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THE QUASI KO*-TYPES
OF WEIGHTED MOD 4 LENS SPACES

Dedicated to Professor Fuichi Uchida on his sixtieth birthday

YASUZO NISHIMURA and ZEN-ICHI YOSIMURA

(Received July 4, 1996)

0. Introduction

Let $KU$ and $KO$ be the complex and the real $K$-spectrum, respectively. For any CW-spectrum $X$ its $KU$-homology group $KU_*X$ is regarded as a (Z/2-graded) abelian group with involution because $KU$ possesses the conjugation $\psi \psi 1$. Given CW-spectra $X$ and $Y$ we say that $X$ is quasi $KO*$-equivalent to $Y$ if there exists an equivalence $f : KO \wedge X \to KO \wedge Y$ of $KO$-module spectra (see [8]). If $X$ is quasi $KO*$-equivalent to $Y$, then $KO_*X$ is isomorphic to $KO_*Y$ as a $KO*$-module, and in addition $KU_*X$ is isomorphic to $KU_*Y$ as an abelian group with involution. In the latter case we say that $X$ has the same $C$-type as $Y$ (cf. [2]). In [10] and [11] we have determined the quasi $KO*$-types of the real projective space $\mathbb{RP}_k$ and its stunted projective space $\mathbb{RP}_k/S\mathbb{RP}_1$. Moreover in [12] we have determined the quasi $KO*$-types of the mod 4 lens space $L^4_{n}$ and its stunted lens space $L^4_{n}/L^4_{n-1}$ where we simply denote by $\mathbb{Z}/4n+1$ the usual $(2n+1)$-dimensional mod 4 lens space $L^4_{n}(4)$ and by $L^2_{n}$ its 2n-skeleton $L^2_{n}(4)$. In this note we shall generally determine the quasi $KO*$-types of a weighted mod 4 lens space $L^4_{n}(4; q_0, \ldots, q_n)$ and its 2n-skeleton $L^2_{n}(4; q_0, \ldots, q_n)$ along the line of [12].

The weighted mod 4 lens space $L^4_{n}(4; q_0, \ldots, q_n)$ is obtained as the fiber of the canonical inclusion $i : P^n(q_0, \ldots, q_n) \to P^{n+1}(4, q_0, \ldots, q_n)$ of weighted projective spaces (see [3]). Using the result of Amrani [1, Theorem 3.1] we can calculate the $KU$-cohomology group $KU_*L^4_{n}(4; q_0, \ldots, q_n)$ and the behavior of the conjugation $\psi^{-1}$ on it. Our calculation asserts that $\Sigma L^4_{n}(4; q_0, \ldots, q_n)$ has the same $C$-type as one of the small spectra $\Sigma^2S\mathbb{Z}/2^r \vee P^r_{s,t}$, $S\mathbb{Z}/2^r \vee P^r_{s,t}$ and $PP^r_{s,t}$, and $\Sigma L^4_{n}(4; q_0, \ldots, q_n)$ has the same $C$-type as one of the small spectra $\Sigma^2M_r \vee P^r_{s,t}$, $M_r \vee P^r_{s,t}$, $MPP^r_{s,t}$ and $\Sigma^2m \vee \Sigma L^4_{n}(4; q_0, \ldots, q_n)$ (see Proposition 3.2). Here $S\mathbb{Z}/2^r$ is the Moore spectrum of type $\mathbb{Z}/2^r$ and $M_r$, $P^r_{s,t}$, $PP^r_{s,t}$ and $MPP^r_{s,t}$ are the small spectra constructed as the cofibers of the maps $i \eta : \Sigma^1 \to S\mathbb{Z}/2^r$, $i \bar{\eta} : \Sigma^1S\mathbb{Z}/2^t \to S\mathbb{Z}/2^t$, $i \bar{\eta} + \bar{\eta}j : \Sigma^1S\mathbb{Z}/2^t \to S\mathbb{Z}/2^*\otimes S\mathbb{Z}/2^*$, $(\bar{\eta}j, i \bar{\eta}) : \Sigma^1S\mathbb{Z}/2^t \to S\mathbb{Z}/2^*\otimes S\mathbb{Z}/2^*$ and $(i, M\bar{\eta}, i \bar{\eta}) : \Sigma^1S\mathbb{Z}/2^t \to M_r \vee S\mathbb{Z}/2^*\otimes S\mathbb{Z}/2^*$, respectively, in which $i : \Sigma^0 \to S\mathbb{Z}/2^r$ and $j : S\mathbb{Z}/2^r \to \Sigma^1$ are the bottom cell inclusion and the top cell
projection, \( i_M : \text{SZ}/2^r \to M_r \) is the canonical inclusion, \( \eta : \Sigma^1 \to \Sigma^0 \) is the stable Hopf map, and \( \eta : \Sigma^1 \text{SZ}/2^r \to \Sigma^0 \) and \( \eta : \Sigma^2 \to \text{SZ}/2^r \) are its extension and coextension satisfying \( \eta i = \eta \) and \( j \eta = \eta \).

In [12, Proposition 3.1 and Theorem 3.3] we have already characterized the quasi \( \text{KO}^* \)-types of spectra having the same \( \text{C} \)-type as \( \text{SZ}/2^r \vee \mathcal{P}_s \) or \( \text{M}_r \vee \mathcal{P}_s \), and calculate their \( \text{KO}^* \)-homology groups \( \text{KO}^*X \) (Propositions 1.5 and 1.7). In §1 we introduce some new small spectra \( X \) having the same \( \text{C} \)-type as \( \text{SZ}/2^r \vee \mathcal{P}_s \) or \( \text{M}_r \vee \mathcal{P}_s \), and calculate their \( \text{KO}^* \)-homology groups \( \text{KO}^*X \) (Propositions 1.5 and 1.7). In §2 we shall characterize the quasi \( \text{KO}^* \)-types of spectra having the same \( \text{C} \)-type as \( \text{SZ}/2^r \vee \mathcal{P}_s \) or \( \text{M}_r \vee \mathcal{P}_s \) (Theorems 2.3 and 2.4) by using the small spectra introduced in §1. Our discussion developed in §2 is quite similar to the one done in [6, §4] in order to characterize the quasi \( \text{KO}^* \)-types of spectra having the same \( \text{C} \)-type as \( \text{SZ}/2^r \vee \text{SZ}/2^r \) (see [6, Theorem 5.3]). In §3 we first calculate the \( \text{KU}^* \)-cohomology group \( \text{KU}^0 L^n(4; q_0, \ldots, q_n) \), and then investigate the behavior of the conjugation \( \psi_{C^{-1}} \) on it (Proposition 3.1). Dualizing this result we study the \( \text{C} \)-types of \( L = L^n(4; q_0, \ldots, q_n) \) and \( L_0^n(4; q_0, \ldots, q_n) \) as is stated above (Proposition 3.2), and moreover calculate the sets \( S(L) = \{ 2i; \text{KO}_2 L = 0 \} \) (Lemma 3.3). Since \( \mathcal{P}'_{s,t} \) and \( \Sigma^2 \mathcal{P}'_{s,t+1} \) have the same \( \text{C} \)-type we can apply Theorems 1.2, 1.3, 2.3 and 2.4 with the aid of Proposition 3.2 and Lemma 3.3 to determine the quasi \( \text{KO}^* \)-types of the weighted mod 4 lens spaces \( L^n(4; q_0, \ldots, q_n) \) and \( L_0^n(4; q_0, \ldots, q_n) \) as our main results (Theorems 3.5 and 3.6).

1. **Small spectra having the same \( \text{C} \)-type as \( \text{SZ}/2^r \vee \mathcal{P}_s \) or \( \text{M}_r \vee \mathcal{P}_s \)**

1.1. Let \( \text{SZ}/2^m \) (\( m \geq 1 \)) be the Moore spectrum of type \( Z/2^m \), and \( i : \Sigma^0 \to \text{SZ}/2^r \) and \( j : \text{SZ}/2^m \to \Sigma^1 \) be the bottom cell inclusion and the top cell projection, respectively. The stable Hopf map \( \eta : \Sigma^1 \to \Sigma^0 \) of order 2 admits an extension \( \eta : \Sigma^1 \text{SZ}/2^r \to \Sigma^0 \) and a coextension \( \eta : \Sigma^1 \to \text{SZ}/2^r \) satisfying \( \eta i = \eta \) and \( j j \eta = \eta \). As in [13] (see [8]) we denote by \( M_m, N_m, P_{m,n}, P'_{m,n}, P''_{m,n}, R_m, R'_m, \) and \( K_m \) the small spectra constructed as the cofibers of the following maps:

\[
\begin{align*}
\Sigma^0 &\xrightarrow{2^m-1} C(\eta) \xrightarrow{i_v} V_m \xrightarrow{j_V} \Sigma^1, \quad \Sigma^0 &\xrightarrow{2^m-1} C(\eta) \xrightarrow{h_M} M_m \xrightarrow{k_M} \Sigma^1
\end{align*}
\]

where \( C(\eta) \) and \( C(\eta) \) are the cofibers of the maps \( \eta : \Sigma^1 \to \Sigma^0 \) and \( \eta : \Sigma^1 \text{SZ}/2 \to \Sigma^0 \), and \( i_P : \Sigma^0 \to C(\eta) \) and \( i : \Sigma^0 \to C(\eta) \) are the bottom cell inclusions. Note that \( C(\eta) \) is quasi \( \text{KO}^* \)-equivalent to \( \Sigma^4 \).

