



Title	素数ノ分布ニ就イテ
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56. 素数ノ分布ニ就イテ

石川平八郎 (東京文理大)

$$\pi(x) > \alpha_1 \frac{x}{\log x}$$

ノ α_1 ノ値ニ就テ13号ニ市原哲治氏カ述ベラレタ御質問ニ對シテ、ソノ後私カ
メタ値ヲ御知ラセシタイ。

$$\pi(x) = \int_2^x \frac{du}{\log u} + O(x e^{-\alpha_2 \sqrt{\log x \log \log x}})$$

ナドノヨク知ラレタ定理ニ依ツテ充分大ナル x ニ對シテ、 $\alpha_1 = 1$ 即チ

$$\pi(x) > \frac{x}{\log x}$$

カ成立スルコトハ明カデアル。併シ(2)ナドノ式ヲ不等式ニ書き改メテ、充分大ナル
ト云フノデナク、アル定マツタ x_0 ヨリ大ナル x ニ對シテ(3)カ成立スル事ヲ證明
シヤウト云フ時、 x_0 ノ値ハ Lehmer ノ素数表位ノモノデ届ク程ニハサクハナラナ
イ。 x_0 カ 10100000 位ニナレバソレ以下ニツイテ(3)カ成立スルコトカ証明出
來ルカラ(後ニ証明ヲ述ベル)ウマイノデアルカソウ行カナイ。 $x \geq 5$ ニ對シ
テ $\alpha_1 = 0.918$ カ成立スルコトハ

$$\vartheta(x) = \sum_{p \leq x} \log p > ax - \frac{12}{5} a \sqrt{x} - \frac{3}{2} \log^2 x - 13 \log x - 15$$

$$\vartheta(x) < \frac{6}{5} ax + 3 \log^2 x + 8 \log x + 5,$$

$$a = \log \frac{2^{\frac{1}{2}} 3^{\frac{1}{3}} 5^{\frac{1}{5}}}{80^{\frac{1}{10}}} = 0.92129$$

ヲ用ヒテ証明シタガ、ソノ方法ハ非常ニ初等的デアツテ、ソレデハ如何ニ精密ニ解イ
テモ、 Ω_1 ガ Ω ヲコエナイ。ソレデ次ニ Math. Zeit. 34 = R. Brunsch ン $x \geq 4$
ニ對シテ

$$\pi\left(\frac{9}{8}x\right) - \pi(x) > 0$$

ヲ証明シテキルソノ方法ヲ用ヒテミタラ次ノ様ニ結果カ得ラレタカラソレヲオ知
ラセスル。

定理: $x \geq 11$ ニ對シテ

$$\pi(x) > 0.93219 \frac{x}{\log x} \quad (5)$$

下エヲ証明スル、尚 $\psi(x)$, $\mathcal{U}(x)$ ハ凡テ Landau / Primzahlen ニアルモノデアル。

補助定理 1: $x \geq 5$, n ハ $\left[\frac{x}{5}\right]$ ヨリ大ナラザル任意ノ整数ナリトス、ニカニ

$$\pi(x) \log x > \psi(x) - 2\psi(\sqrt{x}) + 0.918 \frac{n - \log n + 2}{n} \frac{x}{\log x}$$

