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REMARKS ON PROOF OF A THEOREM OF KATO AND KOBAYASI ON LINEAR EVOLUTION EQUATIONS

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1. Introduction

Let

$$du/dt + A(t)u = f(t), \quad 0 \le t \le T,$$
 (1.1)

be an evolution equation of "hyperbolic" type in a Banach space E with A(t) having a domain containing a fixed dense linear subspace F. T. Kato [1], [2], J.R. Dorroh [3], S. Ishii [4],[5], K. Kobayasi [7] etc. have developed methods of constructing an evolution operator for (1.1). The main theorem due to T. Kato and K. Kobayasi is stated as follows:

Theorem. Let E and F be Banach spaces such that F is densely and continuously embedded in E, and $\{A(t)\}_{0 \le t \le T}$ be a family of closed linear operators in E with the domains

$$D(A(t))\supset F$$
.

Assume that

- $(I) \quad \{A(t)\}_{0 \le t \le T}$ is stable on E,
- (II) $A \in \mathcal{C}([0, T]; \mathcal{L}_s(F; E)),$
- (III) There is family $\{S(t)\}_{0 \le t \le T}$ of isomorphisms from F onto E such that

$$S \in \mathcal{C}^{1}([0, T]; \mathcal{L}_{s}(F; E)),$$

and

$$S(t)A(t)S(t)^{-1} = A(t) + B(t)$$

for each $t \in [0, T]$ with some

$$B \in \mathcal{C}([0, T]; \mathcal{L}_s(E))$$
.

Then we can construct an unique evolution operator $\{U(t, s)\}_{0 \le s \le t \le T}$ with the following properties

- a) $U \in \mathcal{C}(\{(t,s); 0 \leq s \leq t \leq T\}; \mathcal{L}_s(E)),$
- b) $U \in \mathcal{C}(\{(t, s); 0 \leq s \leq t \leq T\}; \mathcal{L}_s(F)),$
- c) $U(t,s)U(s,r)=U(t,r), 0 \le r \le s \le t \le T; U(s,s)=I, 0 \le s \le T,$

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d)
$$U(\cdot, s) \in C^{1}([s, T]; \mathcal{L}_{s}(F; E)), 0 \leq s < T; (\partial/\partial t)U(t, s) = -A(t)U(t, s),$$

e) $U(t, \cdot) \in C^{1}([0, t]; \mathcal{L}_{s}(F; E)), 0 < t \leq T; (\partial/\partial s)U(t, s) = U(t, s)A(s).$

T. Kato [1] first proved this theorem under stronger condition that A(t)is norm continuous in $t: A \in \mathcal{C}([0, T]; \mathcal{L}(F; E))$. J.R. Dorroh [3] then simplified the proof of the differentiability of U(t, s). The author [6] noticed that if E and F are reflexive Banach spaces, then the norm continuity of A(t) is weakened to the strong continuity (II). K. Kobayasi [7] recently eliminated this restriction and proved the theorem for general Banach spaces. He showed that a way of parting intervals used in the case of non-linear evolution equations (e.g. [8]) is available also for this linear problem. In this paper we will notice that though in [7] he used the partition of each [s, T] depending on s, it can be replaced by an appropriate partition of the whole interval [0, T]. We need more detailed consideration than [7] to obtain the partition independent of s. But it makes it possible to utilize the Yoside approximation $A_n(t)$ of A(t) in proof of the theorem. We give in section 3 the proof in this method. Once it is established that the evolution operator $U_n(t, s)$ for $A_n(t)$ is strongly convergent, we can verify more immediately that the limit U(t, s) is really an evolution operator for A(t).