Moreover we denote by \( V P_{m,n}, V R_m, V R'_m, V R''_m, V R_m, M P_{m,n}, M P''_{m,n}, M R_{m,n} \) and \( RM_{m,n} \) the small spectra constructed as the cofibers of the following maps:
respectively, where \( i_V : SZ/2^{m-1} \to V_m \) and \( i_M : \Sigma^0 \to M_m \) are the canonical inclusions, and \( \xi_V : \Sigma^5 \to V_m \) is the map satisfying \( j_V(\xi_V) = \tilde{\eta} \) for the canonical projection \( j_V : V_m \to \Sigma^5SZ/2 \). Here we understand \( i_V\tilde{\eta} = i : \Sigma^0 \to SZ/2 \) and \( \xi_V = \tilde{\eta} : \Sigma^0 \to \Sigma^5SZ/2 \) when \( m = 1 \). According to [6, Proposition 3.2] and its dual the spectra \( V\Sigma_{m,n}, P^V_{m,n}, V\Sigma_{m,n}^{\Sigma^5} \) and \( R^V_{m,n} \) \((m \geq 2)\) are quasi \( KO_*\)-equivalent to \( \Sigma^2P_{n+1,m-1}, \Sigma^6P_{n+1,m-1}, \Sigma^2V'N_{m,n} \) and \( \Sigma^6V'N_{m,n} \), respectively. Here the spectrum \( V'N_{m,n} \) is constructed as the cofiber of the map \( \tilde{\eta}j \vee in^2j : \Sigma^2SZ/2^{m-1} \vee \Sigma^5SZ/2^n \to \Sigma^5SZ/2 \), and it is quasi \( KO_*\)-equivalent to \( \Sigma^2V_{m,n} \vee \Sigma^5SZ/2^n \) if \( m \geq n \). The \( S \)-dual spectrum \( NV_{n,m} \) of \( V'N_{m,n} \) and the spectrum \( VR_{m,n} \) have been introduced in [13, Proposition 3.1], and the spectra \( MP_{m,n} \) and \( PM_{m,n} \) were written as \( MV_{m,n} \) and \( V'M_{m,n} \), respectively, in [12, Propositions 2.3 and 2.4]. On the other hand, the spectra \( MR_{m,n} \) and \( RM_{m,n} \) have the same \( C \)-type as \( M_m \vee SZ/2^n \). Note that \( MR_{m,n} \) is quasi \( KO_*\)-equivalent to \( M_m \vee \Sigma^4SZ/2^n \) if \( m \geq n \), and \( RM_{m,n} \) is quasi \( KO_*\)-equivalent to \( SZ/2^m \vee \Sigma^4M_n \) if \( m > n \). By a routine computation we obtain the \( KO \)-homology groups \( KO_iX \) \((0 \leq i \leq 7)\) of \( X = MR_{m,n} \) \((m < n)\) and \( RM_{m,n} \) \((m \leq n)\) as follows:

\[
\begin{array}{c|cccccccc}
X \setminus i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
MR_{m,n} & Z/2^m \oplus Z/2^n & 0 & Z \oplus Z/2^n & Z/2^n \oplus Z/2^{m+1} & Z/2 & Z \oplus (\ast)_n & Z/2 \\
RM_{m,n} & Z/2^n \oplus Z/2^{m+1} & 0 & Z \oplus (\ast)_m & Z/2 & Z/2^{m-1} \oplus Z/2^n & 0 & Z \oplus Z/2^n & Z/2 \\
\end{array}
\]

where \((\ast)_1 \cong Z/4\) and \((\ast)_k \cong Z/2 \oplus Z/2\) if \( k \geq 2 \).

For any maps \( f : \Sigma^4SZ/2^t \to Z_r \) and \( g : \Sigma^4Z_r \to SZ/2^s \) whose cofibers are denoted by \( X_{r,t} \) and \( Y_{s,r} \), we introduce new small spectra \( XP^i_{r,s,t} \) and \( P^iY_{s,t,r} \) constructed as the cofibers of the following maps:

\[
(f, i_V\tilde{\eta}) : \Sigma^iSZ/2^t \to Z_r \vee \Sigma^{i-1}SZ/2^s,
\]

\[
(i\tilde{\eta} \vee g : \Sigma^iSZ/2^t \vee \Sigma^iZ_r \to SZ/2^s,
\]

respectively. In particular, the spectra \( NP^i_{r,s,t} \) and \( PP^i_{r,s,t} \) are written as \( NV_{r,s,t+1} \) and \( PV_{r,s+1} \) in [13, Proposition 3.1], respectively, and \( R^iP^i_{r,s,t} \) are \( KO_*\)-equivalent in [6, Propositions 3.2 and 3.3] the spectra \( \Sigma^4P^i_{r,s,t} \) and \( \Sigma^4P^i_{r,s,t} \) and \( \Sigma^4P^i_{r,s,t} \) and \( \Sigma^4P^i_{r,s,t} \) and \( \Sigma^4P^i_{r,s,t} \) respectively. On the other hand, the spectrum \( VRP^i_{r,s,t} \) is quasi \( KO_*\)-equivalent to \( R^iR_{r,s+1} \) and \( V_r \vee \Sigma^4V_{s+1} \) according as \( r > s + 1, r = s + 1 \) or \( r \leq s \), and the spectrum \( P^iR^i_{r,t} \) is quasi \( KO_*\)-equivalent
to $\Sigma^4 R_{s+1,r}, R'_r R_{s+1,r}$ or $V_{s+1} V \Sigma^4 V_r$ according as $r > s + 1$, $r = s + 1$ or $r \leq s$.

Here the spectrum $R'_r R_m,n$ has been introduced in [13, Proposition 3.3]. The spectra $PP'_{r,s,t}, VPP'_{r,s,t}, P'_t P_{s,t,r}, MMP'_{r,s,t}$ and $PP'M_{s,t,r}$ were written as $U_{s,r,t}, V_{s,r,t}, U'_s t, r, M U'_{s,t,r}$ and $U'M_{s,t,r}$ in [12], respectively, and their KU-homology groups with the conjugation $\psi_C^{-1}$ and their KO-homology groups have been obtained in [12, Propositions 2.1, 2.2, 2.3 and 2.4].

**Proposition 1.1.**

i) "The $X = PP'_{r,s,t}$ or $VPP'_{r,s,t}$ case"

<table>
<thead>
<tr>
<th>$r &gt; t &gt; s$</th>
<th>$r \geq t \leq s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$KU_0 X \cong \mathbb{Z}/2^r \oplus \mathbb{Z}/2^t \oplus \mathbb{Z}/2^s$</td>
<td>$\mathbb{Z}/2^r \oplus \mathbb{Z}/2^t \oplus (\mathbb{Z}/2^s \oplus \mathbb{Z}/2^{s+1})$</td>
</tr>
<tr>
<td>$\psi_C^{-1} = \begin{pmatrix} 1 &amp; 2^{r-t} &amp; 0 \ 0 &amp; -1 &amp; 0 \ 0 &amp; -1 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 &amp; 2^{r-t+1} &amp; -2^{r-t} \ 0 &amp; -1 &amp; 1 \ 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
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</tbody>
</table>

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<tr>
<th>$r \leq t &gt; s$</th>
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<tbody>
<tr>
<td>$KU_0 X \cong \mathbb{Z}/2^{r-1} \oplus \mathbb{Z}/2^{t+1} \oplus \mathbb{Z}/2^s$</td>
<td>$\mathbb{Z}/2^{r-1} \oplus \mathbb{Z}/2^t \oplus \mathbb{Z}/2^{s+1}$</td>
</tr>
<tr>
<td>$\psi_C^{-1} = \begin{pmatrix} 1 &amp; 0 &amp; 0 \ -2^{r-t+2} &amp; -1 &amp; 0 \ -2^{r-t+1} &amp; -1 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ -2^{r-t+1} &amp; -1 &amp; 1 \ 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
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</table>

ii) "The $X = MMP'_{r,s,t}$ case"

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<th>$r \geq t \leq s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$KU_0 X \cong \mathbb{Z} \oplus \mathbb{Z}/2^r \oplus \mathbb{Z}/2^t \oplus \mathbb{Z}/2^s$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2^r \oplus \mathbb{Z}/2^{t-1} \oplus \mathbb{Z}/2^{s+1}$</td>
</tr>
<tr>
<td>$\psi_C^{-1} = \begin{pmatrix} -1 &amp; 0 &amp; 0 &amp; 0 \ -1 &amp; 1 &amp; 2^{r-t} &amp; 0 \ 0 &amp; 0 &amp; -1 &amp; 0 \ 0 &amp; 0 &amp; -1 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} -1 &amp; 0 &amp; 0 &amp; 0 \ -1 &amp; 1 &amp; 2^{r-t+1} &amp; -2^{r-t} \ 0 &amp; 0 &amp; -1 &amp; 1 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

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<tr>
<th>$r \leq t &gt; s$</th>
<th>$r \leq t \leq s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$KU_0 X \cong \mathbb{Z} \oplus \mathbb{Z}/2^{r-1} \oplus \mathbb{Z}/2^{t+1} \oplus \mathbb{Z}/2^s$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2^{r-1} \oplus \mathbb{Z}/2^t \oplus \mathbb{Z}/2^{s+1}$</td>
</tr>
<tr>
<td>$\psi_C^{-1} = \begin{pmatrix} -1 &amp; 0 &amp; 0 &amp; 0 \ -1 &amp; 1 &amp; 0 &amp; 0 \ 2^{t-r+1} &amp; -2^{t-r+2} &amp; -1 &amp; 0 \ 0 &amp; -2^{t-r+1} &amp; -1 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} -1 &amp; 0 &amp; 0 &amp; 0 \ -1 &amp; 1 &amp; 0 &amp; 0 \ 2^{t-r} &amp; -2^{t-r+1} &amp; -1 &amp; 1 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
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</tbody>
</table>

iii) Their KO-homology groups KO$_i X$ ($0 \leq i \leq 7$) are tabled as follows:

<table>
<thead>
<tr>
<th>$X \setminus i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$PP'_{r,s,t}$</td>
<td>$\mathbb{Z}/2^r \oplus \mathbb{Z}/2^s \oplus \mathbb{Z}/2^{(*)_{l-1,r}} \oplus \mathbb{Z}/2^l \oplus \mathbb{Z}/2^{s+1} \oplus \mathbb{Z}/2^{l+1}$</td>
<td>0</td>
<td>$\mathbb{Z}/2^l$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$VPP'_{r,s,t}$</td>
<td>$\mathbb{Z}/2^{r-1} \oplus \mathbb{Z}/2^s \oplus \mathbb{Z}/2^t \oplus \mathbb{Z}/2^s \oplus \mathbb{Z}/2^{s+1} \oplus \mathbb{Z}/2^l$</td>
<td>0</td>
<td>$\mathbb{Z}/2^l$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$MPP'_{r,s,t}$</td>
<td>$\mathbb{Z}/2^r \oplus \mathbb{Z}/2^s \oplus \mathbb{Z}/2^t \oplus \mathbb{Z}/2^l \oplus \mathbb{Z}/2^s \oplus \mathbb{Z}/2^{s+1} \oplus \mathbb{Z}/2^l$</td>
<td>0</td>
<td>$\mathbb{Z}/2^l$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where $(*)_{k,l} \cong \mathbb{Z}/2^{k+2}$ and $(*)_{k,l} \cong \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2$ if $l \geq 2$. 
For any spectrum $X$ having the same $C$-type as $PP'_{r,s,t}$ or $MPP'_{r,s,t}$ we have already determined its quasi $KO_*$-type in [12, Theorem 3.3].

