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Let $E$ be a $CW$-spectrum and $G$ be an abelian group. Following Kainen [11] we can construct a $CW$-spectrum $\hat{E}(G)$ which has a universal coefficient sequence

$$0 \to \text{Ext}(E_{-*}(X), G) \to \hat{E}(G)^*(X) \to \text{Hom}(E_*(X), G) \to 0.$$ 

In the previous paper [14] with the same title we investigated several properties of $\hat{E}(G)$. But some of our results are restrictive as yet, e.g., Proposition 8 and Theorem 4 in [14]. In this note we continue the investigations to develop and improve our partial results.

First we discuss whether the correspondences $G \to \hat{E}(G)$ as well as $G \to EG$ are functorial in $G$, as analogous discussions were done in [9] and [10]. Next, under some finiteness assumption on $E$ or $G$ we show that $\hat{E}(G)$ and $\hat{E}(R)G$ are homotopy equivalent where $\mathbb{Z} \subset R \subset Q$ (Theorem 1). This result is a satisfactory improvement of [14, Proposition 8]. As an application of the main result of Huber and Meier [10] we can then give a criterion for $EG^*(X)$ being Hausdorff (Theorem 2). Moreover we discuss the uniqueness of $\hat{E}(G)$ again to improve a partial result obtained in [14, Theorem 4]. When $E$ is the sphere spectrum $S$ we have a complete result (Theorem 3), but for a general $E$ we need still some restriction although the finiteness assumptions on $E$ and $G$ can be eliminated in our previous result (Theorem 4). Finally we show that the universal coefficient sequence is pure under some restriction on $E$ or $G$, adopting an argument given in [9].

In this note we shall work in the stable homotopy category of $CW$-spectra (see [1] or [13]).

The author wishes to thank Professors Huber and Meier for sending him their preprint [10] by which he has been motivated to write this sequel.

1. Functoriality of $\hat{E}(G)$

1.1. Let $E$ be a $CW$-spectrum and $G$ be an abelian group. Then there
is a $CW$-spectrum $\hat{E}(G)$ so that $E$ and $\hat{E}(G)$ are related by a universal coefficient sequence

\begin{equation}
0 \to \text{Ext}(E_{n-1}(X), G) \xrightarrow{\eta_G} \hat{E}(G)^*(X) \xrightarrow{\tau_G} \text{Hom}(E_n(X), G) \to 0
\end{equation}

(see [11] and [14]). Let us first recall the construction of $\hat{E}(G)$ involving an injective resolution of $G$. By the representability theorem there is a $CW$-spectrum $\hat{E}(I)$ and a natural equivalence $\tau_I: \hat{E}(I)^*( ) \to \text{Hom}(E_n( ), I)$ for every injective $I$. Take any injective resolution $0 \to G \to I \xrightarrow{\psi} J \to 0$ and denote by $\psi: \hat{E}(I) \to \hat{E}(J)$ the unique map induced by $\psi$. We define $\hat{E}(G)$ to be the fiber of $\psi$, i.e.

$$
\hat{E}(G) \to \hat{E}(I) \xrightarrow{\psi} \hat{E}(J)
$$

is a cofibering. The homotopy type of $\hat{E}(G)$ is independent of the choice of an injective resolution.

Let us denote by $S$ the sphere spectrum. By the exactness of function spectra [13] there is a cofibering

$$
F(E, \hat{S}(G)) \to F(E, \hat{S}(I)) \xrightarrow{\psi} F(E, \hat{S}(J)).
$$

By the aid of Five lemma [13] we obtain

**Proposition 1.** For any abelian group $G$ the spectrum $\hat{E}(G)$ has the same homotopy type as the function spectrum $F(E, \hat{S}(G))$.

Given an abelian group $G$, each map $f: W \to E$ of $CW$-spectra determines the unique map $\hat{f} = F(f, \hat{S}(G))$: $F(E, \hat{S}(G)) \to F(W, \hat{S}(G))$. Thereby Proposition 1 contains the following functorial property.

**Corollary 2.** Fix an abelian group $G$. Then the correspondence $E \mapsto \hat{E}(G) = F(E, \hat{S}(G))$ is a contravariant exact functor.

We may now turn our attention to the spectrum $\hat{S}(G)$. The map $\tau_G$ gives rise to an isomorphism

\begin{equation}
t_G: \pi_0(\hat{S}(G)) \xrightarrow{\tau_G} \text{Hom}(\pi_0(S), G)=G.
\end{equation}

**Lemma 3.** The composition map

$$
\hat{S}(G)^*(X) \xrightarrow{\tau_G} \text{Hom}(\pi_*(X), G) \xrightarrow{t_G} \text{Hom}(\pi_*(X), \pi_0(\hat{S}(G)))
$$

is just the homomorphism $\kappa$ assigning to a map $f$ the induced homomorphism $f_\ast$ in 0-th homotopy groups.
Proof. It is sufficient to show the equality \( t_G = \tau_G(1_{S(G)}) \) for the identity map \( 1_{S(G)} \) of \( S(G) \). Take any element \( f \) of \( \pi_0(S(G)) \), i.e., a map \( f : S \to S(G) \). By the naturality of \( \tau_G \) we have
\[
\tau_G(f)(1_s) = (\tau_G(1_{S(G)}f_*)(1_s)) = \tau_G(1_{S(G)})(f).
\]

Because of Lemma 3 we may employ \( \kappa \) instead of \( \tau_G \). Thus there is a natural exact sequence
\[
0 \to \text{Ext}(\pi_{*-1}(X), G) \xrightarrow{\eta_G} S(G)^{*}(X) \xrightarrow{\kappa} \text{Hom}(\pi_{*}(X), G) \to 0
\]
where \( \pi_0(S(G)) \) is identified with \( G \) via the map \( t_G \).

