Remarks on proof of a theorem of Kato and Kobayasi on linear evolution equations

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Osaka Journal of Mathematics. 17(1) P.233-P.244

1980

publisher

https://doi.org/10.18910/7418

10.18910/7418
REMARKS ON PROOF OF A THEOREM OF KATO AND KOBAYASI ON LINEAR EVOLUTION EQUATIONS

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(Received January 31, 1979)

1. Introduction

Let
\[ \frac{du}{dt} + A(t) u = f(t), \quad 0 \leq t \leq T, \]
be an evolution equation of "hyperbolic" type in a Banach space $E$ with $A(t)$ having a domain containing a fixed dense linear subspace $F$. T. Kato [1], [2], J.R. Dorroh [3], S. Ishii [4],[5], K. Kobayasi [7] etc. have developed methods of constructing an evolution operator for (1.1). The main theorem due to T. Kato and K. Kobayasi is stated as follows:

**Theorem.** Let $E$ and $F$ be Banach spaces such that $F$ is densely and continuously embedded in $E$, and \{A(t)\}_{0 \leq t \leq T}$ be a family of closed linear operators in $E$ with the domains

\[ D(A(t)) \supset F. \]

Assume that

(I) \{A(t)\}_{0 \leq t \leq T} is stable on $E$,

(II) $A \in C([0, T]; \mathcal{L}(F; E))$,

(III) There is family \{S(t)\}_{0 \leq t \leq T} of isomorphisms from $F$ onto $E$ such that

\[ S \in C([0, T]; \mathcal{L}(F; E)), \]

and

\[ S(t)A(t)S(t)^{-1} = A(t) + B(t) \]

for each $t \in [0, T]$ with some

\[ B \in C([0, T]; \mathcal{L}(E)). \]

Then we can construct an unique evolution operator \{U(t, s)\}_{0 \leq s \leq t \leq T} with the following properties

a) $U \in C(\{(t, s); 0 \leq s \leq t \leq T\}; \mathcal{L}(E))$, 

b) $U \in C(\{(t, s); 0 \leq s \leq t \leq T\}; \mathcal{L}(F))$,

c) $U(t, s)U(s, r) = U(t, r), \quad 0 \leq r \leq s \leq t \leq T; \quad U(s, s) = I, \quad 0 \leq s \leq T,$
T. Kato [1] first proved this theorem under stronger condition that $A(t)$ is norm continuous in $t$: $A \in C([0, T]; \mathcal{L}(F; E))$. J.R. Dorroh [3] then simplified the proof of the differentiability of $U(t, s)$. The author [6] noticed that if $E$ and $F$ are reflexive Banach spaces, then the norm continuity of $A(t)$ is weakened to the strong continuity (II). K. Kobayasi [7] recently eliminated this restriction and proved the theorem for general Banach spaces. He showed that a way of parting intervals used in the case of non-linear evolution equations (e.g. [8]) is available also for this linear problem. In this paper we will notice that though in [7] he used the partition of each $[s, T]$ depending on $s$, it can be replaced by an appropriate partition of the whole interval $[0, T]$. We need more detailed consideration than [7] to obtain the partition independent of $s$. But it makes it possible to utilize the Yosida approximation $A_n(t)$ of $A(t)$ in proof of the theorem. We give in section 3 the proof in this method. Once it is established that the evolution operator $U_n(t, s)$ for $A_n(t)$ is strongly convergent, we can verify more immediately that the limit $U(t, s)$ is really an evolution operator for $A(t)$.