**Theorem 1.2.**

i) If a spectrum $X$ has the same $C$-type as $PP'_{r,s,t}$, then it is quasi $KO_*$-equivalent to one of the following small spectra $PP'_{r,s,t}$, $\Sigma^4PP'_{r,s,t}$ and $\Sigma^4VPP'_{r,s,t}$.

ii) If a spectrum $X$ has the same $C$-type as $MPP'_{r,s,t}$, then it is quasi $KO_*$-equivalent to either of the small spectra $MPP'_{r,s,t}$ and $\Sigma^4MPP'_{r,s,t}$.

Applying Theorem 1.2 we see that (1.4) the spectra $P'P_{s,t,r}$, $P'P'_{s,t,r}$ and $P'P'_{s,t,r}$ are quasi $KO_*$-equivalent to $\Sigma^2PP_{r,t-1,s}$, $\Sigma^2VPP_{r,t-1,s}$ and $\Sigma^2MPP_{r,t-1,s}$, respectively (see [12, Corollary 3.4]).

We can also show the following result (see [12, Proposition 3.1]).

**Theorem 1.3.**

i) If a spectrum $X$ has the same $C$-type as $SZ/2^r \vee P''_{s,t}$, then it is quasi $KO_*$-equivalent to one of the following wedge sums $SZ/2^r \vee P''_{s,t}$, $\Sigma^4SZ/2^r \vee P''_{s,t}$, $V_r \vee P''_{s,t}$ and $\Sigma^4V_r \vee P''_{s,t}$.

ii) If a spectrum $X$ has the same $C$-type as $M_r \vee P''_{s,t}$, then it is quasi $KO_*$-equivalent to either of the following wedge sums $M_r \vee P''_{s,t}$ and $\Sigma^4M_r \vee P''_{s,t}$.

1.2. Since $P''_{s,t}$ and $\Sigma^2P_{t-1,s+1}$ have the same $C$-type a routine computation shows

**Proposition 1.4.**

i) The spectra $NP'_{r,s,t}$, $VRP'_{r,s,t}$, $RP'_{r,t-1,s+1}$, $\vee RP'_{r,s,t}$ and $P'R_{r,s,t,r}$ have the same $C$-type as the wedge sum $SZ/2^r \vee P''_{s,t}$.

ii) The spectra $MRP'_{r,t-1,s+1}$ and $P'RM_{s,t,r}$ have the same $C$-type as the wedge sum $M_r \vee P''_{s,t}$.

Note that if $r \geq t$ the spectra $RP'_{r,s,t}$, $VRP'_{r,s,t}$ and $MRP'_{r,s,t}$ are quasi $KO_*$-equivalent to $SZ/2^r \vee \Sigma^2P'_{s,t}$, $V_r \vee \Sigma^2P'_{s,t}$ and $M_r \vee \Sigma^2P'_{s,t}$, respectively, and if $r \leq s$ the spectra $P''R_{s,t,r}$, $P'R'_{s,t,r}$ and $P''RM_{s,t,r}$ are quasi $KO_*$-equivalent to $\Sigma^4SZ/2^r \vee P''_{s,t}$, $\Sigma^4V_r \vee P''_{s,t}$ and $\Sigma^4M_r \vee P''_{s,t}$, respectively. By use of [13, Propositions 2.2 and 3.1] and (1.2) we can easily calculate

**Proposition 1.5.** For the small spectra $X$ listed in Proposition 1.4 the $KO$-homology groups $KO_iX$ ($0 \leq i \leq 7$) are tabulated as follows:
Let $N'_t$, $P'_t$ and $R'_t$ denote the small spectra constructed as the cofibers of the following maps $\eta^2 j, \tilde{\eta} : \Sigma^1 S\mathbb{Z}/2^t \to \Sigma^0$ and $\eta^2 \tilde{\eta} : \Sigma^3 S\mathbb{Z}/2^t \to \Sigma^0$, respectively. Consider the small spectrum $N'P'_t$ constructed as the cofiber of the map $(\eta^2 j, \tilde{\eta}) : \Sigma^1 S\mathbb{Z}/2^t \to \Sigma^0 \vee \Sigma^0$. Then we have two maps $i^t_{N'P} : \Sigma^0 \to N'P'_t$ and $\rho^t_{N'P} : \Sigma^0 \to N'P'_t$ whose cofibers are $N'_t$ and $P'_t$, respectively. These two maps are related by the equality $i^t_{N'P} \tilde{\eta} = \rho^t_{N'P} \eta^2 j$. In particular, $i^t_{N'P} = (2, \tilde{1}) : \Sigma^0 \to \Sigma^0 \vee C(\tilde{\eta})$ and $\rho^t_{N'P} = (1, 0) : \Sigma^0 \to \Sigma^0 \vee C(\tilde{\eta})$ when $t = 1$. We denote by $N'_t r_t, P'_t s_t, t$ and $F'_{t,n,m}$ the spectra constructed as the cofibers of the following maps $2^r \rho^t_{N'P'} + 2^s t_{N'P} \beta$ and $F'_{t,n,m} = 2^m \rho^t_{N'P'} + 2^m t_{N'P} : \Sigma^0 \to N'P'_t$, respectively. In particular, $N'_t r_t = C(\tilde{\eta}) \vee \Sigma^2 S\mathbb{Z}/2^t$ and $P'_{t,s+1,1} = \Sigma^4 R'_{t+1}$ if $n = m + 1$. Whenever $t \geq 2$ we can regard that the induced homomorphisms $\rho^t_{N'P'} : KU_* \to KU_0 N'P'_t$ and $i^t_{N'P'} : KU_0 \to KU_0 N'P'_t$ are
given by \( \rho'_Np'(1) = (1,0,0) \) and \( i'_Np'(1) = (0,2,1) \) in \( KU_0N'P'_t \cong Z \oplus Z \oplus Z/2^{t-1} \) because \( i'_NP \) may be replaced by \( i'_NP + 2q\rho'_NP \) if necessary. Hence it is easily shown that

(1.5) i) the spectra \( N'P'_{t,t} \) and \( P'N'_{t,t} \) have the same \( C \)-type as \( SZ/2^r \vee P'_t \) and \( \Sigma^0 \vee P'_{s,t} \), respectively, and

ii) the spectrum \( F^n_{*,m} \) has the same \( C \)-type as \( SZ/2^r \vee P'_t \) when \( n \leq m \), and as \( \Sigma^0 \vee P'_{m,m} \) when \( n > m \).

By use of [8, Proposition 4.2] and [9, Proposition 2.4] we can easily calculate the \( KO \)-homology groups \( KO_*X \) (0 \( \leq i \leq 7 \)) of \( X = N'P'_{r,t} \), \( P'N'_{s,t} \) and \( F^n_{*,m} \) (\( t \geq 2 \)) as follows:

\[
\begin{array}{cccccccc}
X \backslash i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
N'P'_{r,t} & Z + Z/2^r & Z/2 & Z/2^r \oplus Z/2 & Z/2 & Z \oplus Z/2^{r+1} & Z/2 & Z/2^t & 0 \\
P'N'_{s,t} & Z + Z/2^r & Z/2 & Z/2^r \oplus Z/2 & Z/2 & Z \oplus Z/2^{r+1} & Z/2 & Z/2^t & 0 \\
F^n_{*,m} & Z + Z/2^r & Z/2 & Z/2^r \oplus Z/2 & Z/2 & Z \oplus Z/2^{r+1} & Z/2 & Z/2^t & 0 \\
\end{array}
\]

where \( l = \min\{n, m\} \).

Choose two maps \( h'_N : \Sigma^2 \to N'_{t} \) and \( \bar{\rho}'_N : \Sigma(\bar{\eta}) \to N'_t \) whose cofibers coincide with \( C(\eta^2) \) and \( V'_{t} \), respectively, where \( C(\eta^2) \) is the cofiber of the map \( \eta^2 : \Sigma^2 \to \Sigma^0 \) and \( V'_t = P_{1,t-1} \) which is quasi \( KO_* \)-equivalent to \( \Sigma^0V_t \) (see [13]). Then there exist two maps \( \lambda'_N : \Sigma(\eta) \to N'P'_t \) and \( \bar{\rho}'_N : \Sigma(\eta) \to N'P'_t \) satisfying \( j'_N \lambda'_N = h'_N \eta \bar{\rho}'_N \) and \( j'_N \bar{\rho}'_N = \bar{\rho}'_N \) for the canonical projection \( j'_N : N'P'_t \to N'_t \). In particular, we may choose as \( \lambda'_N = (\bar{\lambda}, 2) : \Sigma(\eta) \to \Sigma^0 \vee C(\eta) \) and \( \bar{\rho}'_N = (0,1) : \Sigma(\eta) \to \Sigma^0 \vee C(\eta) \) when \( t = 1 \). Here the map \( \bar{\lambda} : C(\eta) \to \Sigma^0 \) satisfies the equalities \( \bar{\lambda}_{t} = 4 \) and \( i_{t} \bar{\lambda} = 4 \) (see [13, (1.3)]). Whenever \( t \geq 2 \), we can regard that the induced homomorphisms \( \bar{\rho}'_{N*,\eta} \) and \( \lambda'_N : \Sigma(\eta) \to \Sigma^0 \vee C(\eta) \) are given by \( \bar{\rho}'_{N,\eta}(1) = (1,0,0) \) and \( \lambda'_N(1) = (0,2,1) \) in \( KU_0N'P'_t \cong Z \oplus Z \oplus Z/2^{t-1} \) because \( \bar{\rho}'_N \) and \( \lambda'_N \) may be replaced by \( \bar{\rho}'_{N,\eta} + ki'_P \bar{\lambda} \) and \( \lambda'_N + li'_P \bar{\lambda} \) if necessary. By virtue of [13, Lemma 1.5] we obtain that the cofiber of \( \bar{\rho}'_N \) is quasi \( KO_* \)-equivalent to \( \Sigma^tP'_t \). On the other hand, by use of [13, Lemma 1.2 and Proposition 4.1] (or [9, Theorem 4.2]) we see that the cofiber of \( \lambda'_N \) is quasi \( KO_* \)-equivalent to \( \Sigma^tP'_t \). More generally, the cofibers of the maps \( 2^r\bar{\rho}'_{NP} \) and \( 2^r\lambda'_N \) are quasi \( KO_* \)-equivalent to \( \Sigma^4N'P'_{t,t} \) and \( \Sigma^4P'N'_{s,t} \), respectively, because \( N'P'_t \) and \( \Sigma^4N'P'_{t,t} \) have the same quasi \( KO_* \)-type (see [9, Corollary 4.5]).

