Title	Universal coefficient sequences for cohomology theories of CW-spectra. II
Author(s)	Yosimura, Zen-ichi
Citation	Osaka Journal of Mathematics. 1979, 16(1), p. 201-217
Version Type	VoR
URL	https://doi.org/10.18910/7417
rights	
Note	

Yosimura, Z.
Osaka J. Math.
16 (1979), 201–217

UNIVERSAL COEFFICIENT SEQUENCES FOR COHOMOLOGY THEORIES OF CW-SPECTRA, II

Dedicated to Professor A. Komatu on his 70th birthday

ZEN-ICHI YOSIMURA

(Received October 17, 1977)

Let E be a CW-spectrum and G be an abelian group. Following Kainen [11] we can construct a CW-spectrum $\hat{E}(G)$ which has a universal coefficient sequence

$$0 \to \mathrm{Ext}(E_{*-1}(X), G) \to \hat{E}(G)^*(X) \to \mathrm{Hom}(E_*(X), G) \to 0\,.$$

In the previous paper [14] with the same title we investigated several properties of $\hat{E}(G)$. But some of our results are restrictive as yet, e.g., Proposition 8 and Theorem 4 in [14]. In this note we continue the investigations to develop and improve our partial results.

First we discuss whether the correspondences $G \to \hat{E}(G)$ as well as $G \to EG$ are functorial in G, as analogous discussions were done in [9] and [10]. Next, under some finiteness assumption on E or G we show that $\hat{E}(G)$ and $\hat{E}(R)G$ are homotopy equivalent where $Z \subset R \subset Q$ (Theorem 1). This result is a satisfactory improvement of [14, Proposition 8]. As an application of the main result of Huber and Meier [10] we can then give a criterion for $EG^*(X)$ being Hausdorff (Theorem 2). Moreover we discuss the uniqueness of $\hat{E}(G)$ again to improve a partial result obtained in [14, Theorem 4]. When E is the sphere spectrum S we have a complete result (Theorem 3), but for a general E we need still some restriction although the finiteness assumptions on E and G can be eliminated in our previous result (Theorem 4). Finally we show that the universal coefficient sequence is pure under some restriction on E or G, adopting an argument given in [9].

In this note we shall work in the stable homotopy category of CW-spectra (see [1] or [13]).

The author wishes to thank Professors Huber and Meier for sending him their preprint [10] by which he has been motivated to write this sequel.

1. Functoriality of $\hat{E}(G)$

1.1. Let E be a CW-spectrum and G be an abelian group. Then there is a CW-spectrum $\hat{E}(G)$ so that E and $\hat{E}(G)$ are related by a universal coefficient sequence

$$(1.1)\qquad 0 \to \mathrm{Ext}(E_{*-1}(X), G) \xrightarrow{\eta_G} \hat{E}(G)^*(X) \xrightarrow{\tau_G} \mathrm{Hom}(\hat{E}_*(X), G) \to 0$$

(see [11] and [14]). Let us first recall the eonstruction of $\hat{E}(G)$ involving an injective resolution of G. By the representability theorem there is a CW-spectrum $\hat{E}(I)$ and a natural equivalence τ_I: $\hat{E}(I)^*(\ )\to \mathrm{Hom}(E_*(\ ), I)$ for every injective I. Take any injective resolution $0\to G\to I \xrightarrow{\psi} J\to 0$ and denote by $\hat{\psi}$: $\hat{E}(I)\to\hat{E}(J)$ the unique map induced by ψ. We define $\hat{E}(G)$ to be the fiber of $\hat{\psi}$, i.e.

$$\hat{E}(G) \to \hat{E}(I) \xrightarrow{\hat{\psi}} \hat{E}(J)$$

is a cofibering. The homotopy type of $\hat{E}(G)$ is independent of the choice of an injective resolution.

Let us denote by S the sphere spectrum. By the exactness of function spectra [13] there ls a cofibering

$$F(E,\hat{S}(G)) \to F(E,\hat{S}(I)) \xrightarrow{F(E,\hat{\psi})} F(E), \hat{S}(J)) .$$

By the aid of Five lemma [I3] we obtain

Proposition 1. *For any abelian group G the spectrum $\hat{E}(G)$ has the same homotopy type as the function spectrum $F(E,\hat{S}(G))$.*

Given an abelian group G, each map f: $W\to E$ of CW- spectra determines the unique map $\hat{f}=F(f,\hat{S}(G))$: $F(E,\hat{S}(G))\to F(W,\hat{S}(G))$. Tbereby Proposition 1 contains the following functorial property.

Corollary 2. *Fix an abelian group G. Then the correspondence $E\to\hat{E}(G)=F(E,\hat{S}(G))$ is a contravariant exact functor.*

We may now turn our attention to the spectrum $\hat{S}(G)$. The map τ_G gives rise to an isomorphism

$$(1.2)\qquad t_G\colon\ \pi_0(\hat{S}(G)) \xrightarrow{\tau_G} \mathrm{Hom}(\pi_0(S), G)\cong G .$$

Lemma 3. *The composition map*

$$\hat{S}(G)^*(X) \xrightarrow{\tau_G} \mathrm{Hom}(\pi_*(X), G) \xrightarrow{t_{G*}} \mathrm{Hom}(\pi_*(X), \pi_0(\hat{S}(G)))$$

is just the homomorphism κ assigning to a map f the induced homomorphism f_* in 0-th homotopy groups.

Proof. It is sufficient to show the equality $t_G=\tau_G(1_{\hat{S}(G)})$ for the identity map $1_{\hat{S}(G)}$ of $\hat{S}(G)$. Take any element f of $\pi_0(\hat{S}(G))$, i.e., a map $f\colon S\to\hat{S}(G)$. By the naturality of τ_G we have

$$t_G(f)=\tau_G(f)(1_S)=(\tau_G(1_{\hat{S}(G)})f_*)(1_S)=\tau_G(1_{\hat{S}(G)})(f)\,.$$

Because of Lemma 3 we may employ κ instead of τ_G. Thus there is a natural exact sequence

$$(1.3)\qquad 0\to \mathrm{Ext}(\pi_{*-1}(X),G)\xrightarrow{\eta_G}\hat{S}(G)^*(X)\xrightarrow{\kappa}\mathrm{Hom}(\pi_*(X),G)\to 0$$

where $\pi_0(\hat{S}(G))$ is identified with G via the map t_G.

