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ON THE EXPONENTIAL DECAY OF SOLUTIONS OF THE WAVE EQUATION WITH THE POTENTIAL FUNCTION

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1. Introduction and results. We consider the Schroedinger operator $L=-\Delta+c(x)$ with the potential function $c(x)$ satisfying the following condition (A_1) in the whole 3-dimensional Euclidean space E, where Δ designates the 3-dimensional Laplacian and x a position vector in E with its length $|x|$:

$$(\mathrm{A}_1)\qquad \begin{cases} c(x) \text{ is a real-valued } \mathscr{B}^1 \text{ function defined on } E \text{ and satisfies} \\ \qquad\qquad |c(x)| \leqslant C_1 e^{-\delta|x|} \qquad (x\in E), \\ \text{where } C_1 \text{ and } \delta \text{ are positive constants.} \end{cases}$$

Here $\mathscr{B}^1$ stands for the space of all bounded, continuous functions $f(x)$ defined over E with bounded, continuous first derivatives.

It is well known that under condition (A_1) the symmetric operator L on $\mathscr{D}$ is lower semi-bounded and essentially self-adjoint in $L^2=L^2(E)$, where $\mathscr{D}$ consists of all infinitely many times differentiable functions with compact support in E (see T. Kato [2], Section 6, Theorem 1). Then we denote again by L the unique self-adjoint extension with domain $\mathscr{D}^2_{L^2}$. $\mathscr{D}^m_{L^2}$ $(m=1, 2)$ is the completion of the space $\mathscr{D}$ with respect to the norm $||f||_m=\left(\sum_{|\alpha|\leqslant m}\int_E |D^\alpha f(x)|^2 dx\right)^{1/2}$, or is equivalently the space with the norm $||\cdot||_m$ of all functions $f(x)\in L^2$ whose derivatives $D^\alpha f(x)$ $(|\alpha|\leqslant m)$ in the distribution sense all belong to L^2 ($\alpha=(\alpha_1, \alpha_2, \alpha_3)$ with the α_k's non-negative integers; $D_k=\dfrac{\partial}{\partial x_k}$, $D^\alpha=D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3}$, and $|\alpha|=\alpha_1+\alpha_2+\alpha_3$).

The spectrum of L can be only on the real axis. On the whole positive axis there exists only the essential spectrum, which is, in fact, absolutely continuous, while on the negative axis we have only the discrete point spectrum, if any (see T. Ikebe [1], Chapter 2, Section 7). Here we assume

(A_2) *The operator L has no negative eigenvalues.*

Let B_δ be the space of all continuous functions $f(x)$ defined on E with

$\|f\|_\delta=\sup_{x\in E}|e^{-(\delta/2)|x|}f(x)|<\infty$. Then B_δ becomes a Banach space with the norm $\|f\|_\delta$. Furthermore, we impose another condition on $c(x)$:

$$(\mathrm{A}_3)\quad \begin{cases} \textit{The homogeneous integral equation} \\ \qquad f(x)=-\dfrac{1}{4\pi}\displaystyle\int_E \frac{c(y)}{|x-y|}f(y)dy \\ \textit{has only the trivial solution } f=0 \textit{ in } B_\delta. \end{cases}$$

An appendix will be added for a remark on (A_3).

Let $j_a(t)=\begin{cases}1 & (t\geqslant a)\\ t/a & (0\leqslant t<a)\end{cases}$, where a is any fixed positive number, and ω be real. Now we consider in the free space E the initial value problem for the wave equation

$$(1.1\mathrm{a})\quad \begin{cases} \dfrac{\partial^2}{\partial t^2}v_1(x,t)+Lv_1(x,t)=q(x)e^{i\omega t}j_a(t) \\ v_1(x,0)=0\,,\quad \dfrac{\partial}{\partial t}v_1(x,0)=g(x)\,, \end{cases}$$

and the reduced wave equation

$$(1.2)\qquad Lu(x)=\omega^2u(x)+q(x)$$

under conditions (A_1), (A_2), (A_3),

$$(\mathrm{B}_1)\quad \begin{cases} q(x) \textit{ is a measurable function defined on } E \textit{ and there exists a pair of} \\ \textit{positive numbers } Q_1 \textit{ and } \gamma \textit{ such that for any } x \textit{ in } E \\ \qquad\qquad |q(x)|\leqslant Q_1e^{-\gamma|x|}\,, \end{cases}$$

and

$$(\mathrm{C})\quad \begin{cases} g(x) \textit{ is a } C^2 \textit{ function defined over } E \textit{ and has the estimate} \\ \qquad |D^\alpha g(x)|\leqslant Ge^{-\mu|x|} \qquad (x\in E,\ |\alpha|\leqslant 2)\,, \\ \textit{where } G \textit{ and } \mu \textit{ are positive constants.} \end{cases}$$

Then the initial value problem $(1.1a)$ has a unique solution $v_1(\cdot,t)$ in L^2, which will be set forth more in detail in Proposition 2.1.

Our first result can be stated as follows:

Theorem 1. *Suppose (A_1), (A_2), (A_3), (B_1) and (C). Then the following assertions hold*:

(i) (Limiting amplitude principle) *There exists the limit function $u(x)=\lim_{t\to\infty}v_1(x,t)e^{-i\omega t}$ uniformly on any bounded set in E, which is a solution of* (1.2).

(ii) (Exponential decay) *$v_1(x,t)$ can be expressed as*

$$\begin{cases} v_1(x, t) = u(x)e^{i\omega t} + \delta_1(x, t) \\ |\delta_1(x, t)| \leqslant Ce^{(\delta/2)|x|}e^{-\alpha_1 t}, \end{cases}$$

where α_1 *is some positive constant* $< \min(\delta/2, \gamma, \mu)$, *and* C *is a positive constant depending only on* $c(x)$, $q(x)$, $g(x)$, ω, a *and* α_1.

(iii) (Sommerfeld radiation principle) $u(x)$ *satisfies the Sommerfeld radiation conditions*

$$\begin{cases} u(x) = O(|x|^{-1}) \\ \dfrac{du(x)}{d|x|} + i\omega u(x) = o(|x|^{-1}) \end{cases}$$

as $|x| \to \infty$.

In the sequel the letter C exclusively means a positive constant. C does not always denote the same one.

In addition to (A_1), (A_2), (A_3), (B_1) and (C) assume

$$(\mathrm{B}_2) \quad \begin{cases} q(x) \textit{ has continuous first derivatives satisfying} \\ \qquad |\operatorname{grad} q(x)| \leqslant Q_2 e^{-\nu|x|} \qquad (x \in E), \\ \textit{where } Q_2 \textit{ and } \nu \textit{ are positive contants,} \end{cases}$$

and consider the initial value problem

$$(1.1\mathrm{b}) \quad \begin{cases} \dfrac{\partial^2}{\partial t^2} v_2(x, t) + Lv_2(x, t) = q(x)e^{i\omega t} \\ v_2(x, 0) = 0, \quad \dfrac{\partial}{\partial t} v_2(x, 0) = g(x), \end{cases}$$

where $j_a(t)$ which appeared in (1.1a) has been deleted. Then there exists a unique solution $v_2(\cdot, t) \in L^2$ (see Proposition 2.1), and our second result is as follows:

Theorem 2. *Under assumptions* (A_1), (A_2), (A_3), (B_1), (B_2) *and* (C) *the statements* (*i*), (*ii*) *and* (*iii*) *in Theorem* 1 *are all valid for* (1.1b) *and* (1.2), *if* $v_1(x, t)$ *is replaced there by* $v_2(x, t) = u(x)e^{i\omega t} + \delta_2(x, t)$, *and the estimate for* $\delta_1(x, t)$ *by* $|\delta_2(x, t)| \leqslant Ce^{(\delta/2)|x|}e^{-\alpha_2 t}$, *where* α_2 *is some positive constant* $< \min(\delta/2, \gamma, \nu, \mu)$, *and where* C *depends only on* $c(x)$, $q(x)$, $g(x)$, ω *and* α_2.

O.A. Ladyženskaja [3] has given a proof of Theorem 1 on the basis of the Laplace transformation theory where both $c(x)$ and $q(x)$ are assumed to have a compact support, and (1.1a) to have zero initial data. Fundamentally on the same line as hers we shall prove the two above-mentioned theorems.

Meanwhile, C.S. Morawetz has studied the decay of solutions of the initial-boundary value problem for the wave equation $v_{tt}(x, t) - \Delta v(x, t) = 0$ with zero Dirichlet condition in the exterior of a star-shaped reflecting body in E. In [8] and [9] she has obtained the rate of decay with time t at least like $t^{-1/2}$ and t^{-1} respectively by the Kirchhoff formula and certain estimates derived from so-called energy identities (Friedrichs' a, b, c-method). Moreover, P.D. Lax, C.S. Morawetz and R.S. Phillips [4] have proved the exponential energy decay of the solutions, following the ideas developed in [8] and [5]. Recently C.S. Morawetz [10] has deduced an exponential energy decay from a certain preassumed decay rate for not-necessarily star-shaped domains (she has considered Robin as well as Dirichlet boundary conditions), and, applying this criterion for the exponential decay to the problem in [4], has given another direct proof. [9] also contains the result that the solution v of the initial-boundary value problem for the inhomogeneous equation $v_{tt}(x, t)-\Delta v(x, t)=q(x)e^{i\omega t}$ with zero Dirichlet condition approaches to a solution u of its reduced equation $\Delta u(x)=\omega^2 u(x)+q(x)$ as fast as $t^{-1/2}$.