Throughout this paper, we use the same notation and terminology as in [6]. $||\cdot||_{E}$ is the norm of a normed space E. For two normed spaces E and F, $\mathcal{L}(E; F)$ is the normed space of all bounded linear operators from E to F with the operator norm $||\cdot||_{E,F}$, and $\mathcal{L}_{s}(E; F)$ is the locally convex space $\mathcal{L}(E; F)$ equipped with the strong topology. $\mathcal{L}_{s}(E; E)$ is abbreviated as $\mathcal{L}_{s}(E)$, and $||\cdot||_{E,E}$ as $||\cdot||_{E}$, if there is no fear of confusion. For a locally convex space E, $E-\lim_{\lambda \to \lambda_{0}} x_{\lambda}$ is the limit in E of a convergent family $\{x_{\lambda}\}_{\lambda \in \Lambda}$ of E, $\mathcal{C}(D; E)$ is the set of all continuous mappings from a metric space D to E, and $\mathcal{C}^{1}([a, b]; E)$ is the set of all continuously differentiable functions in the interval [a, b]. C_{1}, C_{2}, \cdots denote constants determined by $\sup_{t} ||\mathcal{A}(t)||_{F,E}$, $\sup_{t} ||\mathcal{S}(t)||_{F,E}$, $\sup_{t} ||\mathcal{S}(t)^{-1}||_{E,F}$, $\sup_{t} ||\mathcal{S}(t)||_{F,E}$, $\sup_{t} ||\mathcal{S}(t)||_{F,E}$, $\sup_{t} ||\mathcal{S}(t)||_{F,E}$, sup $||\mathcal{S}(t)||_{F,E}$, and $\{M, \beta\}$ alone; where c_{0} is a constant such that $||\cdot||_{E} \leq c_{0} ||\cdot||_{F}$, and $\{M, \beta\}$ are the constants of stability of $\{\mathcal{A}(t)\}$ on E. It is known that the part of $\{\mathcal{A}(t)\}$ in F is stable with the constants of stability $\{\tilde{M}, \tilde{\beta}\}$ given by

$$\tilde{M} = M \sup_{t} ||S(t)|| \sup_{t} ||S(t)^{-1}|| \exp \{TM \sup_{t} ||S(t)^{-1}|| \sup_{t} ||dS/dt||\}$$

$$\tilde{\beta} = \beta + M \sup_{t} ||B(t)||$$

(see [1], [9]).

2. Existence of the appropriate partition of [0, T]

For a finite partition $\Delta: 0 = T_0 < T_1 < \cdots < T_N = T$ of [0, T], A_{Δ} denotes a

step function of A

$$A_{\Delta}(t) = \begin{cases} A(T_j), & T_j \leq t < T_{j+1}, \\ A(T_N), & t = T, \end{cases}$$

and $\{U_{\Delta}(t, s)\}_{0 \le s \le t \le T}$ is the evolution operator for A_{Δ}

$$U_{\Delta}(t,s) = \begin{cases} \exp(-(t-s)A(T_{j})), & T_{j} \leq s \leq t \leq T_{j+1}, \\ \exp(-(t-T_{j})A(T_{j})) \cdots \exp(-(T_{i+1}-s)A(T_{i})), \\ & T_{i} \leq s \leq T_{i+1} \cdots T_{j} \leq t \leq T_{j+1}. \end{cases}$$

Proposition 2.1. For any $\varepsilon > 0$ and any $y \in F$, there exists a finite partition Δ of [0,T] such that

$$\sup_{0\leq s\leq t\leq T} || \{A(t)-A_{\Delta}(t)\} U_{\Delta}(t,s)y||_{E} \leq \varepsilon.$$

Proof. We define inductively an increasing sequence $\{T_k\}_{k=0,1,2,\dots}$ of [0, T] in the following way. $T_0=0$. Assume that $\{T_j\}_{0\leq j\leq k}$ is defined so that the estimate

$$\sup_{0 \le s \le t \le T_k} || \{A(t) - A_{\Delta_k}(t)\} U_{\Delta_k}(t, s) y ||_E \le \varepsilon$$
(2.1)

holds for the partition Δ_k : $0=T_0 < \cdots < T_k = T_k$ of $[0, T_k]$. If $T_k < T$, we consider a set J_k of all elements $h \in (0, T-T_k]$ such that

$$\sup_{\substack{T_k \leq t < T_k + h \\ 0 \leq \tau < h}} || \{A(t) - A(T_k)\} \exp\left(-\tau A(T_k)\right) z ||_{\mathcal{E}} \leq \varepsilon$$

holds for every

$$z \in L_k = \{U_{\Delta_k}(t,s)y; \ 0 \leq s \leq t \leq T_k\}.$$

Since L_k is compact in F, J_k is non-empty and has the maximum. Putting $h_k = \operatorname{Max} J_k$, we define $T_{k+1} = T_k + h_k$. Then the estimate