Let \( E \) be a ring spectrum and \( F \) be an (associative) right \( E \)-module spectrum equipped with a structure map \( \mu : F \wedge E \to F \). Then there is a unique map
\[
\bar{\mu}_G : E \wedge \hat{F}(G) \to \hat{F}(G)
\]
such that \( e_{F,G}(1_F \wedge \bar{\mu}_G) = e_{F,G}(\mu \wedge 1_{\hat{F}(G)}) \) where \( e_{F,G} : F \wedge \hat{F}(G) \to S(G) \) is the evaluation map. Thereby \( \hat{F}(G) \) is an (associative) left \( E \)-module spectrum. Using the structure maps \( \mu \) and \( \bar{\mu}_G \) we can give \( \text{Hom}(F_*(\ ), G) \) and \( \hat{F}(G)^{*}(\ ) \) structures of left \( E^{*}(\ ) \)-modules. Thus we have two homomorphisms
\[
\mu_* : E^{*}(Y) \otimes \text{Hom}(F_*(X), G) \to \text{Hom}(F_*(Y \wedge X), G)
\]
\[
\bar{\mu}_* : E^{*}(Y) \otimes \hat{F}(G)^{*}(X) \to \hat{F}(G)^{*}(Y \wedge X)
\]
defined in the obvious way. By virtue of Lemma 3 we have

**Proposition 4.** Let \( E \) be a ring spectrum and \( F \) be a right \( E \)-module spectrum. Then the universal coefficient sequence
\[
0 \to \text{Ext}(\pi_{*-1}(X), G) \xrightarrow{\eta_G} S(G)^{*}(X) \xrightarrow{\kappa} \text{Hom}(\pi_{*}(X), G) \to 0
\]
is an exact sequence of left \( E^{*}(\ ) \)-modules.

Proof. As is easily seen, the induced homotopy homomorphism \( \kappa \) is a map of left \( E^{*}(\ ) \)-modules, i.e., the following square
\[
\begin{array}{ccc}
E^{*}(Y) \otimes \hat{F}(G)^{*}(X) & \xrightarrow{1 \otimes \kappa} & E^{*}(Y) \otimes \text{Hom}(F_*(X), \pi_0(S(G))) \\
\bar{\mu}_* \downarrow & & \downarrow \mu_* \\
\hat{F}(G)^{*}(Y \wedge X) & \xrightarrow{\kappa} & \text{Hom}(F_*(Y \wedge X), \pi_0(S(G)))
\end{array}
\]
is commutative. By a routine computation the result is immediate.

1.2. Take any homomorphism \( \phi : G \to H \) of abelian groups, then there is a (non-unique) map \( \hat{\phi} : \hat{S}(G) \to \hat{S}(H) \) making the diagram below commutative.
0 \to \text{Ext}(\pi_{*-1}(X), G) \xrightarrow{\eta_G} \hat{S}(G)^*(X) \xrightarrow{\tau_G} \text{Hom}(\pi_*(X), G) \to 0
\xrightarrow{\phi_*}
0 \to \text{Ext}(\pi_{*-1}(X), H) \xrightarrow{\eta_H} \hat{S}(H)^*(X) \xrightarrow{\tau_H} \text{Hom}(\pi_*(X), H) \to 0.

Thus the correspondence \( G \to \hat{S}(G) \) is quasi-functorial in \( G \) [11].

**Lemma 5.** If \( 0 \to G \xrightarrow{\phi} H \xrightarrow{\psi} K \to 0 \) is a short exact sequence, then there exist maps \( \phi, \psi: \hat{S}(G) \to \hat{S}(H) \) and \( \hat{S}(H) \to \hat{S}(K) \) which give us a cofibering
\[ \hat{S}(G) \xrightarrow{\phi} \hat{S}(H) \xrightarrow{\psi} \hat{S}(K). \]

**Proof.** Choose an injective resolution \( 0 \to H \to I \to J_1 \to 0 \) and consider commutative exact diagram

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & G & H & K & 0 \\
\downarrow & \downarrow & \downarrow & I = I \\
I & I & I \\
\downarrow & \downarrow & \downarrow \\
0 & K & J_0 & J_1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0
\end{array}
\]

in which there appear three injective resolutions of \( G, H \) and \( K \). By applying Verdier's lemma [6] we obtain a cofibering as desired.

Denote by \( k_{G,H} \) the composition map
\[ \hat{S}(H)^*(\hat{S}(G)) \xrightarrow{\tau_H} \text{Hom}(\pi_*(\hat{S}(G)), H) \xrightarrow{\tau_\phi} \text{Hom}(G, H). \]

It is epic, in fact we observe that
\begin{equation}
(1.4) \quad k_{G,H}(\phi) = \phi_*(1_G) = \phi,
\end{equation}

by making use of the equality \( t_g = \tau_G(1_{\hat{S}(G)}) \). But \( \text{Ker} k_{G,H} \cong \text{Ext}(\pi_-(\hat{S}(G)), H) \cong \text{Ext}(\text{Hom}(Z_2, G), H) \cong \text{Ext}(G, Z_2 \otimes H) \). By an easy computation we verify that

\begin{equation}
(1.5) \quad k_{G,H} \text{ is an isomorphism if and only if either } G \text{ is } 2\text{-torsion free or } H \text{ is } 2\text{-divisible.}
\end{equation}

This implies

**Proposition 6.** If \( G \) is 2-torsion free or if \( H \) is 2-divisible, then \( \phi = F(E, \phi): \hat{E}(G) \to \hat{E}(H) \) is uniquely determined for each \( \phi: G \to H \).
Let us denote by $\eta: \Sigma^1 S \to S$ the Hopf map, i.e., the non-zero element of $\pi_1(S)$. A CW-spectrum $E$ is said to be good if $\eta \wedge 1_E: \Sigma^1 E \to E$ is trivial [9]. For a good $E$ we have the following functorial property.

**Proposition 7.** Assume that a fixed CW-spectrum $E$ is good. Then the composite $G \to \hat{S}(G) \to \hat{E}(G) = F(E, \hat{S}(G))$ is a covariant exact functor.