Throughout this paper, we use the same notation and terminology as in [6]. $\| \cdot \|_E$ is the norm of a normed space $E$. For two normed spaces $E$ and $F$, $\mathcal{L}(E; F)$ is the normed space of all bounded linear operators from $E$ to $F$ with the operator norm $\| \cdot \|_{E,F}$, and $\mathcal{L}_s(E; F)$ is the locally convex space $\mathcal{L}(E; F)$ equipped with the strong topology. $\mathcal{L}_c(E; E)$ is abbreviated as $\mathcal{L}_c(E)$, and $\| \cdot \|_{E,E}$ as $\| \cdot \|_E$, if there is no fear of confusion. For a locally convex space $E$, $E$-$\lim x_\lambda$ is the limit in $E$ of a convergent family $\{x_\lambda\}_{\lambda \in \Lambda}$ of $E$, $C(D; E)$ is the set of all continuous mappings from a metric space $D$ to $E$, and $C^r([a, b]; E)$ is the set of all continuously differentiable functions in the interval $[a, b]$. $C_1, C_2, \ldots$ denote constants determined by $\sup_t \|A(t)\|_{E,F}$, $\sup_t \|S(t)\|_{F,E}$, $\sup_t \|S(t)^{-1}\|_{E,F}$, $\sup_t \|dS/dt\|_{F,F}$, $\sup_t \|B(t)\|_E$, $\alpha_0$ and $\{M, \beta\}$ alone; where $\alpha_0$ is a constant such that $\| \cdot \|_E \leq \alpha_0 \| \cdot \|_F$, and $\{M, \beta\}$ are the constants of stability of $\{A(t)\}$ on $E$. It is known that the part of $\{A(t)\}$ in $F$ is stable with the constants of stability $\{M, \beta\}$ given by

$$\tilde{M} = M \sup_t \|S(t)\| \sup_t \|S(t)^{-1}\| \exp \{TM \sup_t \|S(t)^{-1}\| \sup_t \|dS/dt\|\}$$

$$\tilde{\beta} = \beta + M \sup_t \|B(t)\|$$

(see [1], [9]).

2. Existence of the appropriate partition of $[0, T]$

For a finite partition $\Delta$: $0 = T_0 < T_1 < \cdots < T_N = T$ of $[0, T]$, $A_\Delta$ denotes a
THEOREM OF KATO AND KOBAYASI

step function of $A$

$$A_\Delta(t) = \begin{cases} A(T_j), & T_j \leq t < T_{j+1}, \\ A(T_n), & t = T, \end{cases}$$

and $\{U_\Delta(t, s)\}_{0 \leq s \leq t \leq T}$ is the evolution operator for $A_\Delta$

$$U_\Delta(t, s) = \begin{cases} \exp(- (t-s) A(T_j)), & T_j \leq s \leq t \leq T_{j+1}, \\ \exp(- (t-T_j) A(T_j)) \cdots \exp(- (T_{j+1}-s) A(T_j)), & T_j \leq s \leq T_{j+1} \cdots T_j \leq t \leq T_{j+1}. \end{cases}$$

**Proposition 2.1.** For any $\varepsilon > 0$ and any $y \in F$, there exists a finite partition $\Delta$ of $[0, T]$ such that

$$\sup_{0 \leq s \leq t \leq T} \| \{A(t) - A_\Delta(t)\} U_\Delta(t, s)y \|_E \leq \varepsilon.$$ 

Proof. We define inductively an increasing sequence $\{T_k\}_{k=0}^\infty$ of $[0, T]$ in the following way. $T_0 = 0$. Assume that $\{T_j\}_{0 \leq j \leq k}$ is defined so that the estimate

$$\sup_{0 \leq s \leq t \leq T_k} \| \{A(t) - A_\Delta(t)\} U_\Delta(t, s)y \|_E \leq \varepsilon \tag{2.1}$$

holds for the partition $\Delta_k$: $0 = T_0 < \cdots < T_k = T_k$ of $[0, T_k]$. If $T_k < T$, we consider a set $J_k$ of all elements $h \in (0, T - T_k]$ such that

$$\sup_{T_k \leq t \leq T_k + h} \| \{A(t) - A_\Delta(t)\} \exp(- \tau A(T_k))z \|_E \leq \varepsilon$$

holds for every $z \in L_k = \{ U_\Delta(t, s)y; 0 \leq s \leq t \leq T_k \}$. Since $L_k$ is compact in $F$, $J_k$ is non-empty and has the maximum. Putting $h_k = \text{Max} J_k$, we define $T_{k+1} = T_k + h_k$. Then the estimate

$$\sup_{0 \leq s \leq t \leq T_{k+1}} \| \{A(t) - A_\Delta(t)\} U_\Delta(t, s)y \|_E \leq \varepsilon \tag{2.2}$$

is valid. In fact, (2.2) is trivial if $t = T_{k+1}$. If $T_{k+1} > t \geq s \geq T_k$, $A_{\Delta_{k+1}}(t) = A(T_k)$ and $U_{\Delta_{k+1}}(t, s)y = \exp(- (t-s) A(T_k))y$. Therefore it follows that

$$\| \{A(t) - A_{\Delta_{k+1}}(t)\} U_{\Delta_{k+1}}(t, s)y \|_E \leq \sup_{T_k \leq t \leq T_k + h_k} \| \{A(t) - A(T_k)\} \exp(- \tau A(T_k))y \|_E \leq \varepsilon.$$ 