Using the maps \( f^n_{*,m} \), \( \bar{\rho}'_N \) and \( \lambda'_N \) we introduce new small spectra \( N'P'F^n_{r,t} \) and \( P'N'F^n_{s,t} \) constructed as the cofibers of the following maps

\[
\begin{align*}
\bar{\rho}'_N : \Sigma^0 \vee C(\eta) &\to N'P'_t, \\
\lambda'_N : \Sigma^0 \vee C(\eta) &\to N'P'_t,
\end{align*}
\]

respectively. In particular, \( N'P'F^n_{r,1} \) is equal to \( (C(\eta) \wedge SZ/2^r) \vee SZ/2^m \) if \( n \leq m \), to \( (C(\eta) \wedge SZ/2^r) \vee SZ/2^{m+2} \) if \( n = m > r \), and to \( (C(\eta) \wedge SZ/2^r) \vee SZ/2^{m+1} \)
if \( n > m + 1 > r \). Moreover it is quasi \( KO_* \)-equivalent to \( \Sigma^4 \! V_{r+1} \lor V_{m+1}, \Sigma^4 R_{r,m+2} \) or \( \Sigma^4 R'_{r,m+1} \) according as \( n = m + 1 < r, n = m + 1 = r \) or \( n > m + 1 \leq r \) (use [6, Proposition 3.1]). On the other hand, \( P^i N' F_{s,t}^{n,m} \) is just \( R'_{n,m+1,s+1} \) introduced in [13]. By a routine computation we can easily show

**Proposition 1.6.**

i) The spectrum \( N' P^i F_{r,t}^{n,m} \) \((t \geq 2)\) has the same \( C \)-type as \( SZ/2^n \lor P^t_{m-n+r,t} \) if \( m \geq n < r \), and as \( SZ/2^n \lor P^t_{m,t} \) if otherwise.

ii) The spectrum \( P^i N' F_{s,t}^{n,m} \) \((t \geq 2)\) has the same \( C \)-type as \( SZ/2^n \lor P^t_{s,t} \) if \( n > m \leq s \), and as \( SZ/2^n \lor P^t_{s,t} \) if otherwise.

Using (1.6) we can easily calculate

**Proposition 1.7.** For the spectra \( X = N' P^i F_{r,t}^{n,m} \) and \( P^i N' F_{s,t}^{n,m} \) \((t \geq 2)\) the \( KO \)-homology groups \( KO_i X \) \((0 \leq i \leq 7)\) are tabled as follows:

<table>
<thead>
<tr>
<th>( i ) ( \setminus ) ( X )</th>
<th>( N' P^i F_{r,t}^{n,m} )</th>
<th>( P^i N' F_{s,t}^{n,m} )</th>
</tr>
</thead>
</table>
| 0 \( \{ \)
| \( Z/2^n \lor Z/2^m \) \((m \geq n < r)\) | \( Z/2^n \lor Z/2^{m+1} \) \((n > m \leq s)\) |
| \( Z/2^n \lor Z/2^{m+1} \) \((m \geq n < r)\) | \( Z/2^n \lor Z/2^{m+1} \) \((n > m \leq s)\) |
| 1 | \( Z/2 \) | \( Z/2 \) |
| 2 | \( Z/2 \lor Z/2 \) | \( Z/2 \lor Z/2 \) |
| 3 | \( Z/2 \) | \( Z/2 \) |
| 4 \( \{ \)
| \( Z/2^n \lor Z/2^{m+1} \) \((m \geq n < r)\) | \( Z/2^n \lor Z/2^{m+1} \) \((n > m \leq s)\) |
| \( Z/2^n \lor Z/2^{m+1} \) \((m \geq n < r)\) | \( Z/2^n \lor Z/2^{m+1} \) \((n > m \leq s)\) |
| 5 | \( Z/2 \) | \( Z/2 \) |
| 6 | \( Z/2 \lor Z/2 \) | \( Z/2 \lor Z/2 \) |
| 7 | \( Z/2 \) | \( Z/2 \) |

2. The same quasi \( KO_* \)-type as \( SZ/2^n \lor P^t_{s,t} \) or \( M_r \lor P^t_{s,t} \)

2.1. Let \( X \) be a spectrum having the same \( C \)-type as \( SZ/2^n \lor P^t_{s,t} \). Then its self-conjugate \( K \)-homology group \( KC_i X \) \((0 \leq i \leq 3)\) is given as follows:

\[
KC_i X \cong Z/2^n \lor Z/2^s \lor Z/2, Z/2^n \lor Z/2^{s+1}, Z/2 \lor Z/2 \lor Z/2^{t-1}, Z/2 \lor Z/2^t
\]

according as \( i = 0, 1, 2, 3 \). In addition,

\[
KO_1 X \lor KO_3 X \cong Z/2 \lor Z/2 \quad \text{and} \quad KO_2 X \lor KO_4 X \cong Z/2 \lor Z/2 .
\]

Hence \( KO_{2i+1} X \) \((0 \leq i \leq 3)\) are divided into the nine cases \((A,D)\) with \( A=a, b, c \)
and D=d, e, f as follows:

\[(a) \ KO_1X \cong KO_5X \cong Z/2 \quad (b) \ KO_5X = 0 \quad (c) \ KO_1X = 0 \quad (d) \ KO_3X \cong KO_7X \cong Z/2 \quad (e) \ KO_7X = 0 \quad (f) \ KO_3X = 0. \]

(2.1)

The induced homomorphisms \((-\tau, \tau\pi_C)_*: KC_iX \to KO_{i+1}X \oplus KO_{i+5}X \ (i = 0, 2)\) are represented by the following matrices

\[
\Phi_0 = \varphi_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} : Z/2^r \oplus Z/2^s \oplus Z/2 \to Z/2 \oplus Z/2
\]

\[
\Phi_2 = \varphi_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : Z/2 \oplus Z/2^s \oplus Z/2^{t-1} \to Z/2 \oplus Z/2,
\]

respectively, where \(\varphi_0, \varphi_2 : Z/2^r \oplus Z/2 \to Z/2 \oplus Z/2\) is one of the following matrices:

\[
(2.2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Evidently it is sufficient to take as \(\varphi_0\) or \(\varphi_2\) only the matrix (1) in case of (b), (c), (e) or (f). By using a suitable transformation of \(KU_0X\) similarly to [6, 4.1] we can verify that in case of (a) the matrix (1) as \(\varphi_0\) is replaced by (5), and in case of (d) the matrix (1) as \(\varphi_2\) is replaced by (5) if \(r \leq s\), and by (3) if \(r > s\). Therefore it is sufficient to take as \(\varphi_0\) the matrices (1), (2) and (3) in case of (a), and as \(\varphi_2\) the matrices (1), (2) and (3) in case of (d) and \(r \leq s\), and (1), (2) and (6) in case of (d) and \(r > s\).

Let \(X\) be a spectrum having the same C-type as \(M_r \vee P_{s,t}\). Then its self-conjugate \(K\)-homology group \(KC_iX\ (0 \leq i \leq 3)\) is given as follows:

\[
KC_iX \cong Z/2^r \oplus Z/2^s \oplus Z/2, Z/2^{s+1} \oplus Z/2^{s+1}, Z \oplus Z/2 \oplus Z/2 \oplus Z/2^{t-1}, Z \oplus Z/2^t,
\]

according as \(i = 0, 1, 2, 3\). In addition,

\[
KO_1X \oplus KO_5X \cong Z/2 \quad \text{and} \quad KO_3X \oplus KO_7X \cong Z/2 \oplus Z/2.
\]

Hence \(KO_{2i+1}X\ (0 \leq i \leq 3)\) are divided into the six cases (A, D) with A = b, c and D = d, e, f given in (2.1). The induced homomorphisms \((-\tau, \tau\pi_C)_*: KC_iX \to KO_{i+1}X \oplus KO_{i+5}X \ (i = 0, 2)\) are represented by the following matrices

\[
\Phi_0 = (0, 0, 1) : Z/2^r \oplus Z/2^s \oplus Z/2 \to Z/2
\]

\[
\Phi_2 = \varphi_2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} : Z \oplus Z/2 \oplus Z/2 \oplus Z/2^{t-1} \to Z/2 \oplus Z/2,
\]
where $\varphi_2 : \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is one of the matrices given in (2.2). Evidently it is sufficient to take as $\varphi_2$ only the matrix (1) in case of (e) or (f). On the other hand, it is sufficient to take as $\varphi_2$ the matrices (1), (2) and (3) in case of (d) and $r \leq s$, and (1), (2) and (6) in case of (d) and $r > s$.

Given a spectrum $X$ having the same $C$-type as $SZ/2^r \vee P'_{s,t}$ or $M_r \vee P'_{s,t}$ we define its $\varphi$-type $(A, D, i, j)$ where $A = a, b, c$, $D = d, e, f$ and $1 \leq i, j \leq 6$, using the above notations as in [6, §4].

**Lemma 2.1.**

i) Let $X$ be a spectrum having the same $C$-type as $SZ/2^r \vee P'_{s,t}$. Then its $\varphi$-type is one of the following 25 types: $(a, d, 1, j)$, $(a, e, i, 1)$, $(a, f, i, 1)$, $(b, d, 1, j)$, $(c, d, 1, j)$, $(c, f, 1, 1)$ where $i = 1, 2, 3$, and $j = 1, 2, 3$ if $r \leq s$ and $j = 1, 2, 6$ if $r > s$.

ii) Let $X$ be a spectrum having the same $C$-type as $M_r \vee P'_{s,t}$. Then its $\varphi$-type is one of the following 10 types: $(b, d, 1, j)$, $(c, d, 1, j)$, $(b, e, 1, 1)$, $(b, f, 1, 1)$, $(c, e, 1, 1)$ and $(c, f, 1, 1)$ where $j = 1, 2, 3$ if $r < s$ and $j = 1, 2, 6$ if $r > s$.

2.2. Using [6, Lemmas 4.2 and 4.3] we can easily determine the $\varphi$-types of the small spectra appearing in Propositions 1.3 and 1.5.

