The University of Osaka
Institutional Knowledge Archive

Title	Prime one-sided ideals of a finite normalizing extension
Author(s)	Nakamoto, Taichi
Citation	Osaka Journal of Mathematics. 1986, 23(4), p. 823-833
Version Type	VoR
URL	https://doi.org/10.18910/7401
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/

The University of Osaka

Nakamoto, T.
Osaka J. Math.
23 (1986), 823–833

PRIME ONE-SIDED IDEALS OF A FINITE NORMALIZING EXTENSION

Dedicated to Professor Hisao Tominaga on his 60th birthday

TAICHI NAKAMOTO

(Received November 30, 1984)
(Revised November 27, 1985)

Introduction

Throughout the present paper, R will represent a ring with identity 1. Let I be a right ideal of R, and $b_R(I)=\{r\in R \mid Rr\subset I\}$. Then, $b_R(I)$ is the largest ideal of R contained in I. We shall call that I is a *prime right ideal* provided that if X and Y are right ideals of R with $XY\subset I$, then either $X\subset I$ or $Y\subset I$. It is clear that a maximal right ideal is a prime right ideal. If I is a prime right ideal, then $b_R(I)$ is a prime ideal. Next, let S be a ring extension of a ring R with the same identity 1. S is said to be a *left torsionfree* R-bimodule if $r_S(X)=0$ for every essential ideal X of R, where $r_S(X)$ is the right annihilator of X in S (cf. [1]). Right torsionfree is defined similarly, and S is said to be *torsionfree* if it is both left and right torsionfree. Moreover, S is said to be *fully torsionfree* if, for every prime ideal P of S, S/P is a right torsionfree $R/(P\cap R)$-bimodule (cf. [3]). Furthermore, we say that S is a *finite normalizing extension* (resp. a *liberal extension*) of R if there exists a finite subset $\{a_1, a_2, \cdots, a_n\}$ of S such that $S=\sum_{i=1}^{n} Ra_i$ and $Ra_i=a_iR$ for all $i=1, 2, \cdots, n$ (resp. $ra_i=a_ir$ for all $r\in R$ and for all $i=1, 2, \cdots, n$). A ring extension T of R is said to be an *intermediate normalizing extension* (resp. an *intermediate extension*) if there exists a finite normalizing extension (resp. a liberal extension) S of R containing T.

Recently, Heinicke and Robson [1, 2], Lorenz [5], Jabbour [3] and others, gave some descriptions of the relationship between the prime ideals of R and any intermediate normalizing extension T. In this paper, we shall verify that there is a similar relationship between the prime right ideals of R and T. In Section 1, we shall prove a "lying over" theorem for a liberal extension, and a "lying inside" theorem and a "lying outside" theorem for an intermediate extension. In Sections 2 and 3, we shall prove a "cutting down" theorem for a fully torsionfree finite normalizing extension and an intermediate normalizing extension of a fully torsionfree finite normalizing extension.

1. Prime right ideals of a liberal extension

In this section, we discuss the relationship of prime right ideals of a liberal extension and an intermediate extension.

Theorem 1.1 (Lying over). *Let S be a liberal extension of a ring R. If K is a prime right ideal of R, then there exists a prime right ideal I of S such that $I \cap R = K$. When this is the case, there hold $b_R(I) \cap R = b_R(K)$ and $I \cap R = KS \cap R = K$.*

Proof. Since $b_R(K)$ is a prime ideal, there exists a prime ideal P of S such that $P \cap R = b_R(K)$ and P is a maximal with respect to $P \cap R = b_R(K)$ by [9, Theorem 4.1]. By [9, Lemma 3.2], S/P is a liberal extension of $R/b_R(K)$. Hence we may assume that S is a prime liberal extension of a prime ring R such that $B \cap R \neq 0$ for each non-zero ideal B of S, and K is a prime right ideal of R with $b_R(K)=0$. Since, by [9, Lemma 3.5], there is a non-zero ideal A of S such that $R+A$ is contained in a full matrix ring $M_m(R)$, we have $KA \subset M_m(K)$, and so $KA \cap R \subset K$. Consequently, by Zorn's Lemma, there exists a right ideal I of S which is maximal with respect to $I \cap R \subset K$ and $I \supset KA$. Let X and Y be right ideals of S with $XY \subset I$ and $Y \not\subset I$. Then we have $((X+I) \cap R)(SY \cap R) \subset K$. Since $SY \neq 0$, $SY \cap R$ is a non-zero ideal of R, and so $SY \cap R \not\subset K$. Therefore $(X+I) \cap R \subset K$, and so $X \subset I$. This implies that I is a prime right ideal of S. According to [9, Theorem 4.6], it is clear that $b_S(I)=0$ and $A \not\subset I$. Since $KSA = KA \cap I$, we have $KS \subset I$ and $KS \cap R = I \cap R = K$.