Let E be a ring spectrum and F be an (associative) right E-module spectrum equipped with a structure map $\mu\colon F\wedge E\to F$. Then there is a unique map

$$\bar{\mu}_G\colon E\wedge\hat{F}(G)\to\hat{F}(G)$$

such that $e_{F,G}(1_F\wedge\bar{\mu}_G)=e_{F,G}(\mu\wedge 1_{\hat{F}(G)})$ where $e_{F,G}\colon F\wedge\hat{F}(G)\to\hat{S}(G)$ is the evaluation map. Thereby $\hat{F}(G)$ is an (associative) left E-module spectrum. Using the structure maps μ and $\bar{\mu}_G$ we can give $\mathrm{Hom}(F_*(\ ),G)$ and $\hat{F}(G)^*(\ )$ structures of left $E^*(\ )$-modules. Thus we have two homomorphisms

$$\mu_\#\colon E^*(Y)\otimes\mathrm{Hom}(F_*(X),G)\to\mathrm{Hom}(F_*(Y\wedge X),G)$$
$$\bar{\mu}_\#\colon E^*(Y)\otimes\hat{F}(G)^*(X)\to\hat{F}(G)^*(Y\wedge X)$$

defined in the obvious way. By virtue of Lemma 3 we have

Proposition 4. *Let E be a ring spectrum and F be a right E-module spectrum. Then the universal coefficient sequence*

$$0\to\mathrm{Ext}(F_{*-1}(X),G)\xrightarrow{\eta_G}\hat{F}(G)^*(X)\xrightarrow{\tau_G}\mathrm{Hom}(F_*(X),G)\to 0$$

is an exact sequence of left $E^*(\ )$-modules.

Proof. As is easily seen, the induced homotopy homomorphism κ is a map of left $E^*(\ )$-modules, i.e., the following square

$$\begin{array}{ccc} E^*(Y)\otimes\hat{F}(G)^*(X) & \xrightarrow{1\otimes\kappa} & E^*(Y)\otimes\mathrm{Hom}(F_*(X),\pi_0(\hat{S}(G))) \\ \bar{\mu}_\#\big\downarrow & & \big\downarrow\mu_\# \\ \hat{F}(G)^*(Y\wedge X) & \xrightarrow{\kappa} & \mathrm{Hom}(F_*(Y\wedge X),\pi_0(\hat{S}(G))) \end{array}$$

is commutative. By a routine computation the result is immediate.

1.2. Take any homomorphism $\phi\colon G\to H$ of abelian groups, then there is a (non-unique) map $\hat{\phi}\colon\hat{S}(G)\to\hat{S}(H)$ making the diagram below commutative

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}(\pi_{*-1}(X), G) & \xrightarrow{\eta_G} & \hat{S}(G)^*(X) & \xrightarrow{\tau_G} & \mathrm{Hom}(\pi_*(X), G) & \to 0 \\ & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \hat{\phi}_*} & & \downarrow{\scriptstyle \phi_*} & \\ 0 \to & \mathrm{Ext}(\pi_{*-1}(X), H) & \xrightarrow{\eta_H} & \hat{S}(H)^*(X) & \xrightarrow{\tau_H} & \mathrm{Hom}(\pi_*(X), H) & \to 0\,. \end{array}$$

Thus the correspondence $G \to \hat{S}(G)$ is quasi-functorial in G [11].

Lemma 5. *If* $0 \to G \xrightarrow{\phi} H \xrightarrow{\psi} K \to 0$ *is a short exact sequence, then there exist maps* $\hat{\phi}\colon \hat{S}(G) \to \hat{S}(H)$ *and* $\hat{\psi}\colon \hat{S}(H) \to \hat{S}(K)$ *which give us a cofibering*

$$\hat{S}(G) \xrightarrow{\hat{\phi}} \hat{S}(H) \xrightarrow{\hat{\psi}} \hat{S}(K)\,.$$

Proof. Choose an injective resolution $0 \to H \to I \to J_1 \to 0$ and consider commutative exact diagram

$$\begin{array}{ccccccccc} & & & 0 & & 0 & & & \\ & & & \downarrow & & \downarrow & & & \\ & & 0 \to & G & \to & H & \to K \to 0 & & \\ & & & \downarrow & & \downarrow & & & \\ & & & I & = & I & & & \\ & & & \downarrow & & \downarrow & & & \\ 0 \to & K & \to & J_0 & \to & J_1 & \to 0 & & \\ & & & \downarrow & & \downarrow & & & \\ & & & 0 & & 0 & & & \end{array}$$

in which there appear three injective resolutions of G, H and K. By applying Verdier's lemma [6] we obtain a cofibering as desired.

Denote by $k_{G,H}$ the composition map

$$\hat{S}(H)^0(\hat{S}(G)) \xrightarrow{\tau_H} \mathrm{Hom}(\pi_0(\hat{S}(G)), H) \underset{\cong}{\xleftarrow{t_G^*}} \mathrm{Hom}(G, H)\,.$$

It is epic, in fact we observe that

$$k_{G,H}(\hat{\phi}) = \phi_*(1_G) = \phi\,, \tag{1.4}$$

by making use of the equality $t_G = \tau_G(1_{\hat{S}(G)})$. But $\mathrm{Ker}\, k_{G,H} \cong \mathrm{Ext}(\pi_{-1}(\hat{S}(G)), H) \cong \mathrm{Ext}(\mathrm{Hom}(Z_2, G), H) \cong \mathrm{Ext}(G, Z_2 \otimes H)$. By an easy computation we verify that

(1.5) $k_{G,H}$ is an isomorphism if and only if either G is 2-torsion free or H is 2-divisible.

This implies

Proposition 6. *If G is 2-torsion free or if H is 2-divisible, then* $\hat{\phi} = F(E, \hat{\phi})\colon \hat{E}(G) \to \hat{E}(H)$ *is uniquely determined for each* $\phi\colon G \to H$.

Let us denote by η: $\Sigma^1 S \to S$ the Hopf map, i.e., the non-zero element of $\pi_1(S)$. A CW-spectrum E is said to be *good* if $\eta \wedge 1_E$: $\Sigma^1 E \to E$ is trivial [9]. For a good E we have the following functorial property.

