\mathbb{R}}$ is the natural projection of $G$ onto $G/K$. Since $\widehat{M}$ is a component of $\widehat{M}^{\mathbb{C}} \cap G$ containing $e$ and $(Q' \cap (G \times G)) \cdot e$ is a complete open submanifold of $\widehat{M}^{\mathbb{C}} \cap G$, $\widehat{M}$ is a component of $(Q' \cap (G \times G)) \cdot e$. Therefore we have $\widehat{M} = (Q' \cap (G \times G))_0 \cdot e$, where $(Q' \cap (G \times G))_0$ is the identity component of $Q' \cap (G \times G)$. Set $Q'_{\mathbb{R}} := (Q' \cap (G \times G))_0$. Since $\widehat{M}$ consists of fibres of $\pi_{\mathbb{R}}$, we have $\langle Q'_{\mathbb{R}} \cup (e \times K)\rangle \cdot e = \widehat{M}$, where $\langle Q'_{\mathbb{R}} \cup (e \times K)\rangle$ is the group generated by $Q'_{\mathbb{R}} \cup (e \times K)$. Denote by the same symbol $Q'_{\mathbb{R}}$ the group $\langle Q'_{\mathbb{R}} \cup (e \times K)\rangle$ under abuse of the notation. Set $(Q'_{\mathbb{R}})_1 := \{g_1 \in G \mid \exists g_2 \in G \text{ s.t. } (g_1, g_2) \in Q'_{\mathbb{R}}\}$ and $(Q'_{\mathbb{R}})_2 := \{g_2 \in G \mid \exists g_1 \in G \text{ s.t. } (g_1, g_2) \in Q'_{\mathbb{R}}\}$. Also, set $(Q'_{\mathbb{R}})^{\bullet}_1 := \{g \in G \mid (g, e) \in Q'_{\mathbb{R}}\}$ and $(Q'_{\mathbb{R}})^{\bullet}_2 := \{g \in G \mid (e, g) \in Q'_{\mathbb{R}}\}$. It is clear that $(Q'_{\mathbb{R}})^{\bullet}_i$ is a normal subgroup of $(Q'_{\mathbb{R}})_i$ $(i = 1, 2)$. From $e \times K \subset Q'_{\mathbb{R}}$, we have $K \subset (Q'_{\mathbb{R}})^{\bullet}_2$. Since $K \subset (Q'_{\mathbb{R}})^{\bullet}_2 \subset (Q'_{\mathbb{R}})_2 \subset G$ and $K$ is a maximal subgroup of $G$, we have $(Q'_{\mathbb{R}})_2 = K$ or $G$ and $(Q'_{\mathbb{R}})^{\bullet}_2 = K$ or $G$. Suppose that $(Q'_{\mathbb{R}})^{\bullet}_2 = G$. Then we have $\widehat{M} = G$ and hence $M = G/K$. Thus a contradiction arises. Hence we have $(Q'_{\mathbb{R}})^{\bullet}_2 = K$. Since $K$ is not a normal subgroup of $G$ and it is a normal subgroup of $(Q'_{\mathbb{R}})_2$, we have $(Q'_{\mathbb{R}})_2 \neq G$. Therefore we have $(Q'_{\mathbb{R}})_2 = K$ and hence $Q'_{\mathbb{R}} \subset G \times K$. Set $Q''_{\mathbb{R}} := \{g \in G \mid (\{g\} \times K) \cap Q'_{\mathbb{R}} \neq \emptyset\}$. Then, since $\widehat{M} = Q'_{\mathbb{R}} \cdot e$ and $M = \pi(\widehat{M})$, we have $M = Q''_{\mathbb{R}}(eK)$. Thus $M$ is extrinsically homogeneous. □

## 8. Proof of Theorem C

In this section, we prove Theorem C (Main theorem) by using Theorems A and B. Let $M$ be as in Theorem C and $F$ be its reflective focal submanifold. Without loss of generality, we may assume that $o := eK \in F$. Denote by $A$ the shape tensor of $M$ and $R$ the curvature tensor of $G/K$.

First we prove the following fact by using Theorem A.

**Proposition 8.1.** *The submanifold $M$ satisfies the condition $(*_{\mathbb{C}})$.*

Proof. We prove this statement in the case where $G/K$ is of non-compact type (this statement is proved similarly in the case where $G/K$ is of compact type). Take $Z_0 \in \mathfrak{p}$ with $\mathrm{Exp}\, Z_0 \in M$. Set $x_0 := \mathrm{Exp}\, Z_0$, $\mathfrak{t} := T_o F$, $\mathfrak{t}^{\perp} := T^{\perp}_o F$ and $\mathfrak{b} := (\exp Z_0)^{-1}_{*o}(T^{\perp}_{x_0} M)$. We use the notations in the proof of Theorem A (in Section 6). Take any $v \in T^{\perp}_{x_0} M$. As stated in the proof of Theorem A, the decomposition

$$T_{\mathrm{Exp}\,Z_0}M = \left(\bigoplus_{\beta\in(\triangle_{\mathfrak b})_+^H\cup\{0\}}(\mathfrak{p}_\beta\cap\mathfrak{t})^L_{Z_0}\right)\oplus\left(\bigoplus_{\beta\in(\triangle_{\mathfrak b})_+^V}(\exp\, Z_0)_{*o}(\mathfrak{p}_\beta\cap\mathfrak{t}^\perp)\right)$$

is the common eigenspace decomposition of $A_v$ and $R(v)$. Also, we have $R(v)|_{(\exp\,Z_0)_{*o}(\mathfrak{p}_\beta)} = \beta(v)^2\,\mathrm{id}$ $(\beta\in(\triangle_{\mathfrak b})_+\cup\{0\})$. From (ii) of Proposition 5.3 that

$$(\mathfrak{p}_\beta\cap\mathfrak{t})^L_{Z_0}\subset\mathrm{Ker}(A_v+\beta(\bar v)\tanh(\beta(Z_0))\mathrm{id})\quad(\beta\in(\triangle_{\mathfrak b})^H_+).$$