By making use of the same methods as in the proof of the above theorem, we readily obtain the following

Corollary 1.2 (Going up). *Let S be a liberal extension of R. If $K_0 \supset K$ are prime right ideals of R and I is a prime right ideal of S with $I \cap R = K$, then there exists a prime right ideal I_0 of S such that $I_0 \supset I$ and $I_0 \cap R = K_0$.*

If P and Q are prime ideals of S such that $P \supset Q$ and $P \cap R = Q \cap R$, then $P = Q$ ([1, Theorem 5.10]). We shall now present some examples of liberal extensions in which there does not hold an "incomparability" theorem for prime right ideals.

EXAMPLE 1.3. Let D be a division ring, and $S = \begin{pmatrix} D & D \\ D & D \end{pmatrix}$. Then S is a liberal extension of $D = \left\{ \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \middle| d \in D \right\}$, and $I = \begin{pmatrix} 0 & 0 \\ D & D \end{pmatrix}$ is a maximal right ideal of S with $b_S(I)=0$. But $I \cap D = 0$ which is a prime ideal of D.

EXAMPLE 1.4. Let A be a simple ring with a non-zero maximal right ideal M, and $S=\begin{pmatrix} A & A \\ A & A \end{pmatrix}$. Then $I_1=\begin{pmatrix} M & M \\ A & A \end{pmatrix}$ and $I_2=\begin{pmatrix} M & M \\ M & M \end{pmatrix}$ are prime right ideals of S such that $I_1 \supsetneqq I_2$, but $I_1 \cap A=M=I_2 \cap A$, which is a prime right ideal but not an ideal.

In the rest of this section, we investigate the relationship of the prime right ideals between R and any intermediate extension.

Theorem 1.5 (Lying outside). *Let T be a intermediate extension of R with a liberal extension S of R containing T, and J a prime right ideal of T. Then $J \cap R$ is a prime right ideal of R, and there exists a prime right ideal I of S such that $I \cap T \subset J$ and $b_S(I) \cap R=b_T(J) \cap R=b_R(J \cap R)=b_R(I \cap R)$.*

Proof. By Zorn's Lemma, there exists an ideal P of S which is maximal with respect to the property $P \cap T \subset b_T(J)$. Since $b_T(J)$ is a prime ideal of T, P is prime and $P \cap R=b_T(J) \cap R$ (cf. [7, Theorem 12.3] and [8, Theorem 3.2]). Then, since $S/P \supset T/(P \cap T) \supset R/(P \cap R)$, we may assume that S is a prime liberal extension of a prime ring R, and T is a subring of S containing R, and J is a prime right ideal of T such that $b_T(J) \cap R=0$ and $Q \cap T \not\subset J$ for each non-zero ideal Q of S. By Zorn's Lemma, there is a right ideal I which is maximal with respect to the property $I \cap T \subset J$. It is clear that $b_S(I)=0$. Suppose that X and Y are right ideals of S with $XY \subset I$ and $Y \not\subset I$. Then $((X+I) \cap T) \times (SY \cap T) \subset I \cap T \subset J$ and SY is a non-zero ideal of S. Therefore we obtain $(X+I) \cap T \subset J$, and so $X \subset I$.Thus I is a prime right ideal of S. Next we claim that $J \cap R$ is a prime right ideal of R. To prove this, assume that X_1 and X_2 are right ideals of R with $X_1X_2 \subset J \cap R$ and $X_2 \not\subset J \cap R$. Now, by [8, Proposition 2.5], there exist a liberal extension $S'=\sum_{j=1}^{p} b_jCR$ of CR and a non-zero ideal X of CR such that $XS' \subset CT \subset S' \subset CS$, where C is the center of the complete ring of quotients of R, and $b_1, b_2, \cdots, b_p \in V_{CS}(CR)$. Moreover, by [8, Lemma 4.1], there exist non-zero ideals Y_1, Y_2 of R such that $\sum_{j=1}^{p} b_jY_2$ is a ring (without 1) and $TY_1T \subset \sum_{j=1}^{p} b_jY_2 \subset T$. Then we have $X_1TY_1TX_2Y_2T \subset X_1 \sum_j b_jY_2X_2Y_2T \subset X_1Y_2X_2 \sum_j b_jY_2T \subset X_1X_2T \subset J$. Since $Y_1 \neq 0$, $Y_2 \neq 0$ and $X_2 \neq 0$, $Y_1X_2Y_2$ is a non-zero ideal of R contained in the ideal $TY_1TX_2Y_2T$ of T. Since $b_T(J) \cap R=0$, we have $X_1T \subset J$. Hence $J \cap R$ is a prime right ideal. Once again, using [8, Lemma 4.1], we obtain that $TY_1Tb_R(J \cap R)Y_2T \subset J$, and so $b_R(J \cap R)Y_2T \subset J$. This implies that $b_R(J \cap R)=0$. The rest is clear.

