



Title	A note on a result of Kamae
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Citation	Osaka Journal of Mathematics. 1975, 12(2), p. 283-284
Version Type	VoR
URL	<a href="https://doi.org/10.18910/7387">https://doi.org/10.18910/7387</a>
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## A NOTE ON A RESULT OF KAMAE\*

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(Received August 12, 1974)

In [3] Kamae proved that for each constant  $c$ , there exists a finite string  $y$  such that

$$K(y) - K(y|x) > c$$

for all but finitely many finite strings  $x$ , where  $K(\cdot)$  and  $K(\cdot | \cdot)$  are the unconditional and conditional minimal-program complexity measures respectively of Kolmogorov [4]. By considering infinite sequences we are able to obtain a slightly stronger statement of this result.

Let  $X^\infty$  denote the set of all infinite binary strings. For  $x \in X^\infty$  let  $x^n$  denote the initial segment of  $x$  of length  $n$ , i.e.,  $x^n = x(1) \dots x(n)$ . To simplify matters we will associate with each finite binary string  $y$  the integer  $n$  whose binary representation is  $1y$ . By this means we will consider complexity expressions of the form  $K(x^n)$ ,  $K(x^n|m)$  and  $K(m)$  for  $x \in X^\infty$  and integers  $n$  and  $m$ . Then  $K(x^n|n)$  is Kolmogorov's restricted conditional complexity. By a recursively enumerable sequence we mean the characteristic sequence of a recursively enumerable set. By " $\exists n$ " and " $\forall n$ " we mean respectively "There exist infinitely many integers  $n$ " and "For all but finitely many integers  $n$ ."

**Theorem.** *There exists a recursively enumerable sequence  $x$ , such that*

$$\forall c \exists n \forall m. K(x^n|n) - K(x^n|m) > c.$$

*Proof.* We need the following two lemmas.

**Lemma.** *For every recursively enumerable sequence  $x$ ,*

$$\exists c_1 \forall n \forall m. K(x^n|m) \leq K(n) + c_1.$$

*Proof.* Let  $h$  be a total recursive function which enumerates the 1's of  $x$ , i.e.,  $x(i) = 1 \Leftrightarrow \exists j. h(j) = i$ . Define the total recursive function  $f(n, m)$  as follows:  
 Step 1: Enumerate via  $h$  the first  $m$  1's of the sequence  $x$ , i.e. compute  $h(1), \dots, h(m)$ .  
 Step 2: Output the following finite string  $y$  of length  $n$  —  $y(i) = 1 \Leftrightarrow i$  appears on

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\* This work was supported by NSF Grant GJ-43223.

the list generated in Step 1, i.e.  $\exists j \leq m. h(j) = i$ .

Using  $f$  and the asymptotically optimal algorithm for  $K(\cdot | \cdot)$  the result follows.

**Lemma.** *There exists a recursively enumerable sequence  $x$  such that  $\forall n. K(x^{2^n} | 2^n) > n - 1$ .*

**Proof.** Let  $A$  be the asymptotically optimal algorithm for  $K(\cdot | \cdot)$  and define the sequence  $x$  as follows: For each  $n$  and each  $m$  such that  $1 \leq m \leq 2^{n-1}$ ,  $x(2^{n-1} + m) = 1 \Leftrightarrow m^{\text{th}}$  digit of  $A(m, 2^n)$  is 0. Clearly  $x$  is recursively enumerable and  $x^{2^n} \neq A(y, 2^n)$  for all programs  $y$  of length  $\leq n - 1$ .

Combining these two lemmas we have that there exists a recursively enumerable sequence  $x$  such that

$$\exists c_1 \forall n \tilde{\forall} m. K(x^{2^n} | 2^n) - K(x^{2^n} | m) > n - 1 - c_1 - K(2^n).$$

While it is true that  $K(2^n)$  for most integers of the form  $2^n$  is about  $n$ , there are infinitely many such integers which have descriptions of arbitrarily (in an effective sense) short length relative to  $n$ , i.e.  $\forall c \exists n. n - K(2^n) > c$ . (see [1]) Combining these two inequalities yields the result. Q.E.D.

Recently, Chaitin (see [2]) has shown that a sequence  $x$  is recursive  $\Leftrightarrow \exists c \forall n. K(x^n) \leq K(n) + c$ . Combining this result with the first lemma above we have immediately

**Corollary.** *For every non-recursive recursively enumerable sequence  $x$ ,*

$$\forall c \exists n \tilde{\forall} m. K(x^n) - K(x^n | m) > c.$$

However, the above theorem is the best result obtainable for any recursively enumerable sequence in view of the following result which is proved in [1].

**Theorem.** *For every recursively enumerable sequence  $x$*

$$\exists c \exists n. K(x^n | n) \leq c.$$

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