2. Laplace transformations. Under the hypotheses (A_1), (B_1) and (C) the general existence theorem on the initial value problem for hyperbolic equations (refer *e.g* to S. Mizohata [7], Chapter 6 or [6]) guarantees

Proposition 2.1. *Suppose* (A_1), (B_1) *and* (C). *Then there exists a unique solution* $v_1(x, t)$ *of* (1.1*a*) *such that* $(v_1(x, t), \frac{\partial}{\partial t}v_1(x, t))\in\mathcal{E}^0_{t\geq 0}(\mathcal{D}^2_{L^2})\times\mathcal{E}^0_{t\geq 0}(\mathcal{D}^1_{L^2})$ *and*

$$\|v_1(\cdot, t)\|_2+\left\|\frac{\partial}{\partial t}v_1(\cdot, t)\right\|_1\leqslant Ce^{\beta t}, \tag{2.1}$$

where β *is some positive constant, and* $f(t)\in\mathcal{E}^0_{t\geq 0}(\mathcal{D}^m_{L^2})$ $(m=1, 2)$ *means that* $f(t)\in\mathcal{D}^m_{L^2}$ *and is continuous on the interval* $t\geqslant 0$ *in the topology of* $\mathcal{D}^m_{L^2}$. *The same is true for* $v_2(x, t)$ *of* (1.1*b*).

By Proposition 2.1. the Laplace transform $w_k(x, \lambda)$ of $v_k(x, t)$

$$w_k(x, \lambda)=\int_0^\infty v_k(x, t)e^{-\lambda t}dt \qquad (\operatorname{Re}\lambda>\beta;\ k=1, 2) \tag{2.2}$$

exists in $\mathcal{D}^2_{L^2}$ and is analytic in $\operatorname{Re}\lambda>\beta$, where $\operatorname{Re}\lambda$ denotes the real part of λ. Moreover, the inverse transformation of $w_k(x, \lambda)$

$$v_k(x, t)=\lim_{A\to\infty}\frac{1}{2\pi i}\int_{\sigma-iA}^{\sigma+iA}w_k(x, \lambda)e^{\lambda t}d\lambda \qquad (k=1, 2) \tag{2.3}$$

can be carried out along any path $\operatorname{Re}\lambda=\sigma>\beta$.

Applying the Laplace transformation to the initial value problems (1.1*a*) and (1.1*b*) in $\operatorname{Re}\lambda>\beta$, we have

$$(L+\lambda^2)w_k(x, \lambda)=q(x)f_k(\lambda)+g(x) \qquad (k=1, 2), \tag{2.4}$$

where

$$\begin{cases} f_1(\lambda) = \dfrac{1-e^{a(i\omega-\lambda)}}{a(i\omega-\lambda)^2} \\ f_2(\lambda) = \dfrac{1}{\lambda-i\omega}\,. \end{cases}$$

From (A_2) and the statement on the spectrum of L in Section 1 it follows that every point $-\lambda^2$ with $\mathrm{Re}\,\lambda>0$ becomes a regular point of L. Hence there exists the resolvent $(L+\lambda^2)^{-1}$ carrying L^2 onto $\mathcal{D}^2_{L^2}$ in $\mathrm{Re}\,\lambda>0$. Therefore (2.4) has a unique solution in $\mathcal{D}^2_{L^2}$ which is analytic in $\mathrm{Re}\,\lambda>0$. Then we can consider the Laplace transform $w_k(x, \lambda)$ $(k=1, 2)$ to be extended analytically to the whole half-plane $\mathrm{Re}\,\lambda>0$.

Putting

$$w_k(x, \lambda) = u_k(x, \lambda)f_k(\lambda) \qquad (\mathrm{Re}\,\lambda>0;\ k=1, 2)\,, \tag{2.5}$$

we can see from (2.4) and the discussion following it that $u_k(x, \lambda)$ is a unique solution in $\mathcal{D}^2_{L^2}$ of the equation

$$(L+\lambda^2)u_k(x, \lambda) = q(x)+g(x)f_k^{-1}(\lambda) \qquad (\mathrm{Re}\,\lambda>0;\ k=1, 2)\,, \tag{2.6}$$

which is analytic in $\mathrm{Re}\,\lambda>0$. Since in $\mathrm{Re}\,\lambda>0$ there exists the resolvent $(-\Delta+\lambda^2)^{-1}$, which is an integral operator of Carleman type with the kernel $(4\pi|x-y|)^{-1}e^{-\lambda|x-y|}$, $u_k(x, \lambda)$ in (2.6) satisfies the integral equation

$$\begin{aligned} u_k(x, \lambda) = {}& \frac{1}{4\pi}\int_E \frac{e^{-\lambda|x-y|}}{|x-y|}(q(y)+g(y)f_k^{-1}(\lambda))dy - \\ & -\frac{1}{4\pi}\int_E \frac{e^{-\lambda|x-y|}}{|x-y|}c(y)u_k(y, \lambda)dy \qquad (\mathrm{Re}\,\lambda>0;\ k=1, 2)\,. \end{aligned} \tag{2.7}$$

3. Integral equations. In this section we shall study the unique solvability of equation (2.7) in B_δ, which leads to the analytical extension to $\mathrm{Re}\,\lambda\leqslant 0$ of $u_k(x,\lambda)$ in (2.5) and then to the meromorphical one of $w_k(x,\lambda)$ in (2.5) $(k=1,2)$.

Now let us introduce a domain D_δ defined by $D_\delta=\{\lambda;\ \mathrm{Re}\,\lambda>-\delta/2\}$.

Proposition 3.1. *Under assumption (A_1) the integral operator T_λ, defined by*

$$T_\lambda f(x) = -\frac{1}{4\pi}\int_E \frac{e^{-\lambda|x-y|}}{|x-y|}c(y)f(y)dy\,,$$

is completely continuous on B_δ into itself for each fixed $\lambda\in D_\delta$.

Proof. Let $f\in B_\delta$ and $\lambda\in D_\delta$. When $\mathrm{Re}\,\lambda<0$, we first have

$$|T_\lambda f(x)| \leqslant C\|f\|_\delta \int_E \frac{e^{-\mathrm{Re}\,\lambda|x-y|-(\delta/2)|y|}}{|x-y|}dy = C\|f\|_\delta J\,.$$

To estimate J we note that $|x-y|\leqslant|x|+|y|$, and $|x-y|\geqslant|y|$ when $|y|\leqslant|x|/2$, and $|y|\geqslant(1/3)|x-y|$ when $|y|\geqslant|x|/2$. We then obtain

$$J\leqslant e^{-\mathrm{Re}\,\lambda|x|}\Big(\int_{|y|\leqslant|x|/2}\frac{e^{-(1/2)(\delta+2\mathrm{Re}\,\lambda)|y|}}{|y|}dy+\int_{|y|\geqslant|x|/2}\frac{e^{-(1/6)(\delta+2\mathrm{Re}\,\lambda)|x-y|}}{|x-y|}dy\Big)$$
$$\leqslant C(\delta+2\mathrm{Re}\,\lambda)^{-2}e^{-\mathrm{Re}\,\lambda|x|}.$$

Hence we have

$$|T_\lambda f(x)|\leqslant C(\delta+2\mathrm{Re}\,\lambda)^{-2}\|f\|_\delta e^{-\mathrm{Re}\,\lambda|x|}\qquad(\mathrm{Re}\,\lambda<0)\,,\tag{3.1}$$

where C is independent of x and λ. When $\mathrm{Re}\,\lambda\geqslant0$, the estimate for $T_\lambda f(x)$ can be worked out with more ease in a similar fashion by using the inequality $|e^{-\lambda|x-y|}|\leqslant1$, and we get

$$|T_\lambda f(x)|\leqslant C\|f\|_\delta\qquad(\mathrm{Re}\,\lambda\geqslant0)\,,\tag{3.2}$$

where C does not depend on x nor λ. From (3.1) and (3.2) it follows that

$$e^{-(\delta/2)|x|}|T_\lambda f(x)|\leqslant\begin{cases}C(\delta+2\mathrm{Re}\,\lambda)^{-2}\|f\|_\delta e^{-(\delta/2+\mathrm{Re}\,\lambda)|x|} & (\mathrm{Re}\,\lambda<0)\\ C\|f\|_\delta e^{-(\delta/2)|x|} & (\mathrm{Re}\,\lambda\geqslant0)\,.\end{cases}\tag{3.3}$$