If $T_{k+1} > t \geq T_k > s$, $U_{\Delta_{k+1}}(t, s)y = \exp(- (t-T_k) A(T_k)) U_{\Delta_k}(T_k, s)y$. Similarly $U_{\Delta_k}(T_k, s)y$ is an element of $L_k$. Finally if $T_k > t \geq s$, $U_{\Delta_{k+1}}(t, s)y = U_{\Delta_k}(t, s)y$. (2.2) is nothing but the assumption (2.1). Until $T_k$ reaches $T$, we continue...
the inductive procedure. In order to complete the proof, it remains now to prove that such a procedure finishes within finite times. Suppose the contrary. Then we would have an infinite sequence \( \{T_k\}_{k=0,1,\ldots} \) of \([0, T)\) satisfying (2.1) for each \( k \). To reach a contradiction we will prove that

\[
L = \bigcup_{k=0}^{\infty} L_k
\]

is relatively compact in \( F \) by using the next lemma essentially due to K. Kobayasi [7].

**Lemma 2.2.** There exists a constant \( C_1 \) such that the estimation

\[
\| \prod_{k=1}^{p} \exp \left( -\tau_k A(t_k) \right) z - \prod_{k=1}^{p} \exp \left( -\tau_k A(t_k) \right) \|_F
\]

\[
\leq C_1 \left\{ \sum_{i=r+1}^{p} \tau_i \exp \left( \beta \sum_{k=r+1}^{p} \tau_k \right) \right\} \|x\|_F \tag{2.3}
\]

\[
+ C_1 \exp \left( \beta \sum_{k=r+1}^{p} \tau_k \right) \|S(t_r) \prod_{k=1}^{r} \exp \left( -\tau_k A(t_k) \right) z - x\|_E \tag{2.4}
\]

\[
+ C_1 \{t_p - t_r\} + \sum_{k=r+1}^{p} \tau_k \exp \left( \beta \sum_{k=r+1}^{p} \tau_k \right) \|x\|_E \tag{2.5}
\]

holds for any \( x, z \in F, \tau_k \geq 0 (1 \leq k \leq p), 0 \leq t_1 \leq \cdots \leq t_p \leq T, \) and integers \( p \geq q \geq r \geq 1 \).

**Proof.**

\[
\prod_{k=1}^{p} \exp \left( -\tau_k A(t_k) \right) z - \prod_{k=1}^{p} \exp \left( -\tau_k A(t_k) \right) z
\]

\[
= \left\{ \prod_{k=r+1}^{p} \exp \left( -\tau_k A(t_k) \right) - \prod_{k=r+1}^{p} \exp \left( -\tau_k A(t_k) \right) \right\} S(t_r)^{-1} \times
\]

\[
\times \left\{ S(t_r) \prod_{k=1}^{r} \exp \left( -\tau_k A(t_k) \right) z - x \right\}
\]

\[
+ \left\{ \prod_{k=r+1}^{p} \exp \left( -\tau_k A(t_k) \right) - \prod_{k=r+1}^{p} \exp \left( -\tau_k A(t_k) \right) \right\} S(t_r)^{-1} x
\]

\[
= R_1 + R_2.
\]

\( R_1 \) is estimated by (2.4).

\[
R_2 = S(t_r)^{-1} \{S(t_p) \prod_{k=r+1}^{p} \exp \left( -\tau_k A(t_k) \right) S(t_r)^{-1}
\]

\[
- S(t_r) \prod_{k=r+1}^{p} \exp \left( -\tau_k A(t_k) \right) S(t_r)^{-1} \} x
\]

\[
+ S(t_p)^{-1} \{S(t_r) - S(t_p)\} \prod_{k=r+1}^{p} \exp \left( -\tau_k A(t_k) \right) S(t_r)^{-1} x
\]

\[
= R_3 + R_4.
\]
THEOREM OF KATO AND KOBAYASI

\[ R_4 \] is estimated by (2.5).