Also, since $F$ is reflective and the fibre $M\cap\mathrm{Exp}(\mathfrak{t}^\perp)$ is a principal orbit of the isotropy action of the symmetric space $\mathrm{Exp}(\mathfrak{t}^\perp)$, it follows from (i) of Proposition 5.3 that

$$(\exp\,Z_0)_*(\mathfrak{p}_\beta\cap\mathfrak{t}^\perp)\subset\mathrm{Ker}\left(A_v+\frac{\beta(\bar v)}{\tanh(\beta(Z_0))}\mathrm{id}\right).$$

From these facts, it follows that are not equal the absolute values of the eigenvalues $A_v$ and $R(v)$ on each of the common eigenspaces $(\mathfrak{p}_\beta\cap\mathfrak{t})^L_{Z_0}$'s $(\beta\in(\triangle_{\mathfrak b})^H_+\cup\{0\})$ and $(\exp\,Z_0)_{*o}(\mathfrak{p}_\beta\cap\mathfrak{t}^\perp)$'s $((\triangle_{\mathfrak b})^V_+)$ of $A_v$ and $R(v)$. This implies that $M$ satisfies the condition $(*_{\mathbb C})$. □

From Theorem B and this proposition, we can derive Theorem C.

Proof of Theorem C. Since $M$ satisfies the condition $(*_{\mathbb C})$ by Proposition 8.1, it follows from Theorem B that $M$ is extrinsically homogeneous. Hence it follows from Theorem A of [19] that $M$ is a principal orbit of a (complex) hyperpolar action on $G/K$. See [16] (or [19]) about the definition of a (complex) hyperpolar action. Furthermore, since this action admits a reflective (hence totally geodesic) singular orbit and it is of cohomogeneity greater than one, it follows from Theorem C and Remark 1.1 of [19] that this action is orbit equivalent to a Hermann type action. Therefore $M$ is a principal orbit of a Hermann type action. □

## 9. Classifications

From Theorem C and the list of Hermann type actions in [19], we can classify isoparametric submanifolds as in Theorem C as follows.

**Theorem 9.1.** *Let $M$ be a full irreducible isoparametric $C^\omega$-submanifold of codimension greater than one in an irreducible symmetric space $G/K$ of non-compact type. If $M$ admits a reflective focal submanifold, then it is a principal orbit of the action of one of symmetric subgroups $H$'s of $G$ as in* Tables 1-3.

## 10. Proof of Theorem D

In 1991, G. Thorbergsson ([39]) proved that any full irreducible isoparametric submanifold of codimension greater than two in a Euclidean space is extrinsically homogeneous by using the building theory. In this section, we shall prove Theorem D by defining the topological Tits building of spherical type associated to an isoparametric submanifold as in Theorem D and using it, where we refer the proof in [39]. First we recall the notion of a topological Tits building. Let $\Delta=(\mathcal{V},\mathcal{S})$ be an $r$-dimensional simplicial complex, where $\mathcal{V}$ denotes the set of all vertices and $\mathcal{S}$ denotes the set of all simplices. Each $r$-simplex of $\Delta$ is called a *chamber* of $\Delta$. Let $\mathcal{A}:=\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ be a family of subcomplexes of $\Delta$. The pair

Table 1. List of Hermann type actions.