Next we proceed to show the continuity of $T_\lambda f(x)$ in x for each fixed $\lambda\in D_\delta$. For this purpose we consider the difference

$$T_\lambda f(x)-T_\lambda f(x')=-\frac{1}{4\pi}\int_E\frac{e^{-\lambda|x-y|}-e^{-\lambda|x'-y|}}{|x-y|}c(y)f(y)dy-$$
$$-\frac{1}{4\pi}\int_E\Big(\frac{1}{|x-y|}-\frac{1}{|x'-y|}\Big)e^{-\lambda|x'-y|}c(y)f(y)dy$$
$$=J_1+J_2.$$

Considering (A_1) and the inequality

$$|e^{-\lambda|x-y|}-e^{-\lambda|x'-y|}|\leqslant\begin{cases}e^{-\mathrm{Re}\,\lambda\max(|x-y|,|x'-y|)}|\lambda|\,|x'-x| & (\mathrm{Re}\,\lambda<0)\\ |\lambda|\,|x'-x| & (\mathrm{Re}\,\lambda\geqslant0)\,,\end{cases}$$

we have, when $\mathrm{Re}\,\lambda<0$,

$$|J_1|\leqslant C\|f\|_\delta|x'-x|\Big(\int_{|x'-y|\leqslant|x-y|}\frac{e^{-\mathrm{Re}\,\lambda|x-y|-(\delta/2)|y|}}{|x-y|}dy+$$
$$+\int_{|x'-y|\geqslant|x-y|}\frac{e^{-\mathrm{Re}\,\lambda|x'-y|-(\delta/2)|y|}}{|x-y|}dy\Big)$$
$$\leqslant C\|f\|_\delta|x'-x|(1+e^{-\mathrm{Re}\,\lambda|x'-x|})\int_E\frac{e^{-\mathrm{Re}\,\lambda|x-y|-(\delta/2)|y|}}{|x-y|}dy$$
$$\leqslant C\|f\|_\delta|x'-x|(1+e^{-\mathrm{Re}\,\lambda|x'-x|})e^{-\mathrm{Re}\,\lambda|x|}\,,$$

$$|J_2| \leqslant C\|f\|_\delta |x'-x| \left(\int_E \frac{e^{-2\mathrm{Re}\,\lambda|x'-y|-(1/2)(\delta-2\mathrm{Re}\,\lambda)|y|}}{|x'-y|^2}\,dy\right)^{1/2} \times$$
$$\times \left(\int_E \frac{e^{-(1/2)(\delta+2\mathrm{Re}\,\lambda)|y|}}{|x-y|^2}\,dy\right)^{1/2} \leqslant C\|f\|_\delta |x'-x|\, e^{-\mathrm{Re}\,\lambda|x'|}$$

in a way similar to the one used for obtaining (3.1). Similarly, when $\mathrm{Re}\,\lambda \geqslant 0$, we get

$$|J_1| \leqslant C\|f\|_\delta |x'-x| \int_E \frac{e^{-(\delta/2)|y|}}{|x-y|}\,dy \leqslant C\|f\|_\delta |x'-x|,$$
$$|J_2| \leqslant C\|f\|_\delta |x'-x| \left(\int_E \frac{e^{-(\delta/2)|y|}}{|x-y|^2}\,dy\right)^{1/2} \left(\int_E \frac{e^{-(\delta/2)|y|}}{|x'-y|^2}\,dy\right)^{1/2}$$
$$\leqslant C\|f\|_\delta |x'-x|.$$

So it follows that in $E\times D_\delta$

$$(3.4) \qquad |T_\lambda f(x) - T_\lambda f(x')| \leqslant \begin{cases} C\|f\|_\delta |x'-x|\, e^{-\mathrm{Re}\,\lambda(|x|+|x'-x|)} & (\mathrm{Re}\,\lambda<0) \\ C\|f\|_\delta |x'-x| & (\mathrm{Re}\,\lambda \geqslant 0), \end{cases}$$

where C's depend only on λ. By (3.3) and (3.4) T_λ is a bounded linear operator on B_δ into itself.

Let S be any bounded set in B_δ. We want to show that $T_\lambda S$ is relatively compact in B_δ, which deduces the complete cotinuity of T_λ. Let $\{g_n\}$ ($n=1, 2, \cdots$) be any sequence chosen from $T_\lambda S$. By (3.1), (3.2), (3.4) and the boundedness of S, $\{g_n(x)\}$ must be a uniformly bounded and equicontinuous family of continuous functions on any compact domain of E. Employing the Ascoli-Arzelà selection theorem we can find out a subsequence $\{g_{n'}(x)\}$ converging to a continuous function $g(x)$ uniformly on any compact domain of E. In view of (3.3), $e^{-(\delta/2)|x|}g_{n'}(x)$ tends to 0 uniformly in n' as $|x|\to\infty$. So does $e^{-(\delta/2)|x|}g(x)$. Then $\|g_{n'}-g\|_\delta \to 0$ as $n'\to\infty$, and $g\in B_\delta$, which is the desired result.

Proposition 3.2. *Assume* (A_1), (A_2) *and* (A_3). *Then there exists a positive number* $\alpha(<\delta/2)$ *such that for any* $g\in B_\delta$ *the equation* $(I-T_\lambda)f=g$ *admits one and only one solution* $f\in B_\delta$ *for each fixed* $\lambda\in D=\{\lambda;\ \mathrm{Re}\,\lambda>-\alpha\}$, *which is analytic in D, and* $(I-T_\lambda)^{-1}$ *has the estimate*

$$\|(I-T_\lambda)^{-1}\|_\delta \leqslant C$$

uniformly in D, where $\|T\|_\delta = \sup_{\|f\|_\delta=1} \|Tf\|_\delta$ *for an operator T in* B_δ *and I stands for the identity operator in* B_δ.

Proposition 3.2 will be proved by the following four lemmas.

Lemma 3.1. *Suppose* (A_1), (A_2) *and* (A_3). *Let* $\mathrm{Re}\,\lambda\geqslant 0$ *and* $g\in B_\delta$. *Then the integral equation*

$$f=g+T_\lambda f \tag{3.5}$$

has a unique solution $f\in B_\delta$, *which is analytic on* $\mathrm{Re}\,\lambda\geqslant 0$.

Proof. Let $f\in B_\delta$ be a solution of the homogeneous equation $f=T_\lambda f$. When $\mathrm{Re}\,\lambda>0$, we get

$$|f(x)|=|T_\lambda f(x)|\leqslant C\|f\|_\delta\int_E\frac{e^{-\mathrm{Re}\,\lambda|x-y|-(\delta/2)|y|}}{|x-y|}dy=C\|f\|_\delta J,$$

and, noting that $|x-y|\geqslant|y|$ and $|x-y|\geqslant\frac{|x|}{2}$ when $|y|\leqslant\frac{|x|}{2}$,

$$\begin{aligned}J&\leqslant e^{-(\mathrm{Re}\,\lambda/4)|x|}\int_{|y|\leqslant|x|/2}\frac{e^{-(\mathrm{Re}\,\lambda/2+\delta/2)|y|}}{|y|}dy+e^{-(\delta/4)|x|}\int_{|y|\geqslant|x|/2}\frac{e^{-\mathrm{Re}\,\lambda|x-y|}}{|x-y|}dy\\&\leqslant C\left(e^{-(\mathrm{Re}\,\lambda/4)|x|}\left(\frac{\mathrm{Re}\,\lambda}{2}+\frac{\delta}{2}\right)^{-2}+e^{-(\delta/4)|x|}(\mathrm{Re}\,\lambda)^{-2}\right),\end{aligned}$$

where C's are independent of x and λ. So we can see that

$$|f(x)|=|T_\lambda f(x)|\leqslant C\|f\|_\delta(e^{-(\mathrm{Re}\,\lambda/4)|x|}+e^{-(\delta/4)|x|}),$$

where C is dependent only on λ. Therefore $f\in L^2$ for λ with $\mathrm{Re}\,\lambda>0$. Also when $\mathrm{Re}\,\lambda=0$ and $\lambda\neq 0$, $f\in L^2$. This follows from A. Ya. Povzner [11], Chapter 2, Lemmas 1, 2, 5 and 6. As f fulfills the equation $(L+\lambda^2)f=0$, $-\lambda^2$ ($\mathrm{Re}\,\lambda\geqslant 0$, $\lambda\neq 0$) cannot be an eigenvalue of L by (A_2) and the statement on the spectrum of L in Section 1. Hence, if we also note (A_3) for $\lambda=0$, the equation $f=T_\lambda f$ implies $f=0$ in B_δ for λ with $\mathrm{Re}\,\lambda\geqslant 0$. So by the Riesz-Schauder theory together with Proposition 3.1, equation (3.5) is uniqely solvable in B_δ for any $g\in B_\delta$ and the operator $I-T_\lambda$ has a bounded inverse in B_δ. Moreover, from its definition in Proposition 3.1 T_λ is seen to be analytic for $\mathrm{Re}\,\lambda\geqslant 0$ (*cf. e.g.* K. Yosida [13], Chapter 5, Section 3). Thus $(I-T_\lambda)^{-1}$ is also analytic for $\mathrm{Re}\,\lambda\geqslant 0$, which was to be proved (*cf. e.g. ibid.*, Chapter 8, Section 2).

