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Author(s)	Yosimura, Zen-ichi
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UNIVERSAL COEFFICIENT SEQUENCES FOR COHOMOLOGY THEORIES OF CW-SPECTRA, II

Dedicated to Professor A. Komatu on his 70th birthday

ZEN-ICHI YOSIMURA

(Received October 17, 1977)

Let E be a CW -spectrum and G be an abelian group. Following Kainen [11] we can construct a CW -spectrum $\hat{E}(G)$ which has a universal coefficient sequence

$$0 \rightarrow \text{Ext}(E_{*-1}(X), G) \rightarrow \hat{E}(G)^*(X) \rightarrow \text{Hom}(E_*(X), G) \rightarrow 0.$$

In the previous paper [14] with the same title we investigated several properties of $\hat{E}(G)$. But some of our results are restrictive as yet, e.g., Proposition 8 and Theorem 4 in [14]. In this note we continue the investigations to develop and improve our partial results.

First we discuss whether the correspondences $G \rightarrow \hat{E}(G)$ as well as $G \rightarrow EG$ are functorial in G , as analogous discussions were done in [9] and [10]. Next, under some finiteness assumption on E or G we show that $\hat{E}(G)$ and $\hat{E}(R)G$ are homotopy equivalent where $Z \subset R \subset Q$ (Theorem 1). This result is a satisfactory improvement of [14, Proposition 8]. As an application of the main result of Huber and Meier [10] we can then give a criterion for $EG^*(X)$ being Hausdorff (Theorem 2). Moreover we discuss the uniqueness of $\hat{E}(G)$ again to improve a partial result obtained in [14, Theorem 4]. When E is the sphere spectrum S we have a complete result (Theorem 3), but for a general E we need still some restriction although the finiteness assumptions on E and G can be eliminated in our previous result (Theorem 4). Finally we show that the universal coefficient sequence is pure under some restriction on E or G , adopting an argument given in [9].

In this note we shall work in the stable homotopy category of CW -spectra (see [1] or [13]).

The author wishes to thank Professors Huber and Meier for sending him their preprint [10] by which he has been motivated to write this sequel.

1. Functoriality of $\hat{E}(G)$

1.1. Let E be a CW -spectrum and G be an abelian group. Then there

is a CW -spectrum $\hat{E}(G)$ so that E and $\hat{E}(G)$ are related by a universal coefficient sequence

$$(1.1) \quad 0 \rightarrow \text{Ext}(E_{*-1}(X), G) \xrightarrow{\eta_G} \hat{E}(G)^*(X) \xrightarrow{\tau_G} \text{Hom}(\hat{E}_*(X), G) \rightarrow 0$$

(see [11] and [14]). Let us first recall the construction of $\hat{E}(G)$ involving an injective resolution of G . By the representability theorem there is a CW -spectrum $\hat{E}(I)$ and a natural equivalence $\tau_I: \hat{E}(I)^*(\) \rightarrow \text{Hom}(E_*(\), I)$ for every injective I . Take any injective resolution $0 \rightarrow G \rightarrow I \xrightarrow{\psi} J \rightarrow 0$ and denote by $\hat{\psi}: \hat{E}(I) \rightarrow \hat{E}(J)$ the unique map induced by ψ . We define $\hat{E}(G)$ to be the fiber of $\hat{\psi}$, i.e.

$$\hat{E}(G) \rightarrow \hat{E}(I) \xrightarrow{\hat{\psi}} \hat{E}(J)$$

is a cofiber. The homotopy type of $\hat{E}(G)$ is independent of the choice of an injective resolution.

Let us denote by S the sphere spectrum. By the exactness of function spectra [13] there is a cofiber

$$F(E, \hat{S}(G)) \rightarrow F(E, \hat{S}(I)) \xrightarrow{F(E, \hat{\psi})} F(E, \hat{S}(J)).$$

By the aid of Five lemma [13] we obtain

Proposition 1. *For any abelian group G the spectrum $\hat{E}(G)$ has the same homotopy type as the function spectrum $F(E, \hat{S}(G))$.*

Given an abelian group G , each map $f: W \rightarrow E$ of CW -spectra determines the unique map $\hat{f} = F(f, \hat{S}(G)): F(E, \hat{S}(G)) \rightarrow F(W, \hat{S}(G))$. Thereby Proposition 1 contains the following functorial property.

Corollary 2. *Fix an abelian group G . Then the correspondence $E \rightarrow \hat{E}(G) = F(E, \hat{S}(G))$ is a contravariant exact functor.*

We may now turn our attention to the spectrum $\hat{S}(G)$. The map τ_G gives rise to an isomorphism

$$(1.2) \quad t_G: \pi_0(\hat{S}(G)) \xrightarrow{\tau_G} \text{Hom}(\pi_0(S), G) \cong G.$$

Lemma 3. *The composition map*

$$\hat{S}(G)^*(X) \xrightarrow{\tau_G} \text{Hom}(\pi_*(X), G) \xrightarrow{t_{G^*}} \text{Hom}(\pi_*(X), \pi_0(\hat{S}(G)))$$

is just the homomorphism κ assigning to a map f the induced homomorphism f_ in 0-th homotopy groups.*

Proof. It is sufficient to show the equality $t_G = \tau_G(1_{\hat{S}(G)})$ for the identity map $1_{\hat{S}(G)}$ of $\hat{S}(G)$. Take any element f of $\pi_0(\hat{S}(G))$, i.e., a map $f: S \rightarrow \hat{S}(G)$. By the naturality of τ_G we have

$$t_G(f) = \tau_G(f)(1_S) = (\tau_G(1_{\hat{S}(G)})f*)(1_S) = \tau_G(1_{\hat{S}(G)})(f).$$

Because of Lemma 3 we may employ κ instead of τ_G . Thus there is a natural exact sequence

$$(1.3) \quad 0 \rightarrow \text{Ext}(\pi_{*-1}(X), G) \xrightarrow{\eta_G} \hat{S}(G)^*(X) \xrightarrow{\kappa} \text{Hom}(\pi_*(X), G) \rightarrow 0$$

where $\pi_0(\hat{S}(G))$ is identified with G via the map t_G .

