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A NOTE ON A RESULT OF KAMAE*

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In [3] Kamae proved that for each constant c , there exists a finite string y such that

$$K(y) - K(y|x) > c$$

for all but finitely many finite strings x , where $K(\cdot)$ and $K(\cdot | \cdot)$ are the unconditional and conditional minimal-program complexity measures respectively of Kolmogorov [4]. By considering infinite sequences we are able to obtain a slightly stronger statement of this result.

Let X^∞ denote the set of all infinite binary strings. For $x \in X^\infty$ let x^n denote the initial segment of x of length n , i.e., $x^n = x(1) \dots x(n)$. To simplify matters we will associate with each finite binary string y the integer n whose binary representation is $1y$. By this means we will consider complexity expressions of the form $K(x^n)$, $K(x^n|m)$ and $K(m)$ for $x \in X^\infty$ and integers n and m . Then $K(x^n|n)$ is Kolmogorov's restricted conditional complexity. By a recursively enumerable sequence we mean the characteristic sequence of a recursively enumerable set. By " $\exists n$ " and " $\forall n$ " we mean respectively "There exist infinitely many integers n " and "For all but finitely many integers n ."

Theorem. *There exists a recursively enumerable sequence x , such that*

$$\forall c \exists n \forall m. K(x^n|n) - K(x^n|m) > c.$$

Proof. We need the following two lemmas.

Lemma. *For every recursively enumerable sequence x ,*

$$\exists c_1 \forall n \forall m. K(x^n|m) \leq K(n) + c_1.$$

Proof. Let h be a total recursive function which enumerates the 1's of x , i.e., $x(i) = 1 \Leftrightarrow \exists j. h(j) = i$. Define the total recursive function $f(n, m)$ as follows:
 Step 1: Enumerate via h the first m 1's of the sequence x , i.e. compute $h(1), \dots, h(m)$.
 Step 2: Output the following finite string y of length n — $y(i) = 1 \Leftrightarrow i$ appears on

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the list generated in Step 1, i.e. $\exists j \leq m. h(j) = i$.

Using f and the asymptotically optimal algorithm for $K(\cdot | \cdot)$ the result follows.

Lemma. *There exists a recursively enumerable sequence x such that $\forall n. K(x^{2^n} | 2^n) > n - 1$.*

Proof. Let A be the asymptotically optimal algorithm for $K(\cdot | \cdot)$ and define the sequence x as follows: For each n and each m such that $1 \leq m \leq 2^{n-1}$, $x(2^{n-1} + m) = 1 \Leftrightarrow m^{\text{th}}$ digit of $A(m, 2^n)$ is 0. Clearly x is recursively enumerable and $x^{2^n} \neq A(y, 2^n)$ for all programs y of length $\leq n - 1$.

Combining these two lemmas we have that there exists a recursively enumerable sequence x such that

$$\exists c_1 \forall n \tilde{\forall} m. K(x^{2^n} | 2^n) - K(x^{2^n} | m) > n - 1 - c_1 - K(2^n).$$

While it is true that $K(2^n)$ for most integers of the form 2^n is about n , there are infinitely many such integers which have descriptions of arbitrarily (in an effective sense) short length relative to n , i.e. $\forall c \exists n. n - K(2^n) > c$. (see [1]) Combining these two inequalities yields the result. Q.E.D.

Recently, Chaitin (see [2]) has shown that a sequence x is recursive $\Leftrightarrow \exists c \forall n. K(x^n) \leq K(n) + c$. Combining this result with the first lemma above we have immediately

Corollary. *For every non-recursive recursively enumerable sequence x ,*

$$\forall c \exists n \tilde{\forall} m. K(x^n) - K(x^n | m) > c.$$

However, the above theorem is the best result obtainable for any recursively enumerable sequence in view of the following result which is proved in [1].

Theorem. *For every recursively enumerable sequence x*

$$\exists c \exists n. K(x^n | n) \leq c.$$

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