Lemma 3.2. *Assume* (A_1), (A_2) *and* (A_3). *Then for any* $N'>0$ *one can find a positive* $\alpha'(<\delta/2)$ *such that for* $\lambda\in D'=\{\lambda;\ -\alpha'<\mathrm{Re}\,\lambda,\ |\lambda|<N'\}$ *equation* (3.5) *has a unique solution in* B_δ, *which is analytic in* D', *and the estimate*

$$\|(I-T_\lambda)^{-1}\|_\delta\leqslant C$$

holds in D', *where* C *is dependent only on* D'.

Proof. Clear from Lemma 3.1 and the Heine-Borel theorem.

Lemma 3.3. *Assume* (A_1) *and let* $0<b<1$. *Then, for any* $\varepsilon>0$ *there exists a positive number* N *independent of* $(x,y)\in E\times E$ *such that the kernel of the operator*

T_λ^2

$$\tau(x, y; \lambda) = (4\pi)^{-2}c(y)\int_E \frac{e^{-\lambda|x-s|-\lambda|s-y|}}{|x-s|\,|s-y|}c(s)ds$$

has the estimate

$$e^{-(\delta/2)|x|}|\tau(x, y; \lambda)| < \varepsilon e^{-(1/2+(1-b)/4)\delta|y|}$$

for $(x, y, \lambda) \in E \times E \times \left\{\lambda;\ \mathrm{Re}\,\lambda > -\frac{b}{2}\delta,\ |\lambda| > N\right\}$.

Proof. Let $\mathrm{Re}\,\lambda > -\frac{b}{2}\delta$. We first assume $(x, y) \in (E-K_R) \times E$ or $K_R \times (E-K_R)$, where K_R means the sphere of radius R with its center at the origin. Since by an estimation similar to the one utilized for having (3.1) we have

$$|\tau(x, y; \lambda)| \leqslant Ce^{-\delta|y|}\left(\int_E \frac{e^{b\delta|x-s|-\delta|s|}}{|x-s|^2}ds\right)^{1/2}\left(\int_E \frac{e^{b\delta|s-y|-\delta|s|}}{|s-y|^2}ds\right)^{1/2}$$
$$\leqslant Ce^{-(1-b/2)\delta|y|}e^{(b/2)\delta|x|},$$

we can see that

$$e^{-(\delta/2)|x|}|\tau(x, y; \lambda)| \leqslant Ce^{-(1/2)(1-b)\delta R}e^{-(1-b/2)\delta|y|} \qquad ((x, y) \in (E-K_R) \times E), \tag{3.6}$$

$$e^{-(\delta/2)|x|}|\tau(x, y; \lambda)| \leqslant Ce^{-(1/4)(1-b)\delta R}e^{-(1/2+(1-b)/4)\delta|y|} \quad ((x, y) \in K_R \times (E-K_R)). \tag{3.7}$$

Next, assuming $(x, y) \in K_R \times K_R$, we similarly have

$$e^{-(\delta/2)|x|}|\tau(x, y; \lambda)| \leqslant Ce^{-(\delta/2)|x|-\delta|y|}\left|\int_{K_R} \frac{e^{-\lambda|x-s|-\lambda|s-y|}}{|x-s|\,|s-y|}c(s)ds\right| + \tag{3.8}$$
$$+Ce^{-(\delta/2)|x|-\delta|y|}\left|\int_{E-K_R}\right| = J_1 + J_2,$$

$$J_2 \leqslant Ce^{-(1/2)(1-b)\delta R}e^{-(1-b/2)\delta|y|}\left(\int_E \frac{e^{-(1/2)(1-b)\delta|s|}}{|x-s|^2}ds\right)^{1/2}\left(\int_E \frac{e^{-(1/2)(1-b)\delta|s|}}{|s-y|^2}ds\right)^{1/2} \tag{3.9}$$
$$\leqslant Ce^{-(1/2)(1-b)\delta R}e^{-(1-b/2)\delta|y|}.$$

To estimate J_1 we consider the ellipsoid $|x-s|+|s-y| \leqslant \xi$ with the foci x and y for each fixed $(x, y) \in K_R \times K_R$ and a positive ξ. Let us denote such an ellipsoid by $E_\xi(x, y)$. Then, noting that $E_{4R}(x, y)$ contains K_R for any $(x, y) \in K_R \times K_R$, we have

$$J_1 = Ce^{-(\delta/2)|x|-\delta|y|}\left|\int_{E_{|x-y|+\rho}(x,y)} \frac{e^{-\lambda|x-s|-\lambda|s-y|}}{|x-s|\,|s-y|}c(s)ds + \right. \tag{3.10}$$

$$+\int_{E_{4R}(x,y)-E_{|x-y|+\rho}(x,y)} - \int_{E_{4R}(x,y)-K_R}\Bigg|$$

$$\leqslant e^{-\delta|y|}\Bigg(\Bigg|C\int_{E_{|x-y|+\rho}(x,y)}\Bigg|+\Bigg|C\int_{E_{4R}(x,y)-E_{|x-y|+\rho}(x,y)}\Bigg|\Bigg)+Ce^{-(\delta/2)|x|-\delta|y|}\times$$

$$\times\Bigg|\int_{E_{4R}(x,y)-K_R}\Bigg| = e^{-\delta|y|}(J_{11}+J_{12})+J_{13} \qquad (0<\rho<2R),$$

$$(3.11)\quad J_{13}\leqslant Ce^{-(\delta/2)|x|-\delta|y|}\Bigg|\int_{E-K_R}\Bigg| = J_2\leqslant Ce^{-(1/2)(1-b)\delta R}e^{-(1-b/2)\delta|y|}.$$

Here C's appearing in (3.6) to (3.11) all depend only on b. Now for any $\varepsilon>0$ we can choose a sufficiently large R such that $Ce^{-(1/2)(1-b)\delta R}$ in (3.6), $Ce^{-(1/4)(1-b)\delta R}$ in (3.7), and $Ce^{-(1/2)(1-b)\delta R}$ in (3.9) and (3.11) are all smaller than $\varepsilon/4$. Furthermore

$$J_{11}\leqslant C\int_{\substack{E_{|x-y|+\rho}(x,y)\\ |x-s|\leqslant|s-y|}}\frac{ds}{|x-s|^2},$$

where C depends only on b and R. In $E_{|x-y|+\rho}(x, y)$ we introduce a cylindrical coordinate system (r, θ, t) such that $(0, \theta, 0)$ corresponds to x and the t-axis is directed from x to y. Since $|x-s|^2=r^2+t^2$ for $s=(r, \theta, t)$ and the Jacobian for the coordinate transformation becomes r, a simple evaluation gives

$$(3.12)\qquad J_{11}\leqslant C\int_0^{2\pi}d\theta\int_{-R}^{R}dt\int_0^{C\sqrt{\rho}}\frac{r}{t^2+r^2}dr<\frac{\varepsilon}{4}$$

for a sufficiently small $\rho(<2R)$ uniformly in $(x, y)\in K_R\times K_R$. C's in (3.12) never depend on ρ. For the estimation of J_{12} we first assign the 3-dimensional orthogonal coordinates (s_1, s_2, s_3) to each point s in $E_{4R}(x, y)-E_{|x-y|+\rho}(x, y)$ in such a way that the origin 0 is the middle point of x and y, and the s_1-axis the line directed from 0 to y. Secondly, considering s to be a radius vector which starts from 0, we introduce two angles θ and φ; θ is measured from the s_3-axis toward s, and φ from the s_1-axis toward the projection of s to the (s_1, s_2)-plane. Finally we set $\xi=|x-s|+|s-y|$. Thus for each fixed $(x, y)\in K_R\times K_R$ every point $s=(s_1, s_2, s_3)$ in $E_{4R}(x, y)-E_{|x-y|+\rho}(x, y)$ can be expressed in terms of new coordinates (ξ, θ, φ) $(|x-y|+\rho\leqslant\xi\leqslant 4R,\ 0\leqslant\theta\leqslant\pi,\ 0\leqslant\varphi\leqslant 2\pi)$. Now we have

$$|x-s|\geqslant\frac{\rho}{2},\quad |s-y|\geqslant\frac{\rho}{2};\quad \frac{\partial}{\partial\xi}|x-s|\leqslant C,\quad \frac{\partial}{\partial\xi}|s-y|\leqslant C;$$

$$|J(\xi, \theta, \varphi)|\leqslant C,\quad \frac{\partial}{\partial\xi}|J(\xi, \theta, \varphi)|\leqslant C,$$

where $J(\xi, \theta, \varphi)$ is the Jacobian for the coordinate transformation, and C's depend only on R and ρ. By integration by parts in consideration of (A_1) and these inequalities, J_{12} can be estimated as follows:

$$J_{12}=\left|C\int_{\substack{|x-y|+\rho\leqslant\xi\leqslant 4R\\ 0\leqslant\theta\leqslant\pi,\,0\leqslant\varphi\leqslant 2\pi}}\frac{e^{-\lambda\xi}c(\xi,\theta,\varphi)\,|J(\xi,\theta,\varphi)|}{|x-s(\xi,\theta,\varphi)|\,|s(\xi,\theta,\varphi)-y|}\,d\xi d\theta d\varphi\right|$$

$$=\left|-\frac{C}{\lambda}\int d\theta d\varphi\left[\frac{e^{-\lambda\xi}c(\xi,\theta,\varphi)\,|J(\xi,\theta,\varphi)|}{|x-s|\,|s-y|}\right]_{\xi=|x-x|+\rho}^{\xi=4R}+\right.$$

$$\left.+\frac{C}{\lambda}\int d\theta d\varphi\int e^{-\lambda\xi}\frac{\partial}{\partial\xi}\left(\frac{c(\xi,\theta,\varphi)\,|J(\xi,\theta,\varphi)|}{|x-s|\,|s-y|}\right)d\xi\right|\leqslant\frac{C}{|\lambda|}\,.$$

