



Title	On the kernel of positive definite type
Author(s)	Ninomiya, Nobuyuki
Citation	Osaka Journal of Mathematics. 1980, 17(1), p. 187-198
Version Type	VoR
URL	<a href="https://doi.org/10.18910/7375">https://doi.org/10.18910/7375</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

## ON THE KERNEL OF POSITIVE DEFINITE TYPE

NOBUYUKI NINOMIYA

(Received March 31, 1979)

In a locally compact Hausdorff space, let  $k(P, Q)$  be a real-valued function, continuous for any points  $P$  and  $Q$ , may be  $\infty$  for  $P=Q$  and always finite for  $P \neq Q$ , and  $n(P, Q)$  a real-valued function finite and continuous for any points  $P$  and  $Q$ . A complex-valued function

$$K(P, Q) = k(P, Q) + in(P, Q)$$

is said to be of positive definite type if the double integral (called energy integral)

$$\iint K(P, Q) d\sigma(Q) d\bar{\sigma}(P)$$

of any complex-valued measure  $\sigma$  supported by a relatively compact Borelian set, whenever it is finitely determined, is non-negative. As is well-known, any function  $K(P, Q)$  of positive definite type is symmetric:

$$K(P, Q) = \overline{K(Q, P)} \quad \text{i.e.} \quad k(P, Q) = k(Q, P) \quad \text{and} \quad n(P, Q) = -n(Q, P),$$

and

$$K(P, P) \geq 0 \quad \text{and} \quad |K(P, Q)| \leq \sup K(P, P)$$

for any points  $P$  and  $Q$ . In the real function theory, we see some results which characterize functions of positive definite type. In the present paper, we shall try to characterize functions of positive definite type on the point of view of the potential theory. We shall advance the argument adopting an idea and a method in the previous paper [2].

For any measure  $\alpha$  and  $\beta$  (real-valued or complex-valued) supported by a relatively compact Borelian set, consider the potential taken with respect to a kernel  $K(P, Q)$

$$K(P, \alpha) = \int K(P, Q) d\alpha(Q), \quad K(\alpha, P) = \int K(Q, P) d\alpha(Q)$$

and the double integral (called mutual energy integral)

$$K(\alpha, \beta) = \int d\alpha(P) \int K(P, Q) d\beta(Q).$$

Similarly, we shall consider  $k(P, \alpha)$ ,  $k(\alpha, P)$ ,  $k(\alpha, \beta)$ ,  $n(P, \alpha)$ ,  $n(\alpha, P)$  and  $n(\alpha, \beta)$  with respect to kernels  $k(P, Q)$  and  $n(P, Q)$ . We are going to prove following theorems.

**Theorem 1.** *Suppose that a kernel  $K(P, Q)$  is symmetric. A necessary and sufficient condition that the kernel  $K(P, Q)$  is of positive definite type is that the following property is satisfied:*

[P<sub>1</sub>] *Let  $E_1, E_2, F_1$  and  $F_2$  be compact sets,  $E_1$  and  $E_2$  being disjoint,  $F_1$  and  $F_2$  being disjoint. Let  $\mu_1, \mu_2, \nu_1$  and  $\nu_2$  be positive measures supported by  $E_1, E_2, F_1$  and  $F_2$  respectively and with total mass  $a_1, a_2, b_1$  and  $b_2 (=1)$  respectively. For certain constants  $A$  and  $B$ , if there hold inequalities*

$$k(P, \mu_1) \leq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A$$

*on the support of  $\mu_1$  and*

$$k(P, \nu_1) \leq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B$$

*on the support of  $\nu_1$ , then there holds the inequality*

$$\begin{aligned} k(\mu_2, \mu_1) + k(\nu_2, \nu_1) &\leq k(\mu_2, \mu_2) - n(\nu_1 - \nu_2, \mu_2) \\ &\quad + k(\nu_2, \nu_2) - n(\nu_2, \mu_1 - \mu_2) + a_1 A + b_1 B. \end{aligned}$$

Furthermore, consider the kernel of positive definite type in a stronger form.

**DEFINITION.** A kernel  $K(P, Q)$  of positive definite type is said to satisfy the energy principle, if the double integral (called energy integral)

$$K(\bar{\sigma}, \sigma) = \iint K(P, Q) d\sigma(Q) d\bar{\sigma}(P)$$

of any complex-valued measure  $\sigma$  supported by a relatively compact Borelian set, whenever it is finitely determined, is non-negative and vanishes only when  $\sigma \equiv 0$ .