Corollary 1.6. *Let R, T, S and J be as in the above theorem.If $(J \cap R)S \cap T \subset J$, then there exists a prime right ideal I of S such that $I \cap T \subset J$ and $I \cap R=J \cap R$. In this case, there holds that $b_S(I) \cap R=b_T(J) \cap R=b_R(J \cap R)=b_R(I \cap R)$.*

Let T be an intermediate extension of R, and S a fixed liberal extension of R containing T. Let K be a prime right ideal of R and I a prime right ideal of S with $I \cap R=K$. Then, by Zorn's Lemma, there exists a right ideal J of T which is maximal with respect to the property $J \cap R=K$ and $J \supset I \cap T$. In this situation, we shall prove the following

Lemma 1.7. *$b_R(K)=b_T(J) \cap R=b_S(I) \cap R$ and $b_S(I) \cap T \subset b_T(J)$.*

Proof. Obviously we obtain $b_S(I) \cap T \subset b_T(J)$. Since $b_R(K)S$ is an ideal of S contained in I, this implies $b_R(K) \subset b_R(K)S \cap R \subset b_S(I) \cap R \subset b_T(J) \cap R \subset b_R(K)$.

Proposition 1.8 (Lying inside). *J is a prime right ideal of T if and only if $b_T(J)$ is a prime ideal of T.*

Proof. If $b_T(J)$ is prime, then $b_T(J)$ is an ideal Q of T which is maximal with respect to $Q \cap R=b_R(K)$ and $Q \supset b_S(I) \cap T$ (cf. [7, Theorem 12.7 and 8, Theorem 3.3]). Suppose that X and Y are right ideals of T with $XY \subset I$ and $Y \not\subset J$. Then $TY+b_T(J) \neq b_T(J)$ and $((X+J) \cap R)((TY+b_T(J)) \cap R) \subset J \cap R =K$. Hence it follows from the maximality of $b_T(J)$ that $(X+J) \cap R \subset K$, and so J is a prime right ideal.

The following examples show that whether J is a prime right ideal or not.

EXAMPLE 1.9. Let A, M and S be as in Example 1.4, and let $T=\begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$. Then $I=\begin{pmatrix} M & M \\ A & A \end{pmatrix}$ is a prime right ideal of S with $I \cap A=M$. Since $\begin{pmatrix} M & A \\ 0 & A \end{pmatrix}\begin{pmatrix} A & A \\ 0 & 0 \end{pmatrix} \subset I \cap T$, $I \cap T$ is not prime right. $\begin{pmatrix} M & A \\ 0 & A \end{pmatrix}$ is the required J, and which is a prime right ideal of T.

EXAMPLE 1.10. Let A be a simple ring having at least two maximal right ideals, and let M and N be distinct maximal right ideals. Let us put $S=\begin{pmatrix} A & A & A \\ A & A & A \\ A & A & A \end{pmatrix}$ and $T=\begin{pmatrix} A & A & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$. Since A is two-sided simple, $I=\begin{pmatrix} M & M & M \\ A & A & A \\ N & N & N \end{pmatrix}$ is a prime right ideal of S and $I \cap A=M \cap N$ is a prime right ideal of A. Hence, $\begin{pmatrix} M & A & 0 \\ 0 & A & 0 \\ 0 & 0 & N \end{pmatrix}$ is the required J. However, since $\begin{pmatrix} M & A & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}\begin{pmatrix} A & A & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix} \subset J$, J is not a prime right ideal.

2. Prime right ideals of a finite normalizing extension

In the rest of our study, suppose that S is a finite normalizing extension of R.

Proposition 2.1 (Lying over). *Suppose that S is a finite normalizing extension of R. If K is a prime right ideal of R, then there exists a prime right ideal I of S such that $I \cap R \subset K$ and $b_R(K)$ is a minimal prime ideal over $b_S(I) \cap R$.*

Proof. Since $b_R(K)$ is a prime right ideal of R, there exists a prime ideal Q of S such that $b_R(K)$ is a minimal prime ideal over $Q \cap R$. Hence we may assume that S is a prime finite normalizing extension of R and K is a prime right ideal of R such that $A \cap R \not\subset b_R(K)$ for each non-zero ideal A of S and $b_R(K)$ is minimal prime. We next claim that there is a prime right ideal I of S which satisfies $I \cap R \subset K$ and $b_S(I) \cap R = 0$. By Zorn's Lemma, there exists a right ideal I of S which is maximal with respect to $I \cap R \subset K$. Let X and Y be right ideals of S such that $XY \subset I$ and $Y \not\subset I$. Since $((X+I) \cap R) \times (SY \cap R) \subset I \cap R \subset K$ and SY is a non-zero ideal of S, we have $X \subset I$. Thus I is a prime right ideal of R. Clearly we have $b_S(I) = 0$.

