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THE QUASI KO_*-TYPES OF WEIGHTED MOD 4 LENS SPACES

Dedicated to Professor Fuichi Uchida on his sixtieth birthday

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0. Introduction

Let KU and KO be the complex and the real K-spectrum, respectively. For any CW-spectrum X its KU-homology group KU_*X is regarded as a ($Z/2$-graded) abelian group with involution because KU possesses the conjugation ψ_C^{-1}. Given CW-spectra X and Y we say that X is quasi KO_*-equivalent to Y if there exists an equivalence $f : KO \wedge X \to KO \wedge Y$ of KO-module spectra (see [8]). If X is quasi KO_*-equivalent to Y, then KO_*X is isomorphic to KO_*Y as a KO_*-module, and in addition KU_*X is isomorphic to KU_*Y as an abelian group with involution. In the latter case we say that X has the same $\mathcal{C}$-type as Y (cf. [2]). In [10] and [11] we have determined the quasi KO_*-types of the real projective space RP^k and its stunted projective space RP^k/RP^l. Moreover in [12] we have determined the quasi KO_*-types of the mod 4 lens space L_4^k and its stunted lens space L_4^k/L_4^l where we simply denote by L_4^{2n+1} the usual $(2n+1)$-dimensional mod 4 lens space $L^n(4)$ and by L_4^{2n} its $2n$-skeleton $L_0^n(4)$. In this note we shall generally determine the quasi KO_*-types of a weighted mod 4 lens space $L^n(4; q_0, \cdots, q_n)$ and its $2n$-skeleton $L_0^n(4; q_0, \cdots, q_n)$ along the line of [12].

The weighted mod 4 lens space $L^n(4; q_0, \cdots, q_n)$ is obtained as the fiber of the canonical inclusion $i : P^n(q_0, \cdots, q_n) \to P^{n+1}(4, q_0, \cdots, q_n)$ of weighted projective spaces (see [3]). Using the result of Amrani [1, Theorem 3.1] we can calculate the KU-cohomology group $KU^*L^n(4; q_0, \cdots, q_n)$ and the behavior of the conjugation ψ_C^{-1} on it. Our calculation asserts that $\Sigma^1 L_0^n(4; q_0, \cdots, q_n)$ has the same $\mathcal{C}$-type as one of the small spectra $\Sigma^2 SZ/2^r \vee P'_{s,t}$, $SZ/2^r \vee P''_{s,t}$ and $PP'_{r,s,t}$, and $\Sigma^1 L^n(4; q_0, \cdots, q_n)$ has the same $\mathcal{C}$-type as one of the small spectra $\Sigma^2 M_r \vee P'_{s,t}$, $M_r \vee P''_{s,t}$, $MPP'_{r,s,t}$ and $\Sigma^{2m} \vee \Sigma^1 L_0^n(4; q_0, \cdots, q_n)$ (see Proposition 3.2). Here $SZ/2^r$ is the Moore spectrum of type $Z/2^r$ and M_r, $P'_{s,t}$, $P''_{s,t}$, $PP'_{r,s,t}$ and $MPP'_{r,s,t}$ are the small spectra constructed as the cofibers of the maps $i\eta : \Sigma^1 \to SZ/2^r$, $i\bar{\eta} : \Sigma^1 SZ/2^t \to SZ/2^s$, $i\bar{\eta} + \tilde{\eta} j : \Sigma^1 SZ/2^t \to SZ/2^s$, $(\tilde{\eta} j, i\bar{\eta}) : \Sigma^1 SZ/2^t \to SZ/2^r \vee SZ/2^s$ and $(i_M \tilde{\eta} j, i\bar{\eta}) : \Sigma^1 SZ/2^t \to M_r \vee SZ/2^s$, respectively, in which $i : \Sigma^0 \to SZ/2^r$ and $j : SZ/2^r \to \Sigma^1$ are the bottom cell inclusion and the top cell projection, $i_M : SZ/2^r \to M_r$ is the canonical inclusion, $\eta : \Sigma^1 \to \Sigma^0$ is the stable Hopf map, and $\bar{\eta} : \Sigma^1 SZ/2^r \to \Sigma^0$ and $\tilde{\eta} : \Sigma^2 \to SZ/2^r$ are its extension and coextension satisfying $\bar{\eta}i = \eta$ and $j\tilde{\eta} = \eta$.