Then, we have:

**Theorem 2.** *Suppose that a kernel  $K(P, Q)$  is of positive definite type. A necessary and sufficient condition that the kernel satisfies the energy principle is that the following property is satisfied:*

[P<sub>2</sub>] *Let  $E_1, E_2, F_1$  and  $F_2$  be relatively compact Borelian sets,  $E_1$  and  $E_2$  being disjoint,  $F_1$  and  $F_2$  being disjoint. Let  $\mu_1, \mu_2, \nu_1$  and  $\nu_2$  be positive measures supported by  $E_1, E_2, F_1$  and  $F_2$  respectively and with total mass  $a_1, a_2, b_1$  and  $b_2 (=1)$  respectively. For certain constants  $A$  and  $B$ , if there hold inequalities*

$$k(P, \mu_1) \leq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A$$

almost everywhere with respect to  $\mu_1$  and

$$k(P, \nu_1) \leq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B$$

almost everywhere with respect to  $\nu_1$ , then there holds the inequality

$$\begin{aligned} k(\mu_2, \mu_1) + k(\nu_2, \nu_1) &< k(\mu_2, \mu_2) - n(\nu_1 - \nu_2, \mu_2) \\ &+ k(\nu_2, \nu_2) - n(\nu_2, \mu_1 - \mu_2) + a_1 A + b_1 B. \end{aligned}$$

It is the following lemma that is important to prove the theorems. For any complex-valued measure supported by a relatively compact Borelian set

$$\sigma = \mu + i\nu,$$

we have

$$K(\bar{\sigma}, \sigma) = \iint \{k(P, Q) + in(P, Q)\} \{d\mu(Q) + id\nu(Q)\} \{d\mu(P) - id\nu(P)\}.$$

$K(P, Q)$  being symmetric, i.e.

$$k(P, Q) = k(Q, P) \quad \text{and} \quad n(P, Q) = -n(Q, P),$$

we have further

$$K(\bar{\sigma}, \sigma) = k(\mu, \mu) + k(\nu, \nu) + 2n(\nu, \mu).$$

When  $\mu$  is a real-valued measure of variable sign, its positive part  $\mu^+$  and negative part  $\mu^-$  in Hahn's decomposition are positive measures supported by disjoint relatively compact Borelian sets respectively. This is similar with respect to  $\nu$ , too. Hence, for relatively compact Borelian sets  $E_1$ ,  $E_2$ ,  $F_1$  and  $F_2$ ,  $E_1$  and  $E_2$  being disjoint,  $F_1$  and  $F_2$  being disjoint, each of  $E_1$  and  $E_2$  possibly intersecting with  $F_1$  and  $F_2$ , each of  $F_1$  and  $F_2$  possibly intersecting with  $E_1$  and  $E_2$ , and for positive measures  $\mu_1$ ,  $\mu_2$ ,  $\nu_1$  and  $\nu_2$  supported by  $E_1$ ,  $E_2$ ,  $F_1$  and  $F_2$  respectively, consider the quantity

$$\begin{aligned} I(\mu_1, \mu_2; \nu_1, \nu_2) &= k(\mu_1 - \mu_2, \mu_1 - \mu_2) + k(\nu_1 - \nu_2, \nu_1 - \nu_2) + 2n(\nu_1 - \nu_2, \mu_1 - \mu_2) \\ &= k(\mu_1, \mu_1) - 2k(\mu_1, \mu_2) + k(\mu_2, \mu_2) + k(\nu_1, \nu_1) - 2k(\nu_1, \nu_2) \\ &\quad + k(\nu_2, \nu_2) + 2n(\nu_1 - \nu_2, \mu_1 - \mu_2) \end{aligned}$$

We are going to study the minimum of this quantity. We should like to suppose that each of  $E_1$ ,  $E_2$ ,  $F_1$  and  $F_2$  is of  $k$ -transfinite diameter positive<sup>1)</sup>.