証明

$$\begin{aligned} \mathcal{J}(x) &= \sum_{p \leq x} \log p = \sum_{i=1}^n \sum_{\frac{i-1}{n}x < p \leq \frac{i}{n}x} \log p \\ &\leq \sum_{i=1}^n \left\{ \pi\left(\frac{i}{n}x\right) - \pi\left(\frac{i-1}{n}x\right) \right\} \log \frac{i}{n}x \\ &= \pi(x) \log x - \sum_{i=1}^{n-1} \left(\log \frac{i+1}{n}x - \log \frac{i}{n}x \right) \pi\left(\frac{i}{n}x\right) \\ &= \pi(x) \log x - \sum_{i=1}^{n-1} \log \frac{i+1}{i} \pi\left(\frac{i}{n}x\right) \\ \pi(x) \log x &\geq \mathcal{J}(x) + \sum_{i=1}^{n-1} \log \frac{i+1}{i} \pi\left(\frac{i}{n}x\right) \\ &> \mathcal{J}(x) + \sum_{i=1}^{n-1} \log \frac{i+1}{i} \cdot 0.918 \frac{\frac{i}{n}x}{\log \frac{i}{n}x} \\ &= \mathcal{J}(x) + 0.918 \frac{x}{n} \sum_{i=1}^{n-1} \frac{i \log \frac{i+1}{i}}{\log \frac{i}{n}x} \end{aligned}$$

$$\geq J(x) + 0.918 \frac{x}{n} \sum_{i=1}^{n-1} \frac{1 - \frac{1}{i}}{\log x}$$

$$\geq J(x) + 0.918 \frac{n-1 - \log n - 1}{n} \frac{x}{\log x}$$

シカル =

$$J(x) \geq \psi(x) - 2\psi(\sqrt{x})$$

$$\therefore \pi(x) \log x > \psi(x) - 2\psi(\sqrt{x}) + 0.918 \frac{n - \log n - 2}{n} \frac{x}{\log x}$$

補助定理 2. $\rho = \frac{1}{2} + i\rho_2 \quad \exists \zeta(s)$, 零はトスレバ

$$\frac{2}{3} y^3 \sum_{x < y \leq (1+y)^4 x} a_\nu > x [y^4 - (2+y)^4 \cdot 3.32 \cdot 10^{-8}]$$

$$- \sqrt{x} [(2+y)^4 \cdot 3.32 \cdot 10^{-8} + 2 \sum_{0 < \rho_2 < 200} \frac{|(1+y)^{\rho} - 1|^4}{|\rho|^4} - \frac{1}{12x^2}]$$

$$\text{但シ } a_\nu = \psi(\nu) - \psi(\nu-1)$$

証明 Brunach, 論文参照, Math. Zent., 34. S. 506 - 513.

$$\text{補助定理 3. } \psi((1+y)^4 x) - \psi(x) > x \cdot A(y) - \sqrt{x} B(y) - \frac{1}{8x^2 y^3} \quad (7)$$

$$\text{但シ } A(y) = \frac{3}{2y^3} [y^4 - (2+y)^4 \cdot 3.22 \cdot 10^{-5}] \quad (8)$$

$$B(y) = \frac{3}{2y^3} [(2+y)^4 \cdot 3.22 \cdot 10^{-8} + 2 \sum_{0 < \rho_2 < 200} \frac{|(1+y)^{\rho} - 1|^4}{|\rho|^4}] \quad (9)$$

証明, 補助定理 2 より明カデアル.

補助定理 3 = 於テ $y = \frac{1}{26}, \frac{1}{32}, \frac{1}{28}, \frac{1}{21}$ トオケバ次, 四式ヲ得ル.

$$\left\{ \begin{array}{l} \psi(\frac{x}{6}) - \psi(\frac{x}{7}) \geq \psi((1+\frac{1}{26})^4 \frac{x}{7}) - \psi(\frac{x}{7}) > \frac{x}{7} A(\frac{1}{26}) - \sqrt{\frac{x}{7}} B(\frac{1}{26}) - \frac{7^2 \cdot 26}{8x^2} \\ \psi(\frac{x}{19}) - \psi(\frac{x}{17}) \geq \psi((1+\frac{1}{32})^4 \frac{x}{17}) - \psi(\frac{x}{17}) > \frac{x}{17} A(\frac{1}{32}) - \sqrt{\frac{x}{17}} B(\frac{1}{32}) - \frac{17^2 \cdot 32^3}{8x^2} \\ \psi(\frac{x}{20}) - \psi(\frac{x}{23}) \geq \psi((1+\frac{1}{28})^4 \frac{x}{23}) - \psi(\frac{x}{23}) > \frac{x}{23} A(\frac{1}{28}) - \sqrt{\frac{x}{23}} B(\frac{1}{28}) - \frac{23^2 \cdot 28^3}{8x^2} \\ \psi(\frac{x}{24}) - \psi(\frac{x}{29}) \geq \psi((1+\frac{1}{21})^4 \frac{x}{29}) - \psi(\frac{x}{29}) > \frac{x}{29} A(\frac{1}{21}) - \sqrt{\frac{x}{29}} B(\frac{1}{21}) - \frac{29^2 \cdot 21^3}{8x^2} \end{array} \right.$$