**Proposition 2.2.**

i) The spectra $NP'_{r,s,t}$, $VRP'_{r,s,t}$ ($t \geq 2$), $RP'_{r,t-1,s+1}$, $RP'_{r,t-1,s+1}$, $P'R_{s,t,r}$ and $P'R_{s,t,r}$ ($t \geq 2$) have the following $\varphi$-types $(a, d, 4, 1)$, $(a, d, 4, 1)$, $(a, d, 4, 1)$, $(c, d, 1, 3)$, $(c, d, 1, 3)$, $(c, d, 1, 6)$ and $(a, d, 1, 6)$, respectively.

ii) The spectra $MRP'_{r,t-1,s+1}$ and $P'RM_{s,t,r}$ have the following $\varphi$-types $(c, d, 1, 3)$ and $(c, d, 1, 6)$, respectively.

iii) The spectrum $NP'PF_{r,t}^{m,n}$ ($t \geq 2$) has the following $\varphi$-type $(a, d, 3, 1)$, $(a, d, 3, 1)$ or $(a, d, 4, 2)$ according as $m \geq n < r$, $m \geq n = r$ or otherwise, and the spectrum $P'NP'F_{r,t}^{m,n}$ ($t \geq 2$) has the following $\varphi$-type $(a, d, 3, 2)$, $(a, d, 4, 6)$ or $(a, d, 4, 1)$ according as $n > m < s$, $n > m = s$ or otherwise.

Let $X$ be a spectrum having the same $C$-type as $SZ/2^r \vee P'_{s,t}$ or $M_r \vee P'_{s,t}$. If a spectrum $Y$ has the same $C$-type as $X$, then we can choose a quasi $KU$-equivalence $f : Y \to KU \wedge X$ with $(\psi_{C}^{-1} \wedge 1)f = f$. If there exists a map $h : Y \to KO \wedge X$ satisfying $(\epsilon_U \wedge 1)h = f$ for the complexification map $\epsilon_U : KO \to KU$, then $h$ becomes a quasi $KO$-equivalence (see [8, Proposition 1.1]). After choosing a suitable small spectrum $Y$ having the same $\varphi$-type as $X$ we can prove the following theorems by applying the same method developed in [6], [8] or [9].

**Theorem 2.3.** Let $X$ be a spectrum having the same $C$-type as $SZ/2^r \vee P'_{s,t}$ ($t \geq 2$). Then it is quasi $KO$-equivalent to one of the following small spectra (cf. [6, The-
i) The case of \( r \leq s \): \( Y_r \vee Y_{s,t}, N P^r_{r,s,t}, \Sigma^4 N P^r_{r,s,t}, V R P^r_{r,s,t-1,s+1}, \Sigma^4 V R P^r_{r,s,t-1,s+1}, V R P^r_{r,s,t}, \Sigma^4 V R P^r_{r,s,t}, R P^r_{r,s,t-1,s+1}, \Sigma^4 R P^r_{r,s,t-1,s+1}, N P^r_{r,s,t} \).

ii) The case of \( r > s \): \( Y_r \vee Y_{s,t}, N P^r_{r,s,t}, \Sigma^4 N P^r_{r,s,t}, \Sigma^4 P^r_{s,t,r}, \Sigma^4 P^r_{s,t,r}, P^r_{s,t,r} \).

Here \( Y_r = \frac{S Z}{2^r}, \Sigma^4 S Z/2, V_r \) or \( \Sigma^4 V_r \), and \( Y_{s,t} = P^r_{s,t}, \Sigma^4 P^r_{s,t}, \Sigma^2 P^r_{s,t-1,s+1} \) or \( \Sigma^6 P^r_{s,t-1,s+1} \).

**Theorem 2.4.** Let \( X \) be a spectrum having the same \( C \)-type as \( M_r \vee P^r_{s,t} \). Then it is quasi \( K_{O^*} \)-equivalent to one of the following small spectra:

i) The case of \( r \leq s \): \( Y_r \vee Y_{s,t}, M R P^r_{r,s,t-1,s+1}, \Sigma^4 M R P^r_{r,s,t-1,s+1} \).

ii) The case of \( r > s \): \( Y_r \vee Y_{s,t}, P^r R M_{s,t,r}, \Sigma^4 P^r R M_{s,t,r} \).

Here \( Y_r = M_r \) or \( \Sigma^4 M_r \), and \( Y_{s,t} = P^r_{s,t}, \Sigma^4 P^r_{s,t}, \Sigma^2 P^r_{s,t-1,s+1} \) or \( \Sigma^6 P^r_{s,t-1,s+1} \).

Combining Theorem 2.3 with Proposition 2.2 iii) we get

**Corollary 2.5.**

i) The spectrum \( N^l P^l P^m_{r,t} \) (\( t \geq 2 \)) is quasi \( K_{O^*} \)-equivalent to \( V R P^l_{n,m-n+r,l} \) if \( m \geq n \leq r \), and to \( \Sigma^4 V R P^l_{n,m,t} \) if \( m \geq n \leq r \) or \( m < n \).

ii) The spectrum \( P^l N^l P^m_{s,t} \) (\( t \geq 2 \)) is quasi \( K_{O^*} \)-equivalent to \( \Sigma^4 V R P^l_{n-m+s,m,t} \) if \( n > m \leq s \), and to \( V R P^l_{n,s,t} \) if \( n > m \leq s \) or \( m \leq n \).

3. Weighted mod 4 lens spaces

3.1. Let \( S^{2n+1}(q_0, \ldots, q_n) \) denote the unit sphere \( S^{2n+1} \subset C^{n+1} \) with \( S^1 \)-action defined by \( \lambda \cdot (x_0, \ldots, x_n) = (\lambda q_0 x_0, \ldots, \lambda q_n x_n) \in C^{n+1} \) for any \( \lambda \in S^1 \subset C \). Then we set

\[
P^n(q_0, \ldots, q_n) = S^{2n+1}(q_0, \ldots, q_n)/S^1
\]

\[
L^n(q; q_0, \ldots, q_n) = S^{2n+1}(q_0, \ldots, q_n)/(Z/q)
\]

where \( Z/q \) is the \( q \)-th roots of the unity in \( S^1 \subset C \). Denote by \( L^n_0(q; q_0, \ldots, q_n) \) the subspace of \( L^n(q; q_0, \ldots, q_n) \) defined by

\[
L^n_0(q; q_0, \ldots, q_n) = \{ [x_0, \ldots, x_n] \in L^n(q; q_0, \ldots, q_n) | \text{ } x_n \text{ is real } \geq 0 \}.
\]

Of course, \( P^n(1, \ldots, 1) \), \( L^n(q; 1, \ldots, 1) \) and \( L^n_0(q; 1, \ldots, 1) \) are the usual complex projective space \( CP^n \), the usual mod \( q \) lens space \( L^n(q) \) and its \( 2n \)-skeleton \( L^n_0(q) \), respectively. For a weighted mod 4 lens space \( L^n(q; q_0, \ldots, q_n) \) we may assume that \( q_0 = \cdots = q_{r-1} = 4, \) \( q_r = \cdots = q_{r+s-1} = 2 \) and \( q_{r+s} = \cdots = q_n = 1 \) where \( 0 \leq r \leq r + s \leq n \). For such a tuple \( (q_0, \ldots, q_n) \) we simply set \( P(r,s,t) = P^n(q_0, \ldots, q_n) \), \( L(r,s,t) = L^n(4; q_0, \ldots, q_n) \) and \( L_0(r,s,t) = L^n_0(4; q_0, \ldots, q_n) \) with \( n = r + s + t \).
Moreover we shall omit the “r” as $P(s, t)$, $L(s, t)$ or $L_0(s, t)$ when $r = 0$. Notice that $L(r, s, t) = \Sigma^{2r} L(s, t)$ and $L_0(r, s, t) = \Sigma^{2r} L_0(s, t)$.

Denote by $\gamma$ the canonical line bundle over $CP^n$ and set $a = [\gamma] - 1 \in KU^0 CP^n$. Then it is well known that the (reduced) $KU$-cohomology group $KU^* CP^n_+ \cong Z[a]/(a^{n+1})$ where $CP^n_+$ denotes the disjoint union of $CP^n$ and a point. According to [1, Theorem 3.1] the map $\varphi: CP^n \rightarrow P(r, s, t)$ defined by $\varphi[x_0, \ldots, x_n] = [x_0^0, \ldots, x_n^r]$ with $n = r + s + t$ induces a monomorphism $\varphi^*: KU^* P(r, s, t) \rightarrow KU^* CP^n$ and the free abelian group $KU^* P(r, s, t)$ has the following basis $\{T_i\}$ such that $\varphi^* T_i = a(2)^i$ for $1 \leq i \leq r$, $\varphi^* T_{r+k} = a(2)^i a(1)^k$ for $1 \leq k \leq s$ and $\varphi^* T_{r+s+k} = a(2)^i a(1)^s a^h$ for $1 \leq h \leq t$, where $a(1) = (a + 1)^2 - 1$ and $a(2) = (a + 1)^4 - 1$.

In order to calculate the $KU$-cohomology group $KU^* L(s, t)$ we use the following cofiber sequence

\[
L(s, t) \xrightarrow{\theta} P(s, t) \xrightarrow{i} P(1, s, t)
\]

where $\theta$ is the natural surjection and $i$ is the canonical inclusion (cf. [3, Assertion 1]).

Since $a(2) = 2a(1) + a(1)^2 = 2a(1) + 2a(1)a + a(1)a^2 = 4a + 6a^2 + 4a^3 + a^4$, the induced homomorphism $i^*: KU^* P(1, s, t) \rightarrow KU^* P(s, t)$ is given as follows: $i^* T_k = 2T_k + T_{k+1}$ for $1 \leq k \leq s - 1$, $i^* T_s = 2T_s + 2T_{s+1} + T_{s+2}$, $i^* T_{s+h} = 4T_{s+h} + 6T_{s+h+1} + 4T_{s+h+2} + T_{s+h+3}$ for $1 \leq h \leq t$ and $i^* T_{s+t+1} = 0$. Using the $(n, n)$-matrix $E_k = (e_k, \ldots, e_n, 0, \ldots, 0)$ we here introduce the two $(n, n)$-matrices $A_n = 2E_1 + E_2$ and $B_n = 4E_1 + 6E_2 + 4E_3 + E_4$, where $e_j$ is the unit column vector entered “1” only in the $j$-th component. Moreover we set

\[
C_{s,t} = \begin{pmatrix} A_s & 0 \\ \xi & B_t \end{pmatrix}
\]

Then the induced homomorphism $i^*: KU^0 P(1, s, t) \rightarrow KU^0 P(s, t)$ is expressed as $(C_{s,t}, 0) : \bigoplus_{s+t+1} Z \rightarrow \bigoplus_{s+t} Z$. Therefore $KU^0 L(s, t) \cong \text{Coker} C_{s,t}$ and $KU^1 L(s, t) \cong Z$. In particular, $KU^1 L^n(2) \cong KU^0 L(n, 0) \cong \text{Coker} A_n$ and $KU^0 L^n(4) \cong \text{Coker} B_n$.