$$\sup_{s \le t \le T_{k+1}} || \{A(t) - A_{\Delta_{k+1}}(t)\} U_{\Delta_{k+1}}(t, s) y ||_E \le \varepsilon$$
(2.2)

is valid. In fact, (2.2) is trivial if $t = T_{k+1}$. If $T_{k+1} > t \ge s \ge T_k$, $A_{\Delta_{k+1}}(t) = A(T_k)$ and $U_{\Delta_{k+1}}(t, s)y = \exp(-(t-s)A(T_k))y$. Therefore it follows that

$$||\{A(t)-A_{\Delta_{k+1}}(t)\} U_{\Delta_{k+1}}(t,s)y||_{\mathcal{E}} \leq \sup_{\substack{T_k \leq t < T_k + h_k \\ 0 \leq \tau < h_k}} ||\{A(t)-A(T_k)\} \exp\left(-\tau A(T_k)\right)y||_{\mathcal{E}} \leq \varepsilon$$

If $T_{k+1} > t \ge T_k > s$, $U_{\Delta_{k+1}}(t, s)y = \exp(-(t-T_k)A(T_k))U_{\Delta_k}(T_k, s)y$. Similarly $U_{\Delta_k}(T_k, s)y$ is an element of L_k . Finally if $T_k > t \ge s$, $U_{\Delta_{k+1}}(t, s)y = U_{\Delta_k}(t, s)y$. (2.2) is nothing but the assumption (2.1). Untill T_k reaches T, we continue A. Yagi

the inductive procedure. In order to complete the proof, it remains now to prove that such a procedure finishes within finite times. Suppose the contrary. Then we would have an infinite sequence $\{T_k\}_{k=0,1,2,\cdots}$ of [0, T) satisfying (2.1) for each k. To reach a contradiction we will prove that

$$L = \bigcup_{k=0}^{\infty} L_k$$

is relatively compact in F by using the next lemma essentially due to K. Kobayasi [7].

Lemma 2.2. There exists a constant C_1 such that the estimation

$$\| \prod_{k=1}^{p} \exp (-\tau_{k}A(t_{k}))z - \prod_{k=1}^{q} \exp (-\tau_{k}A(t_{k}))z \|_{F}$$

$$\leq C_{1} \{ \sum_{i=q+1}^{p} \tau_{i} \exp (\tilde{\beta} \sum_{k=r+1}^{i} \tau_{k}) \} \|x\|_{F}$$
(2.3)

$$+C_{1} \exp \left(\tilde{\beta} \sum_{k=r+1}^{p} \tau_{k}\right) ||S(t_{r}) \prod_{k=1}^{r} \exp \left(-\tau_{k} \mathcal{A}(t_{k})\right) z - x||_{E}$$
(2.4)

$$+C_{1}\{(t_{p}-t_{r})+\sum_{k=r+1}^{p}\tau_{k}\} \exp \left(\tilde{\beta}\sum_{k=r+1}^{p}\tau_{k}\right)||x||_{E}$$
(2.5)

holds for any $x, z \in F$, $\tau_k \ge 0$ $(1 \le k \le p)$, $0 \le t_1 \le \cdots \le t_p \le T$, and integers $p \ge q \ge r \ge 1$.

Proof.

$$\begin{split} \prod_{k=1}^{p} \exp((-\tau_{k}A(t_{k}))z - \prod_{k=1}^{q} \exp((-\tau_{k}A(t_{k}))z)z \\ &= \{\prod_{k=r+1}^{p} \exp((-\tau_{k}A(t_{k})) - \prod_{k=r+1}^{q} \exp((-\tau_{k}A(t_{k})))\}S(t_{r})^{-1} \times \\ &\times \{S(t_{r})\prod_{k=1}^{r} \exp((-\tau_{k}A(t_{k}))z - x\} \\ &+ \{\prod_{k=r+1}^{p} \exp((-\tau_{k}A(t_{k})) - \prod_{k=r+1}^{q} \exp((-\tau_{k}A(t_{k})))\}S(t_{r})^{-1}x \\ &= R_{1} + R_{2}. \end{split}$$

 R_1 is estimated by (2.4).