| $G/K$ | $H$ |
|---|---|
| $SL(n,\mathbb{R})/SO(n)$ ($n \geq 6$, $n$ : even) | $SO(n)$, $SO_0(p,n-p)$ $(1 \leq p \leq n-1)$, $Sp(\frac{n}{2},\mathbb{R})$, $SL(\frac{n}{2},\mathbb{C})\cdot U(1)$ $(SL(p,\mathbb{R})\times SL(n-p,\mathbb{R}))\cdot\mathbb{R}_*$ $(2 \leq p \leq n-2)$ |
| $SL(4,\mathbb{R})/SO(4)$ | $SO(4)$, $SO_0(1,3)$, $SO_0(2,2)$, $SL(2,\mathbb{C})\cdot U(1)$, $(SL(2,\mathbb{R})\times SL(2,\mathbb{R}))\cdot\mathbb{R}_*$ |
| $SL(n,\mathbb{R})/SO(n)$ ($n \geq 5$, $n$ : odd) | $SO(n)$, $SO_0(p,n-p)$ $(1 \leq p \leq n-1)$, $(SL(p,\mathbb{R})\times SL(n-p,\mathbb{R}))\cdot\mathbb{R}_*$ $(2 \leq p \leq n-2)$ |
| $SL(3,\mathbb{R})/SO(3)$ | $SO(3)$, $SO_0(1,2)$ |
| $SU^*(2n)/Sp(n)$ $(n \geq 4)$ | $Sp(n)$, $SO^*(2n)$, $Sp(p,n-p)$ $(1 \leq p \leq n-1)$, $SL(n,\mathbb{C})\cdot U(1)$ $SU^*(2p)\times SU^*(2n-2p)\times U(1)$ $(2 \leq p \leq n-2)$ |
| $SU^*(6)/Sp(3)$ | $Sp(3)$, $SO^*(6)$, $Sp(1,2)$ |
| $SU(p,q)/S(U(p)\times U(q))$ ($4 \leq p < q$, $p,q$ : even) | $S(U(p)\times U(q))$, $SO_0(p,q)$, $Sp(\frac{p}{2},\frac{q}{2})$, $S(U(i,j)\times U(p-i,q-j))$ $(1 \leq i \leq p-1,\ 1 \leq j \leq q-1)$ |
| $SU(p,q)/S(U(p)\times U(q))$ ($3 \leq p < q$, $p$ or $q$ : odd) | $S(U(p)\times U(q))$, $SO_0(p,q)$, $S(U(i,j)\times U(p-i,q-j))$ $(1 \leq i \leq p-1,\ 1 \leq j \leq q-1)$ |
| $SU(2,q)/S(U(2)\times U(q))$ $(q \geq 3)$ | $S(U(2)\times U(q))$, $SO_0(2,q)$, $S(U(1,j)\times U(1,q-j))$ $(1 \leq j \leq q-1)$ |
| $SU(p,p)/S(U(p)\times U(p))$ ($p \geq 4$, $p$ : even) | $S(U(p)\times U(p))$, $SO_0(p,p)$, $SO^*(2p)$, $Sp(\frac{p}{2},\frac{p}{2})$, $Sp(p,\mathbb{R})$, $SL(p,\mathbb{C})\cdot U(1)$ $S(U(i,j)\times U(p-i,p-j))$ $(1 \leq i \leq p-1,\ 1 \leq j \leq p-1)$ |
| $SU(2,2)/S(U(2)\times U(2))$ | $S(U(2)\times U(2))$, $SO_0(2,2)$, $SO^*(4)$, $SL(2,\mathbb{C})\cdot U(1)$, $S(U(1,1)\times U(1,1))$ |
| $SU(p,p)/S(U(p)\times U(p))$ ($p \geq 5$, $p$ : odd) | $S(U(p)\times U(p))$, $SO_0(p,p)$, $SO^*(2p)$, $Sp(p,\mathbb{R})$, $SL(p,\mathbb{C})\cdot U(1)$ $S(U(i,j)\times U(p-i,p-j))$ $(1 \leq i \leq p-1,\ 1 \leq j \leq p-1)$ |
| $SU(3,3)/S(U(3)\times U(3))$ | $S(U(3)\times U(3))$, $SO_0(3,3)$, $SO^*(6)$, $SL(3,\mathbb{C})\cdot U(1)$, $S(U(1,1)\times U(2,2))$, $S(U(1,2)\times U(2,1))$ |
| $SL(n,\mathbb{C})/SU(n)$ ($n \geq 6$, $n$ : even) | $SU(n)$, $SO(n,\mathbb{C})$, $SL(n,\mathbb{R})$, $SU(i,n-i)$ $(1 \leq i \leq n-1)$, $Sp(\frac{n}{2},\mathbb{C})$, $SU^*(n)$ $SL(i,\mathbb{C})\times SL(n-i,\mathbb{C})\times U(1)$ $(2 \leq i \leq n-2)$ |
| $SL(4,\mathbb{C})/SU(4)$ | $SU(4)$, $SO(4,\mathbb{C})$, $SL(4,\mathbb{R})$, $SU(i,4-i)$ $(1 \leq i \leq 3)$, $SU^*(4)$ $SL(2,\mathbb{C})\times SL(2,\mathbb{C})\times U(1)$ |
| $SL(n,\mathbb{C})/SU(n)$ ($n \geq 5$, $n$ : odd) | $SU(n)$, $SO(n,\mathbb{C})$, $SL(n,\mathbb{R})$, $SU(i,n-i)$ $(1 \leq i \leq n-1)$ $SL(i,\mathbb{C})\times SL(n-i,\mathbb{C})\times U(1)$ $(2 \leq i \leq n-2)$ |
| $SL(3,\mathbb{C})/SU(3)$ | $SU(3)$, $SO(3,\mathbb{C})$ |

