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PRIME ONE-SIDED IDEALS OF A FINITE NORMALIZING EXTENSION

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Introduction

Throughout the present paper, R will represent a ring with identity 1. Let I be a right ideal of R, and $b_R(I)=\{r\in R\,|\,Rr\subset I\}$. Then, $b_R(I)$ is the largest ideal of R contained in I. We shall call that I is a *prime right ideal* provided that if X and Y are right ideals of R with $XY\subset I$, then either $X\subset I$ or $Y\subset I$. It is clear that a maximal right ideal is a prime right ideal. If I is a prime right ideal, then $b_R(I)$ is a prime ideal. Next, let S be a ring extension of a ring R with the same identity 1. S is said to be a *left torsionfree* R-bimodule if $r_S(X)=0$ for every essential ideal X of R, where $r_S(X)$ is the right annihilator of X in S (cf. [1]). Right torsionfree is defined similarly, and S is said to be *torsionfree* if it is both left and right torsionfree. Moreover, S is said to be *fully torsionfree* if, for every prime ideal P of S, S/P is a right torsionfree $R/(P\cap R)$-bimodule (cf. [3]). Furthermore, we say that S is a *finite normalizing extension* (resp. a *liberal extension*) of R if there exists a finite subset $\{a_1, a_2, \cdots, a_n\}$ of S such that $S=\sum_{i=1}^n Ra_i$ and $Ra_i=a_iR$ for all $i=1, 2, \cdots, n$ (resp. $ra_i=a_ir$ for all $r\in R$ and for all $i=1, 2, \cdots, n$). A ring extension T of R is said to be an *intermediate normalizing extension* (resp. an *intermediate extension*) if there exists a finite normalizing extension (resp. a liberal extension) S of R containing T.

Recently, Heinicke and Robson [1, 2], Lorenz [5], Jabbour [3] and others, gave some descriptions of the relationship between the prime ideals of R and any intermediate normalizing extension T. In this paper, we shall verify that there is a similar relationship between the prime right ideals of R and T. In Section 1, we shall prove a "lying over" theorem for a liberal extension, and a "lying inside" theorem and a "lying outside" theorem for an intermediate extension. In Sections 2 and 3, we shall prove a "cutting down" theorem for a fully torsionfree finite normalizing extension and an intermediate normalizing extension of a fully torsionfree finite normalizing extension.

1. Prime right ideals of a liberal extension

In this section, we discuss the relationship of prime right ideals of a liberal extension and an intermediate extension.

Theorem 1.1 (Lying over). *Let S be a liberal extension of a ring R. If K is a prime right ideal of R, then there exists a prime right ideal I of S such that $I \cap R=K$. When this is the case, there hold $b_R(I) \cap R=b_R(K)$ and $I \cap R=KS \cap R=K$.*

Proof. Since $b_R(K)$ is a prime ideal, there exists a prime ideal P of S such that $P \cap R=b_R(K)$ and P is a maximal with respect to $P \cap R=b_R(K)$ by [9, Theorem 4.1]. By [9, Lemma 3.2], S/P is a liberal extension of $R/b_R(K)$. Hence we may assume that S is a prime liberal extension of a prime ring R such that $B \cap R \neq 0$ for each non-zero ideal B of S, and K is a prime right ideal of R with $b_R(K)=0$. Since, by [9, Lemma 3.5], there is a non-zero ideal A of S such that $R+A$ is contained in a full matrix ring $M_m(R)$, we have $KA \subset M_m(K)$, and so $KA \cap R \subset K$. Consequently, by Zorn's Lemma, there exists a right ideal I of S which is maximal with respect to $I \cap R \subset K$ and $I \supset KA$. Let X and Y be right ideals of S with $XY \subset I$ and $Y \not\subset I$. Then we have $((X+I) \cap R)(SY \cap R) \subset K$. Since $SY \neq 0$, $SY \cap R$ is a non-zero ideal of R, and so $SY \cap R \not\subset K$. Therefore $(X+I) \cap R \subset K$, and so $X \subset I$. This implies that I is a prime right ideal of S. According to [9, Theorem 4.6], it is clear that $b_S(I)=0$ and $A \not\subset I$. Since $KSA=KA \cap I$, we have $KS \subset I$ and $KS \cap R=I \cap R=K$.

By making use of the same methods as in the proof of the above theorem, we readily obtain the following

Corollary 1.2 (Going up). *Let S be a liberal extension of R. If $K_0 \supset K$ are prime right ideals of R and I is a prime right ideal of S with $I \cap R=K$, then there exists a prime right ideal I_0 of S such that $I_0 \supset I$ and $I_0 \cap R=K_0$.*

If P and Q are prime ideals of S such that $P \supset Q$ and $P \cap R=Q \cap R$, then $P=Q$ ([1, Theorem 5.10]). We shall now present some examples of liberal extensions in which there does not hold an "incomparability" theorem for prime right ideals.