次ニヨリ知ラレタ公式

— (10) —

$$\begin{aligned} U(x) &= \psi(x) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{10}\right) + \psi\left(\frac{x}{11}\right) - \psi\left(\frac{x}{12}\right) + \psi\left(\frac{x}{13}\right) - \psi\left(\frac{x}{15}\right) \\ &\quad + \psi\left(\frac{x}{17}\right) - \psi\left(\frac{x}{18}\right) + \psi\left(\frac{x}{19}\right) - \psi\left(\frac{x}{20}\right) + \psi\left(\frac{x}{23}\right) - \psi\left(\frac{x}{24}\right) + \psi\left(\frac{x}{29}\right) - \psi\left(\frac{x}{30}\right) \\ &\leq \psi(x) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{17}\right) - \psi\left(\frac{x}{20}\right) + \psi\left(\frac{x}{23}\right) - \psi\left(\frac{x}{24}\right) + \psi\left(\frac{x}{29}\right) \end{aligned}$$

$$U(x) \geq ax - 5(\log x + 1) \quad \exists \text{リ}$$

$$\begin{aligned} \psi(x) &\geq ax - 5(\log x + 1) \quad \left(\frac{x}{6}\right) - \psi\left(\frac{x}{7}\right) + \psi\left(\frac{x}{15}\right) - \psi\left(\frac{x}{17}\right) + \psi\left(\frac{x}{20}\right) - \psi\left(\frac{x}{23}\right) \\ &\quad + \psi\left(\frac{x}{24}\right) - \psi\left(\frac{x}{29}\right) \end{aligned}$$

ヲ得ル. 一方又

$$\psi(x) < \frac{6}{5}ax + (3\log x + 5)(\log x + 1)$$

$$+ \text{ル} = \exists \text{リ} \quad \psi(\sqrt{x}) < \frac{6}{5}a\sqrt{x} + \frac{3}{4}\log^2 x + 4\log x + 5$$

依ツテ (6), (10), (13), (14) $\exists$ リ

$$\begin{aligned} \pi(x) \log x &> ax - 5(\log x + 1) + \psi\left(\frac{x}{6}\right) - \psi\left(\frac{x}{7}\right) + \psi\left(\frac{x}{15}\right) - \psi\left(\frac{x}{17}\right) + \psi\left(\frac{x}{20}\right) - \psi\left(\frac{x}{23}\right) \\ &\quad + \psi\left(\frac{x}{24}\right) - \psi\left(\frac{x}{29}\right) - 2\left(\frac{6}{5}a\sqrt{x} + \frac{3}{4}\log^2 x + 4\log x + 5\right) + 0.918 \frac{n - \log n - 2}{n} \frac{x}{\log x} \end{aligned}$$