$\mathcal{B} := (\Delta, \mathcal{A})$ is called a *Tits building* if the following conditions hold:

(B1) Each $(r-1)$-dimensional simplex of $\Delta$ is contained in at least three chambers.
(B2) Each $(r-1)$-dimensional simplex in a subcomplex $\mathcal{A}_\lambda$ are contained in exactly two chambers of $\mathcal{A}_\lambda$.
(B3) Any two simplices of $\Delta$ are contained in some $\mathcal{A}_\lambda$.
(B4) If two subcomplexes $\mathcal{A}_{\lambda_1}$ and $\mathcal{A}_{\lambda_2}$ share a chamber, then there is an isomorphism of $\mathcal{A}_{\lambda_1}$ onto $\mathcal{A}_{\lambda_2}$ fixing $\mathcal{A}_{\lambda_1} \cap \mathcal{A}_{\lambda_2}$ pointwisely.

Each subcomplex belonging to $\mathcal{A}$ is called an *apartment* of $\mathcal{B}$. In this appendix, we assume that all Tits building furthermore satisfies the following condition:

Table 2. List of Hermann type actions (continued).

| $G/K$ | $H$ |
|---|---|
| $SO_0(p,q)/SO(p)\times SO(q)$ $(4\leq p<q,\ \ p,q:\text{even})$ | $SO(p)\times SO(q),\ \ SU(\frac{p}{2},\frac{q}{2})\cdot U(1),$ $SO_0(i,j)\times SO_0(p-i,q-j)\ \ (1\leq i\leq p-1,\ 1\leq j\leq q-1)$ |
| $SO_0(2,q)/SO(2)\times SO(q)$ $(4\leq q,\ \ q:\text{even})$ | $SO(2)\times SO(q),\ \ SO_0(1,j)\times SO_0(1,q-j)\ \ (1\leq j\leq q-1)$ |
| $SO_0(p,q)/SO(p)\times SO(q)$ $(2\leq p<q,\ \ p\text{ or }q:\text{odd})$ | $SO(p)\times SO(q),\ \ SO_0(i,j)\times SO_0(p-i,q-j)\ \ (1\leq i\leq p-1,\ 1\leq j\leq q-1)$ |
| $SO_0(p,p)/SO(p)\times SO(p)$ $(p\geq 4,\ \ p:\text{even})$ | $SO(p)\times SO(p),\ \ SO(p,\mathbb{C}),\ \ SU(\frac{p}{2},\frac{p}{2})\cdot U(1),\ \ SL(p,\mathbb{R})\cdot U(1)$ $SO_0(i,j)\times SO_0(p-i,p-j)\ \ (1\leq i\leq p-1,\ 1\leq j\leq p-1)$ |
| $SO_0(2,2)/SO(2)\times SO(2)$ | $SO(2)\times SO(2),\ \ SO(2,\mathbb{C}),\ \ SO_0(1,1)\times SO_0(1,1)$ |
| $SO_0(p,p)/SO(p)\times SO(p)$ $(p\geq 5,\ \ p:\text{odd})$ | $SO(p)\times SO(p),\ \ SO(p,\mathbb{C}),\ \ SL(p,\mathbb{R})\cdot U(1),$ $SO_0(i,j)\times SO_0(p-i,p-j)\ (1\leq i\leq p-1,\ 1\leq j\leq p-1)$ |
| $SO_0(3,3)/SO(3)\times SO(3)$ | $SO(3)\times SO(3),\ \ SO(3,\mathbb{C}),\ \ SO_0(1,1)\times SO_0(2,2)$ $SO_0(1,2)\times SO_0(2,1)$ |
| $SO^*(2n)/U(n)$ $(n\geq 6,\ \ n:\text{even})$ | $U(n),\ \ SO(n,\mathbb{C}),\ \ SU^*(n)\cdot U(1)$ $SO^*(2i)\times SO^*(2n-2i)\ \ (2\leq i\leq n-2),$ $SU(i,n-i)\cdot U(1)\ \ ([\frac{i}{2}]+[\frac{n-i}{2}]\geq 2)$ |
| $SO^*(8)/U(4)$ | $U(4),\ \ SO(4,\mathbb{C}),\ \ SO^*(4)\times SO^*(4),\ \ SU(2,2)\cdot U(1)$ |
| $SO^*(2n)/U(n)$ $(n\geq 5,\ \ n:\text{odd})$ | $U(n),\ \ SO(n,\mathbb{C}),\ \ SO^*(2i)\times SO^*(2n-2i)\ \ (2\leq i\leq n-2),$ $SU(i,n-i)\cdot U(1)\ \ ([\frac{i}{2}]+[\frac{n-i}{2}]\geq 2)$ |
| $SO(n,\mathbb{C})/SO(n)$ $(n\geq 8,\ \ n:\text{even})$ | $SO(n),\ \ SO(i,\mathbb{C})\times SO(n-i,\mathbb{C})\ \ (2\leq i\leq n-2),$ $SO_0(i,n-i)\ \ ([\frac{i}{2}]+[\frac{n-i}{2}]\geq 2),\ \ SL(\frac{n}{2},\mathbb{C})\cdot SO(2,\mathbb{C}),\ \ SO^*(n)$ |
| $SO(6,\mathbb{C})/SO(6)$ | $SO(6),\ \ SO(i,\mathbb{C})\times SO(6-i,\mathbb{C})\ \ (2\leq i\leq 4),$ $SO_0(2,4),\ \ SO_0(3,3),\ \ SO^*(6)$ |
| $SO(4,\mathbb{C})/SO(4)$ | $SO(4),\ \ SO(2,\mathbb{C})\times SO(2,\mathbb{C}),\ \ SO_0(2,2),\ \ SO^*(4)$ |
| $SO(n,\mathbb{C})/SO(n)$ $(n\geq 5,\ \ n:\text{odd})$ | $SO(n),\ \ SO(i,\mathbb{C})\times SO(n-i,\mathbb{C})\ \ (2\leq i\leq n-2),$ $SO_0(i,n-i)\ \ ([\frac{i}{2}]+[\frac{n-i}{2}]\geq 2)$ |
| $Sp(n,\mathbb{R})/U(n)$ $(n\geq 4,\ \ n:\text{even})$ | $U(n),\ \ SU(i,n-i)\cdot U(1)\ \ (1\leq i\leq n-1),\ \ SL(n,\mathbb{R})\cdot U(1),$ $Sp(\frac{n}{2},\mathbb{C}),\ \ Sp(i,\mathbb{R})\times Sp(n-i,\mathbb{R})\ \ (2\leq i\leq n-2)$ |
| $Sp(2,\mathbb{R})/U(2)$ | $U(2),\ \ SU(1,1)\cdot U(1)$ |
| $Sp(n,\mathbb{R})/U(n)$ $(n\geq 5,\ \ n:\text{odd})$ | $U(n),\ \ SU(i,n-i)\cdot U(1)\ \ (1\leq i\leq n-1),\ \ SL(n,\mathbb{R})\cdot U(1),$ $Sp(i,\mathbb{R})\times Sp(n-i,\mathbb{R})\ \ (2\leq i\leq n-2)$ |
| $Sp(3,\mathbb{R})/U(3)$ | $U(3),\ \ SU(1,2)\cdot U(1),\ \ SL(3,\mathbb{R})\cdot U(1)$ |

(B5) Each apartment $\mathcal{A}_\lambda$ is a Coxeter complex.

If $\mathcal{A}_\lambda$ is finite (resp. infinite), then the building $\mathcal{B}$ is said to be *spherical type* (resp. *affine type*). Let $\mathcal{O}$ be a Hausdorff topology of $\mathcal{V}$. The pair $(\mathcal{B},\mathcal{O})$ is called a *topological Tits building* if the following conditions hold:

(TB1) $(\mathcal{B},\mathcal{A})$ is a Tits building.
(TB2) For $k\in\{1,\cdots,r\}$, $\widehat{\mathcal{S}_k}:=\{(x_1,\cdots,x_{k+1})\in\mathcal{V}^{k+1}\mid |x_1\cdots x_{k+1}|\in\mathcal{S}_k\}$ is closed in the product topological space $(\mathcal{V}^{k+1},\mathcal{O}^{k+1})$, where $\mathcal{S}_k$ denotes the set of all $k$-simplices of $\mathcal{S}$ and

Table 3. List of Hermann type actions (continued).