Let E be a ring spectrum and F be an (associative) right E -module spectrum equipped with a structure map $\mu: F \wedge E \rightarrow F$. Then there is a unique map

$$\bar{\mu}_G: E \wedge \hat{F}(G) \rightarrow \hat{F}(G)$$

such that $e_{F,G}(1_F \wedge \bar{\mu}_G) = e_{F,G}(\mu \wedge 1_{\hat{F}(G)})$ where $e_{F,G}: F \wedge \hat{F}(G) \rightarrow \hat{S}(G)$ is the evaluation map. Thereby $\hat{F}(G)$ is an (associative) left E -module spectrum. Using the structure maps μ and $\bar{\mu}_G$ we can give $\text{Hom}(F_*(), G)$ and $\hat{F}(G)^*()$ structures of left $E^*()$ -modules. Thus we have two homomorphisms

$$\begin{aligned} \mu_*: E^*(Y) \otimes \text{Hom}(F_*(X), G) &\rightarrow \text{Hom}(F_*(Y \wedge X), G) \\ \bar{\mu}_*: E^*(Y) \otimes \hat{F}(G)^*(X) &\rightarrow \hat{F}(G)^*(Y \wedge X) \end{aligned}$$

defined in the obvious way. By virtue of Lemma 3 we have

Proposition 4. *Let E be a ring spectrum and F be a right E -module spectrum. Then the universal coefficient sequence*

$$0 \rightarrow \text{Ext}(F_{*-1}(X), G) \xrightarrow{\eta_G} \hat{F}(G)^*(X) \xrightarrow{\tau_G} \text{Hom}(F_*(X), G) \rightarrow 0$$

is an exact sequence of left $E^()$ -modules.*

Proof. As is easily seen, the induced homotopy homomorphism κ is a map of left $E^*()$ -modules, i.e., the following square

$$\begin{array}{ccc} E^*(Y) \otimes \hat{F}(G)^*(X) & \xrightarrow{1 \otimes \kappa} & E^*(Y) \otimes \text{Hom}(F_*(X), \pi_0(\hat{S}(G))) \\ \bar{\mu}_* \downarrow & & \downarrow \mu_* \\ \hat{F}(G)^*(Y \wedge X) & \xrightarrow{\kappa} & \text{Hom}(F_*(Y \wedge X), \pi_0(\hat{S}(G))) \end{array}$$

is commutative. By a routine computation the result is immediate.

1.2. Take any homomorphism $\phi: G \rightarrow H$ of abelian groups, then there is a (non-unique) map $\hat{\phi}: \hat{S}(G) \rightarrow \hat{S}(H)$ making the diagram below commutative

$$\begin{array}{ccccccc}
0 \rightarrow \text{Ext}(\pi_{*-1}(X), G) & \xrightarrow{\eta_G} & \hat{S}(G)^*(X) & \xrightarrow{\tau_G} & \text{Hom}(\pi_*(X), G) & \rightarrow & 0 \\
& \downarrow \phi_* & \downarrow \hat{\phi}_* & & \downarrow \phi_* & & \\
0 \rightarrow \text{Ext}(\pi_{*-1}(X), H) & \xrightarrow{\eta_H} & \hat{S}(H)^*(X) & \xrightarrow{\tau_H} & \text{Hom}(\pi_*(X), H) & \rightarrow & 0.
\end{array}$$

Thus the correspondence $G \rightarrow \hat{S}(G)$ is quasi-functorial in G [11].

Lemma 5. *If $0 \rightarrow G \xrightarrow{\phi} H \xrightarrow{\psi} K \rightarrow 0$ is a short exact sequence, then there exist maps $\hat{\phi}: \hat{S}(G) \rightarrow \hat{S}(H)$ and $\hat{\psi}: \hat{S}(H) \rightarrow \hat{S}(K)$ which give us a cofibering*

$$\hat{S}(G) \xrightarrow{\hat{\phi}} \hat{S}(H) \xrightarrow{\hat{\psi}} \hat{S}(K).$$

Proof. Choose an injective resolution $0 \rightarrow H \rightarrow I \rightarrow J_1 \rightarrow 0$ and consider commutative exact diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \rightarrow & G & \rightarrow & H & \rightarrow & K \rightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & I & = & I & & \\
& & \downarrow & & \downarrow & & \\
0 & \rightarrow & K & \rightarrow & J_0 & \rightarrow & J_1 \rightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

in which there appear three injective resolutions of G , H and K . By applying Verdier's lemma [6] we obtain a cofibering as desired.

Denote by $k_{G,H}$ the composition map

$$\hat{S}(H)^0(\hat{S}(G)) \xrightarrow{\tau_H} \text{Hom}(\pi_0(\hat{S}(G)), H) \xleftarrow[\cong]{t_G^*} \text{Hom}(G, H).$$

It is epic, in fact we observe that

$$(1.4) \quad k_{G,H}(\hat{\phi}) = \phi_*(1_G) = \phi,$$

by making use of the equality $t_G = \tau_G(1_{\hat{S}(G)})$. But $\text{Ker } k_{G,H} \cong \text{Ext}(\pi_{-1}(\hat{S}(G)), H) \cong \text{Ext}(\text{Hom}(Z_2, G), H) \cong \text{Ext}(G, Z_2 \otimes H)$. By an easy computation we verify that

(1.5) $k_{G,H}$ is an isomorphism if and only if either G is 2-torsion free or H is 2-divisible.

This implies

Proposition 6. *If G is 2-torsion free or if H is 2-divisible, then $\hat{\phi} = F(E, \hat{\phi}): \hat{E}(G) \rightarrow \hat{E}(H)$ is uniquely determined for each $\phi: G \rightarrow H$.*

Let us denote by $\eta: \Sigma^1 S \rightarrow S$ the Hopf map, i.e., the non-zero element of $\pi_1(S)$. A CW-spectrum E is said to be *good* if $\eta \wedge 1_E: \Sigma^1 E \rightarrow E$ is trivial [9]. For a good E we have the following functorial property.

