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CLASSIFYING BOUNDED 2-MANIFOLDS IN S^4

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Noguchi has shown that if M_1 and M_2 are closed orientable 2-manifolds in S^4 having the same local oriented knot types, then they are *isoneighboring*, that is, for regular neighborhoods N_1 and N_2 in S^4 of M_1 and M_2 , respectively, there is a homeomorphism of N_1 onto N_2 carrying M_1 onto M_2 [5]. In a later paper, Noguchi showed that one may replace S^4 by an orientable 4-manifold, if one adds the restriction that M_1 and M_2 have the same Stiefel-Whitney numbers [6]. In this paper we show that if M_1 definitely has nonempty boundary (in each of its components), one may drop the orientability requirement (the Stiefel-Whitney numbers are, of course, zero), and obtain the much stronger conclusion that one may ambiently isotope M_2 onto M_1 . The starting point for our proof is the case $N=S^4$ and M a 2-cell, proved by Gugenheim in 1953 [2]. We work throughout in the piecewise linear (*PL*) category, and assume the reader familiar with the elements of *PL* topology. We will also use results of Hudson and Zeeman: the isotopy extension theorem [3], and the theory of relative regular neighborhoods [4], [1].

A *PL* imbedding f of a *PL* manifold M into the interior of another *PL* manifold N is *locally knotted* at $x \in M$ if there is a triangulation J, L of $N, f(M)$ having $f(x)$ as a vertex and such that the ball or sphere $f(lk(x), L)$ is knotted in the sphere $lk(f(x), J)$. The pair $(lk(f(x), J), f(lk(x), L))$ is called the *local knot type* of f at x , and denoted by $\Sigma_f(x)$. If M and N are both orientable, we fix orientations for both, and in this case local knot type is to be understood to mean *oriented* local knot type. It is apparent that an imbedding of a compact 2-manifold into the interior of a 4-manifold can be locally knotted at only finitely many points, all of which are interior points. We say that imbeddings $f_1, f_2: M^2 \rightarrow N^4$ are *locally equivalent* if we may list the local knotting points x_1, x_2, \dots, x_n of f_1 and y_1, \dots, y_n of f_2 in such a way that the local knot type of f_1 at x_i is the same as that of f_2 at y_i , $i=1, 2, \dots, n$. Concealed in Gugenheim's second 1953 paper is the following:

* The work in this paper was carried out in 1966 at Florida State University, where the author was supported by National Science Foundation Grant GP 5458.

Theorem 1. *If D_1 and D_2 are locally equivalently imbedded 2-cells in S^4 , there is a homeomorphism $h : S^4 \rightarrow S^4$ such that $h(D_2) = D_1$.*

Corollary. *If $f_1, f_2 : D^2 \rightarrow \text{int } N^4$ are locally equivalent imbeddings of a 2-cell into a 4-manifold, there is a homeomorphism $h : N^4 \rightarrow N^4$ such that $h(f_2(D)) = f_1(D)$. Moreover h is isotopic to the identity rel ∂N (that is, the isotopy restricts to 1 on ∂N).*

Proof. Let Q_i be a regular neighborhood of $f_i(D)$ in $\text{int } N$ ($i=1, 2$). Then Q_1 and Q_2 are 4-balls in the interior of N , so by Newman's Theorem, we may ambiently isotop (rel ∂N) Q_2 onto Q_1 ; thus we may (and do) assume $Q_1 = Q_2 = Q$. Now we have $f_1(D)$ and $f_2(D)$ both lying inside a 4-ball Q , and by Theorem 1, there will be a homeomorphism $h' : Q \rightarrow Q$ such that $h'|_{\partial Q} = 1$ and $h'f_2(D) = f_1(D)$. Extend h' to $\bar{h} : N \rightarrow N$ by $\bar{h}|_{N-Q} = 1$. Now h' is isotopic to the identity rel ∂Q and we extend this isotopy to all of N by the identity outside of Q , showing that \bar{h} is isotopic (rel ∂N) to the identity.

Our main result is the following:

Theorem 2. *If M_1 and M_2 are homeomorphic 2-manifolds with nonempty boundary, locally equivalently imbedded in the interior of a simply connected 4-manifold N , then there is an ambient isotopy of N rel ∂N carrying M_2 onto M_1 .*

Proof. We give the proof in the case where M is connected (M_1 and M_2 homeomorphic to M); in the other cases of course, one must assume that each component of M has nonempty boundary, but the proof is essentially the same. We also note that if one assumes M_1 and M_2 to be homotopic in N , one need make no connectivity assumptions on N (N may, in fact, be non-orientable); however, the extra technical detail involved does not seem worth the gain.