Therefore for any given $\varepsilon>0$ we can find an $N>0$ independent of $(x, y)\in K_R\times K_R$ such that

$$J_{12}<\frac{\varepsilon}{4}\tag{3.13}$$

holds for $(x, y, \lambda)\in K_R\times K_R\times\left\{\lambda;\ \mathrm{Re}\,\lambda>-\frac{b}{2}\delta,\ |\lambda|>N\right\}$. Considering (3.6), (3.7) and (3.8), together with (3.9) to (3.13) and the statement just below (3.11), we have Lemma 3.3.

Lemma 3.4. *Assume (A_1). Let $0<b<1$. Then there exists an $N''>0$ such that the equation $f=g+T_\lambda f$ has one and only one solution in B_δ for any $g\in B_\delta$ and $\lambda\in D''=\left\{\lambda;\ \mathrm{Re}\,\lambda>-\frac{b}{2}\delta,\ |\lambda|>N''\right\}$, which is analytic in D'', and $(I-T_\lambda)^{-1}$ is uniformly bounded there in the operator norm.*

Proof. By Lemma 3.3, for any η $(0<\eta<1)$ we can find out an $N''>0$ such that the inequality

$$\|T_\lambda^2\|_\delta\leqslant\eta\tag{3.14}$$

holds in $D''=\left\{\lambda;\ \mathrm{Re}\,\lambda>-\frac{b}{2}\delta,\ |\lambda|>N''\right\}$. By (3.14) the series

$$(I+T_\lambda)(I+T_\lambda^2+T_\lambda^4+\cdots+T_\lambda^{2n}+\cdots)$$

converges to a bounded linear operator in B_δ uniformly in D'' in the operator norm. Moreover, multiplication of the series by $I-T_\lambda$ on the left or right gives I, so that the series actually represents $(I-T_\lambda)^{-1}$. Its uniform boundedness and analyticity in D'' follow from the series, (3.14) and the analyticity of T_λ, which proves the assertion of Lemma 3.4.

Combining Lemmas 3.4 and 3.2 under the assumptions of Proposition 3.2, we obtain Proposition 3.2.

Let $a^{(1)}(x, \lambda)$ and $a^{(2)}(x, \lambda)$ be functions given by

$$a^{(1)}(x, \lambda) = \frac{1}{4\pi}\int_E \frac{e^{-\lambda|x-y|}}{|x-y|} q(y)dy\,,$$

$$a^{(2)}(x, \lambda) = \frac{1}{4\pi}\int_E \frac{e^{-\lambda|x-y|}}{|x-y|} g(y)dy\,.$$

Then the integral equation (2.7) can be rewritten in the form

$$(I-T_\lambda)u_k(\cdot, \lambda) = a^{(1)}(\cdot, \lambda)+a^{(2)}(\cdot, \lambda)f_k^{-1}(\lambda) \qquad (k=1, 2)\,. \tag{3.15}$$

Proposition 3.3a. *Suppose* (B_1). *Let* $D_\gamma=\{\lambda;\ \mathrm{Re}\,\lambda>-\gamma\}$. *Then the inequality*

$$|a^{(1)}(x, \lambda)| \leqslant \begin{cases} C(\gamma+\mathrm{Re}\,\lambda)^{-2}e^{-\mathrm{Re}\,\lambda|x|} & (\mathrm{Re}\,\lambda<0) \\ C & (\mathrm{Re}\,\lambda\geqslant 0) \end{cases}$$

holds in $E\times D_\gamma$, *where* C*'s are independent of* x *and* λ.

Proof. Let $\lambda\in D_\gamma$. By virtue of the estimation used for obtaining (3.1) and (3.2), we clearly have, when $\mathrm{Re}\,\lambda<0$,

$$|a^{(1)}(x, \lambda)| \leqslant C\int_E \frac{e^{-\mathrm{Re}\,\lambda|x-y|-\gamma|y|}}{|x-y|}dy$$
$$\leqslant Ce^{-\mathrm{Re}\,\lambda|x|}(\gamma+\mathrm{Re}\,\lambda)^{-2}$$

and, when $\mathrm{Re}\,\lambda\geqslant 0$,

$$|a^{(1)}(x, \lambda)| \leqslant C\int_E \frac{e^{-\gamma|y|}}{|x-y|}dy \leqslant C\,,$$

where C's are all independent of x and λ. This completes the proof.

Proposition 3.3b. *Assume* (B_1) *and* (B_2). *Let* $D_{\gamma,\nu}=\{\lambda;\ \mathrm{Re}\,\lambda>-\min(\gamma,\nu)\}$. *Then* $a^{(1)}(x, \lambda)$ *has the estimate*

$$|a^{(1)}(x, \lambda)| \leqslant \begin{cases} Ce^{-\mathrm{Re}\,\lambda|x|}(1+|x|+(\gamma+\mathrm{Re}\,\lambda)^{-1}+(\nu+\mathrm{Re}\,\lambda)^{-1}|x|+ \\ \qquad +(\nu+\mathrm{Re}\,\lambda)^{-2})|\lambda|^{-1} & (\mathrm{Re}\,\lambda<0) \\ C(1+|x|)|\lambda|^{-1} & (\mathrm{Re}\,\lambda\geqslant 0) \end{cases}$$

in $E\times D_{\gamma,\nu}$, *where* C*'s are not dependent on any of* x *and* λ.

Proof. Let $\lambda\in D_{\gamma,\nu}$. Introducing spherical coordinates, we can express $a^{(1)}(x, \lambda)$ as

$$\begin{aligned} a^{(1)}(x, \lambda) &= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \int_0^\infty e^{-\lambda\rho}\rho q(x+t)d\rho \\ &= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta\cdot J \qquad (y-x=t,\ |t|=\rho)\,. \end{aligned} \tag{3.16}$$

Integrating by parts in view of (B_1) and (B_2), we obtain

$$J=\frac{1}{\lambda}\int_0^\infty e^{-\lambda\rho}\frac{\partial}{\partial\rho}(\rho q(x+t))d\rho$$
$$=\frac{1}{\lambda}\left(\int_0^\infty e^{-\lambda\rho}q(x+t)d\rho+\int_0^\infty e^{-\lambda\rho}\rho\frac{\partial}{\partial\rho}q(x+t)\,d\rho\right),$$

$$|J|\leqslant C\left(\int_0^\infty e^{-\mathrm{Re}\,\lambda\rho}e^{-\gamma|\rho-|x||}d\rho+\int_0^\infty e^{-\mathrm{Re}\,\lambda\rho}\rho e^{-\nu|\rho-|x||}d\rho\right)|\lambda|^{-1} \tag{3.17}$$
$$=C(J_1+J_2)|\lambda|^{-1},$$

where

$$J_1=e^{-\gamma|x|}\int_0^{|x|}e^{(\gamma-\mathrm{Re}\,\lambda)\rho}d\rho+e^{\gamma|x|}\int_{|x|}^\infty e^{-(\gamma+\mathrm{Re}\,\lambda)\rho}d\rho\,,$$
$$J_2=e^{-\nu|x|}\int_0^{|x|}e^{(\nu-\mathrm{Re}\,\lambda)\rho}\rho d\rho+e^{\nu|x|}\int_{|x|}^\infty e^{-(\nu+\mathrm{Re}\,\lambda)\rho}\rho d\rho\,.$$

When $\mathrm{Re}\,\lambda<0$, we get

$$J_1\leqslant Ce^{-\mathrm{Re}\,\lambda|x|}(1+(\gamma+\mathrm{Re}\,\lambda)^{-1})\,,$$
$$J_2\leqslant Ce^{-\mathrm{Re}\,\lambda|x|}(|x|+(\nu+\mathrm{Re}\,\lambda)^{-1}|x|+(\nu+\mathrm{Re}\,\lambda)^{-2})\,.$$