$$\begin{split} R_2 &= S(t_p)^{-1} \{ S(t_p) \prod_{k=r+1}^p \exp\left(-\tau_k A(t_k)\right) S(t_r)^{-1} \\ &\quad -S(t_q) \prod_{k=r+1}^q \exp\left(-\tau_k A(t_k)\right) S(t_r)^{-1} \} x \\ &\quad +S(t_p)^{-1} \{ S(t_q) - S(t_p) \} \prod_{k=r+1}^q \exp\left(-\tau_k A(t_k)\right) S(t_r)^{-1} x \\ &= R_3 + R_4 \,. \end{split}$$

 R_4 is estimated by (2.5).

$$\begin{split} R_{3} &= S(t_{p})^{-1} \{ S(t_{p})_{k=r+1}^{p} \exp\left(-\tau_{k}A(t_{k})\right) S(t_{r})^{-1} - \prod_{k=r+1}^{p} \exp\left(-\tau_{k}A(t_{k})\right) \} x \\ &- S(t_{p})^{-1} \{ S(t_{q})_{k=r+1}^{q} \exp\left(-\tau_{k}A(t_{k})\right) S(t_{r})^{-1} - \prod_{k=r+1}^{q} \exp\left(-\tau_{k}A(t_{k})\right) \} x \\ &+ S(t_{p})^{-1} \{ \prod_{k=r+1}^{p} \exp\left(-\tau_{k}A(t_{k})\right) - \prod_{k=r+1}^{q} \exp\left(-\tau_{k}A(t_{k})\right) \} x \\ &= R_{5} + R_{6} + R_{7} \, . \\ S(t_{p})R_{5} &= \sum_{i=r+1}^{p} \left[\prod_{k=i+1}^{p} \exp\left(-\tau_{k}A(t_{k})\right) \{ S(t_{i}) \exp\left(-\tau_{i}A(t_{i})\right) \\ &- \exp\left(-\tau_{i}A(t_{i})\right) S(t_{i}) \} \prod_{k=r+1}^{i-1} \exp\left(-\tau_{k}A(t_{k})\right) \left[S(t_{r})^{-1} x \\ &+ \sum_{i=r+1}^{p} \left[\prod_{k=i}^{p} \exp\left(-\tau_{k}A(t_{k})\right) \{ S(t_{i}) - S(t_{i-1}) \} \prod_{k=r+1}^{i-1} \exp\left(-\tau_{k}A(t_{k})\right) \right] S(t_{r})^{-1} x \, . \end{split}$$

From this we obtain the estimate of R_5 by (2.5), and similarly that of R_6 . From

$$S(t_{p})R_{7} = \sum_{i=q+1}^{p} \{ \exp(-\tau_{i}A(t_{i})) - I \} \prod_{k=r+1}^{i-1} \exp(-\tau_{k}A(t_{k})) x$$

It follows that R_7 is estimated by (2.3).

Let $T_{\infty} = \lim_{k \to \infty} T_k$. Noting that $U_{\Delta_k}(t, s)y$ coinsides for all k such that $t \leq T_k$, we define

$$U_{\Delta_{\infty}}(t,s)y = \lim_{k \to \infty} U_{\Delta_k}(t,s)y$$

for $0 \leq s \leq t < T_{\infty}$. By the preceding lemma we have the following:

Lemma 2.3. For each $0 \leq s \leq T_{\infty}$ there exists a limit

$$F_{\underset{T_{\infty}>t' \geq s' \geq 0}{(t',s') \to (T_{\infty},s)}} U_{\Delta_{\infty}}(t',s')y.$$
(2.6)

Proof. If $s < T_{\infty}$, $s < T_j < T_{\infty}$ with some j. In this case the limit (2.6) is easily reduced to

$$F-\lim_{t' \to T_{\infty}} U_{\Delta_{\infty}}(t', T_j) z$$
(2.7)

with $z = U_{\Delta_j}(T_j, s)y \in F$. Let $t'' > t' > T_j$ be such that

$$T_{j} < \cdots < T_{j+r-1} < \cdots < T_{j+q-1} \leq t' < T_{j+q} \cdots T_{j+p-2} \leq t'' < T_{j+p-1}$$

with some p > q > r, and apply Lemma 2.2 with

$$t_k = \begin{cases} T_{j+k-1}, & 1 \leq k \leq q, \\ T_{j+k-2}, & q+1 \leq k \leq p, \end{cases}$$