**Lemma.** *Let  $E_1$ ,  $E_2$ ,  $F_1$  and  $F_2$  be relatively compact Borelian sets as stated*

1) For a symmetric kernel  $k(P, Q)$ , a compact set is said to be of  $k$ -transfinite diameter positive if it supports a positive measure with total mass 1 of  $k$ -energy integral finite. A Borelian set is said to be of  $k$ -transfinite diameter positive when it contains a compact set of  $k$ -transfinite diameter positive. Concerning the relation between the transfinite diameter and energy integrals, see for instance [1] (p. 45).

above, and  $\mu_1', \mu_2', \nu_1'$  and  $\nu_2'$  positive measures supported by  $E_1, E_2, F_1$  and  $F_2$  respectively and with total mass  $a_1, a_2, b_1$  and  $b_2 (=1)$  respectively. Suppose that, among all the pairs  $(\mu_1', \mu_2', \nu_1', \nu_2')$  of such measures, a pair  $(\mu_1, \mu_2, \nu_1, \nu_2)$  makes minimum of  $I(\mu_1', \mu_2'; \nu_1', \nu_2')$ :

$$-\infty < I(\mu_1, \mu_2; \nu_1, \nu_2) \leq I(\mu_1', \mu_2'; \nu_1', \nu_2').$$

Then, we have

(E<sub>11</sub>)  $A_1 \leq e_1(P)$  on  $E_1$  except for a set of  $k$ -capacity zero<sup>2)</sup>, where

$$e_1(P) = k(P, \mu_1) - k(P, \mu_2) + n(\nu_1 - \nu_2, P)$$

and

$$a_1 A_1 = k(\mu_1, \mu_1) - k(\mu_1, \mu_2) + n(\nu_1 - \nu_2, \mu_1),$$

(E<sub>12</sub>)  $e_1(P) \leq A_1$  on  $E_1$  almost everywhere with respect to  $\mu_1$ . When  $\bar{E}_1$  and  $\bar{E}_2$  are disjoint, the exceptional set of  $k$ -capacity zero might be replaced by of  $k$ -transfinite diameter zero. Similarly, putting

$$e_2(P) = k(P, \mu_2) - k(P, \mu_1) - n(\nu_1 - \nu_2, P),$$

$$a_2 A_2 = \int e_2(P) d\mu_2(P),$$

$$f_1(P) = k(P, \nu_1) - k(P, \nu_2) + n(P, \mu_1 - \mu_2),$$

$$b_1 B_1 = \int f_1(P) d\nu_1(P),$$

$$f_2(P) = k(P, \nu_2) - k(P, \nu_1) - n(P, \mu_1 - \mu_2),$$

and

$$b_2 B_2 = B_2 = \int f_2(P) d\nu_2(P),$$

we have

(E<sub>21</sub>)  $A_2 \leq e_2(P)$  on  $E_2$  except for a set of  $k$ -capacity zero,

(E<sub>22</sub>)  $e_2(P) \leq A_2$  on  $E_2$  almost everywhere with respect to  $\mu_2$ ,

(F<sub>11</sub>)  $B_1 \leq f_1(P)$  on  $F_1$  except for a set of  $k$ -capacity zero,

(F<sub>12</sub>)  $f_1(P) \leq B_1$  on  $F_1$  almost everywhere with respect to  $\nu_1$ ,

(F<sub>21</sub>)  $B_2 \leq f_2(P)$  on  $F_2$  except for a set of  $k$ -capacity zero,

(F<sub>22</sub>)  $f_2(P) \leq B_2$  on  $F_2$  almost everywhere with respect to  $\nu_2$ .

When  $\bar{E}_1$  and  $\bar{E}_2$  are disjoint and  $\bar{F}_1$  and  $\bar{F}_2$  disjoint, the exceptional sets of  $k$ -capacity zero might be replaced by of  $k$ -transfinite diameter zero.

We are going to prove (E<sub>11</sub>) and (E<sub>12</sub>) only. Take any positive measure

2) A compact set is said to be of  $k$ -capacity positive if it supports a positive measure with total mass 1 whose  $k$ -potential is bounded from above on any compact set. A Borelian set is said to be of  $k$ -capacity positive when it contains a compact set of  $k$ -capacity positive. Evidently, if a set is of  $k$ -capacity positive, it is of  $k$ -transfinite diameter positive.

$\alpha_1$  supported by  $E_1$  with total mass  $a_1$  such that  $k(\alpha_1, \alpha_1) < \infty$  and  $k(\alpha_1, \mu_1) < \infty$ . Then, we have for any positive number  $\varepsilon$  smaller than 1 an inequality

$$I(\mu_1, \mu_2; \nu_1, \nu_2) \leq I((1-\varepsilon)\mu_1 + \varepsilon\alpha_1, \mu_2; \nu_1, \nu_2),$$

which, in consideration of the symmetricity of  $k(P, Q)$ , induces an inequality

$$\begin{aligned} (2-\varepsilon)k(\mu_1, \mu_1) - 2k(\mu_1, \mu_2) + 2n(\nu_1 - \nu_2, \mu_1) \\ \leq 2(1-\varepsilon)k(\alpha_1, \mu_1) - 2k(\alpha_1, \mu_2) + 2n(\nu_1 - \nu_2, \alpha_1) + \varepsilon k(\alpha_1, \alpha_1). \end{aligned}$$