Hence, when $\mathrm{Re}\,\lambda<0$, (3.17) becomes

$$|J|\leqslant Ce^{-\mathrm{Re}\,\lambda|x|}(1+|x|+(\gamma+\mathrm{Re}\,\lambda)^{-1}+(\nu+\mathrm{Re}\,\lambda)^{-1}|x|+(\nu+\mathrm{Re}\,\lambda)^{-2})|\lambda|^{-1}\,, \tag{3.18}$$

where C is independent of x and λ. When $\mathrm{Re}\,\lambda\geqslant 0$, we have

$$J_1\leqslant\begin{cases} e^{-\gamma|x|}\displaystyle\int_0^{|x|}d\rho+e^{\gamma|x|}\int_{|x|}^\infty e^{-\gamma\rho}d\rho\leqslant C & (\mathrm{Re}\,\lambda\geqslant\gamma)\\ e^{-\gamma|x|}\displaystyle\int_0^{|x|}e^{\gamma\rho}d\rho+e^{\gamma|x|}\int_{|x|}^\infty e^{-\gamma\rho}d\rho\leqslant C & (\mathrm{Re}\,\lambda<\gamma)\,,\end{cases}$$

$$J_2\leqslant\begin{cases} e^{-\nu|x|}\displaystyle\int_0^{|x|}\rho d\rho+e^{\nu|x|}\int_{|x|}^\infty e^{-\nu\rho}\rho d\rho\leqslant C(1+|x|) & (\mathrm{Re}\,\lambda\geqslant\nu)\\ e^{-\nu|x|}\displaystyle\int_0^{|x|}e^{\nu\rho}\rho d\rho+e^{\nu|x|}\int_{|x|}^\infty e^{-\nu\rho}\rho d\rho\leqslant C(1+|x|) & (\mathrm{Re}\,\lambda<\nu)\,.\end{cases}$$

Consequently, when $\mathrm{Re}\,\lambda\geqslant 0$, we can see that

$$J_1\leqslant C\,,\qquad J_2\leqslant C(1+|x|)\,,$$

and

$$|J|\leqslant C(1+|x|)|\lambda|^{-1}\,, \tag{3.19}$$

where C's are independent of x and λ.

Thus (3.16), (3.18) and (3.19) give the required estimate for $a^{(1)}(x,\lambda)$, which proves Proposition 3.3*b*.

Proposition 3.4. *Suppose* (C). *Let* $D_\mu=\{\lambda;\ \mathrm{Re}\,\lambda>-\mu\}$. *Then* $a^{(2)}(x,\lambda)$ *satisfies*

$$\text{(i)}\quad |a^{(2)}(x,\lambda)|\leqslant\begin{cases}C(\mu+\mathrm{Re}\,\lambda)^{-2}e^{-\mathrm{Re}\,\lambda|x|} & (\mathrm{Re}\,\lambda<0)\\ C & (\mathrm{Re}\,\lambda\geqslant 0),\end{cases}$$

$$\text{(ii)}\quad |a^{(2)}(x,\lambda)|\leqslant\begin{cases}Ce^{-\mathrm{Re}\,\lambda|x|}(1+|x|+(\mu+\mathrm{Re}\,\lambda)^{-1}(1+|x|)+(\mu+\mathrm{Re}\,\lambda)^{-2})|\lambda|^{-2} & (\mathrm{Re}\,\lambda<0)\\ C(1+|x|)|\lambda|^{-2} & (\mathrm{Re}\,\lambda\geqslant 0)\end{cases}$$

in $E\times D_\mu$, *where C's never depend on x nor* λ.

Proof. Considering (C) and the definition of $a^{(2)}(x,\lambda)$, (i) is obtained by an estimation similar to the one in the proof of Proposition 3.3*a*.

Proceeding as in the proof of Proposition 3.3*b*, we can establish (ii) in $E\times D_\mu$. First we rewrite $a^{(2)}(x,\lambda)$ as

$$\begin{aligned}(3.20)\qquad a^{(2)}(x,\lambda)&=\frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta d\theta\int_0^{\infty}e^{-\lambda\rho}\rho g(x+t)d\rho\\&=\frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta d\theta\cdot J\quad(y-x=t,\ |t|=\rho).\end{aligned}$$

Then, integrating by parts twice in view of (C), we have

$$\begin{aligned}J&=\frac{1}{\lambda}\int_0^{\infty}e^{-\lambda\rho}\frac{\partial}{\partial\rho}(\rho g(x+t))d\rho\\&=\frac{1}{\lambda^2}g(x)+\frac{1}{\lambda^2}\Big(2\int_0^{\infty}e^{-\lambda\rho}\frac{\partial}{\partial\rho}g(x+t)\,d\rho+\int_0^{\infty}e^{-\lambda\rho}\rho\frac{\partial^2}{\partial\rho^2}g(x+t)d\rho\Big),\end{aligned}$$

$$\begin{aligned}(3.21)\quad |J|&\leqslant C\Big(1+\int_0^{\infty}e^{-\mathrm{Re}\,\lambda\rho}e^{-\mu|\rho-|x||}d\rho+\int_0^{\infty}e^{-\mathrm{Re}\,\lambda\rho}\rho\, e^{-\mu|\rho-|x||}d\rho\Big)|\lambda|^{-2}\\&\leqslant\begin{cases}Ce^{-\mathrm{Re}\,\lambda|x|}(1+|x|+(\mu+\mathrm{Re}\,\lambda)^{-1}(1+|x|)+(\mu+\mathrm{Re}\,\lambda)^{-2})|\lambda|^{-2} & (\mathrm{Re}\,\lambda<0)\\ C(1+|x|)|\lambda|^{-2} & (\mathrm{Re}\,\lambda\geqslant 0),\end{cases}\end{aligned}$$

where C's do not depend on any of x and λ. (3.20) and (3.21) immediately prove (ii). This completes the proof of Proposition 3.4.

Proposition 3.5. *Assume* (A_1), (A_2), (A_3), (B_1) *and* (C). *Put* $D_1=\{\lambda;\ \mathrm{Re}\,\lambda>-\min(\alpha,\gamma,\mu)\}$ *with* α *in Proposition* 3.2. *Then it follows that*

(i) $a^{(1)}(\cdot,\lambda)\in B_\delta$, $a^{(2)}(\cdot,\lambda)\in B_\delta$ *for every* $\lambda\in D_1$,

(ii) *Equation* (3.15) *admits a unique solution in* B_δ *for each* $\lambda\in D_1$, *which is analytic in* D_1, *with the estimate* $\|(I-T_\lambda)^{-1}\|_\delta\leqslant C$ *uniformly in* D_1,

(iii) *In terms of this unique solution,* $u_k(x,\lambda)$ $(k=1, 2)$ *in* (2.5) *is extended analytically to* D_1.

Proof. In view of Propositions 3.3*a* and 3.4 (i) we have only to prove the continuity of $a^{(1)}(x, \lambda)$ and $a^{(2)}(x, \lambda)$ in $x \in E$ for every $\lambda \in D_1$ in order to establish (i). This is proved by using (B_1) and (C) if we proceed as in the estimation for having (3.4).

Now (ii) follows directly from Proposition 3.2 with (i) and the analyticity of $a^{(1)}(\cdot, \lambda)$ and $a^{(2)}(\cdot, \lambda)$ in D_1. Let $\tilde{u}_k(x, \lambda)$ be the unique solution of (3.15) $(k=1, 2)$.

Furthermore, $\tilde{u}_k(\cdot, \lambda) \in L^2$ for $\mathrm{Re}\,\lambda > 0$, noting in (3.15) that $a^{(1)}(\cdot, \lambda)$, $a^{(2)}(\cdot, \lambda)$ and $T_\lambda \tilde{u}_k(\cdot, \lambda)$ all belong to L^2 for $\mathrm{Re}\,\lambda > 0$ by (B_1), (C) and (A_1) $(k=1, 2)$ (*cf. e.g.* the estimation of $T_\lambda f(x)$ in the proof of Lemma 3.1). On the other hand, we have already stated in Section 2 that $u_k(\cdot, \lambda)$ $(k=1, 2)$ in (2.5), which is analytic in $\mathrm{Re}\,\lambda > 0$, belongs to L^2 and satisfies (3.15) for $\mathrm{Re}\,\lambda > 0$. By (A_2) and the note on the spectrum of L in Section 1, $f = T_\lambda f$ $(f \in L^2,\ \mathrm{Re}\,\lambda > 0)$ implies that $f=0$. Then $u_k(x, \lambda) = \tilde{u}_k(x, \lambda)$ $(k=1, 2)$ in $E \times \{\lambda;\ \mathrm{Re}\,\lambda > 0\}$ in consideration of their continuity in x for $\mathrm{Re}\,\lambda > 0$. Thus we have (iii), and the proof is complete.