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$$\tau_{k} = \begin{cases} T_{j+k} - T_{j+k-1}, & 1 \leq k \leq q-1 \\ t' - T_{j+q-1}, & k = q \\ T_{j+q} - t', & k = q+1 \\ T_{j+k-1} - T_{j+k-2}, & q+2 \leq k \leq p-1 \\ t'' - T_{j+p-2}, & k = p . \end{cases}$$

Then we get

$$||U_{\Delta_{\infty}}(t'', T_{j})z - U_{\Delta_{\infty}}(t', T_{j})z||_{F}$$

$$\leq C_{1}e^{\tilde{\theta}T}\{(T_{\infty} - T_{j+q-1})||x||_{F} + ||S(t_{r})\prod_{k=1}^{r}\exp(-\tau_{k}A(t_{k}))z - x||_{E}$$

$$+2(T_{\infty} - T_{j+r-1})||x||_{E}\}.$$
(2.8)

For any $\eta > 0$, $T_{\infty} - T_{j+r_0-1} \leq \eta$ with some r_0 , and

$$||S(t_{r_0})\prod_{k=1}^{r_0}\exp(-\tau_kA(t_k))z-x_0||_E \leq \eta$$

with some $x_0 \in F$. $||x_0||_E$ is dominated by

$$||x_0||_E \leq \eta + ||S(t_{r_0}) \prod_{k=1}^{r_0} \exp(-\tau_k A(t_k))z||_E \leq \eta + \tilde{M}e^{\tilde{\beta}T} \sup_t ||S(t)|| ||z||_F.$$

Therefore if $t'' > t' > T_{j+r_0-1}$, (2.8) is smaller than

$$C_2\{(T_{\infty}-T_{j+q-1})||x_0||_F+(1+||z||_F)\eta\}$$
.

If q_0 is large enough for $(T_{\infty} - T_{j+q_0-1})||x_0||_F \leq \eta$, then $t'' > t' > T_{j+q_0-1}$ implies

$$||U_{\Delta_{\infty}}(t'', T_j)z - U_{\Delta_{\infty}}(t', T_j)z||_F \leq C_3(1 + ||z||_F)\eta$$

which shows the existence of (2.7). If $s=T_{\infty}$, we can prove

$$y = F_{-\lim_{(t',s') \to (T_{\infty},T_{\infty})}} U_{\Delta_{\infty}}(t',s')y . \qquad (2.9)$$

Let t' > s' be such that

$$T_j \leq s' < T_{j+1} \cdots T_{j+p-2} \leq t' < T_{j+p-1}$$

with some j and $p \ge 2$, and apply Lemma 2.2 with

$$t_{k} = \begin{cases} T_{j}, & k=1, 2 \\ T_{j+k-2}, & 3 \leq k \leq p, \end{cases}$$

$$\tau_{k} = \begin{cases} 0, & k=1 \\ T_{j+1}-s', & k=2 \\ T_{j+k-1}-T_{j+k-2}, & 3 \leq k \leq p-1 \\ t'-T_{j+p-2}, & k=p, \end{cases}$$

and q = r = 1. Then we get

$$\begin{aligned} ||U_{\Delta_{\infty}}(t',s')y-y||_{F} &\leq C_{4}e^{\beta T}\{(T_{\infty}-T_{j})||x||_{F}+||S(T_{j})y-x||_{E}\}\\ &\leq C_{5}\{(T_{\infty}-T_{j})(||x||_{F}+||y||_{F})+||S(T_{\infty})y-x||_{E}\} \end{aligned}$$

For any $\eta > 0$, $||S(T_{\infty})y - x_0||_E \leq \eta$ with some $x_0 \in F$, and $(T_{\infty} - T_{j_0})(||x_0||_F + ||y||_F) \leq \eta$ with some j_0 . Therefore $t' \geq s' > T_{j_0}$ implies

 $||U_{\Delta_{\infty}}(t',s')y-y||_F \leq C_6 \eta$,

which shows (2.9)

We have known that $U_{\Delta_{\infty}}(t, s)y$ can be extended on $0 \leq s \leq t \leq T_{\infty}$ continuously. Hence L contained in $\{U_{\Delta_{\infty}}(t, s)y; 0 \leq s \leq t \leq T_{\infty}\}$ is a relatively compact set in F.