By the classification of two manifolds, a bounded 2-manifold can be written as a 2-cell D with (possibly twisted) handles H_1, H_2, \dots, H_m attached; let α_i be the indexing arc of H_i , let $\partial\alpha_i = \alpha_i \cap D = \alpha_i \cap \partial D = \{a_{i1}, a_{i2}\}$, and let $H_i \cap D = H_i \cap \partial D = \beta_{i1} \cup \beta_{i2}$, where β_{ij} are disjoint arcs with $a_{ij} \in \text{int } \beta_{ij}$ (see fig. 1). Now we may choose homeomorphisms $f_i : M \rightarrow M_i$ such that all the local knotting occurs at points of the interior of D . The proof of the theorem is by induction on the number n of handles; for $n=0$, we appeal to the corollary to Theorem 1; at this stage, by redefining the imbeddings (but not altering the images), we may assume $f_1|_D = f_2|_D$. Thus let us assume that all but a single handle H with indexing arc α have been "unknotted" (i.e., $f_1|M-H = f_2|M-H$). Now let \bar{N} be a regular neighborhood of $f_1(Cl(M-H))$ mod its boundary, and let $N' = Cl(N - \bar{N})$. Thus we have the two different imbeddings of H dangling inside N' , attached to its boundary in the same two arcs $f_1(\beta_1)$ and $f_1(\beta_2)$ and $f_2(\beta_1)$ and $f_2(\beta_2)$. If we can ambiently isotop $f_1|_H$ to $f_2|_H$ in N' rel $\partial N'$, we could extend this isotopy to all of N by setting it equal to the identity on \bar{N} (hence

leaving fixed what we had already unknotted), and the proof would be complete. First unknot the indexing arcs; one can do this since $f_1|_\alpha$ and $f_2|_\alpha$ are homotopic rel the end points (since N is simply connected) and α lies in the trivial range (i.e., $2(1)+2 \leq 4$).

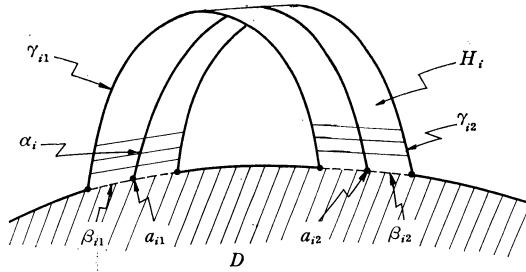


Fig. 1 A twisted handle

We denote by γ_1, γ_2 the complementary arcs to β_1, β_2 in ∂H ; i.e., $Cl(\partial H - \beta_1 - \beta_2) = \gamma_1 \cup \gamma_2$ (see fig. 1). Let us triangulate N' by a complex J having subcomplexes covering $f_1(H)$ and $f_2(H)$, and take the following relative second derived neighborhoods: $Q_i = N(f_i(H) - f_i(\gamma_1 \cup \gamma_2), J')$ ($i=1, 2$). Now Q_1 and Q_2 are also regular neighborhoods of $f_1(\alpha)$ which intersect $\partial N'$ in the same set (namely a regular neighborhood of $f_1(\beta_1 \cup \beta_2)$ mod $f_1(\partial \beta_1 \cup \partial \beta_2)$ in $\partial N'$) and hence we may carry Q_2 onto Q_1 by an ambient isotopy rel $\partial N'$; so we may as well assume that $Q_2 = Q_1 = Q$. Also $(Q, f_1(H))$ and $(Q, f_2(H))$ are locally unknotted proper ball pairs with the big ball collapsing to the smaller one, and hence are unknotted pairs (see [4]). We need to define a homeomorphism of Q onto itself carrying f_2 onto f_1 , which is the identity on $Q \cap \partial N$ and which extends into the rest of N' so as to be isotopic rel $\partial N'$ to the identity. This could all be done if we had an isotopy h_t of $P = Cl(\partial Q - \partial N')$ onto itself which started at a homeomorphism h_0 carrying $f_2|_{(\gamma_1 \cup \gamma_2)}$ to $f_1|_{(\gamma_1 \cup \gamma_2)}$, ended at the identity ($h_1=1$), and stayed the identity on $\partial P = P \cap \partial N'$ at all times. To see why this would do the trick consider the following: extend h_0 to all of ∂Q by setting it the identity on $Q \cap \partial N'$; then extend this to a homeomorphism of Q onto itself carrying f_2 to f_1 by using lemma 18 of [8]. Now P is collared in $Cl(N' - Q)$, and we use the isotopy h_t to extend the homeomorphism of Q to a homeomorphism of the ball $Q' = Q \cup \text{collar}$ on P (see fig. 2) which is the identity on $\partial Q'$ and extend outside Q' by the identity. The homeomorphism of Q' is isotopic rel $\partial Q'$ to the identity and the isotopy may be extended to an isotopy of N' rel $\partial N'$ by setting it equal to the identity on $N' - Q'$. Thus the only thing missing is the construction of the isotopy h_t of P rel ∂P .