Proposition 3.6a. *Suppose* (A_1), (A_2), (A_3), (B_1) *and* (C). *Let* $D_1 = \{\lambda;\ \mathrm{Re}\,\lambda > -\min(\alpha, \gamma, \mu)\}$ *with* α *in Proposition* 3.2. *Then* $w_1(x, \lambda)$ *in* (2.5) *can be defined for* $(x, \lambda) \in E \times D_1$ *in the form* $w_1(x, \lambda) = u_1(x, \lambda) f_1(\lambda)$ *as a meromorphic function in* D_1 *with the sole simple pole* $\lambda = i\omega$, *and has the estimate*

$$|w_1(x, \lambda)| \leqslant \begin{cases} Ce^{(\delta/2)|x|}((\gamma + \mathrm{Re}\,\lambda)^{-2}|\lambda - i\omega|^{-2} + (1 + (\mu + \mathrm{Re}\,\lambda)^{-1} + \\ \qquad + (\mu + \mathrm{Re}\,\lambda)^{-2})|\lambda|^{-2}) \qquad (\mathrm{Re}\,\lambda < 0) \\ Ce^{(\delta/2)|x|}(|\lambda - i\omega|^{-2} + |\lambda|^{-2}) \qquad (\mathrm{Re}\,\lambda \geqslant 0) \qquad ((x, \lambda) \in E \times D_1), \end{cases}$$

where C*'s are not dependent on* x *nor* λ.

Proof. Putting $w_1(x, \lambda) = u_1(x, \lambda) f_1(\lambda)$ for $\mathrm{Re}\,\lambda \leqslant 0$, $w_1(x, \lambda)$ in (2.5) can be extended meromorphically to D_1 by Proposition 3.5 (iii). By Proposition 3.5 (ii) $u_1(x, \lambda)$ may be expressed as

$$(3.22) \qquad u_1(\cdot, \lambda) = (I - T_\lambda)^{-1}[a^{(1)}(\cdot, \lambda) + a^{(2)}(\cdot, \lambda) f_1^{-1}(\lambda)] \qquad (\lambda \in D_1),$$

where $\|(I - T_\lambda)^{-1}\|_\delta \leqslant C$ uniformly in D_1. Using Propositions 3.5 (i), 3.3*a* and 3.4 (ii) we have, in D_1,

$$(3.23) \qquad \|a^{(1)}(\cdot, \lambda)\|_\delta \leqslant \begin{cases} C(\gamma + \mathrm{Re}\,\lambda)^{-2} & (\mathrm{Re}\,\lambda < 0) \\ C & (\mathrm{Re}\,\lambda \geqslant 0), \end{cases}$$

$$(3.24) \qquad \|a^{(2)}(\cdot, \lambda)\|_\delta \leqslant \begin{cases} C(1 + (\mu + \mathrm{Re}\,\lambda)^{-1} + (\mu + \mathrm{Re}\,\lambda)^{-2})|\lambda|^{-2} & (\mathrm{Re}\,\lambda < 0) \\ C|\lambda|^{-2} & (\mathrm{Re}\,\lambda \geqslant 0), \end{cases}$$

where C's are all independent of λ. Hence, noting (3.22) together with (3.23) and (3.24), we get, in $E \times D_1$,

$$|u_1(x,\lambda)|\leqslant Ce^{(\delta/2)|x|}(\|a^{(1)}(\cdot,\lambda)\|_\delta+\|a^{(2)}(\cdot,\lambda)\|_\delta|f_1^{-1}(\lambda)|)$$
$$\leqslant\begin{cases}Ce^{(\delta/2)|x|}((\gamma+\mathrm{Re}\,\lambda)^{-2}+(1+(\mu+\mathrm{Re}\,\lambda)^{-1}+(\mu+\mathrm{Re}\,\lambda)^{-2})\times \\ \qquad\qquad\times|f_1^{-1}(\lambda)|\,|\lambda|^{-2}) & (\mathrm{Re}\,\lambda<0)\\ Ce^{(\delta/2)|x|}(1+|f_1^{-1}(\lambda)|\,|\lambda|^{-2}) & (\mathrm{Re}\,\lambda\geqslant 0)\,,\end{cases}$$

where C's do not depend on any of x and λ. Now, in view of $w_1(x,\lambda)=u_1(x,\lambda)f_1(\lambda)$, we have proved the desired estimate for $w_1(x,\lambda)$ in $E\times D_1$. This completes the proof of the proposition.

Proposition 3.6b. *Assume (B_2) in addition to the conditions in Proposition 3.6a. Let $D_2=\{\lambda;\ \mathrm{Re}\,\lambda>-\min(\alpha,\gamma,\nu,\mu)\}$ with α in Proposition 3.2. Then $w_2(x,\lambda)$ in (2.5) may be extended meromorphically to D_2 in the form $w_2(x,\lambda)=u_2(x,\lambda)f_2(\lambda)$ with the only simple pole $\lambda=i\omega$. Besides, in $E\times D_2$ $w_2(x,\lambda)$ has the estimate*

$$|w_2(x,\lambda)|\leqslant\begin{cases}Ce^{(\delta/2)|x|}(C_1|\lambda-i\omega|^{-1}|\lambda|^{-1}+C_2|\lambda|^{-2}) & (\mathrm{Re}\,\lambda<0)\\ Ce^{(\delta/2)|x|}(|\lambda-i\omega|^{-1}|\lambda|^{-1}+|\lambda|^{-2}) & (\mathrm{Re}\,\lambda\geqslant 0)\,,\end{cases}$$

where

$$\begin{cases}C_1=1+(\gamma+\mathrm{Re}\,\lambda)^{-1}+(\nu+\mathrm{Re}\,\lambda)^{-1}+(\nu+\mathrm{Re}\,\lambda)^{-2}\\ C_2=1+(\mu+\mathrm{Re}\,\lambda)^{-1}+(\mu+\mathrm{Re}\,\lambda)^{-2}\,,\end{cases}$$

and where C's are independent of x and λ.

Proof. Considering that our assumptions in this proposition are more stringent than those in Proposition 3.6*a*, we have (3.24), (3.22) for $u_2(\cdot,\lambda)$ and $f_2^{-1}(\lambda)$ in D_2, and the meromorphical extension of $w_2(x,\lambda)$ in (2.5) to D_2, in a way similar to the one in the proof of the preceding proposition. Now by Propositions 3.5 (i) and 3.3*b* we have the estimate

$$\|a^{(1)}(\cdot,\lambda)\|_\delta\leqslant\begin{cases}C(1+(\gamma+\mathrm{Re}\,\lambda)^{-1}+(\nu+\mathrm{Re}\,\lambda)^{-1}+(\nu+\mathrm{Re}\,\lambda)^{-2})|\lambda|^{-1}\\ \qquad\qquad (\mathrm{Re}\,\lambda<0)\\ C|\lambda|^{-1}\qquad\qquad (\mathrm{Re}\,\lambda\geqslant 0)\end{cases}\tag{3.25}$$

in D_2, where C's do not depend on λ. From (3.24), (3.25) and (3.22) for $u_2(\cdot,\lambda)$ and $f_2^{-1}(\lambda)$ it follows that in $E\times D_2$

$$|u_2(x,\lambda)|\leqslant Ce^{(\delta/2)|x|}(\|a^{(1)}(\cdot,\lambda)\|_\delta+\|a^{(2)}(\cdot,\lambda)\|_\delta|f_2^{-1}(\lambda)|)$$
$$\leqslant\begin{cases}Ce^{(\delta/2)|x|}(C_1|\lambda|^{-1}+C_2|f_2^{-1}(\lambda)|\,|\lambda|^{-2}) & (\mathrm{Re}\,\lambda<0)\\ C(|\lambda|^{-1}+|f_2^{-1}(\lambda)|\,|\lambda|^{-2}) & (\mathrm{Re}\,\lambda\geqslant 0)\,,\end{cases}$$

where C_1 and C_2 are the same as in the proposition, and where C's are independent of x and λ. Noting that $w_2(x,\lambda)=u_2(x,\lambda)f_2(\lambda)$, we complete the proof.

4. Proof of the theorems. Now we assume (A_1), (A_2), (A_3), (B_1) and

(C) for the proof of Theorem 1, and, moreover, (B_2) for that of Theorem 2. On the complex plane we choose a rectangular path Γ_k with vertices $\sigma-iA$, $\sigma+iA$, $-\alpha_k+iA$ and $-\alpha_k-iA$, where $\sigma>\beta$ (see Proposition 2.1), $0<\alpha_1<\min(\alpha, \gamma, \mu)$ and $0<\alpha_2<\min(\alpha, \gamma, \nu, \mu)$ with α in Proposition 3.2, and where A is a positive number large enough for Γ_k to contain the simple pole $\lambda=i\omega$ of $w_k(x, \lambda)$ $(k=1, 2)$. Propositions 3.6*a* $(k=1)$ and 3.6*b* $(k=2)$ enable us to apply the residue theorem to the contour integral of $w_k(x, \lambda)e^{\lambda t}$ along the positively oriented closed path Γ_k. That is

$$\int_{\Gamma_k} w_k(x, \lambda)e^{\lambda t}d\lambda = 2\pi i \operatorname*{Res}_{\lambda=i\omega}[w_k(x, \lambda)e^{\lambda t}] \qquad (k=1, 2).$$

Meanwhile, the left-hand side can be divided as

$$\int_{\Gamma_k} w_k(x, \lambda)e^{\lambda t}d\lambda = \int_{\sigma-iA}^{\sigma+iA} w_k(x, \lambda)e^{\lambda t}d\lambda - \int_{-\alpha_k-iA}^{-\alpha_k+iA} + \int_{\sigma+iA}^{-\alpha_k+iA} + \int_{-\alpha_k-iA}^{\sigma-iA} \qquad (k=1, 2).$$