$$\begin{aligned} &> ax - 5(\log x + 1) + \frac{x}{7} A\left(\frac{1}{26}\right) - \sqrt{\frac{x}{7}} B\left(\frac{1}{26}\right) - \frac{7 \cdot 26^3}{8x^2} + \frac{x}{17} A\left(\frac{1}{32}\right) \\ &\quad - \sqrt{\frac{x}{17}} B\left(\frac{1}{32}\right) - \frac{17 \cdot 32^3}{8x^2} + \frac{x}{23} A\left(\frac{1}{28}\right) - \sqrt{\frac{x}{23}} B\left(\frac{1}{28}\right) - \frac{23 \cdot 28^3}{8x^2} + \frac{x}{29} A\left(\frac{1}{21}\right) - \sqrt{\frac{x}{29}} B\left(\frac{1}{21}\right) \\ &\quad - \frac{29 \cdot 21^3}{8x^2} + 0.918 \frac{n - \log n - 2}{n} \frac{x}{\log x} - 2\left(\frac{6}{5}a\sqrt{x} + \frac{3}{4}\log^2 x + 4\log x + 5\right) \\ &= x \left\{ a + \frac{1}{7} A\left(\frac{1}{26}\right) + \frac{1}{17} A\left(\frac{1}{32}\right) + \frac{1}{23} A\left(\frac{1}{28}\right) + \frac{1}{29} A\left(\frac{1}{21}\right) + 0.918 \frac{n - \log n - 2}{n} \frac{x}{\log x} \right. \\ &\quad \left. - \sqrt{x} \left\{ \frac{1}{\sqrt{7}} B\left(\frac{1}{26}\right) + \frac{1}{\sqrt{17}} B\left(\frac{1}{32}\right) + \frac{1}{\sqrt{23}} B\left(\frac{1}{28}\right) + \frac{1}{\sqrt{29}} B\left(\frac{1}{21}\right) + \frac{12}{5} a \right\} \right. \\ &\quad \left. - \left\{ \frac{3}{2} \log^2 x + 13 \log x + \frac{29732285}{8x^2} + 15 \right\} \right\} \end{aligned}$$

$$\begin{aligned} \therefore \frac{\pi(x) \log x}{x} &> \left\{ a + \frac{1}{7} A\left(\frac{1}{26}\right) + \frac{1}{17} A\left(\frac{1}{32}\right) + \frac{1}{23} A\left(\frac{1}{28}\right) + \frac{1}{29} A\left(\frac{1}{21}\right) \right\} \\ &\quad + 0.918 \frac{n - \log n - 2}{n} \frac{1}{\log x} - \frac{1}{\sqrt{x}} \left\{ \frac{1}{\sqrt{7}} B\left(\frac{1}{26}\right) + \frac{1}{\sqrt{17}} B\left(\frac{1}{32}\right) + \frac{1}{\sqrt{23}} B\left(\frac{1}{28}\right) \right\} \end{aligned}$$

$$+ \frac{1}{\sqrt{21}} B\left(\frac{1}{21}\right) + \frac{12}{5} a \} - \left\{ \frac{3 \log^2 x}{2} + 13 \frac{\log x}{x} + \frac{29732285}{8x^2} + \frac{15}{x} \right\}$$

$$(8) = \text{依ッテ } A\left(\frac{1}{26}\right), A\left(\frac{1}{32}\right), A\left(\frac{1}{28}\right),$$

$$A\left(\frac{1}{26}\right) = 0.042579 \dots$$

$$A\left(\frac{1}{32}\right) = 0.019094 \dots$$

$$A\left(\frac{1}{28}\right) = 0.034796 \dots$$

$$A\left(\frac{1}{21}\right) = 0.063321 \dots$$

終 = 又 (9) = 於テ $y \geq 21$ ナラハ

$$\begin{aligned} |(1+y)^p - 1|^4 &= |(1+y)^{\frac{1}{2}} \cos \rho_2 \log(1+y) - 1 + i(1+y)^{\frac{1}{2}} \sin \rho_2 \log(1+y)|^4 \\ &\leq \left[\left\{ |(1+y)^{\frac{1}{2}} \cos \rho_2 \log(1+y)| + 1 \right\}^2 + \left\{ (1+y)^{\frac{1}{2}} \sin \rho_2 \log(1+y) \right\}^2 \right]^2 \\ &= \left[(1+y) + 1 + 2 |(1+y)^{\frac{1}{2}} \cos \rho_2 \log(1+y)| \right]^2 \\ &\leq [2+y + 2(1+y)^{\frac{1}{2}}] < 17 \end{aligned}$$