Proposition 7. *Assume that a fixed CW-spectrum E is good. Then the composite $G \rightarrow \hat{S}(G) \rightarrow \hat{E}(G) = F(E, \hat{S}(G))$ is a covariant exact functor.*

Proof. We show that the homomorphism

$$F(E, \quad): \{\hat{S}(G), \hat{S}(H)\} \rightarrow \{\hat{E}(G), \hat{E}(H)\}$$

factors through $k_{G,H}$. Recall that $F(E, \quad)$ is given by the composition

$$\{\hat{S}(G), \hat{S}(H)\} \xrightarrow{e_G^*} \{E \wedge \hat{E}(G), \hat{S}(H)\} \xleftarrow{\cong} \{\hat{E}(G), \hat{E}(H)\}$$

where $e_G = e_{E,G}: E \wedge \hat{E}(G) \rightarrow \hat{S}(G)$ is the evaluation map. So it is enough to show that there is a homomorphism λ making the diagram below commutative

$$\begin{array}{ccc} 0 \rightarrow \text{Ext}(\pi_{-1}(\hat{S}(G)), H) & \xrightarrow{\eta_H} & \{\hat{S}(G), \hat{S}(H)\} \xrightarrow{k_{G,H}} \text{Hom}(G, H) \rightarrow 0 \\ \downarrow (e_G^*)^* & & \downarrow e_G^* \quad \swarrow \lambda \\ \text{Ext}(\pi_{-1}(E \wedge \hat{E}(G)), H) & \xrightarrow{\eta_H} & \{E \wedge \hat{E}(G), \hat{S}(H)\} \end{array}$$

Consider the commutative diagram

$$\begin{array}{ccccc} \pi_{-1}(E \wedge \hat{E}(G)) & \xrightarrow{e_G^*} & \pi_{-1}(\hat{S}(G)) & \xrightarrow{\tau_G} & \text{Hom}(\pi_1(S), G) \\ \eta^* \downarrow & & \eta^* \downarrow & & \downarrow (\eta_*)^* \\ \pi_0(E \wedge \hat{E}(G)) & \xrightarrow{e_G^*} & \pi_0(\hat{S}(G)) & \xrightarrow{\tau_G} & \text{Hom}(\pi_0(S), G) \end{array}$$

The left arrow η^* is trivial by our hypothesis on E , and the central one η^* is monic by use of the right square. This implies that the upper arrow e_G^* is trivial. The existence of λ is now immediate. Therefore the correspondence $G \rightarrow \hat{E}(G)$ is a functor which is exact by Lemma 5.

1.3. For each abelian group G we denote by SG the Moore spectrum of type G . Then there is a universal coefficient sequence in the form of a natural exact sequence

$$(1.6) \quad 0 \rightarrow \text{Ext}(G, \pi_{*+1}(X)) \rightarrow \{SG, X\}_* \xrightarrow{\kappa} \text{Hom}(G, \pi_*(X)) \rightarrow 0$$

where κ is just the induced homotopy homomorphism [8]. In particular we have a short exact sequence

$$0 \rightarrow \text{Ext}(G, \pi_1(SH)) \rightarrow \{SG, SH\} \xrightarrow{\kappa} \text{Hom}(G, H) \rightarrow 0.$$

Given a homomorphism $\phi: G \rightarrow H$, there is a (non-unique) map $S\phi: SG \rightarrow SH$ inducing $S\phi_* = \phi: \pi_0(SG) \rightarrow \pi_0(SH)$. Since $\pi_1(SH) \cong H \otimes Z_2$ we have an analogous result to Proposition 6.

Proposition 8. *Assume that G is 2-torsion free or that H is 2-divisible. Then $1_E \wedge S\phi: EG \rightarrow EH$ is uniquely determined for each $\phi: G \rightarrow H$ (see [10, Proposition 3.2]).*

By choosing suitably free resolutions in the dual way to the proof of Lemma 5 we can show that there is a cofibering

$$(1.7) \quad SG \xrightarrow{S\phi} SH \xrightarrow{S\psi} SK$$

if $0 \rightarrow G \xrightarrow{\phi} H \xrightarrow{\psi} K \rightarrow 0$ is a short exact sequence.

Corresponding to [9, Appendix] we obtain

Proposition 9. *Assume that a fixed CW-spectrum E is good. Then the composite $G \rightarrow SG \rightarrow EG$ is a covariant exact functor.*

Proof. The homomorphism $1_E \wedge -: \{SG, SH\} \rightarrow \{EG, EH\}$ is just the composition

$$\{SG, SH\} \xrightarrow{\varepsilon_{H^*}} \{SG, F(E, EH)\} \xrightarrow{\cong} \{EG, EH\}$$

where $\varepsilon_H: SH \rightarrow F(E, EH)$ is the dual of 1_{EH} . So we consider the following commutative diagram

$$\begin{array}{ccc} 0 \rightarrow \text{Ext}(G, \pi_1(SH)) \rightarrow \{SG, SH\} & \xrightarrow{\kappa} & \text{Hom}(G, H) \rightarrow 0 \\ (\varepsilon_{H^*})_* \downarrow & & \downarrow \varepsilon_{H^*} \\ \text{Ext}(G, \pi_1(F(E, EH))) \rightarrow \{SG, F(E, EH)\} & & \end{array}$$

In the commutative square

$$\begin{array}{ccc} \pi_0(SH) & \xrightarrow{\varepsilon_{H^*}} & \pi_0(F(E, EH)) \\ \downarrow \eta^* & & \downarrow \eta_* \\ \pi_1(SH) & \xrightarrow{\varepsilon_{H^*}} & \pi_1(F(E, EH)) \end{array}$$

the left arrow η^* is epic, but the right one η^* is trivial by our hypothesis on E . Hence the lower arrow ε_{H^*} is trivial, too. This claims that $\varepsilon_{H^*}: \{SG, SH\} \rightarrow \{SG, F(E, EH)\}$ factors through κ . Our result is now obvious.

2. Important properties of $\hat{E}(G)$

2.1. Let us denote by R a subring of the rationals \mathbb{Q} and l^c be the set of primes which are invertible in R . A CW-spectrum E is called an R -spectrum if

$p \cdot 1_E: E \rightarrow E$ is a homotopy equivalence for each $p \in \mathcal{I}$. Notice that E is an R -spectrum if and only if $\pi_*(E)$ is an R -module. An R -spectrum E is said to be of *finite type* if $\pi_*(E)$ is of finite type as an R -module.

We now study whether the CW -spectra $\hat{E}(G)$ and $\hat{E}(R)G$ are homotopy equivalent. Assume that an R -spectrum E is of finite type or that an R -module G is finitely generated. Let us first recall our partial result [14] in the special case when G is free. In this case we write P instead of G , i.e., $P = \sum_{\alpha} R$. The canonical injections $i_{\alpha}: R \rightarrow P$ give rise to the map $\bigvee_{\alpha} i_{\alpha}: \bigvee_{\alpha} \hat{E}(R) \rightarrow \hat{E}(P)$ which is unique by Proposition 6. According to [14, Lemma 7] the map $\bigvee i_{\alpha}$ is a homotopy equivalence under our assumption. Consequently the composite map

$$(2.1) \quad \iota_{E,P}: \hat{E}(R)P \leftarrow \bigvee_{\alpha} \hat{E}(R) \rightarrow \hat{E}(P)$$

is a homotopy equivalence, too.