EXAMPLE 1.3. Let D be a division ring, and $S=\begin{pmatrix} D & D \\ D & D \end{pmatrix}$. Then S is a liberal extension of $D=\left\{\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \middle| d \in D\right\}$, and $I=\begin{pmatrix} 0 & 0 \\ D & D \end{pmatrix}$ is a maximal right ideal of S with $b_S(I)=0$. But $I \cap D=0$ which is a prime ideal of D.

EXAMPLE 1.4. Let A be a simple ring with a non-zero maximal right ideal M, and $S=\begin{pmatrix} A & A \\ A & A \end{pmatrix}$. Then $I_1=\begin{pmatrix} M & M \\ A & A \end{pmatrix}$ and $I_2=\begin{pmatrix} M & M \\ M & M \end{pmatrix}$ are prime right ideals of S such that $I_1 \supsetneqq I_2$, but $I_1 \cap A=M=I_2 \cap A$, which is a prime right ideal but not an ideal.

In the rest of this section, we investigate the relationship of the prime right ideals between R and any intermediate extension.

Theorem 1.5 (Lying outside). *Let T be a intermediate extension of R with a liberal extension S of R containing T, and J a prime right ideal of T. Then $J \cap R$ is a prime right ideal of R, and there exists a prime right ideal I of S such that $I \cap T \subset J$ and $b_S(I) \cap R=b_T(J) \cap R=b_R(J \cap R)=b_R(I \cap R)$.*

Proof. By Zorn's Lemma, there exists an ideal P of S which is maximal with respect to the property $P \cap T \subset b_T(J)$. Since $b_T(J)$ is a prime ideal of T, P is prime and $P \cap R=b_T(J) \cap R$ (cf. [7, Theorem 12.3] and [8, Theorem 3.2]). Then, since $S/P \supset T/(P \cap T) \supset R/(P \cap R)$, we may assume that S is a prime liberal extension of a prime ring R, and T is a subring of S containing R, and J is a prime right ideal of T such that $b_T(J) \cap R=0$ and $Q \cap T \not\subset J$ for each non-zero ideal Q of S. By Zorn's Lemma, there is a right ideal I which is maximal with respect to the property $I \cap T \subset J$. It is clear that $b_S(I)=0$. Suppose that X and Y are right ideals of S with $XY \subset I$ and $Y \not\subset I$. Then $((X+I) \cap T) \times (SY \cap T) \subset I \cap T \subset J$ and SY is a non-zero ideal of S. Therefore we obtain $(X+I) \cap T \subset J$, and so $X \subset I$.Thus I is a prime right ideal of S. Next we claim that $J \cap R$ is a prime right ideal of R. To prove this, assume that X_1 and X_2 are right ideals of R with $X_1X_2 \subset J \cap R$ and $X_2 \not\subset J \cap R$. Now, by [8, Proposition 2.5], there exist a liberal extension $S'=\sum_{j=1}^{p} b_jCR$ of CR and a non-zero ideal X of CR such that $XS' \subset CT \subset S' \subset CS$, where C is the center of the complete ring of quotients of R, and $b_1, b_2, \cdots, b_p \in V_{CS}(CR)$. Moreover, by [8, Lemma 4.1], there exist non-zero ideals Y_1, Y_2 of R such that $\sum_{j=1}^{p} b_j Y_2$ is a ring (without 1) and $TY_1T \subset \sum_{j=1}^{p} b_j Y_2 \subset T$. Then we have $X_1TY_1TX_2Y_2T \subset X_1 \sum_j b_j Y_2 X_2 Y_2 T \subset X_1 Y_2 X_2 \sum_j b_j Y_2 T \subset X_1X_2T \subset J$. Since $Y_1 \neq 0$, $Y_2 \neq 0$ and $X_2 \neq 0$, $Y_1X_2Y_2$ is a non-zero ideal of R contained in the ideal $TY_1TX_2Y_2T$ of T. Since $b_T(J) \cap R=0$, we have $X_1T \subset J$. Hence $J \cap R$ is a prime right ideal. Once again, using [8, Lemma 4.1], we obtain that $TY_1Tb_R(J \cap R)Y_2T \subset J$, and so $b_R(J \cap R)Y_2T \subset J$. This implies that $b_R(J \cap R)=0$. The rest is clear.