Making  $\varepsilon \rightarrow 0$ , we have

$$a_1 A_1 \leq \int e_1(P) d\alpha_1(P),$$

hence

$$A_1 \leq e_1(P)$$

almost everywhere with respect to any positive measures  $\alpha_1$  supported by  $E_1$  with total mass  $a_1$  such that  $k(\alpha_1, \alpha_1) < \infty$  and  $k(\alpha_1, \mu_2) < \infty$ . The inequality naturally holds for  $\alpha_1 = \mu_1$  and there is an evident equality

$$a_1 A_1 = \int e_1(P) d\mu_1(P),$$

so we have

$$e_1(P) \leq A_1$$

almost everywhere with respect to  $\mu_1$ . When  $\bar{E}_1$  and  $\bar{E}_2$  are disjoint, the inequality (E<sub>II</sub>) holds with respect to any positive measure  $\alpha_1$  of  $k$ -energy integral finite supported by  $E_1$  with total mass  $a_1$ . Then, the exceptional set might be considered as of  $k$ -transfinite diameter zero.

REMARK. The proof of Lemma never takes need of the anti-symmetricity of  $n(P, Q)$ .

Proof of Theorem 1. First, we shall prove that if a kernel  $K(P, Q)$  is of positive definite type, it satisfies the property [P<sub>1</sub>]. Let  $E_1, E_2, F_1$  and  $F_2$  be compact sets in the property [P<sub>1</sub>] and  $\mu_1, \mu_2, \nu_1$  and  $\nu_2$  positive measures in the same manner. For certain constants  $A$  and  $B$ , suppose that there hold inequalities

$$k(P, \mu_1) \leq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A$$

on the support of  $\mu_1$  and

$$k(P, \nu_1) \leq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B$$

on the support of  $\nu_1$ . We have naturally inequalities

$$k(\mu_1, \mu_1) - k(\mu_1, \mu_2) + n(\nu_1 - \nu_2, \mu_1) \leq a_1 A$$

and

$$k(\nu_1, \nu_2) - k(\nu_1, \nu_2) + n(\nu_1, \mu_1 - \mu_2) \leq b_1 B.$$

Unless we have the result of the property  $[P_1]$ , we have the inequality

$$a_1 A + b_1 B < k(\mu_2, \mu_1) + k(\nu_2, \nu_1) - k(\mu_2, \mu_2) - k(\nu_2, \nu_2) + n(\nu_1 - \nu_2, \mu_2) + n(\nu_2, \mu_1 - \mu_2).$$

Then, we have from those three inequalities

$$I(\mu_1, \mu_2; \nu_1, \nu_2) < 0,$$

which is a contradiction. Next, we shall prove that if a kernel  $K(P, Q)$  satisfies the property  $[P_1]$ , it is of positive definite type. Let  $E_1, E_2, F_1$  and  $F_2$  be relatively compact Borelian sets,  $E_1$  and  $E_2$  being disjoint,  $F_1$  and  $F_2$  being disjoint. We suppose that each of  $E_1, E_2, F_1$  and  $F_2$  contains a compact set of  $k$ -transfinite diameter positive. We are going to prove

$$I(\mu'_1, \mu'_2; \nu'_1, \nu'_2) \geq 0,$$

whenever it is finitely determined, for any pair  $(\mu'_1, \mu'_2, \nu'_1, \nu'_2)$  of positive measures supported by  $E_1, E_2, F_1$  and  $F_2$  respectively and with total mass  $a_1, a_2, b_1$  and  $b_2 (=1)$  respectively. Consider the case when  $E_1, E_2, F_1$  and  $F_2$  all are compact sets. Then, we have a pair that gives minimum of  $I(\mu'_1, \mu'_2; \nu'_1, \nu'_2)$  among all the pairs of four positive measures stated above. Since, putting

$$I = \inf I(\mu'_1, \mu'_2; \nu'_1, \nu'_2),$$

$I$  is finite and there exists a sequence of pairs of positive measures  $\mu_{1n}, \mu_{2n}, \nu_{1n}$  and  $\nu_{2n}$  supported by  $E_1, E_2, F_1$  and  $F_2$  respectively and with total mass  $a_1, a_2, b_1$  and  $b_2 (=1)$  respectively such that