**Proof.** We show that the homomorphism

$$F(E, \_): \{\hat{S}(G), \hat{S}(H)\} \to \{\hat{E}(G), \hat{E}(H)\}$$

factors through $k_{G,H}$. Recall that $F(E, \_)$ is given by the composition

$$\{\hat{S}(G), \hat{S}(H)\} \xrightarrow{e^*_G} \{E \wedge \hat{E}(G), \hat{S}(H)\} \xrightarrow{\sim} \{\hat{E}(G), \hat{E}(H)\}$$

where $e_G = e_{E,G}: E \wedge \hat{E}(G) \to \hat{S}(G)$ is the evaluation map. So it is enough to show that there is a homomorphism $\lambda$ making the diagram below commutative

$$0 \to \text{Ext}(\pi_-(\hat{S}(G)), H) \xrightarrow{\eta^H} \{\hat{S}(G), \hat{S}(H)\} \xrightarrow{k_{G,H}} \text{Hom}(G, H) \to 0$$

$$\text{Ext}(\pi_-(E \wedge \hat{E}(G)), H) \xrightarrow{\eta^H} \{E \wedge \hat{E}(G), \hat{S}(H)\}.$$

Consider the commutative diagram

$$\begin{array}{c}
\pi_-(E \wedge \hat{E}(G)) \xrightarrow{e^*_G} \pi_-(\hat{S}(G)) \xrightarrow{\tau_G} \text{Hom}(\pi_1(S), G) \\
\eta^* \downarrow \quad \eta^* \\
\pi_0(E \wedge \hat{E}(G)) \xrightarrow{e^*_G} \pi_0(\hat{S}(G)) \xrightarrow{\tau_G} \text{Hom}(\pi_0(S), G).
\end{array}$$

The left arrow $\eta^*$ is trivial by our hypothesis on $E$, and the central one $\eta^*$ is monic by use of the right square. This implies that the upper arrow $e^*_G$ is trivial. The existence of $\lambda$ is now immediate. Therefore the correspondence $G \to \hat{E}(G)$ is a functor which is exact by Lemma 5.

1.3. For each abelian group $G$ we denote by $SG$ the Moore spectrum of type $G$. Then there is a universal coefficient sequence in the form of a natural exact sequence

$$0 \to \text{Ext}(G, \pi_{*+1}(X)) \to \{SG, X\}_* \xrightarrow{\kappa} \text{Hom}(G, \pi_*(X)) \to 0$$

where $\kappa$ is just the induced homotopy homomorphism [8]. In particular we have a short exact sequence

$$0 \to \text{Ext}(G, \pi_1(SH)) \to \{SG, SH\} \xrightarrow{\kappa} \text{Hom}(G, H) \to 0.$$
Given a homomorphism $\phi: G \rightarrow H$, there is a (non-unique) map $S\phi: SG \rightarrow SH$ inducing $S\phi_* = \phi: \pi_0(SG) \rightarrow \pi_0(SH)$. Since $\pi_1(SH) \cong H \otimes \mathbb{Z}_2$ we have an analogous result to Proposition 6.

**Proposition 8.** Assume that $G$ is 2-torsion free or that $H$ is 2-divisible. Then $1_E \wedge S\phi: EG \rightarrow EH$ is uniquely determined for each $\phi: G \rightarrow H$ (see [10, Proposition 3.2]).

By choosing suitably free resolutions in the dual way to the proof of Lemma 5 we can show that there is a cofibering

$$ SG \xrightarrow{S\phi} SH \xrightarrow{S\psi} SK $$

if $0 \rightarrow G \xrightarrow{\phi} H \xrightarrow{\psi} K \rightarrow 0$ is a short exact sequence.

Corresponding to [9, Appendix] we obtain

**Proposition 9.** Assume that a fixed CW-spectrum $E$ is good. Then the composite $G \rightarrow SG \rightarrow EG$ is a covariant exact functor.

**Proof.** The homomorphism $1_E \wedge - : \{SG, SH\} \rightarrow \{EG, EH\}$ is just the composition

$$ \{SG, SH\} \xrightarrow{\varepsilon_{H^*}} \{SG, F(E, EH)\} \xrightarrow{\kappa} \{EG, EH\} $$

where $\varepsilon_H: SH \rightarrow F(E, EH)$ is the dual of $1_{EH}$. So we consider the following commutative diagram

$$ 0 \rightarrow \text{Ext}(G, \pi_1(SH)) \rightarrow \{SG, SH\} \xrightarrow{\kappa} \text{Hom}(G, H) \rightarrow 0 $$

$$ \downarrow (\varepsilon_{H^*})_* \quad \downarrow \varepsilon_{H^*} $$

$$ \text{Ext}(G, \pi_1(F(E, EH))) \rightarrow \{SG, F(E, EH)\} . $$

In the commutative square

$$ \begin{array}{ccc}
\pi_0(SH) & \xrightarrow{\varepsilon_{H^*}} & \pi_0(F(E, EH)) \\
\downarrow \eta^* & & \downarrow \eta^* \\
\pi_1(SH) & \xrightarrow{\varepsilon_{H^*}} & \pi_1(F(E, EH))
\end{array} $$

the left arrow $\eta^*$ is epic, but the right one $\eta^*$ is trivial by our hypothesis on $E$. Hence the lower arrow $\varepsilon_{H^*}$ is trivial, too. This claims that $\varepsilon_{H^*}: \{SG, SH\} \rightarrow \{SG, F(E, EH)\}$ factors through $\kappa$. Our result is now obvious.

2. **Important properties of $\hat{E}(G)$**

2.1. Let us denote by $R$ a subring of the rationals $Q$ and $\mathfrak{p}$ be the set of primes which are invertible in $R$. A CW-spectrum $E$ is called an $R$-spectrum if
$p \cdot 1_E: E \to E$ is a homotopy equivalence for each $p \in \mathcal{F}$. Notice that $E$ is an $R$-spectrum if and only if $\pi_*(E)$ is an $R$-module. An $R$-spectrum $E$ is said to be of finite type if $\pi_*(E)$ is of finite type as an $R$-module.

We now study whether the CW-spectra $\hat{E}(G)$ and $\hat{E}(R)G$ are homotopy equivalent. Assume that an $R$-spectrum $E$ is of finite type or that an $R$-module $G$ is finitely generated. Let us first recall our partial result [14] in the special case when $G$ is free. In this case we write $P$ instead of $G$, i.e., $P=\sum R$. The canonical injections $i_a: R \to P$ give rise to the map $\bigvee_a i_a: \bigvee \hat{E}(R) \to \hat{E}(P)$ which is unique by Proposition 6. According to [14, Lemma 7] the map $\bigvee_a i_a$ is a homotopy equivalence under our assumption. Consequently the composite map

\[(2.1) \quad \iota_{E,P}: \hat{E}(R)P \leftarrow \bigvee_a \hat{E}(R) \to \hat{E}(P)\]

is a homotopy equivalence, too.