Notice that the map $\iota_{E,P}$ has a factorization

$$F(E, \hat{S}(R))P \xrightarrow{j} F(E, \hat{S}(R)P) \xrightarrow{F(E, \iota_{S,P})} F(E, \hat{S}(P))$$

whose decomposed maps are both homotopy equivalences. By applying Five lemma we obtain that the canonical map

$$(2.2) \quad j: F(E, \hat{S}(R))G \rightarrow F(E, \hat{S}(R)G)$$

is a homotopy equivalence under our finiteness assumption on E or G .

We here give the following interesting result.

Theorem 1. *Let E be an R -spectrum and G be an R -module. Assume that E is of finite type or that G is finitely generated. Then $\hat{E}(G)$ and $\hat{E}(R)G$ have the same homotopy type.*

Proof. Take a free resolution $0 \rightarrow P_1 \xrightarrow{\phi} P_0 \xrightarrow{\psi} G \rightarrow 0$ of R -modules, and consider the diagram

$$\begin{array}{ccccc} \hat{S}(R)P_1 & \xrightarrow{1 \wedge S\phi} & \hat{S}(R)P_0 & \xrightarrow{1 \wedge S\psi} & \hat{S}(R)G \\ \downarrow \iota_{S,P_1} & & \downarrow \iota_{S,P_0} & & \\ \hat{S}(P_1) & \xrightarrow{\hat{\phi}} & \hat{S}(P_0) & \xrightarrow{\hat{\psi}} & \hat{S}(G) \end{array}$$

involving two cofiberings in (1.7) and Lemma 5. In order to show that the square is commutative, we use the map $\kappa: \hat{S}(P_0)^0(\hat{S}(R)P_1) \rightarrow \text{Hom}(\pi_0(\hat{S}(R)P_1), \pi_0(\hat{S}(P_0)))$ which is an isomorphism. After $\pi_0(\hat{S}(R)P_1)$ and $\pi_0(\hat{S}(P_0))$ are identified with P_1 and P_0 respectively, we compute that

$$\begin{aligned}\kappa(\hat{\phi} \cdot \iota_{S, P_1}) &= \phi_* \kappa(\iota_{S, P_1}) = \phi_*(1_{P_1}) = \phi^*(1_{P_0}) = \phi^* \kappa(\iota_{S, P_0}) \\ &= \kappa(\iota_{S, P_0} \cdot 1 \wedge S\phi),\end{aligned}$$

which claims $\hat{\phi} \cdot \iota_{S, P_1} = \iota_{S, P_0} \cdot 1 \wedge S\phi$. By (2.1) the vertical maps ι_{S, P_i} are both homotopy equivalences. By use of Five lemma we obtain a homotopy equivalence

$$\hat{S}(R)G \rightarrow \hat{S}(G)$$

in the special case $E=S$.

For a general E we use (2.2) to obtain that the composite map

$$F(E, \hat{S}(R)G) \xrightarrow{j} F(E, \hat{S}(R)G) \rightarrow F(E, \hat{S}(G))$$

is a homotopy equivalence.

If E is an R -spectrum of finite type, then it has a nice property that there is a homotopy equivalence

$$(2.3) \quad h_E: E \rightarrow ER \rightarrow \widehat{\hat{E}(R)}(R)$$

(see [14, Theorem 2]). Putting two important results, Theorem 1 and (2.3), together we obtain a natural exact sequence

$$(2.4) \quad 0 \rightarrow \text{Ext}(\hat{E}(R)_{*-1}(X), G) \rightarrow EG^*(X) \rightarrow \text{Hom}(\hat{E}(R)^*(X), G) \rightarrow 0$$

if E is an R -spectrum of finite type. Applying the main result of Huber and Meier [10, Theorem 1.1] we can extend our criterion [14, Theorem 3] for $E^*(X)$ being Hausdorff.

Theorem 2 ([10]). *Assume that E is an R -spectrum of finite type. Then $EG^*(X)$ is Hausdorff if and only if $\text{Pext}(\hat{E}(R)_{*-1}(X), G)=0$.*

2.2. For a CW -spectrum E we denote by $E(-\infty, n]$ ($=E(-\infty, n+1)$) the $(n+1)$ -coconnective Postnikov cofiber of E and by $E(n, \infty)$ ($=E[n+1, \infty)$) the n -connective Postnikov fiber of E (see [3]). Thus $E(-\infty, n]$ is an $(n+1)$ -coconnective CW -spectrum such that there is a map $j_n: E \rightarrow E(-\infty, n]$ which induces an isomorphism $j_{n*}: \pi_r(E) \rightarrow \pi_r(E(-\infty, n])$ for each $r \leq n$, and $E(n, \infty)$ an n -connective CW -spectrum such that there is a map $i_n: E(n, \infty) \rightarrow E$ which induces an isomorphism $i_{n*}: \pi_r(E(n, \infty)) \rightarrow \pi_r(E)$ for each $r > n$. Notice that the sequence

$$E(n, \infty) \xrightarrow{i_n} E \xrightarrow{j_n} E(-\infty, n]$$

is a cofiber sequence.