Recall that the $KU$-cohomology groups $KU^0 L^n(2) \cong Z[\sigma]/(\sigma^{n+1}, \sigma(1))$ and $KU^0 L^n(4) \cong Z[\sigma]/(\sigma^{n+1}, \sigma(2))$ are given as follows (see [4, 5]):

i) $KU^0 L^n(2) \cong Z/2^n$ with generator $\sigma$,

ii) $KU^0 L^{2m+1}(4) \cong Z/2^{2m+1+1} \oplus Z/2^m \oplus Z/2^{m-1}$ with generators $\sigma, \sigma(1)$ and $\sigma(1)\sigma$, $KU^0 L^{2m+1}(4) \cong Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m$ with generators $\sigma, \sigma(1) + 2^{-1+1}\sigma$ and $\sigma(1)\sigma$, where $\sigma = \theta^1 a$ and $\sigma(i) = \theta^i a(i)$.

Therefore the induced homomorphism $\theta^*: KU^0 CP^n \rightarrow KU^0 L^n(2)$ is given by the following row:

\[
\alpha_n = (-1)^{n-1}(1, -2, \ldots, (-2)^{n-1}) : \bigoplus_n Z \rightarrow Z/2^n.
\]
On the other hand, the induced homomorphism $\theta^* : KU^0 CP^n \to KU^0 L^n(4)$ is represented by the following $(3,n)$-matrix $\beta_n$:

\[
\beta_{2m} = \begin{pmatrix}
1 & -2 & 4 - 2m+1 & * \\
0 & 1 & -2 & * \\
0 & 0 & 1 & *
\end{pmatrix}, \quad \beta_{2m+1} = \begin{pmatrix}
1 & -2 - 2m+1 & 4 + 2m+2 & * \\
0 & 1 & -2 & * \\
0 & 0 & 1 & *
\end{pmatrix}.
\]

Notice that $KU^0 L(s,t)$ is isomorphic to the cokernel of

\[
\begin{pmatrix}
A_s \\
\beta_t \xi
\end{pmatrix} : \bigoplus_s Z \to \left( \bigoplus_s Z \right) \oplus \text{Coker} B_t.
\]

Since $\beta_{2m} \xi = (0, \cdots, 0, e_2)$ and $\beta_{2m+1} \xi = (0, \cdots, 0, -2m+1 e_1 + e_2)$, we can easily calculate the $KU$-cohomology group $KU^0 L(s,t)$ for $t \geq 1$ as follows:

\[
\begin{align*}
KU^0 L(s,2m) &\cong Z/2^s + Z/2m+1 + Z/2^m - 1 \\
KU^0 L(s,2m+1) &\cong \begin{cases} 
Z/2^s + Z/2m+2 + Z/2^m & (s \leq m) \\
Z/2^s + Z/2m+1 + Z/2^m & (s > m).
\end{cases}
\]

Moreover we see that the quotient morphism $\delta_{s,t} : \left( \bigoplus_s Z \right) \oplus \text{Coker} B_t \to KU^0 L(s,t)$ is represented by the following matrix:

\[
\begin{pmatrix}
\alpha_s & 0 & -2^s \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
\alpha_s & 0 & -2^s \\
2m-s+1 \alpha_s & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
\alpha_s & 2s-m-1 & 0 \\
0 & 1 & 2m+1 \\
0 & 0 & 1
\end{pmatrix}.
\]

Since the induced homomorphism $\theta^* : KU^0 P(s,t) \to KU^0 L(s,t)$ is expressed as the composition $\delta_{s,t}(1 \oplus \beta_t)$, we can immediately give a basis of $KU^0 L(s,t)$ $(s,t \geq 1)$ as follows:

\[
(3.5) \quad (\sigma(1), \sigma(s,1), \sigma(s,3)) B'_{s,t}
\]

where $\sigma(1) = \theta^* T_1$, $\sigma(s,i) = \theta^* T_{s+i}$, and $B'_{s,t}$ $(s,t \geq 1)$ is the matrix tabled below:

\[
\begin{pmatrix}
(-1)^{s-1} & 0 & (-1)^s 2^{s+1} \\
0 & 1 & 2m+1 - 4 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
B'_{s,2m} = \begin{pmatrix}
(-1)^{s-1} & 0 & (-1)^s 2^{s+1} \\
0 & 1 & 2m+1 - 4 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
B'_{s,2m+1} = \begin{pmatrix}
(-1)^{s-1} & 0 & (-1)^s 2^{s+1} \\
2^{m-s+1} & 1 & -4 \\
0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
(-1)^{s-1} & 0 & (-1)^s 2^{s-m-1} \\
0 & 1 & -4 \\
0 & 0 & 1
\end{pmatrix}.
\]
3.2. Next we shall investigate the behavior of the conjugation $\psi^{-1}_C$ on $KU^0L(s,t)$ $(s, t \geq 1)$. Note that $\psi^{-1}_C a^h = (-1)^h a^h (1 + a)^{-h}$ and $\psi^{-1}_C a(1)^k = (-1)^k a(1)^k (1 + a)^{-2k}$ in $KU^0CP^n$. Since $a(2) = (1 + a(1))^2 - 1$ and $a(1)^s a(2) = a(1)^s (a + 1)^4 - 1$ it follows immediately that

$$
\psi^{-1}_C a(1) \equiv a(1) \mod a(2)
$$

for the following matrix $P_s$:

$$
P_{2n} = \begin{pmatrix}
1 & 3 & 2^n & 3 \cdot 2^{2n+1} \\
0 & -3 & -8 & 0 \\
0 & 1 & 3 & 0
\end{pmatrix}, \quad P_{2n+1} = \begin{pmatrix}
1 & -2^{2n+1} & 2^{2n+2} \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
$$

Consider the following matrix $C_{s,t}$ $(s, t \geq 1)$ representing an automorphism on $KU^0L(s,t)$:

$$
\begin{align*}
C_{s,2m} &= \begin{pmatrix}
1 & 2^{s-1} & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} & \quad C_{s,2m+1} &= \begin{pmatrix}
1 & 2^{s-m-1} & 0 \\
0 & 1 & 0 \\
-2^{m-s} & 0 & 1
\end{pmatrix} \\
C_{s,2m+1} &= \begin{pmatrix}
1 + 2^m & 0 & -2^s \quad s = 2n + 1 \leq m \\
0 & 1 & 0 \\
-2^{m-s} & 0 & 1
\end{pmatrix} & \quad C_{s,2m+1} &= \begin{pmatrix}
1 & 2^{-1}(1 - 2^m) & -2^s(1 - 2^m) \\
2^{2m-s+1} & 1 + 2^{2s}(1 - 2^m) & 0 \\
0 & 0 & 1
\end{pmatrix} \\
C_{s,2m+1} &= \begin{pmatrix}
1 & 0 & 0 \quad s = 2n + 1 > m \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} & \quad C_{s,2m+1} &= \begin{pmatrix}
1 & 2^{s-m} + 2^{s-1} & 2^{2m+1} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \\
C_{s,2m+1} &= \begin{pmatrix}
1 & 0 & 0 \quad s = 2n + 1 > m \geq 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} & \quad C_{s,2m+1} &= \begin{pmatrix}
1 & 0 & 0 \quad s = 2n + 1 > m = 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\end{align*}
$$

In order to express the conjugation $\psi^{-1}_C$ on $KU^0L(s,t)$ plainly we here change the basis of $KU^0L(s,t)$ given in (3.5) slightly as follows:

$$
(\sigma(1), \sigma(s,1), \sigma(s,3))B_{s,t} \text{ where } B_{s,t} = B_{s,t}'C_{s,t}.
$$
Then the conjugation $\psi^{-1}_c$ on $KU^0L(s, t)$ is represented by the composition $B_{s,t}^{-1}PB_{s,t}$. Therefore a routine computation shows

**Proposition 3.1.** On the $KU$-cohomology group $KU^0L(s, t)$ with basis $(\sigma(1), \sigma(s, 1), \sigma(s, 3))B_{s,t}$ ($s, t \geq 1$) the conjugation $\psi^{-1}_c$ behaves as follows:

i) On $KU^0L(s, 2m) \cong \mathbb{Z}/2^{s+m} \oplus \mathbb{Z}/2^{2m+1} \oplus \mathbb{Z}/2^{m-1}$,

$$\psi^{-1}_c = \begin{pmatrix}
1 & -2^s & 2^{s+1} \\
0 & 1 & 0 \\
0 & 1 & -1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & -1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 2^{s+1} \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}.$$

ii) On $KU^0L(s, 2m + 1) \cong \mathbb{Z}/2^{s+m} \oplus \mathbb{Z}/2^{2m+2} \oplus \mathbb{Z}/2^m$ ($s \leq m$),

$$\psi^{-1}_c = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -2^m+1 \\
0 & 1 & -1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}.$$

iii) On $KU^0L(s, 2m + 1) \cong \mathbb{Z}/2^{s+m+1} \oplus \mathbb{Z}/2^{2m+1} \oplus \mathbb{Z}/2^m$ ($s > m$),

$$\psi^{-1}_c = \begin{pmatrix}
1 & -2^s & 2^{s+1} \\
0 & 1 & 0 \\
0 & 1 & -1
\end{pmatrix} \begin{pmatrix}
1 & 2^{s-m} & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.$$

**Remark.** When $t = 0$, the conjugation $\psi^{-1}_c = 1$ on $KU^0L(s, 0) \cong \mathbb{Z}/2^s$ with basis $\sigma(1)$.

We shall use the dual of Proposition 3.1 to study the behavior of the conjugation $\psi^{-1}_c$ on $KU_*L_0(s, t)$ and $KU_*L(s, t)$.