Lemma 2.4. For any $\eta > 0$ there exists $\delta_1 > 0$ such that

$$\sup_{\substack{|t-T_{\infty}|<\delta_{1}\\ 0\leq\tau<\delta_{1}}}||\exp(-\tau A(t))z-z||_{F}\leq\eta$$

for every $z \in L$.

Proof. Let p=2, q=r=1, $t_1=t_2=t$ and $\tau_1=0$, $\tau_2=\tau$ in Lemma 2.2. Then

$$\begin{aligned} ||\exp(-\tau A(t))z - z||_{F} &\leq C_{7}e^{\beta\tau}\{\tau ||x||_{F} + ||S(t)z - x||_{E}\} \\ &\leq C_{8}e^{\beta\tau}\{\tau ||x||_{F} + ||S(T_{\infty})z - x||_{E} + |t - T_{\infty}|||z||_{F}\} \end{aligned}$$

Since $S(T_{\infty})(L)$ is precompact in E and F is dense in E, $S(T_{\infty})(L)$ can be covered with a finite number of open balls $\{B(y_i; \eta/3C_8e^{\tilde{\beta}})\}_{1 \le i \le l}$ with centers $y_i \in F$. Hence for any $z \in L$

$$||\exp(-\tau A(t))z - z||_{F} \leq C_{8} e^{\tilde{\beta}\tau} \{\tau \max_{1 \leq i \leq l} ||y_{i}||_{F} + |t - T_{\infty}|||z||_{F} \} + e^{\tilde{\beta}(\tau-1)}\eta/3.$$

Similarly for any $\eta > 0$ there exists $\delta_2 > 0$ such that

$$\sup_{|t-T_{\infty}|<\delta_{2}}||\{A(t)-A(T_{\infty})\}z||_{E}\leq \eta$$

for every $z \in L$. Put $h = Min \{\delta_1, \delta_2\}$. Then for $T_k > T_{\infty} - h$ the estimation

$$\sup_{\substack{T_k \leq t < T_k + h \\ 0 \leq \tau < h}} || \{A(t) - A(T_k)\} \exp(-\tau A(T_k))z||_E \\ \leq \sup_{\substack{T_k \leq t < T_k + h \\ 0 \leq \tau < h}} [|| \{A(t) - A(T_k)\}z||_E + C_9 || \exp(-\tau A(T_k))z - z||_F] \\ \leq C_{10} \eta$$

holds for every $z \in L \supset L_k$. This shows $h \in J_k$, if we take $\eta = C_{10}^{-1} \mathcal{E}$. But $h \in J_k$ contradicts

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$$h > T_{\infty} - T_k > T_{k+1} - T_k = h_k$$
.

3. Proof of the theorem

For each integer $n > \tilde{\beta}$, $A_n(t)$ is the Yosida approximation of A(t)

$$A_n(t) = n - n(I + n^{-1}A(t))^{-1}.$$
(3.1)

Lemma 3.1. $A_n \in \mathcal{C}([0, T]; \mathcal{L}_s(E)) \cap \mathcal{C}([0, T]; \mathcal{L}_s(F))$.

Proof. In view of (3.1) it suffices to prove

$$(\lambda + A(\cdot))^{-1} \in \mathcal{C}([0, T]; \mathcal{L}_s(E)) \cap \mathcal{C}([0, T]; \mathcal{L}_s(F))$$
(3.2)

for $\lambda > \tilde{\beta}$. For $x \in F$ we can write

$$\{ (\lambda + A(t+h))^{-1} - (\lambda + A(t))^{-1} \} x = -(\lambda + A(t+h))^{-1} \{ A(t+h) - A(t) \} (\lambda + A(t))^{-1} x \, .$$

Together with the uniform boundness of $||(\lambda + A(\cdot))^{-1}||_E$, this shows that $(\lambda + A(\cdot))^{-1}x$ is continuous in $||\cdot||_E$. For general $x \in E$ it follows from the density of F in E. To see the strong continuity in F of (3.2) we have only to show

$$(\lambda + A(\cdot) + B(\cdot))^{-1} \in \mathcal{C}([0, T]; \mathcal{L}_s(E)), \qquad (3.3)$$

since

$$(\lambda + A(t))^{-1} = S(t)^{-1}(\lambda + A(t) + B(t))^{-1}S(t)$$
(3.4)

on F. But (3.3) follows from

$$(\lambda + A(t) + B(t))^{-1} = (\lambda + A(t))^{-1} \{ I + B(t)(\lambda + A(t))^{-1} \}^{-1}$$
(3.5)

and the strong continuity of $(\lambda + A(\cdot))^{-1}$ in E proved above.