In [12, Proposition 3.1 and Theorem 3.3] we have already characterized the quasi KO_*-types of spectra having the same $\mathcal{C}$-type as $SZ/2^r \vee P''_{s,t}$, $M_r \vee P''_{s,t}$, $PP'_{r,s,t}$ or $MPP'_{r,s,t}$ (see Theorems 1.2 and 1.3). In §1 we introduce some new small spectra X having the same $\mathcal{C}$-type as $SZ/2^r \vee P'_{s,t}$ or $M_r \vee P'_{s,t}$, and calculate their KO-homology groups KO_*X (Propositions 1.5 and 1.7). In §2 we shall characterize the quasi KO_*-types of spectra having the same $\mathcal{C}$-type as $SZ/2^r \vee P'_{s,t}$ or $M_r \vee P'_{s,t}$ (Theorems 2.3 and 2.4) by using the small spectra introduced in §1. Our discussion developed in §2 is quite similar to the one done in [6, §4] in order to characterize the quasi KO_*-types of spectra having the same $\mathcal{C}$-type as $SZ/2^r \vee SZ/2^s$ (see [6, Theorem 5.3]). In §3 we first calculate the KU-cohomology group $KU^0L^n(4; q_0, \cdots, q_n)$, and then investigate the behavior of the conjugation ψ_C^{-1} on it (Proposition 3.1). Dualizing this result we study the $\mathcal{C}$-types of $L = L^n(4; q_0, \cdots, q_n)$ and $L_0^n(4; q_0, \cdots, q_n)$ as is stated above (Proposition 3.2), and moreover calculate the sets $S(L) = \{2i; KO_{2i}L = 0 \ (0 \leq i \leq 3)\}$ (Lemma 3.3). Since $P'_{s,t}$ and $\Sigma^2 P'_{t-1,s+1}$ have the same $\mathcal{C}$-type we can apply Theorems 1.2, 1.3, 2.3 and 2.4 with the aid of Proposition 3.2 and Lemma 3.3 to determine the quasi KO_*-types of the weighted mod 4 lens spaces $L^n(4; q_0, \cdots, q_n)$ and $L_0^n(4; q_0, \cdots, q_n)$ as our main results (Theorems 3.5 and 3.6).

1. Small spectra having the same $\mathcal{C}$-type as $SZ/2^r \vee P'_{s,t}$ or $M_r \vee P'_{s,t}$

1.1. Let $SZ/2^m$ $(m \geq 1)$ be the Moore spectrum of type $Z/2^m$, and $i : \Sigma^0 \to SZ/2^m$ and $j : SZ/2^m \to \Sigma^1$ be the bottom cell inclusion and the top cell projection, respectively. The stable Hopf map $\eta : \Sigma^1 \to \Sigma^0$ of order 2 admits an extension $\bar{\eta} : \Sigma^1 SZ/2^m \to \Sigma^0$ and a coextension $\tilde{\eta} : \Sigma^2 \to SZ/2^m$ satisfying $\bar{\eta}i = \eta$ and $j\tilde{\eta} = \eta$. As in [13] (see [8]) we denote by M_m, $N_{m,n}$, $P_{m,n}$, $P'_{m,n}$, $P''_{m,n}$, $R_{m,n}$, $R'_{m,n}$ and $K_{m,n}$ the small spectra constructed as the cofibers of the following maps $i\eta : \Sigma^1 \to SZ/2^m$, $i\eta^2 j$, $\tilde{\eta}j$, $i\bar{\eta}$, $i\bar{\eta}+\tilde{\eta}j : \Sigma^1 SZ/2^n \to SZ/2^m$ and $\tilde{\eta}\eta^2 j$, $i\eta^2\bar{\eta}$, $\tilde{\eta}\bar{\eta} : \Sigma^3 SZ/2^n \to SZ/2^m$, respectively. In particular, $P'_{m-1,1}$ is simply written as V_m. The spectra V_m and M_m are exhibited in the following cofiber sequences:

$$\Sigma^0 \xrightarrow{2^{m-1}\bar{i}} C(\bar{\eta}) \xrightarrow{\bar{i}_V} V_m \xrightarrow{\bar{j}_V} \Sigma^1, \Sigma^0 \xrightarrow{2^m i_P} C(\eta) \xrightarrow{h_M} M_m \xrightarrow{k_M} \Sigma^1$$

where $C(\eta)$ and $C(\bar{\eta})$ are the cofibers of the maps $\eta : \Sigma^1 \to \Sigma^0$ and $\bar{\eta} : \Sigma^1 SZ/2 \to \Sigma^0$, and $i_P : \Sigma^0 \to C(\eta)$ and $\bar{i} : \Sigma^0 \to C(\bar{\eta})$ are the bottom cell inclusions. Note that $C(\bar{\eta})$ is quasi KO_*-equivalent to Σ^4.

Moreover we denote by ${}_VP_{m,n}$, $P^V_{m,n}$, ${}_VR_{m,n}$, $R^V_{m,n}$, $VR_{m,n}$, $MP_{m,n}$, $PM_{m,n}$, $MR_{m,n}$ and $RM_{m,n}$ the small spectra constructed as the cofibers of the following maps:

$$(1.1)\qquad \begin{aligned} i_V\tilde{\eta}j &: \Sigma^1 SZ/2^n \to V_m, & \tilde{\eta}\bar{j}_V &: \Sigma^1 V_n \to SZ/2^m,\\ i_V\tilde{\eta}\eta^2 j &: \Sigma^3 SZ/2^n \to V_m, & \tilde{\eta}\eta^2\bar{j}_V &: \Sigma^3 V_n \to SZ/2^m,\\ \xi_V\eta j &: \Sigma^5 SZ/2^n \to V_m, & &\\ i_M\tilde{\eta}j &: \Sigma^1 SZ/2^n \to M_m, & \tilde{\eta}k_M &: \Sigma^1 M_n \to SZ/2^m,\\ i_M\tilde{\eta}\eta^2 j &: \Sigma^3 SZ/2^n \to M_m, & \tilde{\eta}\eta^2 k_M &: \Sigma^3 M_n \to SZ/2^m, \end{aligned}$$

respectively, where $i_V : SZ/2^{m-1} \to V_m$ and $i_M : \Sigma^0 \to M_m$ are the canonical inclusions, and $\xi_V : \Sigma^5 \to V_m$ is the map satisfying $j_V\xi_V = \tilde{\eta}\eta$ for the canonical projection $j_V : V_m \to \Sigma^2 SZ/2$. Here we understand $i_V\tilde{\eta} = i : \Sigma^0 \to SZ/2$ and $\xi_V = \tilde{\eta}\eta : \Sigma^3 \to SZ/2$ when $m = 1$. According to [6, Proposition 3.2] and its dual the spectra ${}_VP_{m,n}$, $P^V_{m,n}$, ${}_VR_{m,n}$ and $R^V_{m,n}$ ($m \geq 2$) are quasi KO_*-equivalent to $\Sigma^2 P_{n+1,m-1}$, $\Sigma^6 P_{n+1,m-1}$, $\Sigma^2 V'N_{m,n}$ and $\Sigma^6 V'N_{m,n}$, respectively. Here the spectrum $V'N_{m,n}$ is constructed as the cofiber of the map $\tilde{\eta}j \vee i\eta^2 j : \Sigma^1 SZ/2^{m-1} \vee \Sigma^1 SZ/2^n \to SZ/2$, and it is quasi KO_*-equivalent to $\Sigma^6 V_m \vee \Sigma^2 SZ/2^n$ if $m \geq n$. The S-dual spectrum $NV_{n,m}$ of $V'N_{m,n}$ and the spectrum $VR_{m,n}$ have been introduced in [13, Proposition 3.1], and the spectra $MP_{m,n}$ and $PM_{m,n}$ were written as $MV'_{m,n}$ and $V'M_{m,n}$, respectively, in [12, Propositions 2.3 and 2.4]. On the other hand, the spectra $MR_{m,n}$ and $RM_{n,m}$ have the same $\mathcal{C}$-type as $M_m \vee SZ/2^n$. Note that $MR_{m,n}$ is quasi KO_*-equivalent to $M_m \vee \Sigma^4 SZ/2^n$ if $m \geq n$, and $RM_{m,n}$ is quasi KO_*-equivalent to $SZ/2^m \vee \Sigma^4 M_n$ if $m > n$. By a routine computation we obtain the KO-homology groups KO_iX ($0 \leq i \leq 7$) of $X = MR_{m,n}$ ($m < n$) and $RM_{m,n}$ ($m \leq n$) as follows:

(1.2)

$X \setminus i$	0	1	2	3	4	5	6	7
$MR_{m,n}$	$Z/2^m \oplus Z/2^n$	0	$Z \oplus Z/2$	$Z/2$	$Z/2^m \oplus Z/2^{n+1}$	$Z/2$	$Z \oplus (*)_n$	$Z/2$
$RM_{m,n}$	$Z/2^m \oplus Z/2^{n+1}$	$Z/2$	$Z \oplus (*)_m$	$Z/2$	$Z/2^{m-1} \oplus Z/2^{n+1}$	0	$Z \oplus Z/2$	$Z/2$

where $(*)_1 \cong Z/4$ and $(*)_k \cong Z/2 \oplus Z/2$ if $k \geq 2$.

For any maps $f : \Sigma^i SZ/2^t \to Z_r$ and $g : \Sigma^i Z_r \to SZ/2^s$ whose cofibers are denoted by $X_{r,t}$ and $Y_{s,r}$, we introduce new small spectra $XP'_{r,s,t}$ and $P'Y_{s,t,r}$ constructed as the cofibers of the following maps

$$(1.3)\qquad \begin{aligned} &(f, i\tilde{\eta}) : \Sigma^i SZ/2^t \to Z_r \vee \Sigma^{i-1} SZ/2^s,\\ &i\tilde{\eta} \vee g : \Sigma^1 SZ/2^t \vee \Sigma^i Z_r \to SZ/2^s, \end{aligned}$$

respectively. In particular, the spectra $NP'_{r,s,1}$ and $PP'_{r,s,1}$ are written as $NV_{r,s+1}$ and $PV_{r,s+1}$ in [13, Proposition 3.1], respectively, and $RP'_{r,s,1} = SZ/2^r \vee \Sigma^2 V_{s+1}$ and ${}_VRP'_{r,s,1} = V_r \vee \Sigma^2 V_{s+1}$. By virtue of [6, Propositions 3.2 and 3.3] the spectra ${}_VPP'_{r,s,1}$, $P'P_{s,1,r}$, $P'P^V_{s,1,r}$ and $P'R_{s,1,r}$ are quasi KO_*-equivalent to $\Sigma^4 K_{r,s+1}$, $\Sigma^2 P_{r+1,s}$, $\Sigma^4 P_{s+1,r}$ and $\Sigma^2 V'N_{s+1,r}$, respectively. On the other hand, the spectrum $VRP'_{r,s,1}$ is quasi KO_*-equivalent to $R'_{r,s+1}$, $R'R_{r,s+1}$ or $V_r \vee \Sigma^4 V_{s+1}$ according as $r > s+1$, $r = s+1$ or $r \leq s$, and the spectrum $P'R^V_{s,1,r}$ is quasi KO_*-equivalent to $\Sigma^4 R_{s+1,r}$, $R'R_{s+1,r}$ or $V_{s+1} \vee \Sigma^4 V_r$ according as $r > s+1$, $r = s+1$ or $r \leq s$. Here the spectrum $R'R_{m,n}$ has been introduced in [13, Proposition 3.3]. The spectra $PP'_{r,s,t}$, $vPP'_{r,s,t}$, $P'P_{s,t,r}$, $MPP'_{r,s,t}$ and $P'PM_{s,t,r}$ were written as $U_{s,r,t}$, $V_{s,r,t}$, $U'_{s,t,r}$, $MU_{s,r,t}$ and $U'M_{s,t,r}$ in [12], respectively, and their KU-homology groups with the conjugation ψ_C^{-1} and their KO-homology groups have been obtained in [12, Propositions 2.1, 2.2, 2.3 and 2.4].