Notice that the map $\iota_{E,P}$ has a factorization

\[F(E, \hat{S}(R))P \xrightarrow{j} F(E, \hat{S}(R)P) \xrightarrow{F(E, \iota_{S,P})} F(E, \hat{S}(P))\]

whose decomposed maps are both homotopy equivalences. By applying Five lemma we obtain that the canonical map

\[(2.2) \quad j: F(E, \hat{S}(R))G \to F(E, S(R)G)\]

is a homotopy equivalence under our finiteness assumption on $E$ or $G$.

We here give the following interesting result.

**Theorem 1.** Let $E$ be an $R$-spectrum and $G$ be an $R$-module. Assume that $E$ is of finite type or that $G$ is finitely generated. Then $\hat{E}(G)$ and $\hat{E}(G)R$ have the same homotopy type.

Proof. Take a free resolution $0 \to P_1 \xrightarrow{\phi} P_0 \xrightarrow{\psi} G \to 0$ of $R$-modules, and consider the diagram

\[
\begin{array}{ccc}
\hat{S}(R)P_1 & 1 \wedge S\phi & \hat{S}(R)P_0 & 1 \wedge S\psi \\
\iota_{S,P} & & \iota_{S,P_0} & \\
\hat{S}(P_1) & \phi & \hat{S}(P_0) & \psi \\
\end{array}
\]

involving two cofiberings in (1.7) and Lemma 5. In order to show that the square is commutative, we use the map $\kappa: \hat{S}(P_0)^{(\hat{S}(R)P_1)} \to \text{Hom}(\pi_0(\hat{S}(R)P_1), \pi_0(\hat{S}(P_0)))$ which is an isomorphism. After $\pi_0(\hat{S}(R)P_1)$ and $\pi_0(\hat{S}(P_0))$ are identified with $P_1$ and $P_0$ respectively, we compute that
\[ \kappa(\phi \cdot \iota_{S,P}) = \phi_\# \kappa(\iota_{S,P}) = \phi_\#(1_{P_0}) = \phi^* \kappa(\iota_{S,P_0}) = \kappa(\iota_{S,P}, 1 \wedge S\phi), \]
which claims \( \phi \cdot \iota_{S,P} = \iota_{S,P_0} \cdot 1 \wedge S\phi \). By (2.1) the vertical maps \( \iota_{S,P} \) are both homotopy equivalences. By use of Five lemma we obtain a homotopy equivalence

\[ \hat{S}(R)G \rightarrow \hat{S}(G) \]
in the special case \( E = S \).

For a general \( E \) we use (2.2) to obtain that the composite map

\[ F(E, \hat{S}(R)G) \rightarrow F(E, \hat{S}(R)G) \rightarrow F(E, \hat{S}(G)) \]
is a homotopy equivalence.

If \( E \) is an \( R \)-spectrum of finite type, then it has a nice property that there is a homotopy equivalence

\[ (2.3) \quad h_E: E \rightarrow ER \rightarrow \hat{E}(R)(R) \]
(see [14, Theorem 2]). Putting two important results, Theorem 1 and (2.3), together we obtain a natural exact sequence

\[ (2.4) \quad 0 \rightarrow \operatorname{Ext}(\hat{E}(R)_{* - 1}(X), G) \rightarrow EG^*(X) \rightarrow \operatorname{Hom}(\hat{E}(R)^*(X), G) \rightarrow 0 \]
if \( E \) is an \( R \)-spectrum of finite type. Applying the main result of Huber and Meier [10, Theorem 1.1] we can extend our criterion [14, Theorem 3] for \( E^*(X) \) being Hausdorff.

**Theorem 2 ([10]).** Assume that \( E \) is an \( R \)-spectrum of finite type. Then \( EG^*(X) \) is Hausdorff if and only if \( \operatorname{Pext}(\hat{E}(R)^{* - 1}(X), G) = 0 \).

2.2. For a \( CW \)-spectrum \( E \) we denote by \( E(\infty, n] (= E(\infty, n+1)) \) the \((n+1)\)-coconnective Postnikov cofiber of \( E \) and by \( E(n, \infty) (= E[n+1, \infty)) \) the \( n \)-connective Postnikov fiber of \( E \) (see [3]). Thus \( E(\infty, n] \) is an \((n+1)\)-coconnective \( CW \)-spectrum such that there is a map \( j_n: E \rightarrow E(\infty, n] \) which induces an isomorphism \( j_n^*; \pi_r(E) \rightarrow \pi_r(E(\infty, n]) \) for each \( r \leq n \), and \( E(n, \infty) \) an \( n \)-connective \( CW \)-spectrum such that there is a map \( i_n: E(n, \infty) \rightarrow E \) which induces an isomorphism \( i_n^*; \pi_r(E(n, \infty)) \rightarrow \pi_r(E) \) for each \( r > n \). Notice that the sequence

\[ E(n, \infty) \xrightarrow{i_n} E \xrightarrow{j_n} E(\infty, n] \]
is a cofibering.

By routine computations we have
Lemma 10. i) The map $j_n$ induces a homotopy equivalence
\[ E(-\infty, n]G \cong \begin{cases} EG(-\infty, n] & \text{if } \text{Tor}(\pi_n(E), G) = 0 \\ EG(-\infty, n+1] & \text{if } \pi_{n+1}(E) \otimes G = 0 \end{cases} \]
ii) The map $i_n$ induces a homotopy equivalence
\[ E(n, \infty)G \cong \begin{cases} EG(n, \infty) & \text{if } \text{Tor}(\pi_n(E), G) = 0 \\ EG(n+1, \infty) & \text{if } \pi_{n+1}(E) \otimes G = 0 \end{cases} \]

Lemma 11. i) The map $i_n$ induces a homotopy equivalence
\[ \hat{E}(-\infty, n](G) \cong \begin{cases} \hat{E}(G)(-n, \infty) & \text{if } \text{Ext}(\pi_n(E), G) = 0 \\ \hat{E}(G)(-n-1, \infty) & \text{if } \text{Hom}(\pi_{n+1}(E), G) = 0 \end{cases} \]
ii) The map $i_n$ induces a homotopy equivalence
\[ \hat{E}(n, \infty)(G) \cong \begin{cases} \hat{E}(G)(-\infty, -n) & \text{if } \text{Ext}(\pi_n(E), G) = 0 \\ \hat{E}(G)(-\infty, -n-1) & \text{if } \text{Hom}(\pi_{n+1}(E), G) = 0 \end{cases} \]

Combining Theorem 1 with Lemmas 10 and 11 we obtain

Proposition 12. Assume that an $R$-spectrum $E$ is of finite type or that an $R$-module $G$ is finitely generated. If $\text{Ext}(\pi_n(E), G) = 0$, then $\hat{E}(-\infty, n](G)$ has the same homotopy type as $\hat{E}(R)(-n, \infty)G$ and $E(n, \infty)(G)$ does the same as $\hat{E}(R)(-\infty, -n)G$.