$$I(\mu_{1n}, \mu_{2n}; \nu_{1n}, \nu_{2n}) \downarrow I.$$

We may suppose that the sequence  $\{\mu_{1n}\}, \{\mu_{2n}\}, \{\nu_{1n}\}$  and  $\{\nu_{2n}\}$  all are vaguely convergent by means of taking suitable sub-sequences out of them. Let  $\mu_1, \mu_2, \nu_1$  and  $\nu_2$  be their limiting measures respectively. They are positive measures supported by  $E_1, E_2, F_1$  and  $F_2$  respectively and with total mass  $a_1, a_2, b_1$  and  $b_2 (=1)$  respectively. The  $k$ -potential of positive measures with compact supports being lower semi-continuous in the whole space, finite and continuous outside of their supports and the  $n$ -potential being finite and continuous everywhere, we have inequalities

$$k(\mu_1, \mu_1) \leq \varliminf_{n \rightarrow \infty} k(\mu_{1n}, \mu_{1n}),$$

$$k(\mu_2, \mu_2) \leq \varliminf_{n \rightarrow \infty} k(\mu_{2n}, \mu_{2n}),$$

$$k(\mu_1, \mu_2) = \lim_{n \rightarrow \infty} k(\mu_{1n}, \mu_{2n}),$$

those similar with respect to  $\nu_1$  and  $\nu_2$  and further an equality

$$n(\nu_1 - \nu_2, \mu_1 - \mu_2) = \lim_{n \rightarrow \infty} n(\nu_{1n} - \nu_{2n}, \mu_{1n} - \mu_{2n}).$$

So, we have

$$I \leq I(\mu_1, \mu_2; \nu_1, \nu_2) \leq \lim_{n \rightarrow \infty} I(\mu_{1n}, \mu_{2n}; \nu_{1n}, \nu_{2n}) = I$$

Then, by Lemma we have inequalities

$$k(P, \mu_1) \leq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A_1$$

on the support of  $\mu_1$  and

$$k(p, \nu_1) \leq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B_1$$

on the support of  $\nu_1$ . Therefore, we have an inequality

$$\begin{aligned} k(\mu_2, \mu_1) + k(\nu_2, \nu_1) - k(\mu_2, \mu_2) + n(\nu_1 - \nu_2, \mu_2) - k(\nu_2, \nu_2) \\ + n(\nu_2, \mu_1 - \mu_2) \leq a_1 A_1 + b_1 B_1, \end{aligned}$$

similarly, an inequality

$$\begin{aligned} k(\mu_1, \mu_2) + k(\nu_1, \nu_2) - k(\mu_1, \mu_1) - n(\nu_1 - \nu_2, \mu_1) - k(\nu_1, \nu_1) \\ - n(\nu_1, \mu_1 - \mu_2) \leq a_2 A_2 + b_2 B_2. \end{aligned}$$

Thus, we have the inequality looking for:

$$0 \leq 2(a_1 A_1 + b_1 B_1 + a_2 A_2 + b_2 B_2) = 2I(\mu_1, \mu_2; \nu_1, \nu_2).$$

Finally, suppose that some of  $E_1$ ,  $E_2$ ,  $F_1$  and  $F_2$  are not compacts. Let  $\mu_1$ ,  $\mu_2$ ,  $\nu_1$  and  $\nu_2$  be positive measures supported by  $E_1$ ,  $E_2$ ,  $F_1$  and  $F_2$  respectively and with total mass  $a_1$ ,  $a_2$ ,  $b_1$  and  $b_2 (=1)$  respectively. Then, taking sequences of compact sets  $\{E_{1n}\}$ ,  $\{E_{2n}\}$ ,  $\{F_{1n}\}$  and  $\{F_{2n}\}$  such that

$$\begin{aligned} E_{11} \subset E_{12} \subset \dots \subset E_{1n} \subset \dots \subset E_1, & \quad \mu_1(E_{1n}) \uparrow a_1, \\ E_{21} \subset E_{22} \subset \dots \subset E_{2n} \subset \dots \subset E_2, & \quad \mu_2(E_{2n}) \uparrow a_2, \\ F_{11} \subset F_{12} \subset \dots \subset F_{1n} \subset \dots \subset F_1, & \quad \nu_1(F_{1n}) \uparrow b_1, \\ F_{21} \subset F_{22} \subset \dots \subset F_{2n} \subset \dots \subset F_2, & \quad \nu_2(F_{2n}) \uparrow b_2 (=1), \end{aligned}$$