Proposition 1.1.

i) *"The $X = PP'_{r,s,t}$ or $vPP'_{r,s,t}$ case"*

	$r > t > s$	$r \geq t \leq s$
$KU_0X \cong$	$Z/2^r \oplus Z/2^t \oplus Z/2^s$	$Z/2^r \oplus Z/2^{t-1} \oplus Z/2^{s+1}$
$\psi_C^{-1} =$	$\begin{pmatrix} 1 & 2^{r-t} & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2^{r-t+1} & -2^{r-t} \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$
	$r \leq t \geq s$	$r \leq t \leq s$
$KU_0X \cong$	$Z/2^{r-1} \oplus Z/2^{t+1} \oplus Z/2^s$	$Z/2^{r-1} \oplus Z/2^t \oplus Z/2^{s+1}$
$\psi_C^{-1} =$	$\begin{pmatrix} 1 & 0 & 0 \\ -2^{t-r+2} & -1 & 0 \\ -2^{t-r+1} & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ -2^{t-r+1} & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

ii) *"The $X = MPP'_{r,s,t}$ case"*

	$r > t > s$	$r \geq t \leq s$
$KU_0X \cong$	$Z \oplus Z/2^r \oplus Z/2^t \oplus Z/2^s$	$Z \oplus Z/2^r \oplus Z/2^{t-1} \oplus Z/2^{s+1}$
$\psi_C^{-1} =$	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 2^{r-t} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 2^{r-t+1} & -2^{r-t} \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
	$r \leq t \geq s$	$r \leq t \leq s$
$KU_0X \cong$	$Z \oplus Z/2^{r-1} \oplus Z/2^{t+1} \oplus Z/2^s$	$Z \oplus Z/2^{r-1} \oplus Z/2^t \oplus Z/2^{s+1}$
$\psi_C^{-1} =$	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2^{t-r+1} & -2^{t-r+2} & -1 & 0 \\ 0 & -2^{t-r+1} & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2^{t-r} & -2^{t-r+1} & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

iii) *Their KO-homology groups KO_iX $(0 \leq i \leq 7)$ are tabled as follows:*

$X \setminus i$	0	1	2	3	4	5	6	7
$PP'_{r,s,t}$	$Z/2^r \oplus Z/2^s$	$Z/2$	$(*)_{t-1,r} \oplus Z/2$	$Z/2$	$Z/2^{r-1} \oplus Z/2^{s+1}$	0	$Z/2^t$	0
$vPP'_{r,s,t}$	$Z/2^{r-1} \oplus Z/2^s$	0	$Z/2^t \oplus Z/2$	$Z/2$	$Z/2^r \oplus Z/2^{s+1}$	$Z/2$	$(*)_{t-1,r}$	0
$MPP'_{r,s,t}$	$Z/2^r \oplus Z/2^s$	0	$Z \oplus Z/2^t \oplus Z/2$	$Z/2$	$Z/2^r \oplus Z/2^{s+1}$	0	$Z \oplus Z/2^t$	0

where $(*)_{k,1} \cong Z/2^{k+2}$ and $(*)_{k,l} \cong Z/2^{k+1} \oplus Z/2$ if $l \geq 2$.

For any spectrum X having the same $\mathcal{C}$-type as $PP'_{r,s,t}$ or $MPP'_{r,s,t}$ we have already determined its quasi KO_*-type in [12, Theorem 3.3].