For the $BU$, $EO$- and $BSp$- spectrum $K$, $KO$ and $KSp$ we have determined in [14, Theorem 5] (or see [2]) that
\[ (2.5) \quad \hat{K}(G) = KG \quad \text{and} \quad \hat{KSp}(G) = KOG. \]

Applying Proposition 12 we get
\[ (2.6) \quad \hat{K}(0, \infty)(G) = K(-\infty, 0]G, \quad \hat{KSp}(0, \infty)(G) = KO(-\infty, 0]G \quad \text{and so on.} \]

2.3. Let $\tau: F(W, \hat{V}(G))\rightarrow F(V, \hat{W}(G))$ be the homotopy equivalence induced by the switching map $T: W \wedge V \rightarrow V \wedge W$. Putting $V = E$ and $W = \hat{E}(G)$, $\tau$ yields the map
\[ e_{E,G}: E \rightarrow \hat{E}(G)(G) \]
which is the dual of $e_{E,G}T$ where $e_{E,G}: E \wedge \hat{E}(G) \rightarrow \hat{S}(G)$ denotes the evaluation map. Observe that the composition
\[ (2.7) \quad \{W, E\} \xrightarrow{\xi_{E,G}^*} \{W, \hat{E}(G)(G)\} \xrightarrow{\tau_{*}} \{\hat{E}(G), \hat{W}(G)\} \]
Proposition 13. If an R-spectrum E is of finite type, then the map

\[ F(\hat{E}(R), W(E)) : \{W, E\} \rightarrow \{\hat{E}(R), \hat{W}(R)\} \]

is an isomorphism for each W, and equivalently the canonical map \( \varepsilon_{E,R} : E \rightarrow \hat{E}(R)(R) \) is a homotopy equivalence (cf., (2.3)).

Proof. Take a homotopy equivalence \( h_E : E \rightarrow \hat{E}(R)(R) \) of (2.3) and introduce the composite map

\[ \rho_E : F(W, E) \rightarrow F(W, \hat{E}(R)(R)) \rightarrow F(\hat{E}(R), \hat{W}(R)) \]

given by use of \( h_E \), which is functorial with respect to \( W \). We modify the map \( \rho_E \) a bit as it induces the map \( F(\hat{E}(R), W(E)) \). Since \( \rho_{S(R)} : \{W, E\} \rightarrow \{\hat{E}(R), \hat{W}(R)\} \) is an isomorphism, we can find a map \( f : E \rightarrow E \) such that \( \rho_{S(R)}(f) = 1_{\hat{E}(R)} \). The map \( \hat{f} : \hat{E}(R) \rightarrow \hat{E}(R) \) gives rise to a split epic \( \hat{f}^* : \pi_*((\hat{E}(R)) \rightarrow \pi_*((\hat{E}(R)) \) since \( \hat{f}^* \rho_{S(R)}(1_{\hat{E}(R)}) = \rho_{S(R)}(f^*(1_{\hat{E}(R)})) = 1_{\hat{E}(R)} \). But the R-module \( \pi_*((\hat{E}(R)) \) is of finite type, so \( \hat{f}^* \) is isomorphic. This means that the map \( \hat{f} \) is a homotopy equivalence. Consider the composite map

\[ F(W, E) \xrightarrow{\rho_E} F(\hat{E}(R), \hat{W}(R)) \xrightarrow{\hat{f}^*} F(\hat{E}(R), \hat{W}(R)) \]

Obviously the induced isomorphism

\[ \{W, E\} \xrightarrow{\rho_{S(R)}} \{\hat{E}(R), \hat{W}(R)\} \xleftarrow{\hat{f}^*} \{\hat{E}(R), \hat{W}(R)\} \]

conicides with the map \( F(\hat{E}(R), W(E)) \).

We next define a generalization \( \hat{F}_{G,H} : \{W, E\} \rightarrow \{\hat{E}(R)G, \hat{W}(R)H\} \) of the isomorphism \( F(\hat{E}(R), W(E)) \). The evaluation map \( e_{E,R} : E \wedge \hat{E}(R) \rightarrow \hat{S}(R) \) gives us a homomorphism

\[ e_{S(R)} : \{W, E\} \rightarrow \{W \wedge \hat{E}(R)G, \hat{S}(R)H\} \]

defined in the obvious way. On the other hand, if \( W \) is an R-spectrum of finite type or if \( H \) is a finitely generated R-module, then the map \( j : F(W, \hat{S}(R)) \rightarrow F(W, \hat{S}(R)) \) induces an isomorphism

\[ \{\hat{E}(R)G, \hat{W}(R)H\} \rightarrow \{W \wedge \hat{E}(R)G, \hat{S}(R)H\} \]

by (2.2). We compose the above two to obtain a generalization \( \hat{F}_{G,H} \) under the finiteness restriction on \( W \) or \( H \).

Proposition 14. Assume that \( W \) is an R-spectrum of finite type or that \( H \) is a finitely generated R-module. If an R-spectrum \( E \) is of finite type, then the map
$\tilde{F}_{G,H}: \{WG, EH\} \rightarrow \{\hat{E}(R)G, \hat{W}(R)H\}$

is an isomorphism.