**Proposition 3.2.** The weighted mod 4 lens spaces $\Sigma^1L_0(s, t)$ and $\Sigma^1L(s, t)$ ($s \geq 1, t \geq 0$) have the same $C$-types as the small spectra tabled below, respectively (cf. [12, Proposition 5.1]):

<table>
<thead>
<tr>
<th>$s = 2n \leq m$</th>
<th>$s = 2n &gt; m$</th>
<th>$s = 2n + 1 &gt; m \geq 1$</th>
<th>$s = 2n + 1, m \geq 1$</th>
<th>$s = 2n + 1, m = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma^1L_0(s, 2m)$</td>
<td>$\Sigma^1L(s, 2m)$</td>
<td>$\Sigma^1L_0(s, 2m + 1)$</td>
<td>$\Sigma^1L(s, 2m + 1)$</td>
<td>$\Sigma^1L_0(s, 2m + 2)$</td>
</tr>
<tr>
<td>$PP'_2m+1,s+m-1,m$</td>
<td>$MPPP'_2m+1,s+m-1,m \oplus P'_2m+1,m-1,m$</td>
<td>$SZ/2^{s+m} \oplus P'_2m+1,m+1$</td>
<td>$PP'_2m+1,s+m-1,m \oplus P'_2m+1,m+1$</td>
<td>$PP'_2m+1,s+m-1,m \oplus P'_2m+1,m+1$</td>
</tr>
<tr>
<td>$SZ/2^{s+m} \oplus P'_2m+1,m$</td>
<td>$M_{s+m} \oplus P'_2m+1,m$</td>
<td>$SZ/2^m \oplus P'_2m+1,m+1$</td>
<td>$SZ/2^m \oplus P'_2m+1,m+1$</td>
<td>$SZ/2^m \oplus P'_2m+1,m+1$</td>
</tr>
<tr>
<td>$\Sigma^2SZ/2^2m+1 \oplus P'<em>s+s+m-1,m \oplus \Sigma^2M</em>{2m+1} \oplus P'_s+s+m-1,m$</td>
<td>$\Sigma^2M_{2m+1} \oplus P'<em>s+s+m-1,m \oplus \Sigma^2M</em>{2m+1} \oplus P'_s+s+m-1,m$</td>
<td>$\Sigma^2SZ/2^m \oplus P'<em>s+s+m-1,m \oplus \Sigma^2M</em>{2m+1} \oplus P'_s+s+m-1,m$</td>
<td>$\Sigma^2SZ/2^m \oplus P'<em>s+s+m-1,m \oplus \Sigma^2M</em>{2m+1} \oplus P'_s+s+m-1,m$</td>
<td>$\Sigma^2SZ/2^m \oplus P'<em>s+s+m-1,m \oplus \Sigma^2M</em>{2m+1} \oplus P'_s+s+m-1,m$</td>
</tr>
<tr>
<td>$SZ/2^s$</td>
<td>$\Sigma^0 \oplus \Sigma^2SZ/2^s$</td>
<td>$SZ/2^s$</td>
<td>$\Sigma^0 \oplus \Sigma^2SZ/2^s$</td>
<td>$\Sigma^0 \oplus \Sigma^2SZ/2^s$</td>
</tr>
</tbody>
</table>
Moreover $\Sigma^1 L(s, 2m + 1)$ has the same $C$-type as the wedge sum $\Sigma^2 s \vee \Sigma^1 L_0(s, 2m + 1)$.

Proof. By dualizing Proposition 3.1 we can immediately determine the $C$-type of $\Sigma^1 L_0(s, t)$ because $KU_{-1} L_0(s, t) \cong KU^0 L_0(s, t)$ and $KU_0 L_0(s, t) = 0$. On the other hand, Proposition 3.4 below implies that $\Sigma^1 L(s, 2m + 1)$ has the same $C$-type as $\Sigma^2 s \vee \Sigma^1 L_0(s, 2m + 1)$. We shall now investigate the $C$-type of $\Sigma^1 L(s, 2m)$ in case of $s = 2n \leq m$. Note that $KU_{-1} L(s, t) \cong KU_{-1} \Sigma^2 s + 2t + 1 \ominus KU_{-1} L_0(s, t)$ and $KU_0 L(s, t) = 0$. According to the dual of Proposition 3.1 the conjugations $\psi^1_C$ on $KU_{-1} L(s, 2m) \cong Z \oplus Z/2^s + m \oplus Z/2^{m+1} \oplus Z/2^{m-1}$ and $KU_{-1} L_0(s, 2m + 1) \cong Z/2^s + m \oplus Z/2^{m+2} \oplus Z/2^m$ are represented by the following matrices

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-2^{m+1} & 1 & 2^{m+2} \\
c & 0 & -1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 - 2^{m+1} & 2^{m+2} \\
0 & 1 & -1
\end{pmatrix}
\]

for some integers $a, b$ and $c$, respectively. As is easily verified, we may regard that $a = c = 0$ and $b = 0$ or $-1$ after changing the direct sum decomposition of $KU_{-1} L(s, 2m)$ suitably if necessary. Consider the canonical inclusion map $i_{L_0} : L(s, t) \to L_0(s, t + 1)$. By virtue of (3.9) the induced homomorphism $i_{L_0}^* : KU^0 L_0(s, t + 1) \to KU^0 L(s, t)$ is actually represented by the matrix $F_{s, t} = B_{s, t}^{-1} B_{s, t+1}$. Since a routine computation shows that

\[
F_{s, 2m} = \begin{pmatrix}
1 + 2^m \\
-2^{m-s+2}(1 + 2^{m-1}) \\
-2^{m-s+1}
\end{pmatrix}
\]

the induced homomorphism $i_{L_0}^* : KU_{-1} L(s, 2m) \to KU_{-1} L_0(s, 2m + 1)$ is expressed as the following matrix

\[
\begin{pmatrix}
x & 1 + 2^m & -2(1 + 2^{m-1}) & -2^{m+2} \\
y & 0 & 2 & 0 \\
z & -1 & 1 & 2
\end{pmatrix}
\]

for some integers $x, y$ and $z$. Here $y$ must be odd because $i_{L_0}^*$ is an epimorphism. Using the equality $\psi^1_C i_{L_0}^* = i_{L_0}^* \psi_C^1$ we get immediately that $b \equiv y \mod 2^m$, thus $b = -1$. Therefore $\Sigma^1 L(s, 2m)$ has the same $C$-type as $\text{MPP}_2^{s+1, s+m-1, m}$ when $s = 2n \leq m$. In the other three cases the $C$-types of $\Sigma^1 L(s, 2m)$ are similarly obtained.

3.3. Using Proposition 3.2 we can immediately calculate $KO_i X \oplus KO_{i+4} X$ ($i = 0, 2$) for $X = L_0(s, t)$ and $L(s, t)$ ($s \geq 1, t \geq 0$) as tabled below:
where \((**)_0 \cong \mathbb{Z}/2\) and \((**)_m \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2\) if \(m \geq 1\).

**Lemma 3.3.** For \(X = L_0(s, t)\) and \(L(s, t)\) \((s \geq 1, t \geq 0)\) the sets \(S(X) = \{2i; K_{2i}X = 0\} (0 \leq i \leq 3)\) are given as follows:

(i) \(X = L_0(2n, 2m) L(2n, 2m) L_0(2n, 2m + 1) L(2n, 2m + 1)\)
\[
S(X) = \begin{cases} 
\{4, 6\} & \{0, 4, 6\} & \{0, 6\} & n + m : \text{even} \\
\{0, 6\} & \{0, 4, 6\} & \{4, 6\} & n + m : \text{odd}
\end{cases}
\]

(ii) \(X = L_0(2n + 1, 2m) L(2n + 1, 2m) L_0(2n + 1, 2m + 1) L(2n + 1, 2m + 1)\)
\[
S(X) = \begin{cases} 
\{0, 6\} & \{0\} & \{0\} & n, m : \text{even} \\
\{0\} & \{0, 6\} & \{0, 6\} & n, m + 1 : \text{even} \\
\{4, 6\} & \{4, 6\} & \{4, 6\} & n, m + 1 : \text{odd} \\
\{4, 6\} & \{4, 6\} & \{4\} & n, m : \text{odd}
\end{cases}
\]

where \(\{4, 6\}_0 = \{4, 6\}\) and \(\{4, 6\}_m = \{4\}\) if \(m \geq 1\).

Proof. Consider the following (homotopy) commutative diagram

\[
\begin{array}{ccc}
L_0(s, t) & \xrightarrow{\theta} & P(s, t) \\
i_L \downarrow & & \downarrow \tilde{i} \\
L(s, t) & \xrightarrow{\theta} & P(s, t)
\end{array}
\]

with two cofiber sequences, where the maps \(i_L, i\) and \(\tilde{i}\) are the canonical inclusions, and the map \(i_0\) is defined by \(i_0[x_0, \cdots, x_{s+t}] = [x_{s+t}^4, x_0, \cdots, x_{s+t}]\). According to [7, Theorem 2.4] the weighted projective space \(P(s, t)\) is quasi \(KO_*\)-equivalent to the wedge sum \(V_{n+m}C(\eta), \Sigma^{4n+4m+4} \vee (V_{n+m}C(\eta))\), \(\Sigma^{4n+2} \vee (V_{n+m}C(\eta))\) or \(\Sigma^{4n+2} \vee \Sigma^{4n+4m+4} \vee (V_{n+m}C(\eta))\) according as \((s, t) = (2n, 2m), (2n, 2m + 1), (2n + 1, 2m)\) or \((2n + 1, 2m + 1)\). In addition, \(P(1, s, t)\) is quasi \(KO_*\)-equivalent to the wedge sum \(\Sigma^2 \vee \Sigma^2 P(s, t)\). Using the above commutative diagram we can immediately obtain our result.

**Proposition 3.4.** The weighted mod 4 lens space \(L(s, 2m + 1)\) is quasi \(KO_*\)-equivalent to the wedge sum \(\Sigma^{2s+4m+3} \vee L_0(s, 2m + 1)\).
Proof. Consider the following commutative diagram

\[
\begin{array}{ccc}
\Sigma^{2s+4m+3} & \xrightarrow{\tilde{\alpha}} & P(1, s, 2m) \\
\downarrow & & \downarrow \\
\Sigma^{2s+4m+3} & \xrightarrow{\alpha} & \Sigma^1 L_0(s, 2m + 1) \\
\end{array}
\]

with two cofiber sequences. Since the quasi \(KO^*_\)-type of \(P(1, s, t)\) is given as in the proof of Lemma 3.3 we see that the map \(1 \wedge \tilde{\alpha} : \Sigma^{2s+4m+3} KO \to KO \wedge P(1, s, 2m)\) is trivial. Hence our result is immediate.