Lemma 2.2. *Let S be a torsionfree finite normalizing extension of R. If Y is an essential ideal of R, then $b_S(YS) \neq 0$.*

Proof. If X is an R-S-subbimodule of S with $YS \cap X = 0$, then $YX = 0$. Since $Y \neq 0$, there holds $X = 0$. Hence it follows that YS is an essential R-S-subbimodule of S. By [6, Lemma 4], we have $b_S(YS) \neq 0$.

Proposition 2.3. *Let S be a prime torsionfree finite normalizing extension of a prime ring R. If I is a prime right ideal of S with $b_S(I) = 0$, then $I \cap R$ is a prime right ideal of R with $b_R(I \cap R) = 0$.*

Proof. Assume that X and Y are right ideals of R with $XY \subset I \cap R$ and $Y \not\subset I \cap R$. Then we obviously obtain $XSb_S(RYS) \subset I$, and hence we have either $XS \subset I$ or $b_S(RYS) \subset I$. On the other hand, since R is prime, RY is an essential ideal of R, and so, $b_S(RYS)$ is a non-zero ideal of S by Lemma 2.2. Hence there holds $XS \subset I$. Therefore, it follows that $I \cap R$ is a prime right ideal of R. The rest of the proof is clear from Lemma 2.2.

If R is not prime, then it may happen that $I \cap R$ is not a prime right ideal of R.

EXAMPLE 2.4. Let A, M and S be as in Example 1.4. Putting $R = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, S is a prime torsionfree finite normalizing extension of R and R is not prime.

Considering the prime right ideal $I=\begin{pmatrix} M & M \\ M & M \end{pmatrix}$ of S, $I \cap R=\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$ is not a prime right ideal of R.

Let $S=\sum_{i=1}^{n} Ra_i$ $(Ra_i=a_iR)$ be a prime torsionfree finite normalizing extension of R. Let $Q(S)$ be the right Martindale quotient of S. Then, there exist orthogonal idempotents $f_1, f_2, \cdots, f_m$ in $V_{Q(S)}(R)=\{q \in Q(S) \mid rq=qr$ for all $r \in R\}$ such that $f_1+f_2+\cdots+f_m=1$ and $m \leqq n$. We set here $P_i=r_R(f_i)$, $i=1, 2, \cdots, m$. Then, the P_i are m distinct minimal prime ideals of R such that $\cap_{i=1}^{m} P_i=0$, and $R/P_i \cong R/P_1$ for all i. Let us set $D_i=\cap_{j=1, j \neq i}^{m} P_j$, for all i. Then each D_i is a non-zero ideal of R with $f_i d=d$ (for all $d \in D_i$), and so D_i is an essential ideal of f_iR. Since $f_i \in Q(S)$, there exists an essential ideal B of S such that $f_iB \subset S$ for all i (cf. [1] and [5]). By [1, Theorem 5.7], each f_iSf_i $(1 \leqq i \leqq m)$ is a prime torsionfree finite normalizing extension of the prime ring f_iR. Now, let I be a prime right ideal of S with $b_S(I)=0$. Let us set $g_i(I)=\{f_isf_i \in f_iSf_i \mid f_isf_iB \subset I\}$. Then we have $f_iSf_iB \subset f_iB \subset S$ by [1, Proposition 5.5], and so $g_i(I)$ is a right ideal of f_iSf_i. Then $g_i(I)=f_iSf_i$ if and only if $f_iB \supset I$. Under this situation, we shall prove the following

Lemma 2.5. *There exists an f_i such that $g_i(I) \neq f_iSf_i$. Such an f_i is independent of a choice of an essential ideal B.*

Proof. If $g_i(I)=f_iSf_i$ for $i=1, 2, \cdots, m$, then we have $B \subset f_1B+f_2B+\cdots+f_mB \subset I$. This is a contradiction. To prove the rest, for essential ideals B, B' of S, we assume that $f_iB \not\subset I$ and $f_iB' \subset I$. Since $f_iBB' \subset f_iB' \subset I$ and $f_iB \not\subset I$, we have $B' \subset I$, which contradicts $b_S(I)=0$. Hence $f_iB \not\subset I$ if and only if $f_iB \not\subset I$.

By Lemma 2.5, we may assume that $f_iB \not\subset I$ if $i=1, 2, \cdots, t$, and $f_iB \subset I$ if $i=t+1, \cdots, m$.