\[ R_3 = S(t_p)^{-1} \{ S(t_p) \prod_{k=r+1}^{s} \exp (-\tau_k A(t_k)) S(t_r)^{-1} - \prod_{k=r+1}^{s} \exp (-\tau_k A(t_k)) \} \times \]

\[ -S(t_p)^{-1} \{ S(t_q) \prod_{k=r+1}^{s} \exp (-\tau_k A(t_k)) S(t_r)^{-1} - \prod_{k=r+1}^{s} \exp (-\tau_k A(t_k)) \} \times \]

\[ + S(t_p)^{-1} \{ \prod_{k=r+1}^{s} \exp (-\tau_k A(t_k)) - \prod_{k=r+1}^{s} \exp (-\tau_k A(t_k)) \} \times \]

\[ = R_5 + R_6 + R_7. \]

\[ S(t_p)R_5 = \sum_{i=r+1}^{s} \{ \prod_{k=r+1}^{s} \exp (-\tau_k A(t_k)) \{ S(t_i) \exp (-\tau_i A(t_i)) \}

\[ - \exp (-\tau_i A(t_i)) S(t_i) \} \prod_{k=r+1}^{s} \exp (-\tau_k A(t_k)) \} \times \]

\[ + \sum_{i=r+1}^{s} \{ \prod_{k=r+1}^{s} \exp (-\tau_k A(t_k)) \{ S(t_i) - S(t_{i-1}) \} \prod_{k=r+1}^{s} \exp (-\tau_k A(t_k)) S(t_i)^{-1} \times. \]

From this we obtain the estimate of \( R_5 \) by (2.5), and similarly that of \( R_6 \). From

\[ S(t_p)R_7 = \sum_{i=r+1}^{s} \{ \exp (-\tau_i A(t_i)) - I \} \prod_{k=r+1}^{s} \exp (-\tau_k A(t_k)) \times \]

it follows that \( R_7 \) is estimated by (2.3).

Let \( T_\infty = \lim_{k \to \infty} T_k \). Noting that \( U_{\Delta_k}(t, s)y \) coincides for all \( k \) such that \( t \leq T_k \), we define

\[ U_{\Delta_k}(t, s)y = \lim_{k \to \infty} U_{\Delta_k}(t, s)y \]

for \( 0 \leq s \leq t < T_\infty \). By the preceding lemma we have the following:

**Lemma 2.3.** For each \( 0 \leq s \leq T_\infty \) there exists a limit

\[ F_- \lim_{(t', s') \to (T_\infty, s)} U_{\Delta_k}(t', s')y. \]  

(2.6)

**Proof.** If \( s < T_\infty \), \( t < T_j < T_\infty \) with some \( j \). In this case the limit (2.6) is easily reduced to

\[ F_- \lim_{t' \to T_\infty} U_{\Delta_k}(t', T_j)z \]

(2.7)

with \( z = U_{\Delta_j}(T_j, s)y \in F \). Let \( t' > t > T_j \) be such that

\[ T_j < \cdots < T_{j+r-1} < \cdots < T_{j+q-1} \leq t' < T_{j+q} \cdots T_{j+p-2} \leq t' < T_{j+p-1} \]

with some \( p > q > r \), and apply Lemma 2.2 with

\[ t_k = \begin{cases} T_{j+k-1}, & 1 \leq k \leq q, \\ T_{j+k-2}, & q+1 \leq k \leq p, \end{cases} \]
A. Yagi

\[
\tau_k = \begin{cases} 
T_{j+k} - T_{j+k-1}, & 1 \leq k \leq q - 1 \\
t' - T_{j+q-1}, & k = q \\
T_{j+q} - t', & k = q + 1 \\
T_{j+k-1} - T_{j+k-2}, & q + 2 \leq k \leq p - 1 \\
t'' - T_{j+p-2}, & k = p.
\end{cases}
\]