Propositions 3.6*a* $(k=1)$ and 3.6*b* $(k=2)$ also assert that the third and the fourth integrlas on the right-hand side tend to zero as $A\to\infty$ for every $(x, t)\in E\times[0, \infty)$, and that

$$\operatorname*{Res}_{\lambda=i\omega}[w_k(x, \lambda)e^{\lambda t}] = u_k(x, i\omega)e^{i\omega t} \qquad (k=1, 2).$$

In view of equation (3.15) for $\lambda=i\omega$ we can set

$$u_k(x, i\omega) = u(x) \qquad (k=1, 2),$$

which is a solution of the reduced wave equation (1.2) by Proposition 3.5 (ii). Thus, recalling (2.3), we have

$$v_k(x, t) = \lim_{A\to\infty}\frac{1}{2\pi i}\int_{-\alpha_k-iA}^{-\alpha_k+iA} w_k(x, \lambda)e^{\lambda t}d\lambda + u(x)e^{i\omega t} \qquad (k=1, 2). \tag{4.1}$$

Furthermore, by Propositions 3.6*a* $(k=1)$ and 3.6*b* $(k=2)$ we get

$$\left|\int_{-\alpha_k-iA}^{-\alpha_k+iA} w_k(x, \lambda)e^{\lambda t}d\lambda\right| \leqslant \begin{cases} Ce^{(\delta/2)|x|}e^{-\alpha_1 t}\displaystyle\int_{-A}^{A}\left(\frac{1}{(p-\omega)^2+\alpha_1^2}+\frac{1}{p^2+\alpha_1^2}\right)dp & (k=1) \\ Ce^{(\delta/2)|x|}e^{-\alpha_2 t}\displaystyle\int_{-A}^{A}\left(\frac{1}{\sqrt{(p-\omega)^2+\alpha_2^2}\sqrt{p^2+\alpha_2^2}}+\frac{1}{p^2+\alpha_2^2}\right)dp & (k=2), \end{cases}$$

where p is the imaginary part of λ, and where the first C depends on α_1 and the second on α_2, but both are independent of $(x, t)\in E\times[0, \infty)$. As the integrals on the right-hand side converge as $A\to\infty$, (4.1) can be rewritten as

$$|v_k(x, t)-u(x)e^{i\omega t}| \leqslant Ce^{(\delta/2)|x|}e^{-\alpha_k t},$$

where C depends on α_k and ω, but not on x nor t ($k=1, 2$). Therefore we have (i) and (ii) in both theorems.

By its definition $u(x)$ satisfies the integral equation

$$u(x) = \frac{1}{4\pi}\int_E \frac{e^{-i\omega|x-y|}}{|x-y|} q(y)dy - \frac{1}{4\pi}\int_E \frac{e^{-i\omega|x-y|}}{|x-y|} c(y)u(y)dy,$$

where $|q(y)| \leqslant Q_1 e^{-\gamma|y|}$ and $|c(y)u(y)| \leqslant Ce^{-(\delta/2)|y|}$ by means of (B_1), (A_1) and Proposition 3.5 (ii). Then, for the proof of (iii) in both theorems it is sufficient to refer to A. Ya. Povzner [11], Chapter 2, Lemmas 1 and 2.

Thus we have proved Theorems 1 and 2.

Appendix. As was mentioned in Section 1 we remark here that the assumption of (A_3) has a justifiable ground. For we can give an example of $c(x)$ satisfying (A_1) and (A_2), but not (A_3), which means that (A_3) is independent of the others on $c(x)$ (*cf.* also T. Shirota and K. Asano [12]).

Consider

$$c_0(x) = \begin{cases} -1 & \left(|x| < \frac{\pi}{2}\right) \\ 0 & \left(|x| \geqslant \frac{\pi}{2}\right) \end{cases}, \qquad f_0(x) = \begin{cases} \frac{\sin|x|}{|x|} & \left(|x| < \frac{\pi}{2}\right) \\ \frac{1}{|x|} & \left(|x| \geqslant \frac{\pi}{2}\right) \end{cases}.$$

It follows from a simple computation that $f_0(x)$ satisfies the equation

$$f_0(x) = -\frac{1}{4\pi}\int_E \frac{c_0(y)}{|x-y|} f_0(y)dy.$$

However $c_0(x)$ is not smooth. So let $\rho(x)=\rho(|x|)$ be a C^∞ function such that $\rho(x) \geqslant 0$, $\rho(x)=0$ ($|x| \geqslant 1$) and $\int_E \rho(x)dx=1$. Moreover, denoting by $*$ the convolution, we put

$$c_1(x) = \frac{\rho * c_0 f_0(x)}{\rho * f_0(x)}, \qquad f_1(x) = \rho * f_0(x). \tag{1}$$

Then $f_1(x)=f_1(|x|)$ is a strictly postitive solution in $B_\delta \cap C^\infty(E)$ of the equation

$$f_1(x) = -\frac{1}{4\pi}\int_E \frac{c_1(y)}{|x-y|} f_1(y)dy,$$

but it is not in L^2 because it equals $|x|^{-1}$ for $|x| \geqslant \frac{\pi}{2}+1$. Furthermore, the numerator of $c_1(x)$ is a C^∞ function with support in the sphere $|x| \leqslant \frac{\pi}{2}+1$, and so is $c_1(x)=c_1(|x|)$. Hence $c_1(x)$ satisfies (A_1), but not (A_3).

Now we are proving that the operator $L_1=-\Delta+c_1(x)$ has no negative eigenvalue with an L^2 eigenfunction. Let $\lambda<0$. Let $f\in L^2$ be a solution of the equation

$$L_1 f=\lambda f. \tag{2}$$

If we put, in spherical coordinates,

$$g(r, \theta, \varphi)=rf(r, \theta, \varphi) \qquad (|x|=r), \tag{3}$$

g may be expressed in terms of the series

$$g(r, \theta, \varphi)=\sum_{n=0}^{\infty} \sum_{m=0}^{2n} b_{n,m}(r) Y_{n,m}(\theta, \varphi),$$

where $\{Y_{n,m}\}$ is a complete orthonormal system of spherical harmonics, and where

$$b_{n,m}(r)=\int_0^{2\pi} d\varphi \int_0^{\pi} g(r, \theta, \varphi) Y_{n,m}(\theta, \varphi) \sin\theta d\theta, \tag{4}$$

which fulfills the equation

$$\frac{d^2}{dr^2} b_{n,m}(r)+\left(\lambda-c_1(r)-\frac{n(n+1)}{r^2}\right) b_{n,m}(r)=0. \tag{5}$$

Since $\dfrac{n(n+1)}{r^2}$ $(n\geqslant 1)$ represents a positive definite operator, we have only to prove, in view of (5), (3) and (4), that $b(r)\in L^2(0, \infty)$ must identically vanish if $b(r)$ satisfies

$$b''(r)+(\lambda-c_1(r))b(r)=0, \tag{6}$$

$$b(0)=0. \tag{7}$$

By the statement following (1), $b_0(r)=rf_1(r)$ fulfills

$$b_0(r)>0 \ (r>0), \qquad b_0(r)=1 \ \left(r\geqslant \frac{\pi}{2}+1\right), \tag{8}$$

$$b_0''(r)-c_1(r)b_0(r)=0, \tag{9}$$

$$b_0(0)=0, \qquad b_0'(0)>0. \tag{10}$$

Any solution of equtaion (6) with (7) may be determined except for a constant multiple. Hence we can assume

$$b'(0)>b_0'(0). \tag{11}$$

By (7), (10) and (11) we have, for sufficiently small positive values of r,

(12) $$b(r)>b_0(r)\,.$$

Assume that $b(r)$ and $b_0(r)$ have the first common value at $r=r_0$ except the origin. Then we obtain

(13) $$b'(r_0)\leqslant b_0'(r_0)\,.$$

Multiplying (6) and (9) by $b_0(r)$ and $b(r)$ respectively, and subtracting, we have

$$\lambda b_0(r)b(r)=b(r)b_0''(r)-b_0(r)b''(r)\,.$$

Integrating the above equation over $[0, r_0]$, we get, by (7) and (10),

$$\lambda\int_0^{r_0} b_0(r)b(r)dr=b_0(r_0)(b_0'(r_0)-b'(r_0))\qquad(\lambda<0)\,,$$

where the left-hand side is negative by (8) and (12), while the right-hand side is non-negative by (8) and (13). This is a contradiction. So by (12) we have

$$b(r)>b_0(r)\qquad(r>0)\,.$$

Hence, by (8) $b(r)$ does not lie in $L^2(0, \infty)$. That is, $b(r)\in L^2(0, \infty)$ satisfying (6) and (7) must be identically zero, which was to be proved. Therefore (A_2) is the case with $c_1(x)$.

Thus $c_1(x)$ is a required example satisfying (A_1) and (A_2), but not (A_3). In other words, (A_3) is not too unnatural a restriction on the potential function $c(x)$.

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