Applying Theorems 1.2 and 1.3 and Proposition 3.4 with the aid of Proposition 3.2 and Lemma 3.3 we can immediately obtain

**Theorem 3.5.** The weighted mod 4 lens spaces \(\Sigma^1 L_0(2n, t)\) and \(\Sigma^1 L(2n, t)\) for \(n \geq 1\) are quasi \(KO^*_\)-equivalent to the small spectra tabled below, respectively (cf. [12, Theorem 3]):

| \(|n + m|\) | \(\Sigma^1 L_0(2n, 2m)\) | \(\Sigma^1 L(2n, 2m)\) | \(\Sigma^1 L_0(2n, 2m + 1)\) |
|---|---|---|---|
| even | \(PP'_{2m+1,2n+m-1,m}\) \(MPP'_{2m+1,2n+m-1,m}\) \(V_{2n+m} \lor PP'_{2m+1,m+1}\) |
| odd | \(PP'_{2m+1,2n+m-1,m}\) \(MPP'_{2m+1,2n+m-1,m}\) \(SZ/2^{2n+m} \lor PP'_{2m+1,m+1}\) |
| even | \(SV/2^{2n+m} \lor PP'_{2m}m\) \(MV_{2m+n} \lor PP'_{2m,2n+m}\) \(VPP'_{2m+1,2n+m+1}m\) |
| odd | \(SV/2^{2n+m} \lor PP'_{2m}m\) \(MV_{2m+n} \lor PP'_{2m,2n+m}\) \(PP'_{2m+1,2n+m+1}m\) |

in cases when i) \(2n \leq m\) and ii) \(2n > m\). Moreover \(\Sigma^1 L(2n, 2m+1)\) is quasi \(KO^*_\)-equivalent to \(\Sigma^4n+4m+4 \lor \Sigma^1 L_0(2n, 2m + 1)\).

Applying Theorems 2.3 and 2.4 in place of Theorems 1.2 and 1.3 we show

**Theorem 3.6.** The weighted mod 4 lens spaces \(\Sigma^1 L_0(2n + 1, t)\) and \(\Sigma^1 L(2n + 1, t)\) are quasi \(KO^*_\)-equivalent to the small spectra tabled below, respectively:

| \(|n + m|\) | \(\Sigma^1 L_0(2n + 1, 2m)\) | \(\Sigma^1 L(2n + 1, 2m)\) | \(\Sigma^1 L_0(2n + 1, 2m + 1)\) |
|---|---|---|---|
| \(V_{2n+1}\) | \(\Sigma^4 \lor V_{2n+1}\) | \(\Sigma^4 \lor V_{2n+2}\) |
| \(SZ/2^{2n+1} \lor P'_{2n,2n+m,m}\) \(\Sigma^2 M_{2m+1} \lor P'_{2n,2n+m,m}\) \(\Sigma^2 V_{2m} \lor P'_{2n+1,2n+m+1}\) |
| \(\Sigma^6 V_{2n+1} \lor P'_{2n,2n+m,m}\) \(\Sigma^6 M_{2m+1} \lor P'_{2n,2n+m,m}\) \(\Sigma^6 V_{2m} \lor \Sigma^6 P'_{2n+1,2n+m}\) |
| \(\Sigma^6\) \(\lor SZ/2^{2n+1}\) | \(\Sigma^6\) \(\lor SZ/2^{2n+2}\) |

in cases when i) \(n\) is even and \(m = 0\), ii) \(n\) and \(m \geq 2\) are even, iii) \(n\) is even and \(m\) is odd, iv) \(n\) is odd and \(m = 0\), v) \(n\) is odd and \(m \geq 2\) is even, and vi) \(n\) and \(m\) are odd. Moreover \(\Sigma^1 L(2n + 1, 2m + 1)\) is quasi \(KO^*_\)-equivalent to \(\Sigma^4n+4m+6 \lor \Sigma^1 L_0(2n + 1, 2m + 1)\).

Proof. By a quite similar argument to the case of the real projective space $\mathbb{R}P^k$ (cf. [10, Theorem 5]) we can easily determine the quasi $KO_*$-types of $\Sigma^1 L_0(2n + 1, 0)$ and $\Sigma^1 L(2n + 1, 0)$. The quasi $KO_*$-type of $\Sigma^1 L(2n + 1, 2m)$ for $m \geq 1$ is immediately determined by applying Theorem 2.4 ii) with the aid of Proposition 3.2 and Lemma 3.3. On the other hand, the quasi $KO_*$-types of $\Sigma^1 L_0(2n + 1, 2m)$ in cases of ii) and vi) and those of $\Sigma^1 L_0(2n + 1, 2m + 1)$ in cases of iii), iv) and v) are also determined by applying Theorem 2.3 and [6, Theorem 5.3] in place of Theorem 2.4 ii).

We shall now investigate the quasi $KO_*$-types of $\Sigma^1 L_0(2n + 1, 2m - 1)$ and $\Sigma^1 L_0(2n + 1, 2m + 1)$ in case when $n$ is even and $m$ is odd. Consider the following two cofiber sequences

$$
\begin{align*}
\Sigma^{4n+4m} & \xrightarrow{\alpha_0} \Sigma^1 L(2n + 1, 2m - 2) \xrightarrow{\iota_0} \Sigma^1 L_0(2n + 1, 2m - 1) \\
\Sigma^{4n+4m+2} & \xrightarrow{\alpha_0} \Sigma^1 L(2n + 1, 2m - 1) \xrightarrow{\iota_0} \Sigma^1 L_0(2n + 1, 2m)
\end{align*}
$$

where $\Sigma^1 L(2n+1,2m-1)$ is quasi $KO_*$-equivalent to $\Sigma^{4n+4m+2} \vee \Sigma^1 L_0(2n+1,2m-1)$ according to Proposition 3.4. Note that $\Sigma^1 L(2n + 1, 0)$ is quasi $KO_*$-equivalent to $\Sigma^4 \vee V_{2n+1}$. Since $\Sigma^1 L_0(2n + 1, 1)$ has the same $C$-type as $SZ/2 \vee SZ/2^{2n+2}$ by Proposition 3.2, it is quasi $KO_*$-equivalent to $\Sigma^1 L_0(2n + 1, 2m)$ in cases of ii) and i; and those of $\Sigma^1 L_0(2n + 1, 2m + 1)$ in cases of iii), iv) and v) are also determined by applying Theorem 2.3 and [6, Theorem 5.3] in place of Theorem 2.4 ii).

We shall now investigate the quasi $KO_*$-types of $\Sigma^1 L_0(2n + 1, 2m - 1)$ and $\Sigma^1 L_0(2n + 1, 2m + 1)$ in case when $n$ is even and $m$ is odd. Consider the following two cofiber sequences

$$
\begin{align*}
\Sigma^{4n+4m} & \xrightarrow{\alpha_0} \Sigma^1 L(2n + 1, 2m - 2) \xrightarrow{\iota_0} \Sigma^1 L_0(2n + 1, 2m - 1) \\
\Sigma^{4n+4m+2} & \xrightarrow{\alpha_0} \Sigma^1 L(2n + 1, 2m - 1) \xrightarrow{\iota_0} \Sigma^1 L_0(2n + 1, 2m)
\end{align*}
$$

where $\Sigma^1 L(2n+1,2m-1)$ is quasi $KO_*$-equivalent to $\Sigma^{4n+4m+2} \vee \Sigma^1 L_0(2n+1,2m-1)$ according to Proposition 3.4. Note that $\Sigma^1 L(2n + 1, 0)$ is quasi $KO_*$-equivalent to $\Sigma^4 \vee V_{2n+1}$. Since $\Sigma^1 L_0(2n + 1, 1)$ has the same $C$-type as $SZ/2 \vee SZ/2^{2n+2}$ by Proposition 3.2, it is quasi $KO_*$-equivalent to $\Sigma^1 L_0(2n + 1, 2m)$. Hence it is easily calculated that $\alpha_0^*: Z/2 \rightarrow Z/2 \oplus Z/2 \oplus Z/2 \oplus Z/2^{2n+1}$. From Lemma 3.3 we recall that the set $S(X)$ consists of only 0 for $X = L_0(2n + 1, 2m - 1)$ or $L_0(2n + 1, 2m)$ under our assumption on $n$ and $m$. Applying Theorem 2.3 i) and ii) combined with Proposition 3.2 we see that $\Sigma^1 L_0(2n + 1, 2m - 1)$ is quasi $KO_*$-equivalent to one of the three spectra $\Sigma^2 V_{2n+m-1} \vee P'_{2n+m, 2m}$, $\Sigma^2 SZ/2 \vee \Sigma^2 P'_{2m-1, 2n+m+1}$ and $\Sigma^2 N P'_{2m-1, 2n+m+1}$ when $m \geq 3$, and $\Sigma^1 L_0(2n + 1, 2m)$ is quasi $KO_*$-equivalent to one of the three spectra $\Sigma^2 V_{2n+1} \vee P'_{2n+m, 2m}$, $\Sigma^2 SZ/2 \vee \Sigma^2 P'_{2n+m, 2m}$ and $\Sigma^2 N P'_{2n+m, 2m}$ when $m \geq 3$. Therefore $\Sigma^1 L_0(2n + 1, 2m - 1)$ must be quasi $KO_*$-equivalent to $\Sigma^2 V_{2n+m-1} \vee P'_{2n+m, 2m}$ when $m \geq 3$. Hence it is easily calculated that $\Sigma^1 L_0(2n + 1, 2m - 1)$ is quasi $KO_*$-equivalent to $\Sigma^2 M_{2m-1} \vee P'_{2n+m-1, 2n+m-1}$ when $m \geq 3$, it is immediately that $\Sigma^1 L_0(2n + 1, 2m - 1)$ is quasi $KO_*$-equivalent to $\Sigma^2 V_{2n+1} \vee P'_{2n+m, 2m}$ when $m \geq 3$. Therefore $\Sigma^1 L_0(2n + 1, 2m - 1)$ must be quasi $KO_*$-equivalent to $\Sigma^2 V_{2m+1} \vee P'_{2n+m, 2m}$ when $m \geq 3$. Hence it is easily calculated that $\Sigma^1 L_0(2n + 1, 2m) \cong Z/2 \oplus Z/2^{2n+m+1} \oplus Z/2$ and $\Sigma^1 L_0(2n + 1, 2m)$ is isomorphic to the cokernel of $\alpha_0^*: Z/2 \rightarrow Z/2 \oplus Z/2 \oplus Z/2 \oplus Z/2^{2n+m}$. Therefore $\Sigma^1 L_0(2n + 1, 2m)$ must be quasi $KO_*$-equivalent to $\Sigma^2 V_{2m+1} \vee P'_{2n+m, 2m}$ when $m \geq 3$ as well as $m = 1$.

In case when $n$ is odd and $m \geq 2$ is even the quasi $KO_*$-types of $\Sigma^1 L_0(2n + 1, 2m - 1)$ and $\Sigma^1 L_0(2n + 1, 2m)$ are determined by a parallel argument. □

Remark. According to Theorems 3.5 and 3.6, $L_0(s, 0)$ and $L(s, 0)$ are quasi $KO_*$-equivalent to the real projective spaces $\mathbb{R}P^{2s}$ and $\mathbb{R}P^{2s+1}$, respectively.
References


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