Corollary 1.6. *Let R, T, S and J be as in the above theorem.If $(J \cap R)S \cap T \subset J$, then there exists a prime right ideal I of S such that $I \cap T \subset J$ and $I \cap R=J \cap R$. In this case, there holds that $b_S(I) \cap R=b_T(J) \cap R=b_R(J \cap R)= b_R(I \cap R)$.*

Let T be an intermediate extension of R, and S a fixed liberal extension of R containing T. Let K be a prime right ideal of R and I a prime right ideal of S with $I\cap R=K$. Then, by Zorn's Lemma, there exists a right ideal J of T which is maximal with respect to the property $J\cap R=K$ and $J\supset I\cap T$. In this situation, we shall prove the following

Lemma 1.7. *$b_R(K)=b_T(J)\cap R=b_S(I)\cap R$ and $b_S(I)\cap T\subset b_T(J)$.*

Proof. Obviously we obtain $b_S(I)\cap T\subset b_T(J)$. Since $b_R(K)S$ is an ideal of S contained in I, this implies $b_R(K)\subset b_R(K)S\cap R\subset b_S(I)\cap R\subset b_T(J)\cap R\subset b_R(K)$.

Proposition 1.8 (Lying inside). *J is a prime right ideal of T if and only if $b_T(J)$ is a prime ideal of T.*

Proof. If $b_T(J)$ is prime, then $b_T(J)$ is an ideal Q of T which is maximal with respect to $Q\cap R=b_R(K)$ and $Q\supset b_S(I)\cap T$ (cf. [7, Theorem 12.7 and 8, Theorem 3.3]). Suppose that X and Y are right ideals of T with $XY\subset I$ and $Y\not\subset J$. Then $TY+b_T(J)\neq b_T(J)$ and $((X+J)\cap R)((TY+b_T(J))\cap R)\subset J\cap R=K$. Hence it follows from the maximality of $b_T(J)$ that $(X+J)\cap R\subset K$, and so J is a prime right ideal.

The following examples show that whether J is a prime right ideal or not.

Example 1.9. Let A, M and S be as in Example 1.4, and let $T=\begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$. Then $I=\begin{pmatrix} M & M \\ A & A \end{pmatrix}$ is a prime right ideal of S with $I\cap A=M$. Since $\begin{pmatrix} M & A \\ 0 & A \end{pmatrix}\begin{pmatrix} A & A \\ 0 & 0 \end{pmatrix}\subset I\cap T$, $I\cap T$ is not prime right. $\begin{pmatrix} M & A \\ 0 & A \end{pmatrix}$ is the required J, and which is a prime right ideal of T.

Example 1.10. Let A be a simple ring having at least two maximal right ideals, and let M and N be distinct maximal right ideals. Let us put $S=\begin{pmatrix} A & A & A \\ A & A & A \\ A & A & A \end{pmatrix}$ and $T=\begin{pmatrix} A & A & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$. Since A is two-sided simple, $I=\begin{pmatrix} M & M & M \\ A & A & A \\ N & N & N \end{pmatrix}$ is a prime right ideal of S and $I\cap A=M\cap N$ is a prime right ideal of A. Hence, $\begin{pmatrix} M & A & 0 \\ 0 & A & 0 \\ 0 & 0 & N \end{pmatrix}$ is the required J. However, since $\begin{pmatrix} M & A & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}\begin{pmatrix} A & A & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}\subset J$, J is not a prime right ideal.

2. Prime right ideals of a finite normalizing extension

In the rest of our study, suppose that S is a finite normalizing extension of R.

Proposition 2.1 (Lying over). *Suppose that S is a finite normalizing extension of R. If K is a prime right ideal of R, then there exists a prime right ideal I of S such that $I\cap R\subset K$ and $b_R(K)$ is a minimal prime ideal over $b_S(I)\cap R$.*

Proof. Since $b_R(K)$ is a prime right ideal of R, there exists a prime ideal Q of S such that $b_R(K)$ is a minimal prime ideal over $Q\cap R$. Hence we may assume that S is a prime finite normalizing extension of R and K is a prime right ideal of R such that $A\cap R\not\subset b_R(K)$ for each non-zero ideal A of S and $b_R(K)$ is minimal prime. We next claim that there is a prime right ideal I of S which satisfies $I\cap R\subset K$ and $b_S(I)\cap R=0$. By Zorn's Lemma, there exists a right ideal I of S which is maximal with respect to $I\cap R\subset K$. Let X and Y be right ideals of S such that $XY\subset I$ and $Y\not\subset I$. Since $((X+I)\cap R)\times(SY\cap R)\subset I\cap R\subset K$ and SY is a non-zero ideal of S, we have $X\subset I$. Thus I is a prime right ideal of R. Clearly we have $b_S(I)=0$.

Lemma 2.2. *Let S be a torsionfree finite normalizing extension of R. If Y is an essential ideal of R, then $b_S(YS)\neq 0$.*

Proof. If X is an R-S-subbimodule of S with $YS\cap X=0$, then $YX=0$. Since $Y\neq 0$, there holds $X=0$. Hence it follows that YS is an essential R-S-subbimodule of S. By [6, Lemma 4], we have $b_S(YS)\neq 0$.