且ツ $0 < \rho_2 < 200$ ρ_2 値ノ分布ハ

$$14 \leq \rho_2^{(1)} \leq 20 \leq \rho_2^{(2)} \leq 24 \leq \rho_2^{(3)} \leq 30 \leq \rho_2^{(4)}, \rho_2^{(5)} \leq 34 \leq \rho_2^{(6)} \quad \exists \text{ 〃}$$

$$\rho_2^{(10)} \leq 50 \leq \rho_2^{(11)} \quad \exists \text{ 〃} \quad \rho_2^{(29)} \leq 100 \leq \rho_2^{(30)} \quad \exists \text{ 〃} \quad \rho_2^{(99)} < 200$$

$$\begin{aligned} + 17 &= \exists \text{ 〃} \quad \sum_{0 < \rho_2 < 200} \frac{|(1+y)^p - 1|}{|p|^4} < 17 \sum_{0 < \rho_2 < 200} \frac{1}{|p|^4} \\ &< 17 \left\{ \frac{1}{14^4} + \frac{1}{20^4} + \frac{1}{24^4} + \frac{2}{30^4} + \frac{5}{34^4} + \frac{17}{50^4} + \frac{1}{16} \right\} \\ &< 0.00076585 \end{aligned}$$

之ヨリ $B\left(\frac{1}{26}\right), B\left(\frac{1}{32}\right), B\left(\frac{1}{28}\right), B\left(\frac{1}{21}\right)$ ノ値ヲ計算スルハ

$$B\left(\frac{1}{26}\right) < 40.397$$

$$B\left(\frac{1}{32}\right) < 75.304$$

$$B\left(\frac{1}{28}\right) < 50.624$$

$$B\left(\frac{1}{21}\right) < 21.286$$

依ッテ $x > 10^7$ = 對シテ

$$\frac{1}{\sqrt{x}} \left\{ \frac{1}{\sqrt{7}} B\left(\frac{1}{26}\right) + \frac{1}{\sqrt{17}} B\left(\frac{1}{32}\right) + \frac{1}{\sqrt{23}} B\left(\frac{1}{28}\right) + \frac{1}{\sqrt{29}} B\left(\frac{1}{21}\right) + \frac{13}{5} a \right\} \\ + \left\{ \frac{3}{2} \frac{\log^2 x}{x} + 13 \frac{\log x}{x} + \frac{29732285}{8x^3} + \frac{15}{x} \right\} < 0.0175 \quad (20)$$

他方 $n = 10 < \left\lfloor \frac{x}{5} \right\rfloor$ ヲトルバ

$$0.918 \frac{n - \log n - 2}{n} \frac{1}{\log x} > 0.032 \quad (21)$$

依ッテ (15), (16), (20), (21) ヨリ $x > 10^7$ = 對シテ

$$\frac{\pi(x) \log x}{x} > 0.93219$$

ヲ得ル. エデ $x > 10^7$ = 對シテ 定理, 成立スルコトガ分ル. $17 \leq x \leq 10^7$ = 對シテ成立スルコトハ次ノ事柄ニ含マレル.

$17 \leq x < 10100000$ デ (3) ガ成立スル. ソレヲ云フニハ $17 \leq x_1 \leq x_2$ テ

$\pi(x_1) > \frac{x_2}{\log x_2}$ ガ成立スルヲバ $x_1 \leq x \leq x_2$ ナル x = ツイテ (3) ガ成立

スルト云フ論法ヲ用ヒレバヨイ. x_1, x_2 トシテハ次ノ数列ヲトルバ十分デアル.