Proof. For a free $R$-module $P$ we consider the following commutative diagram

$$
\begin{array}{c}
\{W, E\} \otimes P \\
\downarrow e_{\#} \otimes 1 \\
\{W \wedge \hat{E}(R), \hat{S}(R)\} \otimes P \\
\downarrow e_{\#} \\
\{W \wedge \hat{E}(R), \hat{S}(R)P\} \\
\downarrow t_{S, P} \\
\{W \wedge \hat{E}(R), \hat{S}(P)\}.
\end{array}
$$

The upper arrow $e_{\#} \otimes 1$ is an isomorphism by Proposition 13, and two vertical arrows are both isomorphisms for any finite $W$ (use (2.1) and the proof of [14, Lemma 7 ii]). This implies that the lower one $e_{\#}$ is an isomorphism for a general $W$. Now a routine argument shows that $e_{\#}: \{WG, EH\} \rightarrow \{W \wedge \hat{E}(R)G, \hat{S}(R)H\}$ is an isomorphism for any $G$ and $H$, and hence so is the map $\tilde{F}_{G,H}$.

For simplicity we write $\hat{S}$ instead of $\hat{S}(Z)$. When $W=E=S$, $\tilde{F}_{G,H}$ is equal to the map $1_{S \wedge} : \{SG, SH\} \rightarrow \{\hat{S}G, \hat{S}H\}$. So we have

**Corollary 15.** The map

$$1_{S \wedge} : \{SG, SH\} \rightarrow \{\hat{S}G, \hat{S}H\}$$

is an isomorphism for any $G$ and $H$.

### 3. Uniqueness of $\hat{E}(G)$

3.1. We here discuss the uniqueness of $\hat{E}(G)$ as it was done in [14, Theorem 4]. Our attention is first turned to the special case $E=S$. In this case we have the following satisfactory result.

**Theorem 3.** If a CW-spectrum $F$ has a natural exact sequence

$$0 \rightarrow \text{Ext}(\pi_{*}-1(X), G) \rightarrow F^{*}(X) \xrightarrow{\tau} \text{Hom}(\pi_{*}(X), G) \rightarrow 0$$

for a fixed abelian group $G$, then $F$ has the same homotopy type as $\hat{S}(G)$.

Proof. By the same argument as Lemma 3 we may regard $\tau$ as the induced homotopy homomorphism $\kappa$, after $G$ is identified with $\pi_0(F)$ via the isomorphism $t_{F,G}: \pi_0(F) \xrightarrow{\tau} \text{Hom}(\pi_0(S), G) \cong G$. Then there is a map

$$h: \hat{S}(G) \rightarrow F$$

whose induced homomorphism $h_*: \pi_0(\hat{S}(G)) \rightarrow G$ is equal to the isomorphism
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t_
 of (1.2). Using the commutative square
\[
\begin{array}{c}
\pi_{-1}(\hat{S}(G)) \\
\downarrow h_* \\
\pi_{-1}(F)
\end{array} \xrightarrow{\kappa} \begin{array}{c}
\text{Hom}(\pi_{1}(S), \pi_{0}(\hat{S}(G))) \\
\downarrow h_* \\
\text{Hom}(\pi_{1}(S), \pi_{0}(F))
\end{array}
\]
we verify that \( h_*: \pi_{-1}(\hat{S}(G)) \rightarrow \pi_{-1}(F) \) is also an isomorphism. Applying the
natural exact sequences (*) and (1.3) we can see that
\[
h^*: \pi_{-1}(\hat{S}(G)) = \pi_{-1}(F)
\]
is also an isomorphism. Applying the
natural exact sequences (*) and (1.3) we can see that
\[
h^*: F^\circ(F) \rightarrow F^\circ(\hat{S}(G)) \quad \text{and} \quad h^*: \hat{S}(G)^\circ(F) \rightarrow \hat{S}(G)^\circ(\hat{S}(G))
\]
are both isomorphisms. A routine argument shows that \( h \) is a homotopy equivalence.

3.2. In a general case \( E \) we next attempt to weaken some restrictions in
our previous result [14, Theorem 4].

Theorem 4. Let \( G \) be a fixed abelian group and \( D \) be the maximal divisible
subgroup. Assume that a CW-spectrum \( E \) satisfies \( \text{Hom}(t\pi_*(E), G/D) = 0 \) where
\( t\pi_*(E) \) denotes the torsion subgroup of \( \pi_*(E) \). If two CW-spectra \( E \) and \( F \) are
related by a natural exact sequence
\[
0 \rightarrow \text{Ext}(E_{\ast-1}(X), G) \rightarrow F^\ast(X) \xrightarrow{\tau} \text{Hom}(E_{\ast}(X), G) \rightarrow 0
\]
then \( F \) has the same homotopy type as \( \hat{E}(G) \).

Proof. Since the short exact sequence \( 0 \rightarrow D \rightarrow G \rightarrow G/D \rightarrow 0 \) is split we
may choose a map \( f: F \rightarrow \hat{E}(D) \) so that it induces the composition
\[
F^\ast(X) \xrightarrow{\tau} \text{Hom}(E_{\ast}(X), G) \rightarrow \text{Hom}(E_{\ast}(X), D) \xrightarrow{\cong} \hat{E}(D)^{\ast}(X).
\]
Denoting by \( F_{\kappa} \) the fiber of \( f \), the cofibering
\[
F_{\kappa} \rightarrow F \xrightarrow{f} \hat{E}(D)
\]
is split as \( f_*: F^\ast(X) \rightarrow \hat{E}(D)^{\ast}(X) \) is epic. With an application of \( 3 \times 3 \) lemma
as in [14, Theorem 4] we get a natural exact sequence
\[
0 \rightarrow \text{Ext}(E_{\ast-1}(X), G/D) \rightarrow F^\ast_{\kappa}(X) \rightarrow \text{Hom}(E_{\ast}(X), G/D) \rightarrow 0.
\]
Evidently \( \text{Hom}(Q, G/D) = 0 \), i.e., \( G/D \) is reduced (see [7]). Therefore we have
to show that \( F_{\kappa} \) and \( \hat{E}(G/D) \) have the same homotopy type for the reduced \( G/D \).
We may now assume that \( G \) is a reduced group with \( \text{Hom}(t\pi_*(E), G) = 0 \).
Take a free resolution \( 0 \rightarrow P_1 \xrightarrow{\phi} P_0 \xrightarrow{\psi} G \rightarrow 0 \) and proceed our proof as in
[14, Theorem 4]. By Lemma 5 the resolution gives us a cofibering
Evidently \( \text{Hom}(EQ^*(X), P_i) = \text{Hom}(EQ/Z^*(X), P_i) = 0 \) for \( i=1, 2 \) and also \( \text{Hom}(EQ^*(X), G) = 0 \) as \( G \) is reduced. We then obtain maps