Proposition 2.3. *Let S be a prime torsionfree finite normalizing extension of a prime ring R. If I is a prime right ideal of S with $b_S(I)=0$, then $I\cap R$ is a prime right ideal of R with $b_R(I\cap R)=0$.*

Proof. Assume that X and Y are right ideals of R with $XY\subset I\cap R$ and $Y\not\subset I\cap R$. Then we obviously obtain $XSb_S(RYS)\subset I$, and hence we have either $XS\subset I$ or $b_S(RYS)\subset I$. On the other hand, since R is prime, RY is an essential ideal of R, and so, $b_S(RYS)$ is a non-zero ideal of S by Lemma 2.2. Hence there holds $XS\subset I$. Therefore, it follows that $I\cap R$ is a prime right ideal of R. The rest of the proof is clear from Lemma 2.2.

If R is not prime, then it may happen that $I\cap R$ is not a prime right ideal of R.

EXAMPLE 2.4. Let A, M and S be as in Example 1.4. Putting $R=\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, S is a prime torsionfree finite normalizing extension of R and R is not prime.

Considering the prime right ideal $I=\begin{pmatrix} M & M \\ M & M \end{pmatrix}$ of S, $I\cap R=\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$ is not a prime right ideal of R.

Let $S=\sum_{i=1}^{n} Ra_i$ $(Ra_i=a_iR)$ be a prime torsionfree finite normalizing extension of R. Let $Q(S)$ be the right Martindale quotient of S. Then, there exist orthogonal idempotents $f_1, f_2, \cdots, f_m$ in $V_{Q(S)}(R)=\{q\in Q(S) \mid rq=qr$ for all $r\in R\}$ such that $f_1+f_2+\cdots+f_m=1$ and $m\leqq n$. We set here $P_i=r_R(f_i)$, $i=1, 2, \cdots, m$. Then, the P_i are m distinct minimal prime ideals of R such that $\cap_{i=1}^{m} P_i=0$, and $R/P_i\cong R/P_1$ for all i. Let us set $D_i=\cap_{j=1, j\neq i}^{m} P_j$, for all i. Then each D_i is a non-zero ideal of R with $f_id=d$ (for all $d\in D_i$), and so D_i is an essential ideal of f_iR. Since $f_i\in Q(S)$, there exists an essential ideal B of S such that $f_iB\subset S$ for all i (cf. [1] and [5]). By [1, Theorem 5.7], each f_iSf_i $(1\leqq i\leqq m)$ is a prime torsionfree finite normalizing extension of the prime ring f_iR. Now, let I be a prime right ideal of S with $b_S(I)=0$. Let us set $g_i(I)=\{f_isf_i\in f_iSf_i \mid f_isf_iB\subset I\}$. Then we have $f_iSf_iB\subset f_iB\subset S$ by [1, Proposition 5.5], and so $g_i(I)$ is a right ideal of f_iSf_i. Then $g_i(I)=f_iSf_i$ if and only if $f_iB\supset I$. Under this situation, we shall prove the following

Lemma 2.5. *There exists an* f_i *such that* $g_i(I)\neq f_iSf_i$. *Such an* f_i *is independent of a choice of an essential ideal* B.

Proof. If $g_i(I)=f_iSf_i$ for $i=1, 2, \cdots, m$, then we have $B\subset f_1B+f_2B+\cdots+f_mB\subset I$. This is a contradiction. To prove the rest, for essential ideals B, B' of S, we assume that $f_iB\not\subset I$ and $f_iB'\subset I$. Since $f_iBB'\subset f_iB'\subset I$ and $f_iB\not\subset I$, we have $B'\subset I$, which contradicts $b_S(I)=0$. Hence $f_iB\not\subset I$ if and only if $f_iB\not\subset I$.

By Lemma 2.5, we may assume that $f_iB\not\subset I$ if $i=1, 2, \cdots, t$, and $f_iB\subset I$ if $i=t+1, \cdots, m$.

Lemma 2.6. *For each* $i=1, 2, \cdots, t$, $g_i(I)$ *is a prime right ideals of* f_iSf_i *with* $b_{f_iSf_i}(g_i(I))=0$.