| $G/K$ | $H$ |
|---|---|
| $Sp(p,q)/Sp(p)\times Sp(q)$ $(2\le p<q)$ | $Sp(p)\times Sp(q)$, $SU(p,q)\cdot U(1)$, $Sp(i,j)\times Sp(p-i,q-j)$ $(1\le i\le p-1,\ 1\le j\le q-1)$ |
| $Sp(p,p)/Sp(p)\times Sp(p)$ $(p\ge 3)$ | $Sp(p)\times Sp(p)$, $SU(p,p)\cdot U(1)$, $SU^*(2p)\cdot U(1)$, $Sp(p,\mathbb{C})$ $Sp(i,j)\times Sp(p-i,p-j)$ $(1\le i\le p-1,\ 1\le j\le p-1)$ |
| $Sp(2,2)/Sp(2)\times Sp(2)$ | $Sp(2)\times Sp(2)$, $SU(2,2)\cdot U(1)$, $SU^*(4)\cdot U(1)$, $Sp(1,1)\times Sp(1,1)$ |
| $Sp(n,\mathbb{C})/Sp(n)$ $(n\ge 4)$ | $Sp(n)$, $SL(n,\mathbb{C})\cdot SO(2,\mathbb{C})$, $Sp(n,\mathbb{R})$, $Sp(i,n-i)$ $(1\le i\le n-1)$, $Sp(i,\mathbb{C})\times Sp(n-i,\mathbb{C})$ $(2\le i\le n-2)$ |
| $Sp(n,\mathbb{C})/Sp(n)$ $(n=2,3)$ | $Sp(n)$, $SL(n,\mathbb{C})\cdot SO(2,\mathbb{C})$, $Sp(n,\mathbb{R})$, $Sp(i,n-i)$ $(1\le i\le n-1)$ |
| $E_6^6/(Sp(4)/\{\pm 1\})$ | $Sp(4)/\{\pm 1\}$, $Sp(4,\mathbb{R})$, $Sp(2,2)$, $SU^*(6)\cdot SU(2)$, $SL(6,\mathbb{R})\times SL(2,\mathbb{R})$, $SO_0(5,5)\cdot\mathbb{R}$, $F_4^4$ |
| $E_6^2/SU(6)\cdot SU(2)$ | $SU(6)\cdot SU(2)$, $Sp(1,3)$, $Sp(4,\mathbb{R})$, $SU(2,4)\cdot SU(2)$, $SU(3,3)\cdot SL(2,\mathbb{R})$, $SO^*(10)\cdot U(1)$, $SO_0(4,6)\cdot U(1)$ |
| $E_6^{-14}/Spin(10)\cdot U(1)$ | $Spin(10)\cdot U(1)$, $Sp(2,2)$, $SU(2,4)\cdot SU(2)$, $SU(1,5)\cdot SL(2,\mathbb{R})$, $SO^*(10)\cdot U(1)$, $SO_0(2,8)\cdot U(1)$ |
| $E_6^{-26}/F_4$ | $F_4$, $F_4^{-20}$, $Sp(1,3)$ |
| $E_6^{\mathbb{C}}/E_6$ | $E_6$, $E_6^6$, $E_6^2$, $E_6^{-14}$, $Sp(4,\mathbb{C})$, $SL(6,\mathbb{C})\cdot SL(2,\mathbb{C})$, $SO(10,\mathbb{C})\cdot Sp(1)$, $F_4^{\mathbb{C}}\cdot$ $E_6^{-26}$ |
| $E_7^7/(SU(8)/\{\pm 1\})$ | $SU(8)/\{\pm 1\}$, $SL(8,\mathbb{R})$, $SU^*(8)$, $SU(4,4)$, $SO^*(12)\cdot SU(2)$, $SO_0(6,6)\cdot SL(2,\mathbb{R})$, $E_6^6\cdot U(1)$, $E_6^2\cdot U(1)$ |
| $E_7^{-5}/SO'(12)\cdot SU(2)$ | $SO'(12)\cdot SU(2)$, $SU(4,4)$, $SU(2,6)$, $SO^*(12)\cdot SL(2,\mathbb{R})$, $SO_0(4,8)\cdot SU(2)$, $E_6^2\cdot U(1)$, $E_6^{-14}\cdot U(1)$ |
| $E_7^{-25}/E_6\cdot U(1)$ | $E_6\cdot U(1)$, $SU^*(8)$, $SU(2,6)$, $SO^*(12)\cdot SU(2)$, $SO_0(2,10)\cdot SL(2,\mathbb{R})$, $E_6^{-14}\cdot U(1)$, $E_6^{-26}\cdot U(1)$ |
| $E_7^{\mathbb{C}}/E_7$ | $E_7$, $E_7^7$, $E_7^{-5}$, $E_7^{-25}$, $SL(8,\mathbb{C})$, $SO(12,\mathbb{C})\cdot SL(2,\mathbb{C})$, $E_6^{\mathbb{C}}\cdot\mathbb{C}^*$ |
| $E_8^8/SO'(16)$ | $SO'(16)$, $SO^*(16)$, $SO_0(8,8)$, $E_7^{-5}\cdot Sp(1)$, $E_7^7\cdot SL(2,\mathbb{R})$ |
| $E_8^{-24}/E_7\cdot Sp(1)$ | $E_7\cdot Sp(1)$, $E_7^{-5}\cdot Sp(1)$, $E_7^{-25}\cdot SL(2,\mathbb{R})$, $SO^*(16)$, $SO_0(4,12)$ |
| $E_8^{\mathbb{C}}/E_8$ | $E_8$, $E_8^8$, $E_8^{-24}$, $SO(16,\mathbb{C})$, $E_7^{\mathbb{C}}\times SL(2,\mathbb{C})$ |
| $F_4^4/Sp(3)\cdot Sp(1)$ | $Sp(3)\cdot Sp(1)$, $Sp(1,2)\cdot Sp(1)$, $Sp(3,\mathbb{R})\cdot SL(2,\mathbb{R})$ |
| $F_4^{\mathbb{C}}/F_4$ | $F_4$, $F_4^4$, $F_4^{-20}$, $Sp(3,\mathbb{C})\cdot SL(2,\mathbb{C})$ |
| $G_2^2/SO(4)$ | $SO(4)$, $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$, $\alpha(SO(4))$ ($\alpha$ :an outer automorphism of $G_2^2$) |
| $G_2^{\mathbb{C}}/G_2$ | $G_2$, $G_2^2$, $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ |

$|x_1\cdots x_{k+1}|$ denotes the $k$-simplex with vertices $x_1,\cdots,x_{k+1}$.

A homeomorphism $\phi$ of $(\mathcal{V},\mathcal{O})$ is called a *topological automorphism* of the topological Tits building $(\Delta,\mathcal{A},\mathcal{O})$ if the following conditions hold:

(TA1) $\phi$ preserves $\mathcal{S}$ (i.e., "$\sigma=|x_1\cdots x_{k+1}|\in\mathcal{S}\Rightarrow\phi(\sigma):=|\phi(x_1)\cdots\phi(x_{k+1})|\in\mathcal{S}$.
(TA2) $\phi$ preserves $\mathcal{A}$ (i.e., for each $\lambda\in\Lambda$, $\phi(\mathcal{A}_\lambda):=\{\phi(\sigma)\,|\,\sigma\in\mathcal{A}_\lambda\}\in\mathcal{A}$.)
(TA3) For each $k\in\{1,\cdots,r\}$, $\phi$ gives a homeomorphism of $\widehat{\mathcal{S}_k}$ onto oneself.

According to (TA1) (resp. (TA2)), $\phi$ gives a bijection of $\mathcal{S}$ onto oneself (resp. $\mathcal{A}$ onto oneself).