Proposition 7. *Assume that a fixed CW-spectrum E is good. Then the composite $G \to \hat{S}(G) \to \hat{E}(G) = F(E, \hat{S}(G))$ is a covariant exact functor.*

Proof. We show that the homomorphism

$$F(E,\ \ ): \{\hat{S}(G), \hat{S}(H)\} \to \{\hat{E}(G), \hat{E}(H)\}$$

factors through $k_{G,H}$. Recall that $F(E,\ \ )$ is given by the composition

$$\{\hat{S}(G), \hat{S}(H)\} \xrightarrow{e_G^*} \{E \wedge \hat{E}(G), \hat{S}(H)\} \xleftarrow{\cong} \{\hat{E}(G), \hat{E}(H)\}$$

where $e_G = e_{E,G}$: $E \wedge \hat{E}(G) \to \hat{S}(G)$ is the evaluation map. So it is enough to show that there is a homomorphism λ making the diagram below commutative

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(\pi_{-1}(\hat{S}(G)), H) \arrow[r, "\eta_H"] \arrow[d, "(e_{G*})^*"] & \{\hat{S}(G), \hat{S}(H)\} \arrow[r, "k_{G,H}"] \arrow[d, "e_G^*"] & \mathrm{Hom}(G, H) \arrow[r] \arrow[dl, "\lambda"] & 0 \\
& \mathrm{Ext}(\pi_{-1}(E \wedge \hat{E}(G)), H) \arrow[r, "\eta_H"] & \{E \wedge \hat{E}(G), \hat{S}(H)\} & &
\end{tikzcd}$$

Consider the commutative diagram

$$\begin{tikzcd}
\pi_{-1}(E \wedge \hat{E}(G)) \arrow[r, "e_{G*}"] \arrow[d, "\eta^*"] & \pi_{-1}(\hat{S}(G)) \arrow[r, "\tau_G", "\cong"'] \arrow[d, "\eta^*"] & \mathrm{Hom}(\pi_1(S), G) \arrow[d, "(\eta_*)^*"] \\
\pi_0(E \wedge \hat{E}(G)) \arrow[r, "e_{G*}"] & \pi_0(\hat{S}(G)) \arrow[r, "\tau_G", "\cong"'] & \mathrm{Hom}(\pi_0(S), G).
\end{tikzcd}$$

The left arrow η^* is trivial by our hypothesis on E, and the central one η^* is monic by use of the right square. This implies that the upper arrow e_{G*} is trivial. The existence of λ is now immediate. Therefore the correspondence $G \to \hat{E}(G)$ is a functor which is exact by Lemma 5.

1.3. For each abelian group G we denote by SG the Moore spectrum of type G. Then there is a universal coefficient sequence in the form of a natural exact sequence

$$(1.6) \qquad 0 \to \mathrm{Ext}(G, \pi_{*+1}(X)) \to \{SG, X\}_* \xrightarrow{\kappa} \mathrm{Hom}(G, \pi_*(X)) \to 0$$

where κ is just the induced homotopy homomorphism [8]. In particular we have a short exact sequence

$$0 \to \mathrm{Ext}(G, \pi_1(SH)) \to \{SG, SH\} \xrightarrow{\kappa} \mathrm{Hom}(G, H) \to 0\,.$$

Given a homomorphism $\phi: G \to H$, there is a (non-unique) map $S\phi: SG \to SH$ inducing $S\phi_* = \phi: \pi_0(SG) \to \pi_0(SH)$. Since $\pi_1(SH) \cong H \otimes Z_2$ we have an analogous result to Proposition 6.

Proposition 8. *Assume that G is 2-torsion free or that H is 2-divisible. Then $1_E \wedge S\phi: EG \to EH$ is uniquely determined for each $\phi: G \to H$ (see* [10, *Proposition* 3.2]).

By choosing suitably free resolutions in the dual way to the proof of Lemma 5 we can show that there is a cofibering

$$SG \xrightarrow{S\phi} SH \xrightarrow{S\psi} SK \tag{1.7}$$

if $0 \to G \xrightarrow{\phi} H \xrightarrow{\psi} K \to 0$ is a short exact sequence.

Corresponding to [9, Appendix] we obtain

Proposition 9. *Assume that a fixed CW-spectrum E is good. Then the composite $G \to SG \to EG$ is a covariant exact functor.*

Proof. The homomorphism $1_E \wedge -: \{SG, SH\} \to \{EG, EH\}$ is just the composition

$$\{SG, SH\} \xrightarrow{\varepsilon_{H^*}} \{SG, F(E, EH)\} \xrightarrow{\cong} \{EG, EH\}$$

where $\varepsilon_H: SH \to F(E, EH)$ is the dual of 1_{EH}. So we consider the following commutative diagram

$$\begin{array}{ccccccc} 0 \to \mathrm{Ext}(G, \pi_1(SH)) & \to & \{SG, SH\} & \xrightarrow{\kappa} & \mathrm{Hom}(G, H) \to 0 \\ \downarrow{\scriptstyle (\varepsilon_{H*})_*} & & \downarrow{\scriptstyle \varepsilon_{H*}} & & \\ \mathrm{Ext}(G, \pi_1(F(E, EH))) & \to & \{SG, F(E, EH)\}\,. & & \end{array}$$

In the commutative square

$$\begin{array}{ccc} \pi_0(SH) & \xrightarrow{\varepsilon_{H^*}} & \pi_0(F(E, EH)) \\ \downarrow{\scriptstyle \eta^*} & & \downarrow{\scriptstyle \eta_*} \\ \pi_1(SH) & \xrightarrow{\varepsilon_{H^*}} & \pi_1(F(E, EH)) \end{array}$$

the left arrow η^* is epic, but the right one η^* is trivial by our hypothesis on E. Hence the lower arrow ε_{H^*} is trivial, too. This claims that $\varepsilon_{H^*}: \{SG, SH\} \to \{SG, F(E, EH)\}$ factors through κ. Our result is now obvious.

2. Important properties of $\hat{E}(G)$

2.1. Let us denote by R a subring of the rationals Q and l^c be the set of primes which are invertible in R. A CW-spectrum E is called an *R-spectrum* if

$p \cdot 1_E \colon E \to E$ is a homotopy equivalence for each $p \in l^c$. Notice that E is an R-spectrum if and only if $\pi_*(E)$ is an R-module. An R-spectrum E is said to be *of finite type* if $\pi_*(E)$ is of finite type as an R-module.

We now study whether the CW-spectra $\hat{E}(G)$ and $\hat{E}(R)G$ are homotopy equivalent. Assume that an R-spectrum E is of finite type or that an R-module G is finitely generated. Let us first recall our partial result [14] in the special case when G is free. In this case we write P instead of G, i.e., $P = \sum_\alpha R$. The canonical injections $i_\alpha \colon R \to P$ give rise to the map $\bigvee_\alpha \hat{i}_\alpha \colon \bigvee_\alpha \hat{E}(R) \to \hat{E}(P)$ which is unique by Proposition 6. According to [14, Lemma 7] the map $\vee \hat{i}_\alpha$ is a homotopy equivalence under our assumption. Consequently the composite map

$$\iota_{E,P} \colon \hat{E}(R)P \leftarrow \bigvee_\alpha \hat{E}(R) \to \hat{E}(P) \tag{2.1}$$

is a homotopy equivalence, too.