Proof. For $s=\sum_{j=1}^{n} r_ja_j\in S(r_j\in R)$, we put $s^{\sharp(i)}=\sum_{j\in\sharp(i)} r_ja_j$, where $\sharp(i)=\{j \mid f_ia_jf_i\neq 0\}$, and then $f_isf_i=s^{\sharp(i)}f_i$ ([1, Proposition 5.4]). Let $s^{\sharp(i)}f_i$ and $s'^{\sharp(i)}f_i$ be any elements of f_iSf_i such that $s^{\sharp(i)}f_if_iSf_is'^{\sharp(i)}f_i\subset g_i(I)$ and $s'^{\sharp(i)}f_i\notin g_i(I)$. Then we obtain $s^{\sharp(i)}f_iB\cdot Ss'^{\sharp(i)}f_iB\subset s^{\sharp(i)}f_iSs'^{\sharp(i)}f_iB\subset I$. Since $f_is'f_i\notin g_i(I)$, it follows that $Ss'^{\sharp(i)}f_iB\subset I$ and so $f_isf_iB\subset I$. Thus, $g_i(I)$ is a prime right ideal of f_iSf_i. Next, if $f_isf_i\in g_i(I)$ and $f_iSf_if_isf_i\subset g_i(I)$, then $f_iB\cdot Sf_isf_iB\subset f_iSf_isf_iB\subset I$. Since $f_iB\not\subset I$, we have $Sf_isf_iB\subset I$. This implies $Sf_isf_iB=0$, and so $f_isf_iB=0$. Since $f_isf_i\in Q(S)$ and B is an essential ideal of S,it follows that $b_{f_iSf_i}(g_i(I))=0$.

Combining Proposition 2.3 with Lemma 2.6, we obtain the following

Corollary 2.7. *$g_i(I)\cap f_iR$ is a prime right ideal of f_iR such that $b_{f_iR}(g_i(I)\cap f_iR)=0$ for $i=1, 2, \cdots, t$.*

Theorem 2.8 (Cutting down). *Let S be a prime torsionfree finite normalizing extension of R. If I is a prime right ideal of S such that $b_S(I)=0$, then there exist prime right ideals $K_1, K_2, \cdots, K_t$ of R such that $\cap_{i=1}^{t} K_i=I\cap R$, $b_R(K_i)=P_i$ for $i=1, 2, \cdots, t$. In this case, there holds $b_R(I\cap R)=\cap_{i=1}^{t} P_i$.*

Proof. By Lemma 2.5, we may assume that $f_iB\not\subset I$ for $i=1, 2, \cdots, t$, and $f_iB\subset I$ for $i=t+1, \cdots, m$. By Corollary 2.7, $g_i(I)\cap f_iR$ $(1\leqq i\leqq t)$ is a prime right ideal of f_iR such that $b_{f_iR}(g_i(I)\cap f_iR)=0$. Here we set $K_i=\{r\in R \mid f_ir\in g_i(I)\cap f_iR\}$ $(1\leqq i\leqq t)$. Then, it is easily seen that each K_i is a prime right ideal of R such that $b_R(K_i)=P_i$. Now we claim that $\cap_{i=1}^{t} K_i=I\cap R$. Actually, if $r\in\cap_{i=1}^{t} K_i$, then $f_ir\in g_i(I)\cap f_iR$ for $i=1, 2, \cdots, t$, and so $f_irB\subset I$. On the other hand, for $i=t+1, \cdots, m$, $f_irB\subset f_iB\subset I$. Hence $rB\subset f_1rB+f_2rB+\cdots+f_trB+f_{t+1}rB+\cdots+f_mrB\subset I$. Since I is the prime right ideal of S with $b_S(I)=0$, we have therefore $r\in I\cap R$. Thus $\cap_{i=1}^{t} K_i\subset I\cap R$. Conversely, for $r\in I\cap R$, we have $rf_iB\subset I$, which implies that $f_irf_i\in g_i(I)\cap f_iR$ for $i=1, 2, \cdots, t$, and so $r\in\cap_{i=1}^{t} K_i$. Therefore, $\cap_{i=1}^{t} K_i=I\cap R$. The rest of the proof is clear.

Corollary 2.9. *Let S be an arbitrary fully torsionfree finite normalizing extension of R. If I is a prime right ideal of S, then there exist prime right ideals $K_1, K_2, \cdots, K_t$ of R such that $I\cap R=\cap_{i=1}^{t} K_i$, and each $b_R(K_i)$ $(1\leqq i\leqq n)$ is a minimal prime ideal of R over $b_S(I)\cap R$. In this case, there holds $b_R(I\cap R)=\cap_{i=1}^{t} b_R(K_i)\supset b_S(I)\cap R$.*

EXAMPLE 2.10. Let A and M be as in Example 1.4. Let us set $S=\begin{pmatrix} A & A & A \\ A & A & A \\ A & A & A \end{pmatrix}$ and $R=\begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$. Since R is an only essential ideal of R, S is a prime torsionfree finite normalizing extension of R. For the prime right ideal $I=\begin{pmatrix} M & M & M \\ M & M & M \\ A & A & A \end{pmatrix}$ of S with $b_S(I)=0$, we immediately obtain that $I\cap R=\begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & A \end{pmatrix}$ is a right ideal of R which is not prime and not an ideal. On the other hand, $P_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$, $P_2=\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{pmatrix}$ and $P_3=\begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are the all minimal prime ideals of R. Moreover, $K_1=\begin{pmatrix} M & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$ and $K_2=\begin{pmatrix} A & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & A \end{pmatrix}$ are prime right ideals of R such that $I\cap R=K_1\cap K_2$ and $b_R(I\cap R)=b_R(K_1)\cap b_R(K_2)=P_1\cap P_2\neq 0$ and $t=2\neq 3=m$.