Then we get

\[
\| U_{\delta_m}(t'', T_j)z - U_{\delta_m}(t', T_j)z \|_F \\
\leq C_1 e^{\delta T} \{ \| T_m - T_{j+q-1} \| \| x \|_F + ||S(t)\prod_{k=1}^q \exp(-\tau_k A(t_k))z - x||_E \} + 2(T_m - T_{j+p-1})||x||_E.
\]

For any \( \eta > 0 \), \( T_m - T_{j+r_0-1} \leq \eta \) with some \( r_0 \), and

\[
||S(t_{r_0})\prod_{k=1}^q \exp(-\tau_k A(t_k))z - x_0||_E \leq \eta
\]

with some \( x_0 \in F \). \( ||x||_E \) is dominated by

\[
||x||_E \leq \eta + ||S(t_{r_0})\prod_{k=1}^q \exp(-\tau_k A(t_k))z - x_0||_E + \bar{M} e^{\delta T} \sup_t ||S(t)|| ||z||_F.
\]

Therefore if \( t'' > t' > T_{j+r_0-1} \), (2.8) is smaller than

\[
C_1 e^{\delta T} \{ \| T_m - T_{j+q-1} \| \| x_0 \|_F + (1 + ||z||_F)\eta \}.
\]

If \( q_0 \) is large enough for \( \eta > 0 \), then \( t'' > t' > T_{j+q_0-1} \) implies

\[
\| U_{\delta_m}(t'', T_j)z - U_{\delta_m}(t', T_j)z \|_F \leq C_3 (1 + ||z||_F)\eta,
\]

which shows the existence of (2.7). If \( s = T_m \), we can prove

\[
y = F_{-t''} \lim_{s' > s} U_{\delta_m}(t', s')y.
\]

Let \( t'' > s' \) be such that

\[
T_j \leq s' < T_{j+1} \cdots T_{j+p-2} \leq t'' < T_{j+p-1}
\]

with some \( j \) and \( p \geq 2 \), and apply Lemma 2.2 with

\[
t_k = \begin{cases} 
T_j, & k = 1, 2 \\
T_{j+k-2}, & 3 \leq k \leq p,
\end{cases}
\]

\[
\tau_k = \begin{cases} 
0, & k = 1 \\
T_{j+1} - s', & k = 2 \\
T_{j+k-1} - T_{j+k-2}, & 3 \leq k \leq p - 1 \\
t'' - T_{j+p-2}, & k = p.
\end{cases}
\]
and $q=r=1$. Then we get

$$\|U_{t,s}(t', s')y-y\|_F \leq C_q e^{\tilde{c}t} \left\{ (|T_t-T_s|)\|x\|_F + \|S(T_t)y-x\|_E \right\}$$

$$\leq C_q \left\{ (|T_t-T_s|)\|x\|_F + \|y\|_F + \|S(T_t)y-x\|_E \right\}.$$ 

For any $\eta>0$, $\|S(T_t)y-x_0\|_E \leq \eta$ with some $x_0 \in F$, and $(T_t-T_s)(|x_0|_F + \|y\|_F) \leq \eta$ with some $j_0$. Therefore $t'-t^* > t_{j_0}$ implies

$$\|U_{t,s}(t', s')y-y\|_F \leq C_q \eta,$$

which shows (2.9)

We have known that $U_{t,s}(t, s)y$ can be extended on $0 \leq s \leq t \leq T_\infty$ continuously. Hence $L$ contained in $\{U_{t,s}(t, s)y; 0 \leq s \leq t \leq T_\infty \}$ is a relatively compact set in $F$.

**Lemma 2.4.** For any $\eta>0$ there exists $\delta_1>0$ such that

$$\sup_{\|x\|_F < \delta_1} \|\exp(-\tau A(t))z-z\|_F \leq \eta$$

for every $z \in L$.