By routine computations we have

Lemma 10. i) *The map j_n induces a homotopy equivalence*

$$E(-\infty, n]G \xrightarrow{\cong} \begin{cases} EG(-\infty, n] & \text{if } \text{Tor}(\pi_n(E), G) = 0 \\ EG(-\infty, n+1] & \text{if } \pi_{n+1}(E) \otimes G = 0. \end{cases}$$

ii) *The map i_n induces a homotopy equivalence*

$$E(n, \infty)G \xrightarrow{\cong} \begin{cases} EG(n, \infty) & \text{if } \text{Tor}(\pi_n(E), G) = 0 \\ EG(n+1, \infty) & \text{if } \pi_{n+1}(E) \otimes G = 0. \end{cases}$$

Lemma 11. i) *The map \hat{i}_n induces a homotopy equivalence*

$$\widehat{E(-\infty, n]}(G) \xrightarrow{\cong} \begin{cases} \hat{E}(G)[-n, \infty) & \text{if } \text{Ext}(\pi_n(E), G) = 0 \\ \hat{E}(G)[-n-1, \infty) & \text{if } \text{Hom}(\pi_{n+1}(E), G) = 0. \end{cases}$$

ii) *The map \hat{i}_n induces a homotopy equivalence*

$$\widehat{E(n, \infty)}(G) \xrightarrow{\cong} \begin{cases} \hat{E}(G)(-\infty, -n) & \text{if } \text{Ext}(\pi_n(E), G) = 0 \\ \hat{E}(G)(-\infty, -n-1) & \text{if } \text{Hom}(\pi_{n+1}(E), G) = 0. \end{cases}$$

Combining Theorem 1 with Lemmas 10 and 11 we obtain

Proposition 12. *Assume that an R -spectrum E is of finite type or that an R -module G is finitely generated. If $\text{Ext}(\pi_n(E), G) = 0$, then $\widehat{E(-\infty, n]}(G)$ has the same homotopy type as $\hat{E}(R)[-n, \infty)G$ and $\widehat{E(n, \infty)}(G)$ does the same as $\hat{E}(R)(-\infty, -n)G$.*

For the BU -, EO - and BSp -spectrum K , KO and KSp we have determined in [14, Theorem 5] (or see [2]) that

$$(2.5) \quad \hat{K}(G) = KG \quad \text{and} \quad \widehat{KSp}(G) = KOG.$$

Applying Proposition 12 we get

$$(2.6) \quad \widehat{K[0, \infty)}(G) = K(-\infty, 0]G, \quad \widehat{KSp[0, \infty)}(G) = KO(-\infty, 0]G \quad \text{and so on.}$$

2.3. Let $\tau: F(W, \hat{V}(G)) \rightarrow F(V, \hat{W}(G))$ be the homotopy equivalence induced by the switching map $T: W \wedge V \rightarrow V \wedge W$. Putting $V=E$ and $W=\hat{E}(G)$, τ yields the map

$$\varepsilon_{E,G}: E \rightarrow \widehat{\hat{E}(G)}(G)$$

which is the dual of $e_{E,G}T$ where $e_{E,G}: E \wedge \hat{E}(G) \rightarrow \hat{S}(G)$ denotes the evaluation map. Observe that the composition

$$(2.7) \quad \{W, E\} \xrightarrow{\varepsilon_{E,G}^*} \{W, \widehat{\hat{E}(G)}(G)\} \xleftarrow[\cong]{\tau_{\#}} \{\hat{E}(G), \hat{W}(G)\}$$

is just the map $F(, \hat{S}(G))$.

Proposition 13. *If an R -spectrum E is of finite type, then the map*

$$F(, \hat{S}(R)): \{W, E\} \rightarrow \{\hat{E}(R), \hat{W}(R)\}$$

is an isomorphism for each W , and equivalently the canonical map $\varepsilon_{E,R}: E \rightarrow \widehat{\hat{E}(R)}(R)$ is a homotopy equivalence (cf., (2.3)).

Proof. Take a homotopy equivalence $h_E: E \rightarrow \widehat{\hat{E}(R)}(R)$ of (2.3) and introduce the composite map

$$\rho_E: F(W, E) \rightarrow F(W, \widehat{\hat{E}(R)}(R)) \xrightarrow{\tau} F(\hat{E}(R), \hat{W}(R)),$$

given by use of h_E , which is functorial with respect to W . We modify the map ρ_E a bit as it induces the map $F(, \hat{S}(R))$. Since $\rho_{E*}: \{W, E\} \rightarrow \{\hat{E}(R), \hat{W}(R)\}$ is an isomorphism, we can find a map $f: E \rightarrow \hat{E}(R)$ such that $\rho_{E*}(f) = 1_{\hat{E}(R)}$. The map $\hat{f}: \hat{E}(R) \rightarrow \hat{E}(R)$ gives rise to a split epic $\hat{f}_*: \pi_*(\hat{E}(R)) \rightarrow \pi_*(\hat{E}(R))$ since $\hat{f}_* \cdot \rho_{E*}(1_E) = \rho_{E*}(f^*(1_E)) = 1_{\hat{E}(R)}$. But the R -module $\pi_*(\hat{E}(R))$ is of finite type, so \hat{f}_* is isomorphic. This means that the map \hat{f} is a homotopy equivalence. Consider the composite map

$$F(W, E) \xrightarrow{\rho_E} F(\hat{E}(R), \hat{W}(R)) \xleftarrow{F(\hat{f}, \hat{W}(R))} F(\hat{E}(R), \hat{W}(R)).$$

Obviously the induced isomorphism

$$\{W, E\} \xrightarrow{\rho_{E*}} \{\hat{E}(R), \hat{W}(R)\} \xleftarrow{\hat{f}_*} \{\hat{E}(R), \hat{W}(R)\}$$

conicides with the map $F(, \hat{S}(R))$.

We next define a generalization $\hat{F}_{G,H}: \{WG, EH\} \rightarrow \{\hat{E}(R)G, \hat{W}(R)H\}$ of the isomorphism $F(, \hat{S}(R))$. The evaluation map $e_{E,R}: E \wedge \hat{E}(R) \rightarrow \hat{S}(R)$ gives us a homomorphism

$$e_{E*}: \{WG, EH\} \rightarrow \{W \wedge \hat{E}(R)G, \hat{S}(R)H\}$$

defined in the obvious way. On the other hand, if W is an R -spectrum of finite type or if H is a finitely generated R -module, then the map $j: F(W, \hat{S}(R))H \rightarrow F(W, \hat{S}(R)H)$ induces an isomorphism

$$\{\hat{E}(R)G, \hat{W}(R)H\} \rightarrow \{W \wedge \hat{E}(R)G, \hat{S}(R)H\}$$

by (2.2). We compose the above two to obtain a generalization $\hat{F}_{G,H}$ under the finiteness restriction on W or H .

Proposition 14. *Assume that W is an R -spectrum of finite type or that H is a finitely generated R -module. If an R -spectrum E is of finite type, then the map*

$$\hat{F}_{G,H}: \{WG, EH\} \rightarrow \{\hat{E}(R)G, \hat{W}(R)H\}$$

is an isomorphism.