and taking restrictions  $\mu_{1n}$ ,  $\mu_{2n}$ ,  $\nu_{1n}$  and  $\nu_{2n}$  of  $\mu_1$ ,  $\mu_2$ ,  $\nu_1$  and  $\nu_2$  to  $E_{1n}$ ,  $E_{2n}$ ,  $F_{1n}$  and  $F_{2n}$  respectively, we have

$$I(\mu_{1n}, \mu_{2n}; \nu_{1n}, \nu_{2n}) \geq 0,$$

making  $n \rightarrow \infty$



$$I(\mu_1, \mu_2; \nu_1, \nu_2) \geq 0.$$

Proof of Theorem 2. First, we shall prove that if a kernel  $K(P, Q)$  is of positive definite type and satisfies the energy principle, it satisfies the property  $[P_2]$ . Let  $E_1, E_2, F_1$  and  $F_2$  be relatively compact sets in the property  $[P_2]$  and  $\mu_1, \mu_2, \nu_1$  and  $\nu_2$  positive measures in the same manner. For certain constants  $A$  and  $B$ , suppose that there hold inequalities

$$k(P, \mu_1) \leq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A$$

almost everywhere with respect to  $\mu_1$  and

$$k(P, \nu_1) \leq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B$$

almost everywhere with respect to  $\nu_1$ . We have naturally inequalities

$$k(\mu_1, \mu_1) - k(\mu_1, \mu_2) + n(\nu_1 - \nu_2, \mu_1) \leq a_1 A$$

and

$$k(\nu_1, \nu_1) - k(\nu_1, \nu_2) + n(\nu_1, \mu_1 - \mu_2) \leq b_1 B.$$

Unless we have the result of the property  $[P_2]$ , we have the inequality

$$a_1 A + b_1 B \leq k(\mu_2, \mu_1) + k(\nu_2, \nu_1) - k(\mu_2, \mu_2) + n(\nu_1 - \nu_2, \mu_2) - k(\nu_2, \nu_2) + n(\nu_2, \mu_1 - \mu_2).$$

Then, we have from those three inequalities

$$I(\mu_1, \mu_2; \nu_1, \nu_2) \leq 0,$$

which is a contradiction. Next, we shall prove that, if a kernel  $K(P, Q)$  is of positive definite type and satisfies the property  $[P_2]$ , it satisfies the energy integral. Let  $E_1, E_2, F_1$  and  $F_2$  be relatively compact sets,  $E_1$  and  $E_2$  being disjoint,  $F_1$  and  $F_2$  being disjoint. Let  $\mu_1, \mu_2, \nu_1$  and  $\nu_2$  be positive measures supported by  $F_1, E_2, F_1$  and  $F_2$  respectively and with total mass  $a_1, a_2, b_1$  and  $b_2 (=1)$  respectively. We are going to prove

$$I(\mu_1, \mu_2; \nu_1, \nu_2) > 0$$

whenever it is finitely determined. If

$$I(\mu_1, \mu_2; \nu_1, \nu_2) = 0,$$

the pair  $(\mu_1, \mu_2, \nu_1, \nu_2)$  is a minimal pair in Lemma and there hold inequalities

$$k(P, \mu_1) \leq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A_1$$

almost everywhere with respect to  $\mu_1$  and

$$k(P, \nu_1) \leq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B_1$$

almost everywhere with respect to  $\nu_1$ . Then, by the property  $[P_2]$  we have an inequality

$$k(\mu_2, \mu_1) + k(\nu_2, \nu_1) < k(\mu_2, \mu_2) - n(\nu_1 - \nu_2, \mu_2) + k(\nu_2, \nu_2) \\ - n(\nu_2, \mu_1 - \mu_2) + a_1 A_1 + b_1 B_1.$$

Accordingly, we have

$$I(\mu_1, \mu_2; \nu_1, \nu_2) > 0,$$

which is a contradiction.

Let us consider the kernel of positive definite type in a weaker form.

DEFINITION. A complex-valued function

$$K(P, Q) = k(P, Q) + in(P, Q)$$

is said to be of positive definite type in restricted sense, if the double integral (called energy integral)

$$\iint K(P, Q) d\sigma(Q) d\bar{\sigma}(P)$$

of any complex-valued measure  $\sigma$  supported by a relatively compact Borelian set with total mass 0, whenever it is finitely determined, is non-negative.

DEFINITION. A kernel  $K(P, Q)$  of positive definite type in restricted sense is said to satisfy the energy principle, if the double integral (called energy integral)

$$\iint K(P, Q) d\sigma(Q) d\bar{\sigma}(P)$$

of any complex-valued measure  $\sigma$  supported by a relatively compact Borelian set with total mass 0, whenever it is finitely determined, is non-negative and vanishes only when  $\sigma \equiv 0$ .