Proof. Let $p=2$, $q=r=1$, $t_1=t_2=t$ and $\tau_1=0$, $\tau_2=\tau$ in Lemma 2.2. Then

$$\|\exp(-\tau A(t))z-z\|_F \leq C_q e^{\tilde{c}t} \left\{ \tau \|x\|_F + \|S(T_t)z-x\|_E \right\}$$

$$\leq C_q e^{\tilde{c}t} \left\{ \tau \|x\|_F + \|S(T_t)z-x\|_E + \|T_t-T_\infty\|_F \right\}.$$ 

Since $S(T_t)(L)$ is precompact in $E$ and $F$ is dense in $E$, $S(T_t)(L)$ can be covered with a finite number of open balls $\{B(y_i; \eta/3C_q e^{\tilde{c}t})\}_{1 \leq i \leq l}$ with centers $y_i \in F$. Hence for any $z \in L$

$$\|\exp(-\tau A(t))z-z\|_F \leq C_q e^{\tilde{c}t} \left\{ \tau \max_{1 \leq i \leq l} \|y_i\|_F + \tau \|T_t-T_\infty\|_F \right\} + e^{(t-1)\eta}/3.$$ 

Similarly for any $\eta>0$ there exists $\delta_2>0$ such that

$$\sup_{\|x\|_F < \delta_2} \|\{A(t)-A(T_\infty)\}z\|_E \leq \eta$$

for every $z \in L$. Put $h=\text{Min}\{\delta_1, \delta_2\}$. Then for $T_k > T_\infty - h$ the estimation

$$\sup_{T_k \leq t \leq T_k+h} \|\{A(t)-A(T_k)\} \exp(-\tau A(T_k))z\|_E$$

$$\leq \sup_{T_k \leq t \leq T_k+h} \|\{A(t)-A(T_k)\}z\|_E + C_0 \|\exp(-\tau A(T_k))z-z\|_F$$

$$\leq C_0 \eta$$

holds for every $z \in L$. This shows $h \in J_k$, if we take $\eta=C_1 e^{\tilde{c}t}$. But $h \in J_k$ contradicts
\[ h > T_k - T_{k+1} = h_k. \]

3. **Proof of the theorem**

For each integer \( n > \beta \), \( A_n(t) \) is the Yosida approximation of \( A(t) \)
\[ A_n(t) = n - n(I + n^{-1}A(t))^{-1}. \quad (3.1) \]

**Lemma 3.1.** \( A_n \in \mathcal{C}([0, T]; \mathcal{L}_s(E) \cap \mathcal{C}([0, T]; \mathcal{L}_s(F)). \)

**Proof.** In view of (3.1) it suffices to prove
\[ (\lambda + A(\cdot))^{-1} \in \mathcal{C}([0, T]; \mathcal{L}_s(E) \cap \mathcal{C}([0, T]; \mathcal{L}_s(F)) \] for \( \lambda > \beta \). For \( x \in F \) we can write
\[
\{ (\lambda + A(t+h))^{-1} - (\lambda + A(t))^{-1} \} x
= - (\lambda + A(t+h))^{-1} \{ A(t+h) - A(t) \} (\lambda + A(t))^{-1} x.
\]
Together with the uniform boundness of \( |(\lambda + A(\cdot))^{-1}|_E \), this shows that
\((\lambda + A(\cdot))^{-1} x\) is continuous in \( \| \cdot \|_E \). For general \( x \in E \) it follows from the density of \( F \) in \( E \). To see the strong continuity in \( F \) of (3.2) we have only to show
\[ (\lambda + A(\cdot) + B(\cdot))^{-1} \in \mathcal{C}([0, T]; \mathcal{L}_s(E)), \] (3.3)
since
\[ (\lambda + A(t))^{-1} = S(t)^{-1} (\lambda + A(t) + B(t))^{-1} S(t) \] (3.4)
on \( F \). But (3.3) follows from
\[ (\lambda + A(t) + B(t))^{-1} = (\lambda + A(t))^{-1} \{ I + B(t) (\lambda + A(t))^{-1} \}^{-1} \] (3.5)
and the strong continuity of \( (\lambda + A(\cdot))^{-1} \) in \( E \) proved above.

**Lemma 3.2.** \( \{ A_n(t) \}_{0 \leq t \leq T} \) is stable on \( E \) (resp. \( F \)) with constants of stability \( \{ M, \beta n(n - \beta)^{-1} \} \) (resp. \( \{ \tilde{M}, \tilde{\beta} n(n - \tilde{\beta})^{-1} \} \)).