3. Prime right ideals of an intermediate normalizing extension

In this section, we shall prove a "cutting down" theorem for prime right ideals of an intermediate normalizing extension which corresponds to that of Section 2. Throughout this section, suppose that T is an intermediate normalizing extension of R, and S is a fixed finite normalizing extension of R containing T.

Lemma 3.1. *Let S be a torsionfree R-bimodule. If Y is an essential ideal of R, then YT is an essential R-submodule of T and there exists a non-zero ideal A of S with $0\neq A\cap T\subset b_T(YT)$.*

Proof. Since Y be an essential ideal of R, by making use of the same methods as in the proof of Lemma 2.2, we readily obtain that YT is an essential R-subbimodule of T. Let T^* be a relative complement of T in the R-bimodule S. Then, by [6, Lemma 4], $YT+T^*$ contains a non-zero ideal A of S which is an essential R-subbimodule of S, and so $0\neq A\cap T\subset b_T(YT)$.

Now, let Q be a prime ideal of T. Then, by [3, Proposition 5.6], there exists a prime ideal P of S such that $P\cap T\subset Q$ and $A\cap T\not\subset Q$ for all ideals $A\supsetneqq P$ of S. Obviously, S/P is a finite normalizing extension of $R/(P\cap R)$ and $Q/(P\cap T)$ is a prime ideal of an intermediate normalizing extension $T/(P\cap T)$ of $R/(P\cap R)$ such that $B/P\cap T/(P\cap T)\not\subset Q/(P\cap T)$ for each non-zero ideal B/P of S/P. As in [2], Q will be called a *standard setting* if S is a prime ring and Q satisfies $B\cap T\not\subset Q$ for each non-zero ideal B of S.

Proposition 3.2. *Let S be a prime torsionfree finite normalizing extension of a prime ring R. If J is a prime right ideal of T such that $b_T(J)$ is a standard setting, then $J\cap R$ is a prime right ideal of R with $b_R(J\cap R)=0$.*

Proof. Let X and Y be right ideals of R with $XY\subset J\cap R$ and $Y\not\subset J\cap R$. Since R is prime, RY is an essential ideal of R, and so, by Lemma 3.1, there exist a non-zero ideal A of S with $0\neq A\cap T\subset b_T(RYT)$. Hence, we have $b_T(RYT)\not\subset J$ since $b_T(J)$ is a standard setting. Noting $XTb_T(RYT)\subset J$, we obtain $X\subset XT\cap R\subset J\cap R$. The assertion $b_R(J\cap R)=0$ is clear by Lemma 3.1.

Throughout the rest of our study, we assume that S is a prime torsionfree finite normalizing extension of R. The notations in Section 2 will be used again here. As was seen, each f_iSf_i $(1\leqq i\leqq m)$ is a prime torsionfree finite normalizing extension of the prime ring f_iR. Now, by $T_{[i]}$, we denote the subring of the prime ring f_iSf_i which is generated by f_iTf_i. Then, by [3, Proposition 5.1 (2)], there exists an ideal $V_{(i)}$ of $T_{[i]}$ such that $V_{(i)}\subset T$ and $V_{(i)}$ is an essential f_iR-subbimodule of $T_{[i]}$. Then $V_{(i)}$ can be regarded as an essential R-subbimodule of $T_{[i]}$. Hence, $\sum_{i=1}^m V_{(i)}=\sum_{i=1}^m\oplus V_{(i)}$ is an essential R-subbimodule of $\sum_{i=1}^m\oplus T_{[i]}$. It is obvious that $(\sum_{i=1}^T T_{[i]})\cap R=R$. Moreover, for a prime right ideal J of T such that $b_T(J)$ is a standard setting, we set $h_i(J)=\{q\in T_{[i]}\,|\,qV_{(i)}\subset J\}$. Then, $h_i(J)=T_{[i]}$ if and only if $V_{(i)}T\subset J$. Using a similar argument to Lemma 2.5 making use of the above remark and Lemma 3.1, we obtain the following

Lemma 3.3. $(\sum_{i=1}^m V_{(i)})\cap R$ *is an essential R-subbimodule of R, and* $V_{(i)}T\not\subset J$ *for some* f_i.