Notice that the map $\iota_{E,P}$ has a factorization

$$F(E, \hat{S}(R))P \xrightarrow{j} F(E, \hat{S}(R)P) \xrightarrow{F(E,\, \iota_{S,P})} F(E, \hat{S}(P))$$

whose decomposed maps are both homotopy equivalences. By applying Five lemma we obtain that the canonical map

$$j \colon F(E, \hat{S}(R))G \to F(E, S(R)G) \tag{2.2}$$

is a homotopy equivalence under our finiteness assumption on E or G.

We here give the following interesting result.

Theorem 1. *Let E be an R-spectrum and G be an R-module. Assume that E is of finite type or that G is finitely generated. Then $\hat{E}(G)$ and $\hat{E}(G)R$ have the same homotopy type.*

Proof. Take a free resolution $0 \to P_1 \xrightarrow{\phi} P_0 \xrightarrow{\psi} G \to 0$ of R-modules, and consider the diagram

$$\begin{tikzcd}
\hat{S}(R)P_1 \arrow[r, "1 \wedge S\phi"] \arrow[d, "\iota_{S,P_1}"'] & \hat{S}(R)P_0 \arrow[r, "1 \wedge S\psi"] \arrow[d, "\iota_{S,P_0}"] & \hat{S}(R)G \\
\hat{S}(P_1) \arrow[r, "\hat{\phi}"'] & \hat{S}(P_0) \arrow[r, "\hat{\psi}"'] & \hat{S}(G)
\end{tikzcd}$$

involving two cofiberings in (1.7) and Lemma 5. In order to show that the square is commutative, we use the map $\kappa \colon \hat{S}(P_0)^0(\hat{S}(R)P_1) \to \mathrm{Hom}(\pi_0(\hat{S}(R)P_1), \pi_0(\hat{S}(P_0)))$ which is an isomorphism. After $\pi_0(\hat{S}(R)P_1)$ and $\pi_0(\hat{S}(P_0))$ are identified with P_1 and P_0 respectively, we compute that

$$\kappa(\hat{\phi}\cdot\iota_{S,P_1}) = \phi_*\kappa(\iota_{S,P_1}) = \phi_*(1_{P_1}) = \phi^*(1_{P_0}) = \phi^*\kappa(\iota_{S,P_0})$$
$$= \kappa(\iota_{S,P_0}\cdot 1\wedge S\phi)\,,$$

which claims $\hat{\phi}\cdot\iota_{S,P_1}=\iota_{S,P_0}\cdot 1\wedge S\phi$. By (2.1) the vertical maps ι_{S,P_i} are both homotopy equivalences. By use of Five lemma we obtain a homotopy equivalence

$$\hat{S}(R)G \to \hat{S}(G)$$

in the special case $E=S$.

For a general E we use (2.2) to obtain that the composite map

$$F(E,\, \hat{S}(R)G) \xrightarrow{j} F(E,\, \hat{S}(R)G) \to F(E,\, \hat{S}(G))$$

is a homotopy equivalence.

If E is an R-spectrum of finite type, then it has a nice property that there is a homotopy equivalence

$$\text{(2.3)} \qquad h_E\colon E \to ER \to \widehat{\hat{E}(R)(R)}$$

(see [14, Theorem 2]). Putting two important results, Theorem 1 and (2.3), together we obtain a natural exact sequence

$$\text{(2.4)} \quad 0 \to \mathrm{Ext}(\hat{E}(R)_{*-1}(X),\, G) \to EG^*(X) \to \mathrm{Hom}(\hat{E}(R)^*(X),\, G) \to 0$$

if E is an R-spectrum of finite type. Applying the main result of Huber and Meier [10, Theorem 1.1] we can extend our criterion [14, Theorem 3] for $E^*(X)$ being Hausdorff.

Theorem 2 ([10]). *Assume that E is an R-spectrum of finite type. Then $EG^*(X)$ is Hausdorff if and only if* $\mathrm{Pext}(\hat{E}(R)_{*-1}(X),\, G)=0$.

2.2. For a CW-spectrum E we denote by $E(-\infty,\, n]$ $(=E(-\infty,\, n+1))$ the $(n+1)$-coconnective Postnikov cofiber of E and by $E(n,\, \infty)$ $(=E[n+1,\, \infty))$ the n-connective Postnikov fiber of E (see [3]). Thus $E(-\infty,\, n]$ is an $(n+1)$-coconnective CW-spectrum such that there is a map $j_n\colon E\to E(-\infty,\, n]$ which induces an isomorphism $j_{n*};\ \pi_r(E)\to\pi_r(E(-\infty,\, n])$ for each $r\leqq n$, and $E(n,\, \infty)$ an n-connective CW-spectrum such that there is a map $i_n\colon E(n,\, \infty)\to E$ which induces an isomorphism $i_{n*};\ \pi_r(E(n,\, \infty))\to\pi_r(E)$ for each $r>n$. Notice that the sequence

$$E(n,\, \infty) \xrightarrow{i_n} E \xrightarrow{j_n} E(-\infty,\, n]$$

is a cofibering.

By routine computations we have

Lemma 10. i) *The map j_n induces a homotopy equivalence*

$$E(-\infty, n]G \overset{\simeq}{\rightarrow} \begin{cases} EG(-\infty, n] & \text{if } \operatorname{Tor}(\pi_n(E), G) = 0 \\ EG(-\infty, n+1] & \text{if } \pi_{n+1}(E) \otimes G = 0 . \end{cases}$$

ii) *The map i_n induces a homotopy equivalence*

$$E(n, \infty)G \overset{\simeq}{\rightarrow} \begin{cases} EG(n, \infty) & \text{if } \operatorname{Tor}(\pi_n(E), G) = 0 \\ EG(n+1, \infty) & \text{if } \pi_{n+1}(E) \otimes G = 0 . \end{cases}$$

Lemma 11. i) *The map $\hat{i}_n$ induces a homotopy equivalence*

$$\widehat{E(-\infty, n]}(G) \overset{\simeq}{\rightarrow} \begin{cases} \hat{E}(G)[-n, \infty) & \text{if } \operatorname{Ext}(\pi_n(E), G) = 0 \\ \hat{E}(G)[-n-1, \infty) & \text{if } \operatorname{Hom}(\pi_{n+1}(E), G) = 0 . \end{cases}$$

ii) *The map $\hat{i}_n$ induces a homotopy equivalence*

$$\widehat{E(n, \infty)}(G) \overset{\simeq}{\rightarrow} \begin{cases} \hat{E}(G)(-\infty, -n) & \text{if } \operatorname{Ext}(\pi_n(E), G) = 0 \\ \hat{E}(G)(-\infty, -n-1) & \text{if } \operatorname{Hom}(\pi_{n+1}(E), G) = 0 . \end{cases}$$

Combining Theorem 1 with Lemmas 10 and 11 we obtain

Proposition 12. *Assume that an R-spectrum E is of finite type or that an R-module G is finitely generated. If* $\operatorname{Ext}(\pi_n(E), G) = 0$, *then* $\widehat{E(-\infty, n]}(G)$ *has the same homotopy type as* $\hat{E}(R)\,[-n, \infty)G$ *and* $\widehat{E(n, \infty)}(G)$ *does the same as* $\hat{E}(R)(-\infty, -n)G$.