Lemma 3.2. $\{A_n(t)\}_{0 \le t \le T}$ is stable on E (resp. F) with constants of stability $\{M, \beta n(n-\beta)^{-1}\}$ (resp. $\{\tilde{M}, \tilde{\beta}n(n-\tilde{\beta})^{-1}\}$).

Proof. The stability of $\{A_n(t)\}$ is observed directly by

$$(\lambda + A_n(t))^{-1} = \frac{1}{\lambda + n} + \left(\frac{n}{\lambda + n}\right)^2 \left(\frac{\lambda n}{\lambda + n} + A(t)\right)^{-1}.$$

For $n > \tilde{\beta}$ let $\{U_n(t, s)\}_{0 \le s \le t \le T}$ be the evolution operator for $\{A_n(t)\}_{0 \le t \le T}$. From Lemma 3.1 and 3.2 we conclude

$$U_n \in \mathcal{C}(\{(t,s); 0 \le s \le t \le T\}; \mathcal{L}_s(E)), ||U_n(t,s)||_E \le M e^{\beta_n(t-s)}$$
(3.6)

with $\beta_n = \beta n(n-\beta)^{-1}$ and

$$U_{n} \in \mathcal{C}(\{(t,s); 0 \leq s \leq t \leq T\}; \mathcal{L}_{s}(F)), ||U_{n}(t,s)||_{F} \leq \tilde{M}e^{\tilde{\theta}_{n}(t-s)}$$
(3.7)

with $\tilde{\beta}_n = \tilde{\beta} n (n - \tilde{\beta})^{-1}$.

Let $y \in F$ be arbitrarily fixed, ε be any positive number and Δ be the partition of [0, T] satisfying

$$\sup_{0 \le s \le t \le T} || \{A(t) - A_{\Delta}(t)\} U_{\Delta}(t, s) y ||_{E} \le \varepsilon$$
(3.8)

whose existence is guaranteed by Proposition 2.1. We can estimate the difference between $U_n(t, s)y$ and $U_{\Delta}(t, s)y$ by the following:

Proposition 3.3. There exists an integer N such that for any $n \ge N$

$$\sup_{0\leq s\leq t\leq T}||U_n(t,s)y-U_{\Delta}(t,s)y||_E\leq 2MT\varepsilon e^{\beta_n T}$$

Proof.

$$\begin{aligned} \{U_{\Delta}(t,s) - U_{n}(t,s)\} y &= \int_{s}^{t} U_{n}(t,\tau) \{A_{n}(\tau) - A_{\Delta}(\tau)\} U_{\Delta}(\tau,s) y d\tau \\ &= \int_{s}^{t} U_{n}(t,\tau) \{A_{n}(\tau) - A(\tau)\} U_{\Delta}(\tau,s) y d\tau \\ &+ \int_{s}^{t} U_{n}(t,\tau) \{A(\tau) - A_{\Delta}(\tau)\} U_{\Delta}(\tau,s) y d\tau . \end{aligned}$$

The second term is evaluated by (3.8). Hence our proposition follows from the next lemma.

Lemma 3.4. For any compact set K of E, there exists an integer N such that for any $n \ge N$

$$\sup_{0 \le t \le T} ||(I + n^{-1}A(t))^{-1}x - x||_E \le \varepsilon$$

holds for every $x \in K$.