Let $M$ be a full irreducible curvature-adapted isoparametric submanifold of codimension $r(\geq 2)$ in an irreducible symmetric space $G/K$ of non-compact type. Assume that $M$ satisfies the condition $(*'_{\mathbb{R}})$. Set $\mathfrak{p} := T_{eK}(G/K)$ and $\mathfrak{b} := T^{\perp}_{eK}M$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}\,(\subset \mathfrak{g})$ containing $\mathfrak{b}$ and $\mathfrak{p} = \mathfrak{a} \oplus \left(\underset{\alpha\in\triangle_+}{\oplus} \mathfrak{p}_\alpha\right)$ be the root space decomposition with respect to $\mathfrak{a}$, that is, $\mathfrak{p}_\alpha := \{X \in \mathfrak{p}\,|\,\mathrm{ad}(a)^2(X) = \alpha(a)^2 X\ (\forall\, a \in \mathfrak{a})\}$ and $\triangle_+$ is the positive root system of the root system $\triangle := \{\alpha \in \mathfrak{a}^* \setminus \{0\}\,|\,\mathfrak{p}_\alpha \neq \{0\}\}$ under a lexicographic ordering of $\mathfrak{a}^*$. Set $\triangle_{\mathfrak{b}} := \{\alpha|_{\mathfrak{b}}\,|\,\alpha \in \triangle \text{ s.t. } \alpha|_{\mathfrak{b}} \neq 0\}$ and let $\mathfrak{p} = \mathfrak{z}_{\mathfrak{p}}(\mathfrak{b}) \oplus \left(\underset{\beta\in(\triangle_{\mathfrak{b}})_+}{\oplus} \mathfrak{p}_\beta\right)$ be the root space decomposition with respect to $\mathfrak{b}$, where $\mathfrak{z}_{\mathfrak{p}}(\mathfrak{b})$ is the centralizer of $\mathfrak{b}$ in $\mathfrak{p}$, $\mathfrak{p}_\beta = \underset{\alpha\in\triangle_+ \text{ s.t. } \alpha|_{\mathfrak{b}}=\pm\beta}{\oplus} \mathfrak{p}_\alpha$ and $(\triangle_{\mathfrak{b}})_+$ is the positive root system of the root system $\triangle_{\mathfrak{b}}$ under a lexicographic ordering of $\mathfrak{b}^*$. For convenience, we denote $\mathfrak{z}_{\mathfrak{p}}(\mathfrak{b})$ by $\mathfrak{p}_0$. Denote by $A$ the shape tensor of $M$ and $R$ the curvature tensor of $G/K$. Let $m_A := \max\limits_{v\in\mathfrak{b}\setminus\{0\}} \sharp \mathrm{Spec}\, A_v$ and $m_R := \max\limits_{v\in\mathfrak{b}\setminus\{0\}} \sharp \mathrm{Spec}\, R(v)$, where $\sharp(\cdot)$ is the cardinal number of $(\cdot)$. Note that $m_R = \sharp(\triangle_{\mathfrak{b}})_+$. Let $U := \{v \in \mathfrak{b} \setminus \{0\}\,|\,\sharp\mathrm{Spec} A_v = m_A,\ \sharp\mathrm{Spec}\, R(v) = m_R\}$, which is an open dense subset of $\mathfrak{b} \setminus \{0\}$. Fix $v \in U$. Note that $\mathrm{Spec}\, R(v) = \{-\beta(v)^2\,|\,\beta \in (\triangle_{\mathfrak{b}})_+\}$. From $v \in U$, $\beta(v)^2$'s $(\beta \in (\triangle_{\mathfrak{b}})_+)$ are mutually distinct. Let $\mathrm{Spec} A_v = \{\lambda^v_1, \cdots, \lambda^v_{m_A}\}$ $(\lambda^v_1 > \cdots > \lambda^v_{m_A})$. Set

$$I^v_0 := \{i\,|\,\mathfrak{p}_0 \cap \mathrm{Ker}(A_v - \lambda^v_i \mathrm{id}) \neq \{0\}\},\quad I^v_\beta := \{i\,|\,\mathfrak{p}_\beta \cap \mathrm{Ker}(A_v - \lambda^v_i \mathrm{id}) \neq \{0\}\},$$

$$(I^v_\beta)^+ := \{i \in I^v_\beta\,|\,|\lambda^v_i| > |\beta(v)|\},\quad (I^v_\beta)^- := \{i \in I^v_\beta\,|\,|\lambda^v_i| < |\beta(v)|\},\quad (I^v_\beta)^0 := \{i \in I^v_\beta\,|\,|\lambda^v_i| = |\beta(v)|\}.$$

Since $M$ is curvature-adapted and satisfies the condition $(*'_{\mathbb{R}})$, we have $I^0_v = \emptyset$, $(I^v_\beta)^- = \emptyset$ (i.e., $I^v_\beta = (I^v_\beta)^+$) $(\beta \in (\triangle_{\mathfrak{b}})_+)$ and $\mathfrak{a} = \mathfrak{b}$ (hence $\triangle = \triangle_{\mathfrak{b}}$). In similar to the fact (2.2) stated in Section 2, we have

$$\mathcal{F}R^{\mathbb{R}}_{M,v} = \left\{ \frac{1}{\beta(v)} \mathrm{arctanh} \frac{\beta(v)}{\lambda^v_i} \,\middle|\, \beta \in \triangle_+,\ i \in I^v_\beta \right\}. \tag{10.1}$$

From the arbitrariness of $v$ and the fact that $U$ is open and dense in $\mathfrak{b}$, the relation (10.1) holds for any $v \in \mathfrak{b}$. Hence the tangential focal set $\mathcal{F}^{\mathbb{R}}_{M,eK}$ of $M$ at $eK$ is given by

$$\mathcal{F}^{\mathbb{R}}_{M,eK} = \bigcup_{v\in T^{\perp}_x M \text{ s.t. } \|v\|=1} \left\{ \frac{1}{\beta(v)} \mathrm{arctanh} \frac{\beta(v)}{\lambda^v_i} \cdot v \,\middle|\, \beta \in \triangle_+,\ i \in I^v_\beta \right\}. \tag{10.2}$$