For a kernel of positive definite type in restricted sense, Theorems 1 and 2 are expressed in the following styles.

**Theorem 1'.** Suppose that a kernel  $K(P, Q)$  is symmetric. A necessary and sufficient condition that the kernel  $K(P, Q)$  is of positive definite type in restricted sense is that the following property is satisfied:

$[P'_1]$  Let  $E_1, E_2, F_1$  and  $F_2$  be compact sets,  $E_1$  and  $E_2$  being disjoint,  $F_1$  and  $F_2$  being disjoint. Let  $\mu_1$  and  $\mu_2$  be positive measures supported by  $E_1$  and  $E_2$  with total mass  $a$  ( $> 0$ ) respectively and  $\nu_1$  and  $\nu_2$  positive measures supported by  $F_1$  and  $F_2$  with total mass 1 respectively. For certain constants  $A$  and  $B$ , if there hold inequalities

$$k(P, \mu_1) \leq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A$$

on the support of  $\mu_1$  and

$$k(P, \nu_1) \leq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B$$

on the support of  $\nu_1$ , then there holds the inequality

$$\begin{aligned} k(\mu_2, \mu_1) + k(\nu_2, \nu_1) &\leq k(\mu_2, \mu_2) - n(\nu_1 - \nu_1, \mu_2) \\ &\quad + k(\nu_2, \nu_2) - n(\nu_2, \mu_1 - \mu_2) + aA + B. \end{aligned}$$

**Theorem 2'.** Suppose that a kernel  $K(P, Q)$  is of positive definite type in restricted sense. A necessary and sufficient condition that the kernel  $K(P, Q)$  satisfies the energy principle is that the following property is satisfied:

[P<sub>2'</sub>] Let  $E_1, E_2, F_1$  and  $F_2$  be relatively compact Borelian sets,  $E_1$  and  $E_2$  being disjoint,  $F_1$  and  $F_2$  being disjoint. Let  $\mu_1$  and  $\mu_2$  be positive measures supported by  $E_1$  and  $E_2$  with total mass  $a$  ( $>0$ ) respectively and  $\nu_1$  and  $\nu_2$  positive measures supported by  $F_1$  and  $F_2$  with total mass 1 respectively. For certain constants  $A$  and  $B$ , if there hold inequalities

$$k(P, \mu_1) \leq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A$$

almost everywhere with respect to  $\mu_1$  and

$$k(P, \nu_1) \leq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B$$

almost everywhere with respect to  $\nu_1$ , then there holds the inequality

$$\begin{aligned} k(\mu_2, \mu_1) + k(\nu_2, \nu_1) &< k(\mu_2, \mu_2) - n(\nu_1 - \nu_2, \mu_2) \\ &\quad + k(\nu_2, \nu_2) - n(\nu_2, \mu_1 - \mu_2) + aA + B. \end{aligned}$$

**Corollary.** A kernel  $K(P, Q)$ , which is symmetric, is of positive definite type in restricted sense if the following property is satisfied:

[P\*] Let  $E_1, E_2, F_1$  and  $F_2$  be relatively compact Borelian sets,  $E_1$  and  $E_2$  being disjoint,  $F_1$  and  $F_2$  being disjoint. Let  $\mu_1$  and  $\mu_2$  be positive measures supported by  $E_1$  and  $E_2$  with total mass  $a$  ( $>0$ ) respectively and  $\nu_1$  and  $\nu_2$  positive measures supported by  $F_1$  and  $F_2$  with total mass 1 respectively. For certain constants  $A$  and  $B$ , if there hold inequalities

$$k(P, \mu_1) \leq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A$$

on the support of  $\mu_1$  and

$$k(P, \nu_1) \leq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B$$

on the support of  $\nu_1$ , then these two inequalities hold at the same time in the whole space.

Using this corollary, we should like to terminate the paper presenting a simple example of a kernel satisfying the above property [P\*], therefore of positive definite type in restricted sense.

EXAMPLE. On the plane  $R^2$ , let

$$P = (x_1, x_2), \quad Q = (y_1, y_2),$$

$$k(P, Q) = \log \frac{1}{PQ} = \log \frac{1}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}}$$

and

$$n(P, Q) = c_1(x_1 - y_1) + c_2(x_2 - y_2),$$

$c_1$  and  $c_2$  being any real constants. A complex-valued function, which is symmetric,

$$K(P, Q) = k(P, Q) + in(P, Q)$$

is of positive definite type in restricted sense.