Proof. K is covered with a finite number of open balls $\{B(y_i; \varepsilon/2(M+1))\}_{1 \le i \le l}$ in E with centers $y_i \in F$. Hence for any $x \in K$, taking some y_i ,

$$||(I+n^{-1}A(t))^{-1}x-x||_{E} \leq ||\{(I+n^{-1}A(t))^{-1}-I\}(x-y_{i})||_{E}+||\{(I+n^{-1}A(t))^{-1}-I\}y_{i}||_{E} \leq (\mathcal{E}/2)n(n-\beta)^{-1}+M(n-\beta)^{-1}\max_{1\leq i\leq l}||A(t)y_{i}||_{E}.$$

We can now prove that $\{U_n(t,s)\}_{n>\tilde{\beta}}$ is convergent in $\mathcal{L}_s(E)$ uniformly in (t,s). In fact we have

$$\sup_{0\leq s\leq t\leq T} ||U_m(t,s)y-U_n(t,s)y||_{\mathcal{E}} \leq 2MT\varepsilon(e^{\beta_m T}+e^{\beta_n T})$$

for any $m, n \ge N$ by the mediation of $U_{\Delta}(t, s)y$. $\{U_n(t, s)y\}_{n>\tilde{\beta}}$ is convergent in *E* uniformly in (t, s). Since $y \in F$ was arbitrary and $||U_n(t, s)||_E$ is uniformly bounded by (3.6), $\{U_n(t, s)x\}_{n>\tilde{\beta}}$ is uniformly convergent in E for any $x \in E$. Thus the operator U(t, s) is defined by

$$U(t,s) = \mathcal{L}_{s}(E) - \lim_{n \to \infty} U_{n}(t,s) .$$
(3.9)

Obviously U(t, s) satisfies a) and c). To see the remaining properties we introduce bounded operators on E

$$W_n(t,s) = S(t)U_n(t,s)S(s)^{-1}, \quad 0 \le s \le t \le T,$$

for each $n > \tilde{\beta}$ analogously to [1]. By (3.7)

$$W_n \in \mathcal{C}(\{(t,s); 0 \leq s \leq t \leq T\}; \mathcal{L}_s(E)).$$

 $W_n(t, s)$ is connected with $U_n(t, s)$ by

$$W_n(t,s) - U_n(t,s) = \int_s^t \frac{\partial}{\partial \tau} U_n(t,\tau) S(\tau) U_n(\tau,s) S(s)^{-1} d\tau$$
$$= \int_s^t U_n(t,\tau) C_n(\tau) W_n(\tau,s) d\tau$$

with

$$C_n(t) = A_n(t) - S(t)A_n(t)S(t)^{-1} + \frac{dS}{dt}(t)S(t)^{-1}, \quad 0 \leq t \leq T.$$

Lemma 3.5.

$$\mathcal{L}_{s}(E) - \lim_{n \to \infty} C_{n}(t) = -B(t) + \frac{dS}{dt}(t)S(t)^{-1}$$
(3.10)

uniformly in t.

Proof. Clearly (3.10) is equivalent to

$$\mathcal{L}_{s}(E) - \lim_{n \to \infty} \{ S(t) A_{n}(t) S(t)^{-1} - A_{n}(t) \} = B(t)$$
(3.11)

uniformly in t. By (3.1), (3.4) and (3.5)

(3.11) is reduced to

$$\mathcal{L}_{s}(E) - \lim_{n \to \infty} (I + n^{-1}A(t))^{-1} = I,$$

but this has already been established (Lemma 3.4).

Let $\{W(t, s)\}_{0 \le s \le t \le T}$ be a solution of the integral equation

$$W(t,s) = U(t,s) + \int_{s}^{t} U(t,\tau)C(\tau)W(\tau,s)d\tau$$

in $\mathcal{L}_{s}(E)$ with the kernel (3.10)

$$C(t) = -B(t) + \frac{dS}{dt}(t)S(t)^{-1}$$

Obviously

$$W \in \mathcal{C}(\{(t,s); 0 \leq s \leq t \leq T\}; \mathcal{L}_s(E)).$$

We can deduce from (3.9) and (3.10)

$$W(t,s) = \mathcal{L}_s(E) - \lim_{n \to \infty} W_n(t,s)$$

uniformly in (t, s). In other words

$$S(t)^{-1}W(t,s)S(s) = \mathcal{L}_s(F) - \lim_{n \to \infty} U_n(t,s)$$
(3.12)

uniformly in (t, s). We know that all other properties are immediate consequences of (3.12) ([9]).

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