On the other hand, H. Ewert ([6]) showed that the tangential focal set of an isoparametric submanifold in a symmetric spaces of non-compact type at any point consists of finitely many (real) hyperplanes (which are called *focal hyperplanes*) in the normal space at the point and the reflections with respect to the hyperplanes generates a Weyl group (see [6] for example), where we note that he ([6]) treated not only an isoparametric submanifold(=equifocal submanifold) but also a submanifold with parallel focal structure (whose sections are not necessarily flat). Denote by $\mathcal{W}$ this Weyl group. Note that the focal hyperplanes are not parallel pairwisely because the Weyl group is a finite Coxeter group. From this fact and (10.2), we see that, for any $\beta \in \triangle_+$, $\sharp\, I^v_\beta = 1$ and $\frac{\beta(v)}{\lambda^v_i}$ is independent of the choice of $v$ and furthermore $\frac{\beta(v)}{\lambda^v_i} = \frac{2\beta(v)}{\lambda^v_j}$ holds when $\beta, 2\beta \in \triangle_+$, where $\{i\} = I_\beta$ and $\{j\} = I_{2\beta}$. So we set

$c_\beta := \frac{\beta(v)}{\lambda_i^v}$ ($\beta \in \triangle'_+$) and furthermore $\hat{c}_\beta :=$ arctanh $c_\beta$. Also, set $\triangle'_+ := \{\beta \in \triangle_+ \,|\, 2\beta \notin \triangle_+\}$ and $k := \sharp\,\triangle'_+$. Then $\mathcal{F}^{\mathbb{R}}_{M,eK}$ is given by

$$\mathcal{F}^{\mathbb{R}}_{M,eK} = \bigcup_{\beta\in\triangle'_+} \beta^{-1}(\hat{c}_\beta). \tag{7.3}$$

This fact implies that $\mathcal{W}$ is isomorphic to the Weyl group of $G/K$ (that is, the Coxeter group of the principal orbits (which are isoparametric submanifolds) of the s-representation of $G/K$). Since $M$ is full and irreducible, we can show that $\mathcal{W}$ is of rank $r$ and irreducible. Therefore $G/K$ is irreducible and its rank is equal to $r$. For the simplicity, set $l_\beta := \beta^{-1}(\hat{c}_\beta)$. It is clear that $\underset{\beta\in\triangle_+}{\cap} l_\beta$ is a one-point set. Denote by $v_0$ this point and set $p_0 := \exp^\perp(v_0)$ and $r_0 := \|v_0\|$. It is clear that the section $\Sigma_x$ of $M$ through any $x \in M$ passes through $p_0$. Let $S(r_0)$ be the unit sphere of radius $r_0$ centered at 0 in $T_{p_0}(G/K)$. It is easy to show that $M$ is included by the geodesic sphere $\exp_{p_0}(S(r_0))$ in $G/K$. Let $\{l_i^x \,|\, i = 1,\cdots,k\}$ be the set of all focal hyperplanes of $M$ at $x(\in M)$, that is, $\overset{k}{\underset{i=1}{\cup}} l_i^x = \mathcal{F}^{\mathbb{R}}_{M,x}$. Set $\bar{l}_i^x := \exp^\perp(l_i^x)$, $\widetilde{l}_i^x := \exp_{p_0}^{-1}(\bar{l}_i^x)$ and $\widetilde{\Sigma}_x := \exp_{p_0}^{-1}(\Sigma_x)$, where we note that $\widetilde{\Sigma}_x$ is an $r$-dimensional affine subspace in $T_{p_0}(G/K)$ through 0 because $\Sigma_x$ is a flat totally geodesic submanifold in $G/K$, and that $\widetilde{l}_i^x$ is an (affine) hyperplane in $\widetilde{\Sigma}_x$ through 0. It is clear that $\widetilde{l}_i^x \cap S(r_0)$'s ($i \in I_x$) and their intersections give a Coxeter complex in $\widetilde{\Sigma}_x \cap S(r_0)$. Denote by $\mathcal{A}_x$ this Coxeter complex. Let $\mathcal{V}_x$ (resp. $\mathcal{S}_x$) be the set of all vertices (resp. simplices) of $\mathcal{A}_x$. Set $\mathcal{V}_M := \cup_{x\in M}\mathcal{V}_x$, $\mathcal{S}_M := \cup_{x\in M}\mathcal{S}_x$ and $\mathcal{A}_M := \{\mathcal{A}_x \,|\, x \in M\}$. Also, set $\Delta_M := (\mathcal{V}_M, \mathcal{S}_M)$. Give $\mathcal{V}_M$ the relative topology (which we denote by $\mathcal{O}$) of $T_{p_0}(G/K)$. Note that $\exp_{p_0}(\mathcal{V}_M)$ is equal to the sum of some lower dimensional submanifold $F_1,\cdots,F_l$. It is shown that $F_1,\cdots,F_l$ are focal submanifolds of $M$. For example, see Figure 6 about the case where $\mathcal{A}_x$ is a Coxeter complex of type $(A_2)$. We have the following fact:

($\sharp$) $\mathcal{F}^{\mathbb{R}}_{M,eK} = \underset{\beta\in\triangle'_+}{\cup} l_\beta = \underset{\beta\in\triangle'_+}{\cup} \beta^{-1}(\hat{c}_\beta)$, the nullity space corresponding to the focal hyperplane

$l_\beta$ is equal to $\mathfrak{p}_\beta$ and $A_v = \lambda_i^v\,\mathrm{id} = \dfrac{\beta(v)}{c_\beta}\,\mathrm{id}$ on $\mathfrak{p}_\beta$.

Set $M' := \exp_{p_0}^{-1}(M)(\subset T_{p_0}(G/K))$. It is clear that $M'$ is included by $S(r_0)$. Also, we can show that $M'$ meets $\widetilde{\Sigma}_x$'s ($x \in M$) orthogonally by calculating the Jacobi vector fields along each radial geodesic starting from $p_0$ and reaching $M$, where we use the fact that the sections $\Sigma_x$'s ($x \in M$) are flat. Assume that $r \geq 3$. Let $L$ be a principal orbit of the s-representation of $G/K$, which is a full irreducible isoparametric submanifold of codimension $r$ in $T_{eK}(G/K)$. It is clear that the same fact as ($\sharp$) holds at any point of $M$ (other than $eK$). Hence it is shown that the above $\mathcal{B}_M := (\Delta_M, \mathcal{A}_M, \mathcal{O})$ essentially coincides with the topological Tits building of spherical type associated to the full irreducible isoparametric submanifold $L$ constructed in [39] by comparing their constructions. Thus $\mathcal{B}_M$ is a topological Tits building of spherical type.