Lemma 2.6. *For each $i=1, 2, \cdots, t$, $g_i(I)$ is a prime right ideals of f_iSf_i with $b_{f_iSf_i}(g_i(I))=0$.*

Proof. For $s=\sum_{j=1}^{n} r_ja_j \in S (r_j \in R)$, we put $s^{\sharp(i)}=\sum_{j \in \sharp(i)} r_ja_j$, where $\sharp(i)=\{j \mid f_ia_jf_i \neq 0\}$, and then $f_isf_i=s^{\sharp(i)}f_i$ ([1, Proposition 5.4]). Let $s^{\sharp(i)}f_i$ and $s'^{\sharp(i)}f_i$ be any elements of f_iSf_i such that $s^{\sharp(i)}f_if_iSf_is'^{\sharp(i)}f_i \subset g_i(I)$ and $s'^{\sharp(i)}f_i \notin g_i(I)$. Then we obtain $s^{\sharp(i)}f_iB \cdot Ss'^{\sharp(i)}f_iB \subset s^{\sharp(i)}f_iSs'^{\sharp(i)}f_iB \subset I$. Since $f_is'f_i \notin g_i(I)$, it follows that $Ss'^{\sharp(i)}f_iB \subset I$ and so $f_isf_iB \subset I$. Thus, $g_i(I)$ is a prime right ideal of f_iSf_i. Next, if $f_isf_i \in g_i(I)$ and $f_iSf_if_isf_i \subset g_i(I)$, then $f_iB \cdot Sf_isf_iB \subset f_iSf_isf_iB \subset I$. Since $f_iB \not\subset I$, we have $Sf_isf_iB \subset I$. This implies $Sf_isf_iB=0$, and so $f_isf_iB=0$. Since $f_isf_i \in Q(S)$ and B is an essential ideal of S,it follows that $b_{f_iSf_i}(g_i(I))=0$.

Combining Proposition 2.3 with Lemma 2.6, we obtain the following

Corollary 2.7. *$g_i(I)\cap f_iR$ is a prime right ideal of f_iR such that $b_{f_iR}(g_i(I)\cap f_iR)=0$ for $i=1, 2, \cdots, t$.*

Theorem 2.8 (Cutting down). *Let S be a prime torsionfree finite normalizing extension of R. If I is a prime right ideal of S such that $b_S(I)=0$, then there exist prime right ideals $K_1, K_2, \cdots, K_t$ of R such that $\cap_{i=1}^t K_i=I\cap R$, $b_R(K_i)=P_i$ for $i=1, 2, \cdots, t$. In this case, there holds $b_R(I\cap R)=\cap_{i=1}^t P_i$.*

Proof. By Lemma 2.5, we may assume that $f_iB\not\subset I$ for $i=1, 2, \cdots, t$, and $f_iB\subset I$ for $i=t+1, \cdots, m$. By Corollary 2.7, $g_i(I)\cap f_iR$ $(1\leqq i\leqq t)$ is a prime right ideal of f_iR such that $b_{f_iR}(g_i(I)\cap f_iR)=0$. Here we set $K_i=\{r\in R \mid f_ir\in g_i(I)\cap f_iR\}$ $(1\leqq i\leqq t)$. Then, it is easily seen that each K_i is a prime right ideal of R such that $b_R(K_i)=P_i$. Now we claim that $\cap_{i=1}^t K_i=I\cap R$. Actually, if $r\in\cap_{i=1}^t K_i$, then $f_ir\in g_i(I)\cap f_iR$ for $i=1, 2, \cdots, t$, and so $f_irB\subset I$. On the other hand, for $i=t+1, \cdots, m$, $f_irB\subset f_iB\subset I$. Hence $rB\subset f_1rB+f_2rB+\cdots+f_trB+f_{t+1}rB+\cdots+f_mrB\subset I$. Since I is the prime right ideal of S with $b_S(I)=0$, we have therefore $r\in I\cap R$. Thus $\cap_{i=1}^t K_i\subset I\cap R$. Conversely, for $r\in I\cap R$, we have $rf_iB\subset I$, which implies that $f_irf_i\in g_i(I)\cap f_iR$ for $i=1, 2, \cdots, t$, and so $r\in\cap_{i=1}^t K_i$. Therefore, $\cap_{i=1}^t K_i=I\cap R$. The rest of the proof is clear.

Corollary 2.9. *Let S be an arbitrary fully torsionfree finite normalizing extension of R. If I is a prime right ideal of S, then there exist prime right ideals $K_1, K_2, \cdots, K_t$ of R such that $I\cap R=\cap_{i=1}^t K_i$, and each $b_R(K_i)$ $(1\leqq i\leqq n)$ is a minimal prime ideal of R over $b_S(I)\cap R$. In this case, there holds $b_R(I\cap R)=\cap_{i=1}^t b_R(K_i)\supset b_S(I)\cap R$.*

EXAMPLE 2.10. Let A and M be as in Example 1.4. Let us set $S=\begin{pmatrix} A & A & A \\ A & A & A \\ A & A & A \end{pmatrix}$ and $R=\begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$. Since R is an only essential ideal of R, S is a prime torsionfree finite normalizing extension of R. For the prime right ideal $I=\begin{pmatrix} M & M & M \\ M & M & M \\ A & A & A \end{pmatrix}$ of S with $b_S(I)=0$, we immediately obtain that $I\cap R=\begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & A \end{pmatrix}$ is a right ideal of R which is not prime and not an ideal. On the other hand, $P_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$, $P_2=\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{pmatrix}$ and $P_3=\begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are the all minimal prime ideals of R. Moreover, $K_1=\begin{pmatrix} M & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$ and $K_2=\begin{pmatrix} A & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & A \end{pmatrix}$ are prime right ideals of R such that $I\cap R=K_1\cap K_2$ and $b_R(I\cap R)=b_R(K_1)\cap b_R(K_2)=P_1\cap P_2\neq 0$ and $t=2\neq 3=m$.