\[
\tilde{\varphi}: F(SQ, \hat{E}(P_0)) \rightarrow F(SQ, F) \quad \text{and} \quad \tilde{\varphi}: F(SQ/Z, \hat{E}(P_0)) \rightarrow F(SQ/Z, F)
\]

which make the diagrams below commutative

\[
\begin{array}{c}
0 \rightarrow \text{Ext}(EQ^*_{-1}(X), P_i) \xrightarrow{\phi_*} \text{Ext}(EQ^*_{-1}(X), P_0) \xrightarrow{\psi_*} \text{Ext}(EQ^*_{-1}(X), G) \rightarrow 0 \\
0 \rightarrow \hat{E}(P_i)^*(XQ) \xrightarrow{\phi_*} \hat{E}(P_0)^*(XQ) \xrightarrow{\psi_*} F^*(XQ) \rightarrow 0
\end{array}
\]

and

\[
\begin{array}{c}
\text{Ext}(EQ/Z^*_{-1}(X), P_i) \xrightarrow{\phi_*} \text{Ext}(EQ/Z^*_{-1}(X), P_0) \xrightarrow{\psi_*} \text{Ext}(EQ/Z^*_{-1}(X), G) \rightarrow 0 \\
\hat{E}(P_i)^*(XQ/Z) \xrightarrow{\phi_*} \hat{E}(P_0)^*(XQ/Z) \xrightarrow{\psi_*} F^*(XQ/Z)
\end{array}
\]

By easy diagram chases we observe that two bottom sequences in the above diagrams are exact. In particular, the composite maps \( \tilde{\varphi} \circ \hat{\phi} \) and \( \tilde{\psi} \circ \hat{\phi} \) are both trivial where \( F(SQ, \hat{\phi}) \) and \( F(SQ/Z, \hat{\phi}) \) are abbreviated as \( \hat{\phi} \)'s. Then there are two maps

\[
\begin{align*}
\tilde{h}: F(SQ, \hat{E}(G)) & \rightarrow F(SQ, F), \quad \tilde{h}: F(SQ/Z, \hat{E}(G)) \rightarrow F(SQ/Z, F)
\end{align*}
\]

such that \( \tilde{h} \circ \tilde{\varphi} = \tilde{\psi} \) and \( \tilde{h} \circ \tilde{\psi} = \tilde{\varphi} \). As is easily seen, the map \( \tilde{h} \) is a homotopy equivalence. On the other hand, our assumption means that \( \text{Hom}(\pi_*(EQ), G) = 0 \) since the map \( \text{Hom}(\text{Tor}(\pi_*\pi(EQ), Q/Z), G) \rightarrow \text{Hom}(\pi_*(EQ/Z), G) \) is an isomorphism for any reduced \( G \). Therefore the coefficients sequence

\[
(3.1) \quad 0 \rightarrow \hat{E}(P_i)^*(SQ/Z) \xrightarrow{\phi_*} \hat{E}(P_0)^*(SQ/Z) \xrightarrow{\psi_*} F^*(SQ/Z) \rightarrow 0
\]

is short exact. By means of [15, Lemma A] (see [5]) we find that the map \( \tilde{h} \) is a homotopy equivalence, too.

Corresponding to the injective resolution \( 0 \rightarrow Z \rightarrow Q \rightarrow Q/Z \rightarrow 0 \) there is a cofibering

\[
S \xrightarrow{i} SQ \xrightarrow{j} SQ/Z.
\]

It is easy to see that the maps \( \tilde{\varphi}, \tilde{\psi} \) and \( \tilde{\psi} \)'s are compatible with \( j \)'s. Consequently we have the following diagram:
in which all but the right square are commutative, and the maps $h$ and $\tilde{h}$ are homotopy equivalences. The map $\psi$ induces a monomorphism

$$\psi^*: F^*(F(SQ/Z, \hat{E}(G))Q) \to F^*(F(SQ/Z, \hat{E}(P_0))Q)$$

because $\psi^*: \pi Q_*(F(SQ/Z, \hat{E}(P_0))) \to \pi Q_*(F(SQ/Z, \hat{E}(G)))$ is epic by (3.1). Hence we get immediately that the right square is commutative like the rest. Thereby we have a homotopy equivalence

$$h: \hat{E}(G) \to F$$

by applying Five lemma.

Note that $\text{Hom}(tA, G)=0$ if $\text{Tor}(A, G)=0$. The above theorem asserts that the finiteness restrictions on $G$ and $E$ may be eliminated in [14, Theorem 4].

4. Purity of the universal coefficient sequence

4.1. We now study whether the universal coefficient sequence (1.1) is pure as Huber and Meier [10] tried. But our method owes to Mislin [12] rather than Hilton and Deleanu [9, Theorem 3.2]. Consider first the universal coefficient sequence of the form

(4.1) \[ 0 \to E_*(X) \otimes Z_q \to EZ(r)(X) \to \text{Tor}(E_{*-1}(X), Z_q) \to 0. \]

According to Araki and Toda [4, Theorem 2.7] (or [9]) we have that

(4.2) \textit{the universal coefficient sequence (4.1) is split if } q \equiv 2 \text{ mod } 4 \textit{ or if } E \textit{ is good.}

An abelian group $G$ is said to be 2-high if the homomorphism $\text{Tor}(G, Z_4) \to \text{Tor}(G, Z_2)$, induced by the projection $Z_4 \to Z_2$, is epic [9]. If a 2-high group $G$ is finitely generated, then it doesn’t contain $Z_2$ as a direct summand. Any 2-high group is certainly the union of all finitely generated 2-high subgroups. Even if $E$ is not good, we still have the following nice result by adopting the argument in [9, Theorem 4.3].
(4.3) If \( E_n(X) \) is 2-high, then the exact sequence (4.1) is split in the \( n \)-th and \((n+1)\)-th dimensions.