**Proof.** The stability of \( \{ A_n(t) \} \) is observed directly by
\[ (\lambda + A_n(t))^{-1} = \frac{1}{\lambda + n} + \left( \frac{n}{\lambda + n} \right)^2 \left( \frac{\lambda n}{\lambda + n} + A(t) \right)^{-1}. \]

For \( n > \beta \) let \( \{ U_n(t, s) \}_{0 \leq s \leq t \leq T} \) be the evolution operator for \( \{ A_n(t) \}_{0 \leq t \leq T} \).
From Lemma 3.1 and 3.2 we conclude
\[ U_n \in \mathcal{C}(\{(t, s); 0 \leq s \leq t \leq T\}; \mathcal{L}_s(E)), \| U_n(t, s) \|_E \leq M \tilde{e}^{\beta_n(t-s)} \] (3.6)
with \( \beta_n = \beta n(n - \beta)^{-1} \) and
\[ U_n \in \mathcal{C}(\{(t, s); 0 \leq s \leq t \leq T\}; \mathcal{L}_s(F)), \| U_n(t, s) \|_F \leq \tilde{M} \tilde{e}^{\tilde{\beta_n(t-s)}} \] (3.7)
THEOREM OF KATO AND KOBAYASHI

241

Let $\beta_n = \beta n(n-\beta)^{-1}$.

Let $y \in F$ be arbitrarily fixed, $\epsilon$ be any positive number and $\Delta$ be the partition of $[0, T]$ satisfying

$$\sup_{0 \leq t \leq T} \| \{ A(t) - A_\Delta(t) \} U_\Delta(t, s)y \|_E \leq \epsilon \quad (3.8)$$

whose existence is guaranteed by Proposition 2.1. We can estimate the difference between $U_n(t, s)y$ and $U_\Delta(t, s)y$ by the following:

**Proposition 3.3.** There exists an integer $N$ such that for any $n \geq N$

$$\sup_{0 \leq t \leq T} \| U_n(t, s)y - U_\Delta(t, s)y \|_E \leq 2MT\epsilon^m \epsilon^{\rho T}.$$ 

Proof.

$$\{ U_\Delta(t, s) - U_n(t, s) \} y = \int_s^t U_n(t, \tau) \{ A_n(\tau) - A_\Delta(\tau) \} U_\Delta(\tau, s)y d\tau \leq \int_s^t U_n(t, \tau) \{ A_n(\tau) - A(\tau) \} U_\Delta(\tau, s)y d\tau + \int_s^t U_n(t, \tau) \{ A(\tau) - A_\Delta(\tau) \} U_\Delta(\tau, s)y d\tau.$$ 

The second term is evaluated by (3.8). Hence our proposition follows from the next lemma.

**Lemma 3.4.** For any compact set $K$ of $E$, there exists an integer $N$ such that for any $n \geq N$

$$\sup_{0 \leq t \leq T} \| (I + n^{-1}A(t))^{-1}x - x \|_E \leq \epsilon$$

holds for every $x \in K$.

Proof. $K$ is covered with a finite number of open balls $\{ B(y_i; \epsilon/2(M+1) \}_{1 \leq i} \in E$ with centers $y_i \in F$. Hence for any $x \in K$, taking some $y_0$

$$\| (I + n^{-1}A(t))^{-1}x - x \|_E = \leq \| \{ (I + n^{-1}A(t))^{-1} - I \}(x - y_0) \|_E + \| \{ (I + n^{-1}A(t))^{-1} - I \} y_0 \|_E$$

$$\leq (\epsilon/2)n(n-\beta)^{-1} + M(n-\beta)^{-1} \max_{1 \leq i \leq i} \| A(t)y_i \|_E.$$ 

We can now prove that $\{ U_n(t, s) \}_{\epsilon \geq \beta}$ is convergent in $L_s(E)$ uniformly in $(t, s)$. In fact we have

$$\sup_{0 \leq t \leq T} \| U_n(t, s)y - U_\Delta(t, s)y \|_E \leq 2MT\epsilon^m \epsilon^{\rho T}$$

for any $m, n \geq N$ by the mediation of $U_\Delta(t, s)y$. $\{ U_n(t, s)y \}_{\epsilon \geq \beta}$ is convergent in $E$ uniformly in $(t, s)$. Since $y \in F$ was arbitrary and $\| U_n(t, s) \|_E$ is uniformly
bounded by (3.6), \( \{U_n(t, s)x\}_{n>\bar{n}} \) is uniformly convergent in \( E \) for any \( x \in E \).