3. Prime right ideals of an intermediate normalizing extension

In this section, we shall prove a "cutting down" theorem for prime right ideals of an intermediate normalizing extension which corresponds to that of Section 2. Throughout this section, suppose that T is an intermediate normalizing extension of R, and S is a fixed finite normalizing extension of R containing T.

Lemma 3.1. *Let S be a torsionfree R-bimodule. If Y is an essential ideal of R, then YT is an essential R-submodule of T and there exists a non-zero ideal A of S with $0\neq A\cap T\subset b_T(YT)$.*

Proof. Since Y be an essential ideal of R, by making use of the same methods as in the proof of Lemma 2.2, we readily obtain that YT is an essential R-subbimodule of T. Let T^* be a relative complement of T in the R-bimodule S. Then, by [6, Lemma 4], $YT+T^*$ contains a non-zero ideal A of S which is an essential R-subbimodule of S, and so $0\neq A\cap T\subset b_T(YT)$.

Now, let Q be a prime ideal of T. Then, by [3, Proposition 5.6], there exists a prime ideal P of S such that $P\cap T\subset Q$ and $A\cap T\not\subset Q$ for all ideals $A\supsetneqq P$ of S. Obviously, S/P is a finite normalizing extension of $R/(P\cap R)$ and $Q/(P\cap T)$ is a prime ideal of an intermediate normalizing extension $T/(P\cap T)$ of $R/(P\cap R)$ such that $B/P\cap T/(P\cap T)\not\subset Q/(P\cap T)$ for each non-zero ideal B/P of S/P. As in [2], Q will be called a *standard setting* if S is a prime ring and Q satisfies $B\cap T\not\subset Q$ for each non-zero ideal B of S.

Proposition 3.2. *Let S be a prime torsionfree finite normalizing extension of a prime ring R. If J is a prime right ideal of T such that $b_T(J)$ is a standard setting, then $J\cap R$ is a prime right ideal of R with $b_R(J\cap R)=0$.*

Proof. Let X and Y be right ideals of R with $XY\subset J\cap R$ and $Y\not\subset J\cap R$. Since R is prime, RY is an essential ideal of R, and so, by Lemma 3.1, there exist a non-zero ideal A of S with $0\neq A\cap T\subset b_T(RYT)$. Hence, we have $b_T(RYT)\not\subset J$ since $b_T(J)$ is a standard setting. Noting $XTb_T(RYT)\subset J$, we obtain $X\subset XT\cap R\subset J\cap R$. The assertion $b_R(J\cap R)=0$ is clear by Lemma 3.1.

Throughout the rest of our study, we assume that S is a prime torsionfree finite normalizing extension of R. The notations in Section 2 will be used again here. As was seen, each f_iSf_i $(1\leqq i\leqq m)$ is a prime torsionfree finite normalizing extension of the prime ring f_iR. Now, by $T_{[i]}$, we denote the subring of the prime ring f_iSf_i which is generated by f_iTf_i. Then, by [3, Proposition 5.1 (2)], there exists an ideal $V_{(i)}$ of $T_{[i]}$ such that $V_{(i)}\subset T$ and $V_{(i)}$ is an essential f_iR-subbimodule of $T_{[i]}$. Then $V_{(i)}$ can be regarded as an essential R-subbimodule of $T_{[i]}$. Hence, $\sum_{i=1}^m V_{(i)}=\sum_{i=1}^m\oplus V_{(i)}$ is an essential R-subbimodule of $\sum_{i=1}^m\oplus T_{[i]}$. It is obvious that $(\sum_{i=1}^T T_{[i]})\cap R=R$. Moreover, for a prime right ideal J of T such that $b_T(J)$ is a standard setting, we set $h_i(J)=\{q\in T_{[i]}\,|\,qV_{(i)}\subset J\}$. Then, $h_i(J)=T_{[i]}$ if and only if $V_{(i)}T\subset J$. Using a similar argument to Lemma 2.5 making use of the above remark and Lemma 3.1, we obtain the following

Lemma 3.3. $(\sum_{i=1}^m V_{(i)})\cap R$ *is an essential R-subbimodule of R, and* $V_{(i)}T\not\subset J$ *for some* f_i.

By Lemma 3.3, we may assume that $V_{(i)}T\not\subset J$ for $i=1, 2, \cdots, s$, and $V_{(i)}T\subset J$ for $i=s+1, \cdots, m$. In this situation, we shall prove the following

Lemma 3.4. $b_T(J)\cap R\subset P_1\cap P_2\cap\cdots\cap P_s$.