In fact, let  $E_1, E_2, F_1$  and  $F_2$  be compact sets,  $E_1$  and  $E_2$  being disjoint,  $F_1$  and  $F_2$  being disjoint. Let  $\mu_1$  and  $\mu_2$  be positive measures with total mass  $a(>0)$  whose supports are  $E_1$  and  $E_2$  respectively, and  $\nu_1$  and  $\nu_2$  be positive measures with total mass 1 whose supports are  $F_1$  and  $F_2$  respectively. For certain constants  $A$  and  $B$ , suppose that there hold inequalities

$$k(P, \mu_1) \leq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A$$

on  $E_1$  and

$$k(P, \nu_1) \leq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B$$

on  $E_2$ . We are going to prove that the former inequality holds everywhere. Then, we shall see that the latter one holds everywhere, too.  $E_1$  being a compact set of logarithmic capacity positive, let  $\lambda$  be the equilibrium measure on  $E_1$ . That is,  $\lambda$  is a positive measure supported by  $E_1$  with total mass 1 such that

$$k(P, \lambda) = \int \log \frac{1}{PQ} d\lambda(Q)$$

is equal to a constant  $V$  on  $E_1$  except for a set of logarithmic capacity zero, and  $\leq V$  everywhere. The logarithmic potential of a positive measures with compact support is superharmonic in the whole plane and harmonic in each component outside of the support, and the  $n$ -potential is harmonic in the whole plane. Then,  $\varepsilon$  being any positive number, consider the function

$$g(P) = k(P, \mu_1) + \varepsilon k(P, \lambda) - k(P, \mu_2) + n(\nu_1 - \nu_2, P) - (A + \varepsilon V).$$

This is subharmonic in each component outside of  $E_1$ , and we have at each boundary point  $M$  of  $E_1$

$$\overline{\lim}_{P \rightarrow M} g(P) \leq \overline{\lim}_{P' \rightarrow M} g(P') \leq 0^3),$$

$P$  being outside of  $E_1$  and  $P'$  being in  $E_1$ . We should like to prove

$$\lim_{P \rightarrow \infty} g(P) = -\infty.$$

First, we have

$$\lim_{P \rightarrow \infty} k(P, \lambda) = -\infty.$$

Next, we have, 0 denoting the origin,

$$\lim_{P \rightarrow \infty} \{k(P, \mu_1) - k(P, \mu_2)\} = \lim_{P \rightarrow \infty} \left\{ \log \frac{PO}{PQ} d\mu_1(Q) - \int \log \frac{PO}{PQ} d\mu_2(Q) \right\} = 0$$

Finally, let us notice that  $n(\nu_1 - \nu_2, P)$  is bounded. Since, taking the Dirac measure  $\varepsilon$  at the origin 0, we have

$$\begin{aligned} |n(\nu_1 - \nu_2, P)| &= |n(P, \nu_1) - n(P, \nu_2)| \leq |n(P, \nu_1) - n(P, \varepsilon)| + |n(P, \nu_2) - n(P, \varepsilon)| \\ &\leq \int \{|c_1 y_1| + |c_2 y_2|\} \{d\nu_1(Q) + d\nu_2(Q)\}. \end{aligned}$$

So, we have

$$g(P) \leq 0$$

in each component outside of  $E_1$ . Making  $\varepsilon \rightarrow 0$ , we have

$$k(P, \mu_1) \leq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A$$

everywhere.

QUESTION. A kernel  $K(P, Q)$  which is of positive definite type in restricted sense but is not symmetric, does it exist ?

### References

- [1] O. Frostman: *Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions*, Medd. Lunds Univ. Mat. Sem. **3** (1935), 1–118.
- [2] N. Ninomiya: *On the potential taken with respect to complex-valued and symmetric kernels*, Osaka J. Math. **9** (1972), 1–9.

Department of Mathematics, Osaka City University  
Sugimoto-cho, Sumiyoshi-ku, Osaka 558, Japan

3) For a kernel  $k(P, Q)$  logarithmic on  $R^2$ , Newtonian in  $R^3$  or generally satisfying the maximum principle of Frostman, the potential  $k(P, \mu)$  of a positive measure  $\mu$  with compact support satisfies at each boundary point  $M$  of the support of  $\mu$  an inequality

$$\overline{\lim}_{P \rightarrow M} k(P, \mu) \leq \overline{\lim}_{P' \rightarrow M} k(P', \mu),$$

$P$  being outside of the support of  $\mu$  and  $P'$  being on the support of  $\mu$  (cf.: [1], p. 69).