A moments reflection will show that we have the following situation: two different unknotted 1-spheres ($f_i(\partial H)$) in the three sphere (∂Q) which agree on a pair of subarcs β_1 and β_2 ; two 3-balls $B_1, B_2 (Q \cap \partial N = B_1 \cup B_2)$ containing

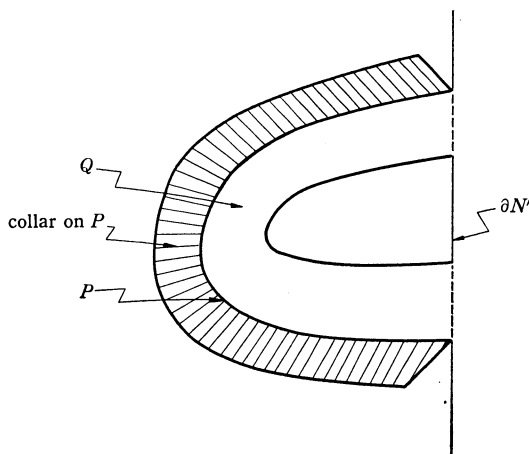


Fig. 2

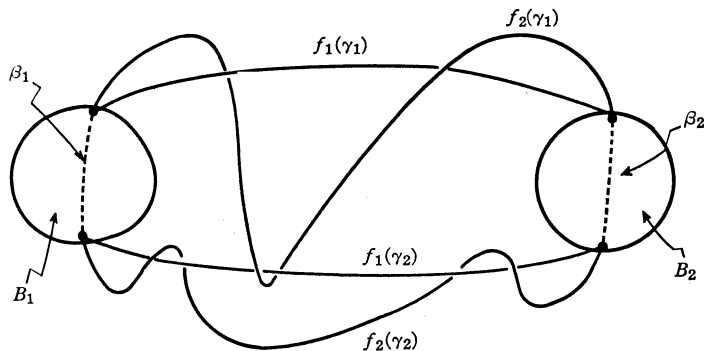


Fig. 3

β_1, β_2 (respectively) such that B_i intersects both 1-spheres in β_i only (see fig. 3). P thus is an annulus, $P = Cl(S^3 - B_1 - B_2)$. The classical method of showing that any arc is unknotted keeping its end points fixed shows that we may isotop $f_2|_{\gamma_1}$ to $f_1|_{\gamma_1}$ by an isotopy of $S^3 (= \partial Q)$ which is the identity at all times on $B_1 \cup B_2 (= Q \cap \partial N')$, so this isotopy restricts to an isotopy of $P \text{ rel } \partial P$. Thus we may assume that $f_2|_{\gamma_1} = f_1|_{\gamma_1}$. To unknot $f_2|_{\gamma_2}$, proceed as follows: let R be a regular neighborhood of $f_1(\gamma_1)$ missing both $f_1(\gamma_2)$ and $f_2(\gamma_2)$. Then if we let $W = Cl(P - R)$, we see that $(W, f_1(\gamma_2))$ and $(W, f_2(\gamma_2))$ are both 3, 1 ball pair subsets of an unknotted 3, 1 sphere pair, and hence are unknotted. Moreover, they have the same boundary pairs, so we can isotop $f_2|_{\gamma_2}$ to $f_1|_{\gamma_1}$ in W by an ambient isotopy of $W \text{ rel } \partial W$, and if we extend this isotopy to all of P by setting it equal to the identity on $R = Cl(P - W)$, we will have ambient isotoped $f_2|_{\gamma_2}$ to $f_1|_{\gamma_1}$ in $P \text{ rel } \partial P$ without moving $f_1(\gamma_1)$. Hence we have

ambient isotoped $f_2|_{\gamma_1 \cup \gamma_2}$ to $f_2|_{\gamma_1 \cup \gamma_2}$ rel ∂P , completing the proof of Theorem 2.

REMARK. The techniques used in this paper have been used by the author to prove the following: homotopic imbeddings of a manifold M^m in the interior of another manifold N^n are ambient isotopic (rel ∂N) if M has a spine of dimension $p < n - m$ and $n - m \geq 3$ [7]. Also, there is no new difficulty to extending the results of the present paper to 1-flat imbeddings of balls with (index 1) handles in codimension 2.

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