Proof. For a free R -module P we consider the following commutative diagram

$$\begin{array}{ccccc} \{W, E\} \otimes P & \xrightarrow{e_{E\#} \otimes 1} & \{W \wedge \hat{E}(R), \hat{S}(R)\} \otimes P & & \\ \downarrow & & \downarrow & \searrow & \\ \{W, EP\} & \xrightarrow{e_{E\#}} & \{W \wedge \hat{E}(R), \hat{S}(R)P\} & \xrightarrow{\iota_{S,P\#}} & \{W \wedge \hat{E}(R), \hat{S}(P)\} . \end{array}$$

The upper arrow $e_{E\#} \otimes 1$ is an isomorphism by Proposition 13, and two vertical arrows are both isomorphisms for any finite W (use (2.1) and the proof of [14, Lemma 7 ii]). This implies that the lower one $e_{E\#}$ is an isomorphism for a general W . Now a routine argument shows that $e_{E\#}: \{WG, EH\} \rightarrow \{W \wedge \hat{E}(R)G, \hat{S}(R)H\}$ is an isomorphism for any G and H , and hence so is the map $\hat{F}_{G,H}$.

For simplicity we write \hat{S} instead of $\hat{S}(Z)$. When $W=E=S$, $\hat{F}_{G,H}$ is equal to the map $1_{\hat{S}} \wedge -: \{SG, SH\} \rightarrow \{\hat{S}G, \hat{S}H\}$. So we have

Corollary 15. *The map*

$$1_{\hat{S}} \wedge -: \{SG, SH\} \rightarrow \{\hat{S}G, \hat{S}H\}$$

is an isomorphism for any G and H .

3. Uniqueness of $\hat{E}(G)$

3.1. We here discuss the uniqueness of $\hat{E}(G)$ as it was done in [14, Theorem 4]. Our attention is first turned to the special case $E=S$. In this case we have the following satisfactory result.

Theorem 3. *If a CW-spectrum F has a natural exact sequence*

$$(*) \quad 0 \rightarrow \text{Ext}(\pi_{*-1}(X), G) \rightarrow F^*(X) \xrightarrow{\tau} \text{Hom}(\pi_*(X), G) \rightarrow 0$$

for a fixed abelian group G , then F has the same homotopy type as $\hat{S}(G)$.

Proof. By the same argument as Lemma 3 we may regard τ as the induced homotopy homomorphism κ , after G is identified with $\pi_0(F)$ via the isomorphism $t_{F,G}: \pi_0(F) \xrightarrow{\tau} \text{Hom}(\pi_0(S), G) \cong G$. Then there is a map

$$h: \hat{S}(G) \rightarrow F$$

whose induced homomorphism $h_*: \pi_0(\hat{S}(G)) \rightarrow G$ is equal to the isomorphism

t_G of (1.2). Using the commutative square

$$\begin{array}{ccc} \pi_{-1}(\hat{S}(G)) & \xrightarrow{\kappa} & \text{Hom}(\pi_1(S), \pi_0(\hat{S}(G))) \\ \downarrow h_* & & \downarrow h_* \\ \pi_{-1}(F) & \xrightarrow{\kappa} & \text{Hom}(\pi_1(S), \pi_0(F)) \end{array}$$

we verify that $h_*: \pi_{-1}(\hat{S}(G)) \rightarrow \pi_{-1}(F)$ is also an isomorphism. Applying the natural exact sequences (*) and (1.3) we can see that

$$h^*: F^0(F) \rightarrow F^0(\hat{S}(G)) \quad \text{and} \quad h^*: \hat{S}(G)^0(F) \rightarrow \hat{S}(G)^0(\hat{S}(G))$$

are both isomorphisms. A routine argument shows that h is a homotopy equivalence.

3.2. In a general case E we next attempt to weaken some restrictions in our previous result [14, Theorem 4].

Theorem 4. *Let G be a fixed abelian group and D be the maximal divisible subgroup. Assume that a CW-spectrum E satisfies $\text{Hom}(t\pi_*(E), G/D)=0$ where $t\pi_*(E)$ denotes the torsion subgroup of $\pi_*(E)$. If two CW-spectra E and F are related by a natural exact sequence*

$$0 \rightarrow \text{Ext}(E_{*-1}(X), G) \rightarrow F^*(X) \xrightarrow{\tau} \text{Hom}(E_*(X), G) \rightarrow 0,$$

then F has the same homotopy type as $\hat{E}(G)$.

Proof. Since the short exact sequence $0 \rightarrow D \rightarrow G \rightarrow G/D \rightarrow 0$ is split we may choose a map $f: F \rightarrow \hat{E}(D)$ so that it induces the composition

$$F^*(X) \xrightarrow{\tau} \text{Hom}(E_*(X), G) \rightarrow \text{Hom}(E_*(X), D) \xleftarrow{\cong} \hat{E}(D)^*(X).$$

Denoting by F_R the fiber of f , the cofibering

$$F_R \rightarrow F \xrightarrow{f} \hat{E}(D)$$

is split as $f_*: F^*(X) \rightarrow \hat{E}(D)^*(X)$ is epic. With an application of 3×3 lemma as in [14, Theorem 4] we get a natural exact sequence

$$0 \rightarrow \text{Ext}(E_{*-1}(X), G/D) \rightarrow F_R^*(X) \rightarrow \text{Hom}(E_*(X), G/D) \rightarrow 0.$$

Evidently $\text{Hom}(Q, G/D)=0$, i.e., G/D is reduced (see [7]). Therefore we have to show that F_R and $\hat{E}(G/D)$ have the same homotopy type for the reduced G/D .

We may now assume that G is a reduced group with $\text{Hom}(t\pi_*(E), G)=0$.