For the BU-, BO- and BSp- spectrum K, KO and KSp we have determined in [14, Theorem 5] (or see [2]) that

$$\hat{K}(G) = KG \quad \textit{and} \quad \widehat{KSp}(G) = KOG . \tag{2.5}$$

Applying Proposition 12 we get

$$\widehat{K[0, \infty)}(G) = K(-\infty, 0]G,\ \widehat{KSp}[0, \infty)(G) = KO(-\infty, 0]G \textit{ and so on.} \tag{2.6}$$

2.3. Let τ: $F(W, \hat{V}(G)) \rightarrow F(V, \hat{W}(G))$ be the homotopy equivalence induced by the switching map T: $W \wedge V \rightarrow V \wedge W$. Putting $V = E$ and $W = \hat{E}(G)$, τ yields the map

$$\varepsilon_{E,G}\colon E \rightarrow \widehat{\hat{E}(G)}(G)$$

which is the dual of $e_{E,G}T$ where $e_{E,G}$: $E \wedge \hat{E}(G) \rightarrow \hat{S}(G)$ denotes the evaluation map. Observe that the composition

$$\{W, E\} \xrightarrow{\varepsilon_{E,G*}} \{W, \widehat{\hat{E}(G)}(G)\} \underset{\simeq}{\xleftarrow{\tau_\#}} \{\hat{E}(G), \hat{W}(G)\} \tag{2.7}$$

is just the map $F(\ ,\hat{S}(G))$.

Proposition 13. *If an R-spectrum E is of finite type, then the map*

$$F(\ ,\hat{S}(R))\colon \{W, E\} \to \{\hat{E}(R), \hat{W}(R)\}$$

is an isomorphism for each W, and equivalently the canonical map $\varepsilon_{E,R}\colon E \to \widehat{\hat{E}(R)}(R)$ *is a homotopy equivalence* (cf., (2.3)).

Proof. Take a homotopy equivalence $h_E\colon E \to \widehat{\hat{E}(R)}(R)$ of (2.3) and introduce the composite map

$$\rho_E\colon F(W, E) \to F(W, \widehat{\hat{E}(R)}(R)) \xrightarrow{T} F(\hat{E}(R), \hat{W}(R)),$$

given by use of h_E, which is functorial with respect to W. We modify the map ρ_E a bit as it induces the map $F(\ ,\hat{S}(R))$. Since $\rho_{E\sharp}\colon \{W, E\} \to \{\hat{E}(R), \hat{W}(R)\}$ is an isomorphism, we can find a map $f\colon E \to E$ such that $\rho_{E\sharp}(f) = 1_{\hat{E}(R)}$. The map $\hat{f}\colon \hat{E}(R) \to \hat{E}(R)$ gives rise to a split epic $\hat{f}_*\colon \pi_*(\hat{E}(R)) \to \pi_*(\hat{E}(R))$ since $\hat{f}\cdot\rho_{E\sharp}(1_E) = \rho_{E\sharp}(f^*(1_E)) = 1_{\hat{E}(R)}$. But the R-module $\pi_*(\hat{E}(R))$ is of finite type, so $\hat{f}_*$ is isomorphic. This means that the map $\hat{f}$ is a homotopy equivalence. Consider the composite map

$$F(W, E) \xrightarrow{\rho_E} F(\hat{E}(R), \hat{W}(R)) \xleftarrow{F(\hat{f}, \hat{W}(R))} F(\hat{E}(R), \hat{W}(R)).$$

Obiously the induced isomorphism

$$\{W, E\} \xrightarrow{\rho_{E\sharp}} \{\hat{E}(R), \hat{W}(R)\} \xleftarrow{\hat{f}^*} \{\hat{E}(R), \hat{W}(R)\}$$

conicides with the map $F(\ ,\hat{S}(R))$.

We next define a generalization $\hat{F}_{G,H}\colon \{WG, EH\} \to \{\hat{E}(R)G, \hat{W}(R)H\}$ of the isomorphism $F(\ ,\hat{S}(R))$. The evaluation map $e_{E,R}\colon E \wedge \hat{E}(R) \to \hat{S}(R)$ gives us a homomorphism

$$e_{E\sharp}\colon \{WG, EH\} \to \{W \wedge \hat{E}(R)G, \hat{S}(R)H\}$$

defined in the obvious way. On the other hand, if W is an R-spectrum of finite type or if H is a finitely generated R-module, then the map $j\colon F(W, \hat{S}(R))H \to F(W, \hat{S}(R)H)$ induces an isomorphism

$$\{\hat{E}(R)G, \hat{W}(R)H\} \to \{W \wedge \hat{E}(R)G, \hat{S}(R)H\}$$

by (2.2). We compose the above two to obtain a generalization $\hat{F}_{G,H}$ under the finiteness restriction on W or H.

Proposition 14. *Assume that W is an R-spectrum of finite type or that H is a finitely generated R-module. If an R-spectrum E is of finite type, then the map*

$$\hat{F}_{G,H}\colon \{WG,\ EH\} \to \{\hat{E}(R)G,\ \hat{W}(R)H\}$$

is an isomorphism.

Proof. For a free R-module P we consider the following commutative diagram

$$\begin{tikzcd} \{W, E\}\otimes P \arrow[r, "e_{E\#}\otimes 1"] \arrow[d] & \{W\wedge \hat{E}(R),\ \hat{S}(R)\}\otimes P \arrow[d] \arrow[dr] & \\ \{W, EP\} \arrow[r, "e_{E\#}"'] & \{W\wedge \hat{E}(R),\ \hat{S}(R)P\} \arrow[r, "\iota_{S,P*}"'] & \{W\wedge \hat{E}(R),\ \hat{S}(P)\}\ . \end{tikzcd}$$

The upper arrow $e_{E\#}\otimes 1$ is an isomorphism by Proposition 13, and two vertical arrows are both isomorphisms for any finite W (use (2.1) and the proof of [14, Lemma 7 ii)]). This implies that the lower one $e_{E\#}$ is an isomorphism for a general W. Now a routine argument shows that $e_{E\#}\colon \{WG, EH\}\to \{W\wedge \hat{E}(R)G,\ \hat{S}(R)H\}$ is an isomorphism for any G and H, and hence so is the map $\hat{F}_{G,H}$.