Now we prove Theorem D by using this topological Tits building $\mathcal{B}_M$ of spherical type.

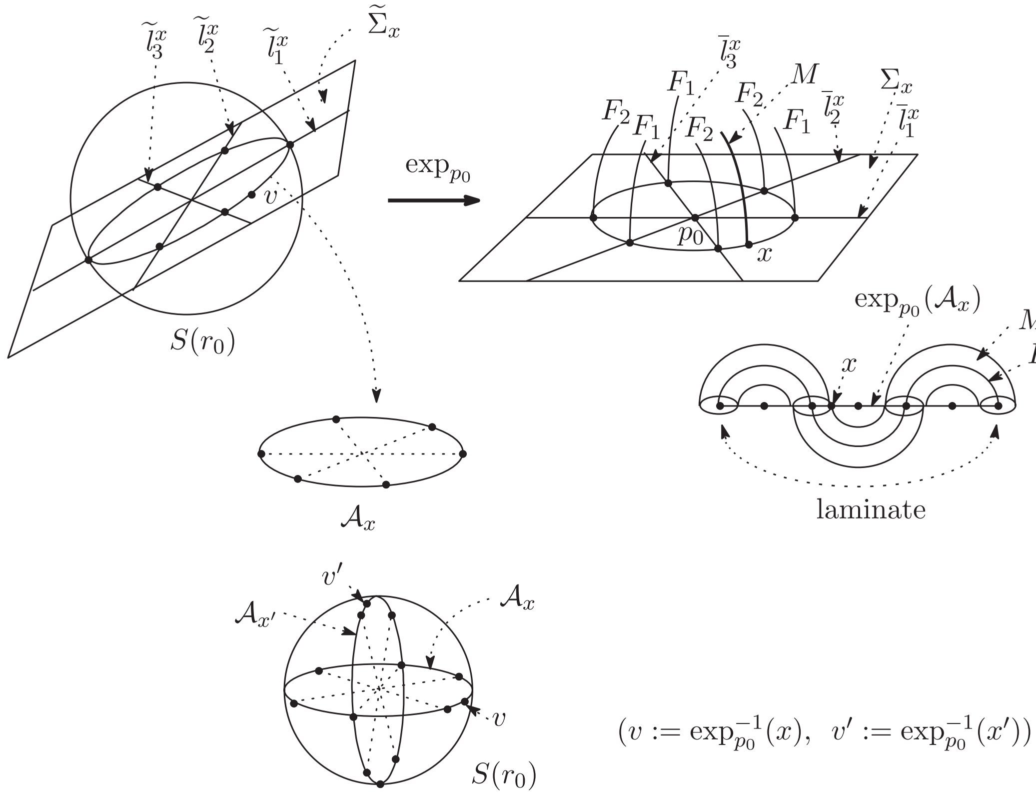



Fig. 6. The topological Tits building of an isoparametric submanifold as in Theorem D

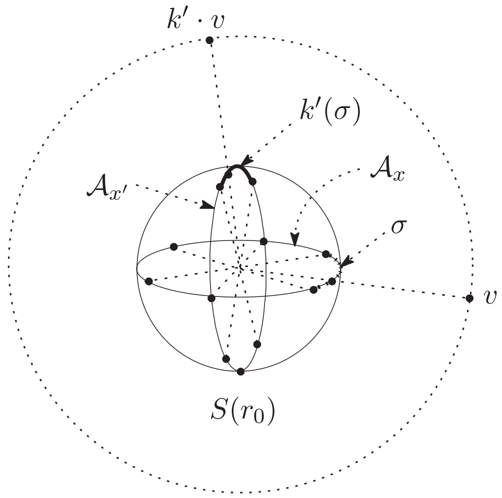



Fig. 7. The action defined by the topological Tits building of an isoparametric submanifold

Proof of Theorem D. Let $G'$ be the topological automorphism group of $\mathcal{B}_M$ and $G'_0$ be its identity component. Then, by the result in [4], it is shown that $G'_0$ is a semi-simple Lie group. Define an involution $s$ of $\mathcal{S}_M$ by $s(\sigma) := \{-p \,|\, p \in \sigma\}$ $(\sigma \in \mathcal{S}_M)$. Let $K'$ be the subgroup consisting of all elements of $G'_0$ commuting with $s$. It is shown that $K'$ is a maximal compact subgroup of $G'_0$. We identify $T_{eK'}(G'_0/K')$ with $T_{p_0}(G/K)$ and denote these by the same symbol $\mathfrak{p}'$. We consider the action of $K'$ on $\mathfrak{p}'$ constructed as in the second paragraph of Section 4 (Page 444) of [39]. That is, we consider the action of $K'$ on $\mathfrak{p}'$ constructed as follows. Take $k' \in K'$ and $v \in \mathfrak{p}'$. Let $\sigma$ be the element of $\mathcal{S}_M$ including $\frac{r_0}{||v||}v$. Let $w(k', v)$ be the element of $k'(\sigma)$ having the same barycentric coordinate as the barycentric coordinate of $\frac{r_0}{||v||}v$ with respect to $\sigma$. We define the $K'$-action on $\mathfrak{p}'$ by

$$k' \cdot v := \frac{||v||}{r_0} w(k', v) \qquad (k' \in K', \ v \in \mathfrak{p}')$$

(see Figure 7). From this construction, it is clear that this action $K' \curvearrowright \mathfrak{p}'$ has $M'$ as its orbit. It is shown that this action is a polar action on $\mathfrak{p}'$ by using the discussion in Pages 444-445 of [39], where we use also the fact that $M'$ meets $\widetilde{\Sigma}_x$'s $(x \in M)$ orthogonally. Hence it follows that this action is orbit equivalent to the $s$-representation of $G'_0/K'$. Furthermore, since the same fact as (♯) holds at any $x \in M$ other than $eK$, this action $K' \curvearrowright \mathfrak{p}'$ is orbit equivalent to the $s$-representation of $G/K$. Therefore $M'$ is a principal orbit of the $s$-representation of $G/K$ and hence $M$ is a principal orbit of the isotropy action $K \curvearrowright G/K$. □

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