Take a free resolution $0 \rightarrow P_1 \xrightarrow{\phi} P_0 \xrightarrow{\psi} G \rightarrow 0$ and proceed our proof as in [14, Theorem 4]. By Lemma 5 the resolution gives us a cofibering

$$\hat{E}(P_1) \xrightarrow{\hat{\phi}} \hat{E}(P_0) \xrightarrow{\hat{\psi}} \hat{E}(G).$$

Evidently $\text{Hom}(EQ_*(X), P_i) = \text{Hom}(EQ/Z_*(X), P_i) = 0$ for $i=1, 2$ and also $\text{Hom}(EQ_*(X), G) = 0$ as G is reduced. We then obtain maps

$$\tilde{\psi}_r: F(SQ, \hat{E}(P_0)) \rightarrow F(SQ, F) \quad \text{and} \quad \bar{\psi}_r: F(SQ/Z, \hat{E}(P_0)) \rightarrow F(SQ/Z, F)$$

which make the diagrams below commutative

$$\begin{array}{ccccccc} 0 \rightarrow \text{Ext}(EQ_{*-1}(X), P_1) & \xrightarrow{\phi_*} & \text{Ext}(EQ_{*-1}(X), P_0) & \xrightarrow{\psi_*} & \text{Ext}(EQ_{*-1}(X), G) \rightarrow 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 \rightarrow \hat{E}(P_1)^*(XQ) & \xrightarrow{\hat{\phi}_*} & \hat{E}(P_0)^*(XQ) & \xrightarrow{\tilde{\psi}_\#} & F^*(XQ) \rightarrow 0 \end{array}$$

and

$$\begin{array}{ccccccc} \text{Ext}(EQ/Z_{*-1}(X), P_1) & \xrightarrow{\phi_*} & \text{Ext}(EQ/Z_{*-1}(X), P_0) & \xrightarrow{\psi_*} & \text{Ext}(EQ/Z_{*-1}(X), G) \rightarrow 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \hat{E}(P_1)^*(XQ/Z) & \xrightarrow{\hat{\phi}_*} & \hat{E}(P_0)^*(XQ/Z) & \xrightarrow{\bar{\psi}_\#} & F^*(XQ/Z) \end{array}$$

By easy diagram chases we observe that two bottom sequences in the above diagrams are exact. In particular, the composite maps $\tilde{\psi}_r \cdot \hat{\phi}$ and $\bar{\psi}_r \cdot \hat{\phi}$ are both trivial where $F(SQ, \hat{\phi})$ and $F(SQ/Z, \hat{\phi})$ are abbreviated as $\hat{\phi}$'s. Then there are two maps

$$\tilde{h}: F(SQ, \hat{E}(G)) \rightarrow F(SQ, F), \quad \bar{h}: F(SQ/Z, \hat{E}(G)) \rightarrow F(SQ/Z, F)$$

such that $\tilde{h} \cdot \hat{\psi} = \tilde{\psi}_r$ and $\bar{h} \cdot \hat{\psi} = \bar{\psi}_r$. As is easily seen, the map \tilde{h} is a homotopy equivalence. On the other hand, our assumption means that $\text{Hom}(\pi_*(EQ/Z), G) = 0$ since the map $\text{Hom}(\text{Tor}(\pi_{*-1}(E), Q/Z), G) \rightarrow \text{Hom}(\pi_*(EQ/Z), G)$ is an isomorphism for any reduced G . Thereby the coefficients sequence

$$(3.1) \quad 0 \rightarrow \hat{E}(P_1)^*(SQ/Z) \xrightarrow{\hat{\phi}_*} \hat{E}(P_0)^*(SQ/Z) \xrightarrow{\bar{\psi}_\#} F^*(SQ/Z) \rightarrow 0$$

is short exact. By means of [15, Lemma A] (see [5]) we find that the map \bar{h} is a homotopy equivalence, too.

Corresponding to the injective resolution $0 \rightarrow Z \rightarrow Q \rightarrow Q/Z \rightarrow 0$ there is a cofiber

$$S \xrightarrow{i} SQ \xrightarrow{j} SQ/Z.$$

It is easy to see that the maps $\tilde{\psi}_r$, $\bar{\psi}_r$ and $\hat{\psi}$'s are compatible with j 's. Consequently we have the following diagram

$$\begin{array}{ccccc}
 F(SQ/Z, \hat{E}(P_0)) & & \xrightarrow{\hat{\psi}} & & F(SQ/Z, F) \\
 \downarrow j & \searrow \hat{\psi} & & \nearrow \bar{h} & \downarrow j \\
 & F(SQ/Z, \hat{E}(G)) & & & \\
 & \downarrow j & & & \\
 & F(SQ, \hat{E}(G)) & & & \\
 \nearrow \hat{\psi} & & \searrow \tilde{h} & & \\
 F(SQ, \hat{E}(P_0)) & & \xrightarrow{\tilde{\psi}} & & F(SQ, F)
 \end{array}$$

in which all but the right square are commutative, and the maps \bar{h} and \tilde{h} are homotopy equivalences. The map $\hat{\psi}$ induces a monomorphism

$$\hat{\psi}_*: F^*(F(SQ/Z, \hat{E}(G))Q) \rightarrow F^*(F(SQ/Z, \hat{E}(P_0))Q)$$

because $\hat{\psi}_*: \pi Q_*(F(SQ/Z, \hat{E}(P_0))) \rightarrow \pi Q_*(F(SQ/Z, \hat{E}(G)))$ is epic by (3.1). Hence we get immediately that the right square is commutative like the rest. Thereby we have a homotopy equivalence

$$h: \hat{E}(G) \rightarrow F$$

by applying Five lemma.

Note that $\text{Hom}(tA, G) = 0$ if $\text{Tor}(A, G) = 0$. The above theorem asserts that the finiteness restrictions on G and E may be eliminated in [14, Theorem 4].

4. Purity of the universal coefficient sequence

4.1. We now study whether the universal coefficient sequence (1.1) is pure as Huber and Meier [10] tried. But our method owes to Mislin [12] rather than Hilton and Deleanu [9, Theorem 3.2]. Consider first the universal coefficient sequence of the form

$$(4.1) \quad 0 \rightarrow E_*(X) \otimes Z_q \rightarrow EZ_{q*}(X) \rightarrow \text{Tor}(E_{*-1}(X), Z_q) \rightarrow 0.$$

According to Araki and Toda [4, Theorem 2.7] (or [9]) we have that

$$(4.2) \quad \text{the universal coefficient sequence (4.1) is split if } q \equiv 2 \pmod{4} \text{ or if } E \text{ is good.}$$

An abelian group G is said to be 2-high if the homomorphism $\text{Tor}(G, Z_4) \rightarrow \text{Tor}(G, Z_2)$, induced by the projection $Z_4 \rightarrow Z_2$, is epic [9]. If a 2-high group G is finitely generated, then it doesn't contain Z_2 as a direct summand. Any 2-high group is certainly the union of all finitely generated 2-high subgroups. Even if E is not good, we still have the following nice result by adopting the argument in [9, Theorem 4.3].