A short exact sequence \( 0 \to A \to B \to C \to 0 \) is called 2-high pure if the induced homomorphism \( A \otimes Z_q \to B \otimes Z_q \) is monic for any \( q \equiv 2 \mod 4 \). Evidently an exact sequence \( 0 \to A \to B \to C \to 0 \) is 2-high pure if and only if the induced homomorphisms \( A \otimes G \to B \otimes G \) are monic for all 2-high \( G \).

Assume that there is a natural exact sequence

\[
0 \to \Ext(E^{*}(X), G) \to F^{*}(X) \to \Hom(E^{*}(X), G) \to 0.
\]

Of course we may introduce \( \hat{E}(G) \) as \( F \) if necessary. Consider the commutative square

\[
\begin{array}{ccc}
\Ext(E^{*}(X), G) \otimes Z_q & \to & F^{*}(X) \otimes Z_q \\
\eta \otimes 1 & \downarrow & \tau \\
\Ext(EZ_{\cdot}(X), G) & \to & F^{*+1}(XZ_{\cdot}).
\end{array}
\]

The upper arrow \( \eta \otimes 1 \) is monic if and only if the left vertical arrow is monic. The latter condition is equivalent to say that the sequence \( 0 \to \Ext(Tor(E^{*}(X), Z_q), G) \to \Ext(EZ_{\cdot}(X), G) \to \Ext(E^{*}(X) \otimes Z_q, G) \to 0 \) induced by (4.1) is exact. Hence we obtain

(4.4) the natural exact sequence (**) is always 2-high pure, and it is pure whenever the exact sequence (4.1) with \( q=2 \) is split.

Moreover we notice

(4.5) the purity of the natural exact sequence (**) doesn't depend on the choice of \( F \).

4.2. We here compute the group \( \{\hat{S}(G), \hat{S}(H)\} \).

**Lemma 16.** If either \( G \) or \( H \) is 2-high, then

\[
\{\hat{S}(G), \hat{S}(H)\} \approx \Hom(G, H) \oplus \Ext(G, H \otimes Z_2).
\]

**Proof.** First assume that \( G \) is 2-high. Then the exact sequence

\[
0 \to \pi_0(\hat{S}(G)) \otimes Z_q \to \pi_0(\hat{S}(G)Z_q) \to \Tor(\pi_0(\hat{S}(G)), Z_q) \to 0
\]

is split by (4.3). Because of (4.4) the exact sequence

\[
0 \to \Ext(\pi_{-1}(\hat{S}(G)), H) \to \hat{S}(H)(\hat{S}(G)) \to \Hom(\pi_0(\hat{S}(G)), H) \to 0
\]

is pure. \( \Ext(\pi_{-1}(\hat{S}(G)), H) \) is bounded, and hence it is algebraically compact (see [7]). So the pure exact sequence is split.
We next assume that $H$ is 2-high. By use of Corollary 15 and Theorem 1 we get an isomorphism $\{SG, SH\} \rightarrow \{\hat{S}(G), \hat{S}(H)\}$. So we use the exact sequence

$$0 \rightarrow \text{Ext}(G, \pi_1(SH)) \rightarrow SH^0(SG) \rightarrow \text{Hom}(G, \pi_0(SH)) \rightarrow 0.$$ 

Consider the commutative square

$$\begin{array}{ccc}
\text{Ext}(G, \pi_1(SH)) \otimes Z_\eta & \rightarrow & SH^0(SG) \otimes Z_\eta \\
\downarrow & & \downarrow \\
\text{Ext}(G, \pi_1(SHZ_\eta)) & \rightarrow & SHZ_\eta^0(SG).
\end{array}$$

The left vertical arrow is monic since the exact sequence

$$0 \rightarrow \pi_1(SH) \otimes Z_\eta \rightarrow \pi_1(SHZ_\eta) \rightarrow \text{Tor}(\pi_0(SH), Z_\eta) \rightarrow 0$$

is split by (4.3). Thus the above exact sequence is pure. Thereby it is split as $\text{Ext}(G, \pi_1(SH))$ is bounded.

We now show the purity of the exact sequence (**) under some restriction on either $E$ or $G$.

**Theorem 5.** Assume that there is a natural exact sequence

$$0 \rightarrow \text{Ext}(E_{*+1}(X), G) \rightarrow F^*(X) \rightarrow \text{Hom}(E_{*}(X), G) \rightarrow 0.$$ 

If the CW-spectrum $E$ is good or if the abelian group $G$ is 2-high, then the above exact sequence is pure. (Cf., [10, Corollary 3.4]).

**Proof.** When $E$ is good, the purity follows from (4.2) and (4.4) Assume that $G$ is 2-high, then $\{\hat{S}(G), \hat{S}(G \otimes Z_\eta)\}$ is a $Z_\eta$-module by Lemma 16. So we have a commutative square

$$\begin{array}{ccc}
\text{Ext}(E_{*+1}(X), G) \otimes Z_\eta & \eta_G \otimes 1 & \hat{E}(G)^*(X) \otimes Z_\eta \\
\eta_G & \rightarrow & 0 \\
\text{Ext}(E_{*+1}(X), G \otimes Z_\eta) & \eta_G \otimes Z_\eta & \hat{E}(G \otimes Z_\eta)^*(X).
\end{array}$$

The upper arrow $\eta_G \otimes 1$ is monic, and hence the universal coefficient sequence

$$0 \rightarrow \text{Ext}(E_{*+1}(X), G) \rightarrow \hat{E}(G)^*(X) \rightarrow \text{Hom}(E_{*}(X), G) \rightarrow 0$$

is pure. By virtue of (4.5) we get the purity of our exact sequence.

Huber and Meier [10] gave several conditions under which each pure exact sequence of the form (**) is split. In particular, we have

**Corollary 18** ([10]). Assume that $E$ is good or that $G$ is 2-high. If
\[ \text{Pext}(Q/Z, tG) = 0, \text{ e.g., the torsion subgroup } tG \text{ is algebraically compact, then a natural exact sequence} \]

\[ 0 \to \text{Ext}(E_{n-1}(X), G) \to F^*(Z) \to \text{Hom}(E_*(X), G) \to 0 \]

is split.

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\section*{References}