Proof. Let $1\leqq i\leqq s$. Since $V_{(i)}T\not\subset J$, we obtain $TV_{(i)}T\not\subset J$ and so $TV_{(i)}T\not\subset b_T(J)$. If $TV_{(i)}V_{(i)}T\subset b_T(J)$, then we have $TV_{(i)}TV_{(i)}T\subset TV_{(i)}f_iTf_iV_{(i)}T\subset TV_{(i)}V_{(i)}T\subset b_T(J)$ and so $TV_{(i)}T\subset b_T(J)$, which contradicts $TV_{(i)}T\not\subset b_T(J)$. Hence we have $TV_{(i)}V_{(i)}T\not\subset b_T(J)$. We set here $P'_{(i)}=\{t_i\in T_{[i]}\,|\,TV_{(i)}t_iV_{(i)}T\subset b_T(J)\}$. Then, by the correspondence of prime ideals in a Morita contest

$$C_i=\begin{pmatrix} T & TV_{(i)} \\ V_{(i)}T & T_{[i]} \end{pmatrix},$$

$P'_{(i)}$ is a prime ideal of $T_{[i]}$ such that $P'_{(i)}\not\supset V_{(i)}TV_{(i)}$. We now claim that $A'\cap T_{[i]}\not\subset P'_{(i)}$ for all non-zero ideals A' of f_iSf_i. Let A' be a non-zero ideal of f_iSf_i such that $A'\cap T_{[i]}\subset P'_{(i)}$, and let $A=\{s\in S\,|\,f_iSsSf_i\subset A'\}$. Then A is an ideal of S. Since $f_iSf_iA'f_iB\subset f_iSf_iB\subset f_iB\subset S$ and $f_iS(f_iSf_iA'f_iB)Sf_i\subset f_iSf_iA'f_iBf_i\subset A'$, we have $f_iSf_iA'f_iB\subset A$. By the Morita context C_i, $b_T(J)$ is the prime ideal of T corresponding to the prime ideal $P'_{(i)}$ of $T_{[i]}$. Clearly, $V_{(i)}T(A\cap T)TV_{(i)}\subset f_iSASf_i\cap T_{[i]}\subset A'\cap T_{[i]}\subset P'_{(i)}$. This implies $A\cap T\subset b_T(J)$. Since $b_T(J)$ is a standard setting, we have $A=0$, and so $f_iSf_iA'f_iBf_i=0$. Recalling that f_iSf_i is a prime ring, we have $A'=0$, which is contradictory to $A'\neq 0$. Hence we obtain that $A'\cap T_{[i]}\not\subset P'_{(i)}$ for all non-zero ideals A' of f_iSf_i. If $P'_{(i)}\cap f_iR\neq 0$, then, by Lemma 3.1, there exists a non-zero ideal A' of f_iSf_i such that $0\neq A'\cap T_{[i]}\subset(P'_{(i)}\cap f_iR)T_{[i]}\subset P'_{(i)}$, which is a contradiction. Therefore we have $P'_{(i)}\cap f_iR=0$. Since $TV_{(i)}f_i(b_T(J)\cap R)f_iV_{(i)}T\subset b_T(J)$, it follows that $f_i(b_T(J)\cap R)f_i\subset P'_{(i)}\cap f_iR=0$, and hence $b_T(J)\cap R\subset r_R(f_i)=P_i$. This implies $b_T(J)\cap R\subset P_1\cap P_2\cap\cdots\cap P_s$, completing the proof.

Lemma 3.5. *Let J be a prime right ideal of T such that $b_T(J)$ is a standard setting. Then, for each $i=1, 2, \cdots, s$, $h_i(J)$ is a prime right ideal of $T_{[i]}$ such that $b_{T_{[i]}}(h_i(J))$ is a standard setting in the extension f_iSf_i of f_iR.*

Proof. It is clear that $h_i(J)$ is a right ideal of $T_{[i]}$. Let X and Y be right ideals of $T_{[i]}$ with $XY \subset h_i(J)$ and $Y \not\subset h_i(J)$. Then, we have $YV_{(i)}T \subset J$ and $XV_{(i)}TYV_{(i)}T \subset J$. Hence $XV_{(i)} \subset J$, and so $X \subset h_i(J)$. Therefore $h_i(J)$ is a prime right ideal of $T_{[i]}$. Next we shall show that $b_{T_{[i]}}(h_i(J)) \cap f_iR = 0$. Now, let f_ir be an arbitrary element in $b_{T_{[i]}}(h_i(J)) \cap f_iR$ $(r \in R)$. Then $T_{[i]}f_irV_{(i)}T \subset J$, and $V_{(i)}Tf_i = V_{(i)}f_iTf_i \subset T_{[i]}$. Hence we have $V_{(i)}TrV_{(i)}T = V_{(i)}Tf_irV_{(i)}T \subset J$, and so $rV_{(i)} \subset TrV_{(i)}T \subset b_T(J) \subset J$. Since the ideal D_i of R is an essential ideal of f_iR, we obtain $r(D_i \cap V_{(i)}) \subset b_T(J) \cap R \subset P_i$ by Lemma 3.4. Noting that $D_i \cap V_{(i)} \neq 0$ and $f_ir(D_i \cap V_{(i)}) \subset f_iP_i = 0$, we have $f_ir = 0$. Thus $b_{T_{[i]}}h_i(J) \cap f_iR = 0$. If $b_{T_{[i]}}(h_i(J))$ is not a standard setting, then there exists a non-zero ideal A of f_iSf_i with $A \cap T_{[i]} \subset b_{T_{[i]}}(h_i(J))$. By [1, Theorem 5.10], $A \cap f_iR \neq 0$, this is a contradiction to $b_{T_{[i]}}(h_i(J)) \cap f_iR = 0$. This completes the proof.