For simplicity we write $\hat{S}$ instead of $\hat{S}(Z)$. When $W=E=S$, $\hat{F}_{G,H}$ is equal to the map $1_{\hat{S}}\wedge -\colon \{SG, SH\}\to\{\hat{S}G, \hat{S}H\}$. So we have

Corollary 15. *The map*

$$1_{\hat{S}}\wedge -\colon \{SG,\ SH\} \to \{\hat{S}G,\ \hat{S}H\}$$

is an isomorphism for any G and H.

3. Uniqueness of $\hat{E}(G)$

3.1. We here discuss the uniqueness of $\hat{E}(G)$ as it was done in [14, Theorem 4]. Our attention is first turned to the special case $E=S$. In this case we have the following satisfactory result.

Theorem 3. *If a CW-spectrum F has a natural exact sequence*

$$(*)\qquad 0\to \mathrm{Ext}(\pi_{*-1}(X),\ G)\to F^*(X)\xrightarrow{\ \tau\ }\mathrm{Hom}(\pi_*(X),\ G)\to 0$$

for a fixed abelian group G, then F has the same homotopy type as $\hat{S}(G)$.

Proof. By the same argument as Lemma 3 we may regard τ as the induced homotopy homomorphism κ, after G is identified with $\pi_0(F)$ via the isomorphism $t_{F,G}\colon \pi_0(F)\xrightarrow{\ \tau\ }\mathrm{Hom}(\pi_0(S),\ G)\cong G$. Then there is a map

$$h\colon \hat{S}(G)\to F$$

whose induced homomorphism $h_*\colon \pi_0(\hat{S}(G))\to G$ is equal to the isomorphism

t_G of (1.2). Using the commutative square

$$\begin{array}{ccc}\pi_{-1}(\hat{S}(G)) & \xrightarrow{\kappa} & \mathrm{Hom}(\pi_1(S),\ \pi_0(\hat{S}(G))) \\ \downarrow{h_*} & & \downarrow{h_*} \\ \pi_{-1}(F) & \xrightarrow{\kappa} & \mathrm{Hom}(\pi_1(S),\ \pi_0(F))\end{array}$$

we verify that $h_*\colon \pi_{-1}(\hat{S}(G))\to\pi_{-1}(F)$ is also an isomorphism. Applying the natural exact sequences $(*)$ and (1.3) we can see that

$$h^*\colon F^0(F)\to F^0(\hat{S}(G)) \quad \text{and} \quad h^*\colon \hat{S}(G)^0(F)\to \hat{S}(G)^0(\hat{S}(G))$$

are both isomorphisms. A routine argument shows that h is a homotopy equivalence.

3.2. In a general case E we next attempt to weaken some restrictions in our previous result [14, Theorem 4].

Theorem 4. *Let G be a fixed abelian group and D be the maximal divisible subgroup. Assume that a CW-spectrum E satisfies* $\mathrm{Hom}(t\pi_*(E),\ G/D)=0$ *where $t\pi_*(E)$ denotes the torsion subgroup of $\pi_*(E)$. If two CW-spectra E and F are related by a natural exact sequence*

$$0\to \mathrm{Ext}(E_{*-1}(X),\ G)\to F^*(X)\xrightarrow{\tau}\mathrm{Hom}(E_*(X),\ G)\to 0\,,$$

then F has the same homotopy type as $\hat{E}(G)$.

Proof. Since the short exact sequence $0\to D\to G\to G/D\to 0$ is split we may choose a map $f\colon F\to\hat{E}(D)$ so that it induces the composition

$$F^*(X)\xrightarrow{\tau}\mathrm{Hom}(E_*(X),\ G)\to\mathrm{Hom}(E_*(X),\ D)\xleftarrow{\cong}\hat{E}(D)^*(X)\,.$$

Denoting by F_R the fiber of f, the cofibering

$$F_R\to F\xrightarrow{f}\hat{E}(D)$$

is split as $f_*\colon F^*(X)\to\hat{E}(D)^*(X)$ is epic. With an application of 3×3 lemma as in [14, Theorem 4] we get a natural exact sequence

$$0\to\mathrm{Ext}(E_{*-1}(X),\ G/D)\to F_R^*(X)\to\mathrm{Hom}(E_*(X),\ G/D)\to 0\,.$$

Evidently $\mathrm{Hom}(Q,\ G/D)=0$, i.e., G/D is reduced (see [7]). Therefore we have to show that F_R and $\hat{E}(G/D)$ have the same homotopy type for the reduced G/D.

We may now assume that G is a reduced group with $\mathrm{Hom}(t\pi_*(E),\ G)=0$. Take a free resolution $0\to P_1\xrightarrow{\phi}P_0\xrightarrow{\psi}G\to 0$ and proceed our proof as in [14, Theorem 4]. By Lemma 5 the resolution gives us a cofibering

$$\hat{E}(P_1) \xrightarrow{\hat{\phi}} \hat{E}(P_0) \xrightarrow{\hat{\psi}} \hat{E}(G)\,.$$

Evidently $\mathrm{Hom}(EQ_*(X), P_i)=\mathrm{Hom}(EQ/Z_*(X), P_i)=0$ for $i=1, 2$ and also $\mathrm{Hom}(EQ_*(X), G)=0$ as G is reduced. We then obtain maps

$$\tilde{\psi}\colon F(SQ, \hat{E}(P_0)) \to F(SQ, F) \quad \text{and} \quad \bar{\psi}\colon F(SQ/Z, \hat{E}(P_0)) \to F(SQ/Z, F)$$

which make the diagrams below commutative

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(EQ_{*-1}(X), P_1) \arrow[r, "\phi_*"] \arrow[d, "\cong"] & \mathrm{Ext}(EQ_{*-1}(X), P_0) \arrow[r, "\psi_*"] \arrow[d, "\cong"] & \mathrm{Ext}(EQ_{*-1}(X), G) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \hat{E}(P_1)^*(XQ) \arrow[r, "\hat{\phi}_*"] & \hat{E}(P_0)^*(XQ) \arrow[r, "\tilde{\psi}_\#"] & F^*(XQ) \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Ext}(EQ/Z_{*-1}(X), P_1) \arrow[r, "\phi_*"] \arrow[d, "\cong"] & \mathrm{Ext}(EQ/Z_{*-1}(X), P_0) \arrow[r, "\psi_*"] \arrow[d, "\cong"] & \mathrm{Ext}(EQ/Z_{*-1}(X), G) \arrow[r] \arrow[d] & 0 \\
\hat{E}(P_1)^*(XQ/Z) \arrow[r, "\hat{\phi}_*"] & \hat{E}(P_0)^*(XQ/Z) \arrow[r, "\bar{\psi}_\#"] & F^*(XQ/Z) &
\end{tikzcd}$$