Theorem 1.2.

i) *If a spectrum X has the same $\mathcal{C}$-type as $PP'_{r,s,t}$, then it is quasi KO_*-equivalent to one of the following small spectra $PP'_{r,s,t}$, $\Sigma^4 PP'_{r,s,t}$, $\nu PP'_{r,s,t}$ and $\Sigma^4\nu PP'_{r,s,t}$.*

ii) *If a spectrum X has the same $\mathcal{C}$-type as $MPP'_{r,s,t}$, then it is quasi KO_*-equivalent to either of the small spectra $MPP'_{r,s,t}$ and $\Sigma^4 MPP'_{r,s,t}$.*

Applying Theorem 1.2 we see that

(1.4) the spectra $P'P_{s,t,r}$, $P'P^V_{s,t,r}$ and $P'PM_{s,t,r}$ are quasi KO_*-equivalent to $\Sigma^2 PP'_{r+1,t-1,s}$, $\Sigma^2\nu PP'_{r+1,t-1,s}$ and $\Sigma^2 MPP'_{r+1,t-1,s}$, respectively (see [12, Corollary 3.4]).

We can also show the following result (see [12, Proposition 3.1]).

Theorem 1.3.

i) *If a spectrum X has the same $\mathcal{C}$-type as $SZ/2^r \vee P''_{s,t}$, then it is quasi KO_*-equivalent to one of the following wedge sums $SZ/2^r \vee P''_{s,t}$, $\Sigma^4 SZ/2^r \vee P''_{s,t}$, $V_r \vee P''_{s,t}$ and $\Sigma^4 V_r \vee P''_{s,t}$.*

ii) *If a spectrum X has the same $\mathcal{C}$-type as $M_r \vee P''_{s,t}$, then it is quasi KO_*-equivalent to either of the following wedge sums $M_r \vee P''_{s,t}$ and $\Sigma^4 M_r \vee P''_{s,t}$.*

1.2. Since $P'_{s,t}$ and $\Sigma^2 P'_{t-1,s+1}$ have the same $\mathcal{C}$-type a routine computation shows

Proposition 1.4.

i) *The spectra $NP'_{r,s,t}$, $VRP'_{r,s,t}$, $RP'_{r,t-1,s+1}$, $\nu RP'_{r,t-1,s+1}$, $P'R_{s,t,r}$ and $P'R^V_{s,t,r}$ have the same $\mathcal{C}$-type as the wedge sum $SZ/2^r \vee P'_{s,t}$.*

ii) *The spectra $MRP'_{r,t-1,s+1}$ and $P'RM_{s,t,r}$ have the same $\mathcal{C}$-type as the wedge sum $M_r \vee P'_{s,t}$.*

Note that if $r \geq t$ the spectra $RP'_{r,s,t}$, $\nu RP'_{r,s,t}$ and $MRP'_{r,s,t}$ are quasi KO_*-equivalent to $SZ/2^r \vee \Sigma^2 P'_{s,t}$, $V_r \vee \Sigma^2 P'_{s,t}$ and $M_r \vee \Sigma^2 P'_{s,t}$, respectively, and if $r \leq s$ the spectra $P'R_{s,t,r}$, $P'R^V_{s,t,r}$ and $P'RM_{s,t,r}$ are quasi KO_*-equivalent to $\Sigma^4 SZ/2^r \vee P'_{s,t}$, $\Sigma^4 V_r \vee P'_{s,t}$ and $\Sigma^4 M_r \vee P'_{s,t}$, respectively. By use of [13, Propositions 2.2 and 3.1] and (1.2) we can easily calculate

Proposition 1.5. *For the small spectra X listed in* Proposition 1.4 *the KO-homology groups KO_iX $(0 \leq i \leq 7)$ are tabled as follows:*