By Lemma 3.3, we may assume that $V_{(i)}T\not\subset J$ for $i=1, 2, \cdots, s$, and $V_{(i)}T\subset J$ for $i=s+1, \cdots, m$. In this situation, we shall prove the following

Lemma 3.4. $b_T(J)\cap R\subset P_1\cap P_2\cap\cdots\cap P_s$.

Proof. Let $1\leqq i\leqq s$. Since $V_{(i)}T\not\subset J$, we obtain $TV_{(i)}T\not\subset J$ and so $TV_{(i)}T\not\subset b_T(J)$. If $TV_{(i)}V_{(i)}T\subset b_T(J)$, then we have $TV_{(i)}TV_{(i)}T\subset TV_{(i)}f_iTf_iV_{(i)}T\subset TV_{(i)}V_{(i)}T\subset b_T(J)$ and so $TV_{(i)}T\subset b_T(J)$, which contradicts $TV_{(i)}T\not\subset b_T(J)$. Hence we have $TV_{(i)}V_{(i)}T\not\subset b_T(J)$. We set here $P'_{(i)}=\{t_i\in T_{[i]}\,|\,TV_{(i)}t_iV_{(i)}T\subset b_T(J)\}$. Then, by the correspondence of prime ideals in a Morita contest

$$C_i=\begin{pmatrix} T & TV_{(i)} \\ V_{(i)}T & T_{[i]}\end{pmatrix},$$

$P'_{(i)}$ is a prime ideal of $T_{[i]}$ such that $P'_{(i)}\not\supset V_{(i)}TV_{(i)}$. We now claim that $A'\cap T_{[i]}\not\subset P'_{(i)}$ for all non-zero ideals A' of f_iSf_i. Let A' be a non-zero ideal of f_iSf_i such that $A'\cap T_{[i]}\subset P'_{(i)}$, and let $A=\{s\in S\,|\,f_iSsSf_i\subset A'\}$. Then A is an ideal of S. Since $f_iSf_iA'f_iB\subset f_iSf_iB\subset f_iB\subset S$ and $f_iS(f_iSf_iA'f_iB)Sf_i\subset f_iSf_iA'f_iBf_i\subset A'$, we have $f_iSf_iA'f_iB\subset A$. By the Morita context C_i, $b_T(J)$ is the prime ideal of T corresponding to the prime ideal $P'_{(i)}$ of $T_{[i]}$. Clearly, $V_{(i)}T(A\cap T)TV_{(i)}\subset f_iSASf_i\cap T_{[i]}\subset A'\cap T_{[i]}\subset P'_{(i)}$. This implies $A\cap T\subset b_T(J)$. Since $b_T(J)$ is a standard setting, we have $A=0$, and so $f_iSf_iA'f_iBf_i=0$. Recalling that f_iSf_i is a prime ring, we have $A'=0$, which is contradictory to $A'\neq 0$. Hence we obtain that $A'\cap T_{[i]}\not\subset P'_{(i)}$ for all non-zero ideals A' of f_iSf_i. If $P'_{(i)}\cap f_iR\neq 0$, then, by Lemma 3.1, there exists a non-zero ideal A' of f_iSf_i such that $0\neq A'\cap T_{[i]}\subset(P'_{(i)}\cap f_iR)T_{[i]}\subset P'_{(i)}$, which is a contradiction. Therefore we have $P'_{(i)}\cap f_iR=0$. Since $TV_{(i)}f_i(b_T(J)\cap R)f_iV_{(i)}T\subset b_T(J)$, it follows that $f_i(b_T(J)\cap R)f_i\subset P'_{(i)}\cap f_iR=0$, and hence $b_T(J)\cap R\subset r_R(f_i)=P_i$. This implies $b_T(J)\cap R\subset P_1\cap P_2\cap\cdots\cap P_s$, completing the proof.

Lemma 3.5. *Let J be a prime right ideal of T such that $b_T(J)$ is a standard setting. Then, for each $i=1, 2, \cdots, s$, $h_i(J)$ is a prime right ideal of $T_{[i]}$ such that $b_{T_{[i]}}(h_i(J))$ is a standard setting in the extension f_iSf_i of f_iR.*