By easy diagram chases we observe that two bottom sequences in the above diagrams are exact. In particular, the composite maps $\tilde{\psi}\cdot\hat{\phi}$ and $\bar{\psi}\cdot\hat{\phi}$ are both trivial where $F(SQ, \hat{\phi})$ and $F(SQ/Z, \hat{\phi})$ are abbreviated as $\hat{\phi}$'s. Then there are two maps

$$\tilde{h}\colon F(SQ, \hat{E}(G)) \to F(SQ, F), \quad \bar{h}\colon F(SQ/Z, \hat{E}(G)) \to F(SQ/Z, F)$$

such that $\tilde{h}\cdot\hat{\psi}=\tilde{\psi}$ and $\bar{h}\cdot\hat{\psi}=\bar{\psi}$. As is easily seen, the map $\tilde{h}$ is a homotopy equivalence. On the other hand, our assumption means that $\mathrm{Hom}(\pi_*(EQ/Z), G)=0$ since the map $\mathrm{Hom}\,(\mathrm{Tor}\,(\pi_{*-1}(E), Q/Z), G) \to \mathrm{Hom}(\pi_*(EQ/Z), G)$ is an isomorphism for any reduced G. Thereby the coefficients sequence

$$(3.1) \qquad 0 \to \hat{E}(P_1)^*(SQ/Z) \xrightarrow{\hat{\phi}_*} \hat{E}(P_0)^*(SQ/Z) \xrightarrow{\bar{\psi}_\#} F^*(SQ/Z) \to 0$$

is short exact. By means of [15, Lemma A] (see [5]) we find that the map $\bar{h}$ is a homotopy equivalence, too.

Corresponding to the injective resolution $0 \to Z \to Q \to Q/Z \to 0$ there is a cofibering

$$S \xrightarrow{i} SQ \xrightarrow{j} SQ/Z\,.$$

It is easy to see that the maps $\tilde{\psi}$, $\bar{\psi}$ and $\hat{\psi}$'s are compatible with j's. Consequently we have the following diagram

$$\begin{tikzcd}
F(SQ/Z, \hat{E}(P_0)) \arrow[rr, "\bar{\psi}"] \arrow[dr, "\hat{\psi}"] \arrow[ddd, "j"] & & F(SQ/Z, F) \arrow[ddd, "j"] \\
& F(SQ/Z, \hat{E}(G)) \arrow[ur, "\bar{h}"] \arrow[d, "j"] & \\
& F(SQ, \hat{E}(G)) \arrow[dr, "\tilde{h}"] & \\
F(SQ, \hat{E}(P_0)) \arrow[ur, "\hat{\psi}"] \arrow[rr, "\tilde{\psi}"'] & & F(SQ, F)
\end{tikzcd}$$

in which all but the right square are commutative, and the maps $\bar{h}$ and $\tilde{h}$ are homotopy equivalences. The map $\hat{\psi}$ induces a monomorphism

$$\hat{\psi}^*: F^*(F(SQ/Z, \hat{E}(G))Q) \to F^*(F(SQ/Z, \hat{E}(P_0))Q)$$

because $\hat{\psi}_*: \pi Q_*(F(SQ/Z, \hat{E}(P_0))) \to \pi Q_*(F(SQ/Z, \hat{E}(G)))$ is epic by (3.1). Hence we get immediately that the right square is commutative like the rest. Thereby we have a homotopy equivalence

$$h: \hat{E}(G) \to F$$

by applying Five lemma.

Note that Hom $(tA, G)=0$ if Tor $(A, G)=0$. The above theorem asserts that the finiteness restrictions on G and E may be eliminated in [14, Theorem 4].

4. Purity of the universal coefficient sequence

4.1. We now study whether the universal coefficient sequence (1.1) is pure as Huber and Meier [10] tried. But our method owes to Mislin [12] rather than Hilton and Deleanu [9, Theorem 3.2]. Consider first the universal coefficient sequence of the form

$$(4.1) \qquad 0 \to E_*(X) \otimes Z_q \to EZ_{q*}(X) \to \mathrm{Tor}\,(E_{*-1}(X), Z_q) \to 0 .$$

According to Araki and Toda [4, Theorem 2.7] (or [9]) we have that

(4.2) *the universal coefficient sequence* (4.1) *is split if* $q \not\equiv 2 \bmod 4$ *or if E is good.*

An abelian group G is said to be 2-*high* if the homomorphism Tor $(G, Z_4) \to \mathrm{Tor}\,(G, Z_2)$, induced by the projection $Z_4 \to Z_2$, is epic [9]. If a 2-high group G is finitely generated, then it doesn't contain Z_2 as a direct summand. Any 2-high group is certainly the union of all finitely generated 2-high subgroups. Even if E is not good, we still have the following nice result by adopting the argument in [9, Theorem 4.3].

(4.3) *If* $E_n(X)$ *is* 2-*high, then the exact sequence* (4.1) *is split in the n-th and* $(n+1)$-*th dimensions.*

A short exact sequence $0 \to A \to B \to C \to 0$ is called 2-*high pure* if the induced homomorphism $A \otimes Z_q \to B \otimes Z_q$ is monic for any $q \not\equiv 2 \bmod 4$. Evidently an exact sequence $0 \to A \to B \to C \to 0$ is 2-high pure if and only if the induced homomorphisms $A \otimes G \to B \otimes G$ are monic for all 2-high G.

Assume that there is a natural exact sequence

$$(**) \qquad 0 \to \mathrm{Ext}(E_{*-1}(X), G) \xrightarrow{\eta} F^*(X) \xrightarrow{\tau} \mathrm{Hom}(E_*(X), G) \to 0 .$$

Of course we may introduce $\hat{E}(G)$ as F if necessary. Consider the commutative square

$$\begin{array}{ccc} \mathrm{Ext}(E_{*-1}(X), G) \otimes Z_q & \xrightarrow{\eta \otimes 1} & F^*(X) \otimes Z_q \\ \downarrow & & \downarrow \\ \mathrm{Ext}(EZ_{q^*}(X), G) & \xrightarrow{\eta} & F^{*+1}(XZ_q) . \end{array}$$

The upper arrow $\eta \otimes 1$ is monic if and only if the left vertical arrow is monic. The latter condition is equivalent to say that the sequence

$$0 \to \mathrm{Ext}(\mathrm{Tor}(E_{*-1}(X), Z_q), G) \to \mathrm{Ext}(EZ_{q^*}(X), G) \to \mathrm{Ext}(E_*(X) \otimes Z_q, G) \to 0$$

induced by (4.1) is exact. Hence we obtain

(4.4) *the natural exact sequence* $(**)$ *is always* 2-*high pure, and it is pure whenever the exact sequence* (4.1) *with* $q=2$ *is split.*

Moreover we notice

(4.5) *the purity of the natural exact sequence* $(**)$ *doesn't depend on the choice of* F.