Combining Lemma 3.5 with Proposition 3.2, we obtain the following

Corollary 3.6. *If J is a prime right ideal of T such that $b_T(J)$ is a standard setting, then $h_i(J) \cap f_iR$ is a prime right ideal of f_iR with $b_{f_iR}h_i(J(\cap f_iR) = 0$ for all $i=1, 2, \cdots, s$.*

Now we arrived at the position to prove the following theorem which corresponds to Theorem 2.8.

Theorem 3.7 (Cutting down). *Let S be a prime torsionfree finite normalizing extension of a ring R, and T a ring with $R \subset T \subset S$. If J is a prime right ideal of T such that $b_T(J)$ is a standard setting, then there exist prime right ideals $K_1, K_2, \cdots, K_s$ of R such that $J \cap R = \cap_{i=1}^s K_i$, $b_R(K_i) = P_i$ for $i=1, 2, \cdots, s$, and $b_R(J \cap R) = \cap_{i=1}^s P_i \supset b_T(J) \cap R$.*

Proof. By Lemma 3.3, we may assume that $V_{(i)}T \not\subset J$ for $i=1, 2, \cdots, s$, and $V_{(i)}T \subset J$ for $i=s+1, \cdots, m$. Then, by Lemma 3.4, we have $b_R(J \cap R) \subset \cap_{i=1}^s P_i$. Let us set $K_i = \{r \in R \mid f_ir \in h_i(J) \cap f_iR\}$ for $i=1, 2, \cdots, s$. Then, by Corollary 3.6, $h_i(J) \cap f_iR$ is a prime right ideal of f_iR with $b_{f_iR}(h_i(J) \cap f_iR) = 0$. Hence it follows that K_i is a prime right ideal of R with $b_R(K_i) = P_i$. By making use of the same methods as in the proof of Theorem 2.8, we obtain $J \cap R = \cap_{i=1}^s K_i$.

Corollary 3.8. *Let S be an arbitrary fully torsionfree finite normalizing extension of R, and T a ring with $R \subset T \subset S$. If J is a prime right ideal of T,* *then there exist prime right ideals* $K_1, K_2, \cdots, K_r$ *of* R *such that* $J \cap R = \cap_{i=1}^{s} K_i$; $b_R(J \cap R) = \cap_{i=1}^{s} b_R(K_i) \supset b_T(J) \cap R$, $b_R(K_i) = P_i$ *for all* $i = 1, 2, \cdots, s$, *and the* P_i *are minimal prime over* $b_T(J) \cap R$.

Acknowledgments. This paper was written while the author visited the Department of Mathematics of Osaka City University. The author wishes to express his gratitude to the members of the Department for their hospitality, particularly to Professor M. Harada. He is also extremely grateful to the referee for helpful comments and suggestions leading to a reorganization of the paper into the present version.

References

[1] A.G. Heinicke and J.C. Robson: *Normalizing extensions: Prime ideals and incomparability*, J. Algebra **72** (1981), 237–268.

[2] A.G. Heinicke and J.C. Robson: *Intermediate normalizing extension*, Trans. Amer. Math. Soc. **282** (1984), 645–667.

[3] S. Jabbour: *Intermediate normalizing extensions*, Comm. Algebra **11** (1983), 1159–1602.

[4] K. Koh: *On one side ideals of a prime type*, Proc. Amer. Math. Soc. **28** (1971), 321–329.

[5] M. Lorenz: *Finite normalizing extensions of rings*, Math. Z. **176** (1981), 447–484.

[6] R. Resco: *Radicals of finite normalizing extensions*, Comm. Algebra **9** (1981), 713–725.

[7] J.C. Robson: Some results on ring extension, Lecture Notes, University of Essen, 1979.

[9] J.C. Robson and L.W. Small: *Liberal extensions*, Proc. London Math. Soc. **42** (1981), 87–103.

Department of Applied Mathematics
Okayama University of Science
Ridai-cho, Okayama 700
Japan