$101 \cdot 10^5, 94 \cdot 10^5, 88 \cdot 10^5, 82 \cdot 10^5, 77 \cdot 10^5, 72 \cdot 10^5, 67 \cdot 10^5, 63 \cdot 10^5,$
 $585 \cdot 10^4, 545 \cdot 10^4, 505 \cdot 10^4, 470 \cdot 10^4, 435 \cdot 10^4, 405 \cdot 10^4, 375 \cdot 10^4, 350 \cdot 10^4,$
 $325 \cdot 10^4, 300 \cdot 10^4, 280 \cdot 10^4, 260 \cdot 10^4, 240 \cdot 10^4, 225 \cdot 10^4, 210 \cdot 10^4, 195 \cdot 10^4,$
 $180 \cdot 10^4, 170 \cdot 10^4, 160 \cdot 10^4, 150 \cdot 10^4, 140 \cdot 10^4, 130 \cdot 10^4, 120 \cdot 10^4, 111 \cdot 10^4,$
 $102 \cdot 10^4, 94 \cdot 10^4, 861 \cdot 10^3, 788 \cdot 10^3, 721 \cdot 10^3, 657 \cdot 10^3, 600 \cdot 10^3, 548 \cdot 10^3,$
 $500 \cdot 10^3, 456 \cdot 10^3, 415 \cdot 10^3, 378 \cdot 10^3, 344 \cdot 10^3, 313 \cdot 10^3, 284 \cdot 10^3, 258 \cdot 10^3,$
 $234 \cdot 10^3, 212 \cdot 10^3, 191 \cdot 10^3, 173 \cdot 10^3, 156 \cdot 10^3, 140 \cdot 10^3, 126 \cdot 10^3, 114 \cdot 10^3,$
 $103 \cdot 10^3, 94 \cdot 10^3, 84300, 75500, 67500, 60200, 53800, 48000,$
 $42600, 37800, 33600, 29800, 26400, 23200, 21200, 18700,$

16500,	14600,	12800,	11200,	9800,	8600,	7500,	6500,
5700,	5000,	3700,	3200,	2700,	2300,	2000,	1700,
1500,	1300,	1100,	930,	800,	660,	560,	470,
400,	340,	280,	230,	200,	170,	140,	110,
90,	73,	61,	47,	41,	37,	31,	29,
23,	19,	17					

11 ≤ x < 17 で (5) が成立スルコトモ容易ニ分ル。之ヲ定理ハ完全ニ証明サレタ。
 尚 $\psi(\frac{x}{10}) - \psi(\frac{x}{11})$, $\psi(\frac{x}{12}) - \psi(\frac{x}{13})$ 等ノ項ヲ加ヘレバモットヨイ結果ガ出ル
 ベキデアルガ、 $A(y)$ が負トナツテ Breusch ノ方法ガ用ヒラレナイ。 $B(y)$ ノテ
 (17) (18) 等ノ計算カ ρ ノ値ヲ精密ニ吟味スル事ナドニ依ツテ遙ニ精密ニ値ガ出ル
 カラエ等ノ項ヲ加エテモ何等差支ヘナイ。 $A(y)$ が早ク負ニナツテエ等ノ項ヲ
 加ヘル事ガ意味ヲナサナクナル、ハ $(2+y)^4$ ノ係數ガ大き過ぎルカラデアル。シカ
 シ Breusch ノ論文ノ前半ノ大部分ガエヲ計算スル、ニ用ヒラレテアルノデスガ
 尚ヨクナルナラバ当前(4)モヨイ結果デオキカハラレルワケデアル。エハナカナ
 容易デハナイト思フ。

トマレモ少シヨイ結果ガ出ソウナモノデアルガコシテ方法デハ α_1 が 1ニ等シク
 ナル事ハ絶対ニナイ、ガカラ止ムヲ得ナイカモシレナイ。

(11月10日 受取)