4.2. We here compute the group $\{\hat{S}(G), \hat{S}(H)\}$.

Lemma 16. *If either* G *or* H *is* 2-*high, then*

$$\{\hat{S}(G), \hat{S}(H)\} \cong \mathrm{Hom}(G, H) \oplus \mathrm{Ext}(G, H \otimes Z_2) .$$

Proof. First assume that G is 2-high. Then the exact sequence

$$0 \to \pi_0(\hat{S}(G)) \otimes Z_q \to \pi_0(\hat{S}(G)Z_q) \to \mathrm{Tor}(\pi_{-1}(\hat{S}(G)), Z_q) \to 0$$

is split by (4.3). Because of (4.4) the exact sequence

$$0 \to \mathrm{Ext}(\pi_{-1}(\hat{S}(G)), H) \to \hat{S}(H)^0(\hat{S}(G)) \to \mathrm{Hom}(\pi_0(\hat{S}(G)), H) \to 0$$

is pure. $\mathrm{Ext}(\pi_{-1}(\hat{S}(G)), H)$ is bounded, and hence it is algebraically compact (see [7]). So the pure exact sequence is split.

We next assume that H is 2-high. By use of Corollary 15 and Theorem 1 we get an isomorphism $\{SG, SH\} \to \{\hat{S}(G), \hat{S}(H)\}$. So we use the exact sequence

$$0 \to \mathrm{Ext}(G, \pi_1(SH)) \to SH^0(SG) \to \mathrm{Hom}(G, \pi_0(SH)) \to 0\,.$$

Consider the commutative square

$$\begin{array}{ccc} \mathrm{Ext}(G, \pi_1(SH))\otimes Z_q & \to & SH^0(SG)\otimes Z_q \\ \downarrow & & \downarrow \\ \mathrm{Ext}(G, \pi_1(SHZ_q)) & \to & SHZ_q^0(SG)\,. \end{array}$$

The left vertical arrow is monic since the exact sequence

$$0 \to \pi_1(SH)\otimes Z_q \to \pi_1(SHZ_q) \to \mathrm{Tor}(\pi_0(SH), Z_q) \to 0$$

is split by (4.3). Thus the above exact sequence is pure. Thereby it is split as $\mathrm{Ext}(G, \pi_1(SH))$ is bounded.

We now show the purity of the exact sequence (**) under some restriction on either E or G.

Theorem 5. *Assume that there is a natural exact sequence*

$$0 \to \mathrm{Ext}(E_{*-1}(X), G) \to F^*(X) \to \mathrm{Hom}(E_*(X), G) \to 0\,.$$

If the CW-spectrum E is good or if the abelian group G is 2-high, then the above exact sequence is pure. (Cf., [10, Corollary 3.4]).

Proof. When E is good, the purity follows from (4.2) and (4.4) Assume that G is 2-high, then $\{\hat{S}(G), \hat{S}(G\otimes Z_q)\}$ is a Z_q-module by Lemma 16. So we have a commutative square

$$\begin{array}{ccc} \mathrm{Ext}(E_{*-1}(X), G)\otimes Z_q & \xrightarrow{\eta_G\otimes 1} & \hat{E}(G)^*(X)\otimes Z_q \\ \cong\downarrow & & \downarrow \\ \mathrm{Ext}(E_{*-1}(X), G\otimes Z_q) & \xrightarrow{\eta_{G\otimes Z_q}} & \hat{E}(G\otimes Z_q)^*(X)\,. \end{array}$$

The upper arrow $\eta_G\otimes 1$ is monic, and hence the universal coefficient sequence

$$0 \to \mathrm{Ext}(E_{*-1}(X), G) \xrightarrow{\eta_G} \hat{E}(G)^*(X) \xrightarrow{\tau_G} \mathrm{Hom}(E_*(X), G) \to 0$$

is pure. By virtue of (4.5) we get the purity of our exact sequence.

Huber and Meier [10] gave several conditions under which each pure exact sequence of the form (**) is split. In particular, we have

Corollary 18 ([10]). *Assume that E is good or that G is 2-high. If*

Pext$(Q/Z, tG)=0$, e.g., *the torsion subgroup tG is algebraically compact, then a natural exact sequence*

$$0 \to \mathrm{Ext}(E_{*-1}(X), G) \to F^*(X) \to \mathrm{Hom}(E_*(X), G) \to 0$$

is split.

OSAKA CITY UNIVERSITY

References

[1] J.F. Adams: Stable homotopy and generalized homology, Chicago Lecture in Math., Univ. of Chicago Press, 1974.

[2] D.W. Anderson: Universal coefficient theorems for K-theory, mimeographed notes, Berkeley.

[3] S. Araki: Generalized cohomology (in Japanese), Kinokuniya Book Store, 1975.

[4] S. Araki and H. Toda: *Multiplicative structures in mod q cohomology theories*, I, Osaka J. Math. **2** (1965), 71–115.

[5] N.A. Baas: *On bordism theory of manifolds with singularities,* Math. Scand. **33** (1973), 279–302.

[6] J. Boardman: Stable homotopy theory, mimeographed notes, Univ. of Warwick, 1965.

[7] L. Fuchs: Infinite abelian groups I, Academic Press, 1970.

[8] P.J. Hilton: Homotopy theory and duality, Gordon and Breach, 1965.

[9] P.J. Hilton and A. Deleanu: *On the splitting of universal coefficient sequences,* Proc. Adv. Study Inst. on Alg. Top., vol. I, Aarhus (1970), 180–201.

[10] M. Huber and W. Meier: *Cohomology theories and infinite CW-complexes,* Comment. Math. Helv. **53** (1978), 239–257.

[11] P.C. Kainen: *Universal coefficient theorems for generalized homology and stable cohomotopy,* Pacific J. Math. **37** (1971), 337–407.

[12] G. Mislin: *The splitting of the Künneth sequence for generalized cohomology,* Math. Z. **122** (1971), 237–245.

[13] R. Vogt: Boardman's stable homotopy category, Lecture notes series 21, Aarhus Univ., 1970.

[14] Z. Yosimura: *Universal coefficient sequences for cohomology theories of CW-spectra,* Osaka J. Math. **12** (1975), 305–323.

[15] Z. Yosimura: *A note on the relation of Z_2-graded complex cobordism to complex K-theory,* Osaka J. Math. **12** (1975), 583–595.