Thus the operator \( U(t, s) \) is defined by

\[
U(t, s) = \mathcal{L}_s(E) \text{-lim}_{n \to \infty} U_n(t, s). \tag{3.9}
\]

Obviously \( U(t, s) \) satisfies a) and c). To see the remaining properties we introduce bounded operators on \( E \)

\[
W_n(t, s) = S(t)U_n(t, s)S(s)^{-1}, \quad 0 \leq s \leq t \leq T,
\]

for each \( n > \bar{n} \) analogously to \([1]\). By (3.7)

\[
W_n \in \mathcal{C}(\{(t, s); 0 \leq s \leq t \leq T\}; \mathcal{L}_s(E)).
\]

\( W_n(t, s) \) is connected with \( U_n(t, s) \) by

\[
W_n(t, s) - U_n(t, s) = \int_s^t \frac{\partial}{\partial \tau} U_n(t, \tau)S(\tau)U_n(\tau, s)S(s)^{-1}d\tau
\]

with

\[
C_n(t) = A_n(t) - S(t)A_n(t)S(t)^{-1} + \frac{dS(t)}{dt}S(t)^{-1}, \quad 0 \leq t \leq T.
\]

**Lemma 3.5.**

\[
\mathcal{L}_s(E) \text{-lim}_{n \to \infty} C_n(t) = -B(t) + \frac{dS(t)}{dt}S(t)^{-1} \tag{3.10}
\]

uniformly in \( t \).

Proof. Clearly (3.10) is equivalent to

\[
\mathcal{L}_s(E) \text{-lim}_{n \to \infty} \{S(t)A_n(t)S(t)^{-1} - A_n(t)\} = B(t) \tag{3.11}
\]

uniformly in \( t \). By (3.1), (3.4) and (3.5)

\[
S(t)A_n(t)S(t)^{-1}
\]

\[
= (A(t) + B(t))\{I + n^{-1}(A(t) + B(t))\}^{-1}
\]

\[
= (A(t) + B(t))(I + n^{-1}A(t))^{-1}\{I + n^{-1}B(t)(I + n^{-1}A(t))^{-1}\}^{-1}
\]

\[
= \{A_n(t) + B(t)(I + n^{-1}A(t))^{-1}\} \{I + n^{-1}B(t)(I + n^{-1}A(t))^{-1}\}^{-1}
\]

\[
= \{A_n(t) + (n^{-1}A_n(t) + (I + n^{-1}A(t))^{-1}B(t)(I + n^{-1}A(t))^{-1}\} \times
\]

\[
\times \{I + n^{-1}B(t)(I + n^{-1}A(t))^{-1}\}^{-1}
\]

\[
= A_n(t) + (I + n^{-1}A(t))^{-1}B(t)(I + n^{-1}A(t))^{-1} \times
\]

\[
\times \{I + n^{-1}B(t)(I + n^{-1}A(t))^{-1}\}^{-1}.
\]
(3.11) is reduced to
\[ \mathcal{L}_s(E) - \lim_{n \to \infty} (I + n^{-1}A(t))^{-1} = I, \]
but this has already been established (Lemma 3.4).

Let \( W(t, s) \) be a solution of the integral equation
\[ W(t, s) = U(t, s) + \int_s^t U(t, \tau)C(\tau)W(\tau, s)d\tau \]
in \( \mathcal{L}_s(E) \) with the kernel (3.10)
\[ C(t) = -B(t) + \frac{dS}{dt} (t)S(t)^{-1}. \]
Obviously
\[ W \in C(\{(t, s); 0 \leq s \leq t \leq T\}; \mathcal{L}_s(E)). \]

We can deduce from (3.9) and (3.10)
\[ W(t, s) = \mathcal{L}_s(E) - \lim_{n \to \infty} W_n(t, s) \]
uniformly in \( (t, s) \). In other words
\[ S(t)^{-1}W(t, s)S(s) = \mathcal{L}_s(F) - \lim_{n \to \infty} U_n(t, s) \]  (3.12)
uniformly in \( (t, s) \). We know that all other properties are immediate consequences of (3.12) ([9]).

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Bibliography


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