(4.3) *If $E_n(X)$ is 2-high, then the exact sequence (4.1) is split in the n -th and $(n+1)$ -th dimensions.*

A short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is called *2-high pure* if the induced homomorphism $A \otimes Z_q \rightarrow B \otimes Z_q$ is monic for any $q \not\equiv 2 \pmod{4}$. Evidently an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is 2-high pure if and only if the induced homomorphisms $A \otimes G \rightarrow B \otimes G$ are monic for all 2-high G .

Assume that there is a natural exact sequence

$$(**) \quad 0 \rightarrow \text{Ext}(E_{*-1}(X), G) \xrightarrow{\eta} F^*(X) \xrightarrow{\tau} \text{Hom}(E_*(X), G) \rightarrow 0.$$

Of course we may introduce $\hat{E}(G)$ as F if necessary. Consider the commutative square

$$\begin{array}{ccc} \text{Ext}(E_{*-1}(X), G) \otimes Z_q & \xrightarrow{\eta \otimes 1} & F^*(X) \otimes Z_q \\ \downarrow & & \downarrow \\ \text{Ext}(EZ_q^*(X), G) & \xrightarrow{\eta} & F^{*+1}(XZ_q). \end{array}$$

The upper arrow $\eta \otimes 1$ is monic if and only if the left vertical arrow is monic. The latter condition is equivalent to say that the sequence

$$0 \rightarrow \text{Ext}(\text{Tor}(E_{*-1}(X), Z_q), G) \rightarrow \text{Ext}(EZ_q^*(X), G) \rightarrow \text{Ext}(E_*(X) \otimes Z_q, G) \rightarrow 0$$

induced by (4.1) is exact. Hence we obtain

(4.4) *the natural exact sequence (**) is always 2-high pure, and it is pure whenever the exact sequence (4.1) with $q=2$ is split.*

Moreover we notice

(4.5) *the purity of the natural exact sequence (**) doesn't depend on the choice of F .*

4.2. We here compute the group $\{\hat{S}(G), \hat{S}(H)\}$.

Lemma 16. *If either G or H is 2-high, then*

$$\{\hat{S}(G), \hat{S}(H)\} \cong \text{Hom}(G, H) \oplus \text{Ext}(G, H \otimes Z_2).$$

Proof. First assume that G is 2-high. Then the exact sequence

$$0 \rightarrow \pi_0(\hat{S}(G)) \otimes Z_q \rightarrow \pi_0(\hat{S}(G)Z_q) \rightarrow \text{Tor}(\pi_{-1}(\hat{S}(G)), Z_q) \rightarrow 0$$

is split by (4.3). Because of (4.4) the exact sequence

$$0 \rightarrow \text{Ext}(\pi_{-1}(\hat{S}(G)), H) \rightarrow \hat{S}(H)^0(\hat{S}(G)) \rightarrow \text{Hom}(\pi_0(\hat{S}(G)), H) \rightarrow 0$$

is pure. $\text{Ext}(\pi_{-1}(\hat{S}(G)), H)$ is bounded, and hence it is algebraically compact (see [7]). So the pure exact sequence is split.

We next assume that H is 2-high. By use of Corollary 15 and Theorem 1 we get an isomorphism $\{SG, SH\} \rightarrow \{\hat{S}(G), \hat{S}(H)\}$. So we use the exact sequence

$$0 \rightarrow \text{Ext}(G, \pi_1(SH)) \rightarrow SH^0(SG) \rightarrow \text{Hom}(G, \pi_0(SH)) \rightarrow 0.$$

Consider the commutative square

$$\begin{array}{ccc} \text{Ext}(G, \pi_1(SH)) \otimes Z_q & \rightarrow & SH^0(SG) \otimes Z_q \\ \downarrow & & \downarrow \\ \text{Ext}(G, \pi_1(SHZ_q)) & \rightarrow & SHZ_q^0(SG). \end{array}$$

The left vertical arrow is monic since the exact sequence

$$0 \rightarrow \pi_1(SH) \otimes Z_q \rightarrow \pi_1(SHZ_q) \rightarrow \text{Tor}(\pi_0(SH), Z_q) \rightarrow 0$$

is split by (4.3). Thus the above exact sequence is pure. Thereby it is split as $\text{Ext}(G, \pi_1(SH))$ is bounded.

We now show the purity of the exact sequence $(**)$ under some restriction on either E or G .

Theorem 5. *Assume that there is a natural exact sequence*

$$0 \rightarrow \text{Ext}(E_{*-1}(X), G) \rightarrow F^*(X) \rightarrow \text{Hom}(E_*(X), G) \rightarrow 0.$$

If the CW-spectrum E is good or if the abelian group G is 2-high, then the above exact sequence is pure. (Cf., [10, Corollary 3.4]).

Proof. When E is good, the purity follows from (4.2) and (4.4). Assume that G is 2-high, then $\{\hat{S}(G), \hat{S}(G \otimes Z_q)\}$ is a Z_q -module by Lemma 16. So we have a commutative square

$$\begin{array}{ccc} \text{Ext}(E_{*-1}(X), G) \otimes Z_q & \xrightarrow{\eta_G \otimes 1} & \hat{E}(G)^*(X) \otimes Z_q \\ \cong \downarrow & & \downarrow \\ \text{Ext}(E_{*-1}(X), G \otimes Z_q) & \xrightarrow{\eta_{G \otimes Z_q}} & \hat{E}(G \otimes Z_q)^*(X). \end{array}$$

The upper arrow $\eta_G \otimes 1$ is monic, and hence the universal coefficient sequence

$$0 \rightarrow \text{Ext}(E_{*-1}(X), G) \xrightarrow{\eta_G} \hat{E}(G)^*(X) \xrightarrow{\tau_G} \text{Hom}(E_*(X), G) \rightarrow 0$$

is pure. By virtue of (4.5) we get the purity of our exact sequence.

Huber and Meier [10] gave several conditions under which each pure exact sequence of the form $(**)$ is split. In particular, we have

Corollary 18 ([10]). *Assume that E is good or that G is 2-high. If*

$\text{Pext}(Q/Z, tG)=0$, e.g., *the torsion subgroup tG is algebraically compact, then a natural exact sequence*

$$0 \rightarrow \text{Ext}(E_{*-1}(X), G) \rightarrow F^*(X) \rightarrow \text{Hom}(E_*(X), G) \rightarrow 0$$

is split.

OSAKA CITY UNIVERSITY

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