$i \backslash X$	$NP'_{r,s,t}$ $(t \geq 2)$	$RP'_{r,s,t}$ $(r < t)$	$vRP'_{r,s,t}$ $(r < t)$	$VRP'_{r,s,t}$ $(t \geq 2)$
0	$Z/2^r \oplus Z/2^s$	$Z/2^r \oplus Z/2^t$	$Z/2^{r-1} \oplus Z/2^t$	$Z/2^r \oplus Z/2^{s+1}$
1	$Z/2$	$Z/2$	0	$Z/2$
2	$Z/2^t \oplus Z/2 \oplus Z/2$	$Z/2^s \oplus (*)_r$	$Z/2^s \oplus Z/2$	$Z/2^t \oplus Z/2$
3	$Z/2 \oplus Z/2$	$Z/2$	$Z/2$	$Z/2$
4	$Z/2^{r+1} \oplus Z/2^{s+1}$	$Z/2^{r-1} \oplus Z/2^t \oplus Z/2$	$Z/2^r \oplus Z/2^t \oplus Z/2$	$Z/2^{r+1} \oplus Z/2^s$
5	$Z/2$	$Z/2$	$Z/2 \oplus Z/2$	$Z/2$
6	$Z/2^t$	$Z/2^{s+1} \oplus Z/2$	$Z/2^{s+1} \oplus (*)_r$	$Z/2^t \oplus Z/2$
7	0	$Z/2$	$Z/2$	$Z/2$

$i \backslash X$	$P'R_{s,t,r}$ $(s < r, t \geq 2)$	$P'R^V_{s,t,r}$ $(s < r, t \geq 2)$	$MRP'_{r,s,t}$ $(r < t)$	$P'RM_{s,t,r}$ $(s < r)$
0	$Z/2^s \oplus Z/2^r$	$Z/2^s \oplus Z/2^{r+1}$	$Z/2^r \oplus Z/2^t$	$Z/2^s \oplus Z/2^{r+1}$
1	0	$Z/2$	0	0
2	$Z/2^{t-1} \oplus Z/2$	$Z/2^{t-1} \oplus Z/2 \oplus Z/2$	$Z \oplus Z/2^s \oplus Z/2$	$Z \oplus Z/2^{t-1} \oplus Z/2$
3	$Z/2$	$Z/2$	$Z/2$	$Z/2$
4	$(*)_{s-1,t} \oplus Z/2^{r+1}$	$(*)_{s-1,t} \oplus Z/2^r$	$Z/2^r \oplus Z/2^s \oplus Z/2$	$(*)_{s-1,t} \oplus Z/2^{r+1}$
5	$Z/2 \oplus Z/2$	$Z/2$	$Z/2$	$Z/2$
6	$Z/2^t \oplus Z/2 \oplus Z/2$	$Z/2^t \oplus Z/2$	$Z \oplus Z/2^{s+1} \oplus Z/2$	$Z \oplus Z/2^t \oplus Z/2$
7	$Z/2$	$Z/2$	$Z/2$	$Z/2$

where $(*)_{k,1} \cong Z/2^{k+2}$ *and* $(*)_{k,l} \cong Z/2^{k+1} \oplus Z/2$ *if* $l \geq 2$, *and* $(*)_{0,l}$ *is abbreviated as* $(*)_l$.