Proof. It is clear that $h_i(J)$ is a right ideal of $T_{[i]}$. Let X and Y be right ideals of $T_{[i]}$ with $XY \subset h_i(J)$ and $Y \not\subset h_i(J)$. Then, we have $YV_{(i)}T \subset J$ and $XV_{(i)}TYV_{(i)}T \subset J$. Hence $XV_{(i)} \subset J$, and so $X \subset h_i(J)$. Therefore $h_i(J)$ is a prime right ideal of $T_{[i]}$. Next we shall show that $b_{T_{[i]}}(h_i(J)) \cap f_iR = 0$. Now, let f_ir be an arbitrary element in $b_{T_{[i]}}(h_i(J)) \cap f_iR$ $(r \in R)$. Then $T_{[i]}f_irV_{(i)}T \subset J$, and $V_{(i)}Tf_i = V_{(i)}f_iTf_i \subset T_{[i]}$. Hence we have $V_{(i)}TrV_{(i)}T = V_{(i)}Tf_irV_{(i)}T \subset J$, and so $rV_{(i)} \subset TrV_{(i)}T \subset b_T(J) \subset J$. Since the ideal D_i of R is an essential ideal of f_iR, we obtain $r(D_i \cap V_{(i)}) \subset b_T(J) \cap R \subset P_i$ by Lemma 3.4. Noting that $D_i \cap V_{(i)} \neq 0$ and $f_ir(D_i \cap V_{(i)}) \subset f_iP_i = 0$, we have $f_ir = 0$. Thus $b_{T_{[i]}}h_i(J) \cap f_iR = 0$. If $b_{T_{[i]}}(h_i(J))$ is not a standard setting, then there exists a non-zero ideal A of f_iSf_i with $A \cap T_{[i]} \subset b_{T_{[i]}}(h_i(J))$. By [1, Theorem 5.10], $A \cap f_iR \neq 0$, this is a contradiction to $b_{T_{[i]}}(h_i(J)) \cap f_iR = 0$. This completes the proof.

Combining Lemma 3.5 with Proposition 3.2, we obtain the following

Corollary 3.6. *If J is a prime right ideal of T such that $b_T(J)$ is a standard setting, then $h_i(J) \cap f_iR$ is a prime right ideal of f_iR with $b_{f_iR}h_i(J(\cap f_iR) = 0$ for all $i=1, 2, \cdots, s$.*

Now we arrived at the position to prove the following theorem which corresponds to Theorem 2.8.

Theorem 3.7 (Cutting down). *Let S be a prime torsionfree finite normalizing extension of a ring R, and T a ring with $R \subset T \subset S$. If J is a prime right ideal of T such that $b_T(J)$ is a standard setting, then there exist prime right ideals $K_1, K_2, \cdots, K_s$ of R such that $J \cap R = \cap_{i=1}^{s} K_i$, $b_R(K_i) = P_i$ for $i=1, 2, \cdots, s$, and $b_R(J \cap R) = \cap_{i=1}^{s} P_i \supset b_T(J) \cap R$.*

Proof. By Lemma 3.3, we may assume that $V_{(i)}T \not\subset J$ for $i=1, 2, \cdots, s$, and $V_{(i)}T \subset J$ for $i=s+1, \cdots, m$. Then, by Lemma 3.4, we have $b_R(J \cap R) \subset \cap_{i=1}^{s} P_i$. Let us set $K_i = \{r \in R \mid f_ir \in h_i(J) \cap f_iR\}$ for $i=1, 2, \cdots, s$. Then, by Corollary 3.6, $h_i(J) \cap f_iR$ is a prime right ideal of f_iR with $b_{f_iR}(h_i(J) \cap f_iR) = 0$. Hence it follows that K_i is a prime right ideal of R with $b_R(K_i) = P_i$. By making use of the same methods as in the proof of Theorem 2.8, we obtain $J \cap R = \cap_{i=1}^{s} K_i$.

Corollary 3.8. *Let S be an arbitrary fully torsionfree finite normalizing extension of R, and T a ring with $R \subset T \subset S$. If J is a prime right ideal of T,* *then there exist prime right ideals* $K_1, K_2, \cdots, K_r$ *of* R *such that* $J\cap R=\cap_{i=1}^{s}K_i$; $b_R(J\cap R)=\cap_{i=1}^{s}b_R(K_i)\supset b_T(J)\cap R$, $b_R(K_i)=P_i$ *for all* $i=1,2,\cdots,s$, *and the* P_i *are minimal prime over* $b_T(J)\cap R$.

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References

[1] A.G. Heinicke and J.C. Robson: *Normalizing extensions: Prime ideals and incomparability*, J. Algebra **72** (1981), 237–268.
[2] A.G. Heinicke and J.C. Robson: *Intermediate normalizing extension*, Trans. Amer. Math. Soc. **282** (1984), 645–667.
[3] S. Jabbour: *Intermediate normalizing extensions*, Comm. Algebra **11** (1983), 1159–1602.
[4] K. Koh: *On one side ideals of a prime type*, Proc. Amer. Math. Soc. **28** (1971), 321–329.
[5] M. Lorenz: *Finite normalizing extensions of rings*, Math. Z. **176** (1981), 447–484.
[6] R. Resco: *Radicals of finite normalizing extensions*, Comm. Algebra **9** (1981), 713–725.
[7] J.C. Robson: Some results on ring extension, Lecture Notes, University of Essen, 1979.
[9] J.C. Robson and L.W. Small: *Liberal extensions*, Proc. London Math. Soc. **42** (1981), 87–103.

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