Let N'_t, P'_t and R'_t denote the small spectra constructed as the cofibers of the following maps $\eta^2 j$, $\bar{\eta} : \Sigma^1 SZ/2^t \to \Sigma^0$ and $\eta^2\bar{\eta} : \Sigma^3 SZ/2^t \to \Sigma^0$, respectively. Consider the small spectrum $N'P'_t$ constructed as the cofiber of the map $(\eta^2 j, \bar{\eta}) : \Sigma^1 SZ/2^t \to \Sigma^0 \vee \Sigma^0$. Then we have two maps $i'_{NP} : \Sigma^0 \to N'P'_t$ and $\rho'_{NP} : \Sigma^0 \to N'P'_t$ whose cofibers are N'_t and P'_t, respectively. These two maps are related by the equality $i'_{NP}\bar{\eta} = \rho'_{NP}\eta^2 j : \Sigma^1 SZ/2^t \to N'P'_t$. In particular, $i'_{NP} = (2, \bar{i}) : \Sigma^0 \to \Sigma^0 \vee C(\bar{\eta})$ and $\rho'_{NP} = (1, 0) : \Sigma^0 \to \Sigma^0 \vee C(\bar{\eta})$ when $t = 1$. We denote by $N'P'_{r,t}$, $P'N'_{s,t}$ and $F^{n,m}_t$ the spectra constructed as the cofibers of the following maps $2^r\rho'_{NP}$, $2^s i'_{NP}$ and $f^{n,m}_t = 2^n\rho'_{NP} + 2^m i'_{NP} : \Sigma^0 \to N'P'_t$, respectively. In particular, $N'P'_{r,1} = C(\bar{\eta}) \vee SZ/2^r$ and $P'N'_{s,1}$ is quasi KO_*-equivalent to $\Sigma^4 R'_{s+1}$. On the other hand, $F^{n,m}_1 = C(\bar{\eta}) \vee SZ/2^n$ if $n \leq m$, $F^{n,m}_1 = C(\bar{\eta}) \vee V_{m+1}$ if $n = m+1$, and it is quasi KO_*-equivalent to $\Sigma^4 R'_{m+1}$ if $n > m+1$. Whenever $t \geq 2$ we can regard that the induced homomorphisms ρ'_{NP*} and $i'_{NP*} : KU_0\Sigma^0 \to KU_0 N'P'_t$ are given by $\rho'_{NP*}(1) = (1,0,0)$ and $i'_{NP*}(1) = (0,2,1)$ in $KU_0N'P'_t \cong Z \oplus Z \oplus Z/2^{t-1}$ because i'_{NP} may be replaced by $i'_{NP} + 2q\rho'_{NP}$ if necessary. Hence it is easily shown that

(1.5) i) the spectra $N'P'_{r,t}$ and $P'N'_{s,t}$ have the same $\mathcal{C}$-type as $SZ/2^r \vee P'_t$ and $\Sigma^0 \vee P'_{s,t}$, respectively, and

ii) the spectrum $F_t^{n,m}$ has the same $\mathcal{C}$-type as $SZ/2^n \vee P'_t$ when $n \leq m$, and as $\Sigma^0 \vee P'_{m,t}$ when $n > m$.

By use of [8, Proposition 4.2] and [9, Proposition 2.4] we can easily calculate the KO-homology groups KO_iX $(0 \leq i \leq 7)$ of $X = N'P'_{r,t}$, $P'N'_{s,t}$ and $F_t^{n,m}$ $(t \geq 2)$ as follows:

(1.6)

$X \setminus i$	0	1	2	3	4	5	6	7
$N'P'_{r,t}$	$Z \oplus Z/2^r$	$Z/2$	$Z/2^t \oplus Z/2$	$Z/2$	$Z \oplus Z/2^{r+1}$	$Z/2$	$Z/2^t$	0
$P'N'_{s,t}$	$Z \oplus Z/2^s$	$Z/2$	$Z/2^t \oplus Z/2$	$Z/2$	$Z \oplus Z/2^{s+1}$	$Z/2$	$Z/2^t$	0
$F_t^{n,m}$	$Z \oplus Z/2^l$	$Z/2$	$Z/2^t \oplus Z/2$	$Z/2$	$Z \oplus Z/2^{l+1}$	$Z/2$	$Z/2^t$	0

where $l = \min\{n, m\}$.

Choose two maps $h'_N : \Sigma^2 \to N'_t$ and $\bar{\rho}'_N : C(\bar{\eta}) \to N'_t$ whose cofibers coincide with $C(\eta^2)$ and V'_t, respectively, where $C(\eta^2)$ is the cofiber of the map $\eta^2 : \Sigma^2 \to \Sigma^0$ and $V'_t = P_{1,t-1}$ which is quasi KO_*-equivalent to $\Sigma^6 V_t$ (see [13]). Then there exist two maps $\lambda'_{NP} : C(\bar{\eta}) \to N'P'_t$ and $\bar{\rho}'_{NP} : C(\bar{\eta}) \to N'P'_t$ satisfying $j'_{NP}\lambda'_{NP} = h'_N\eta j\bar{j}$ and $j'_{NP}\bar{\rho}'_{NP} = \bar{\rho}'_N$ for the canonical projection $j'_{NP} : N'P'_t \to N'_t$. In particular, we may choose as $\lambda'_{NP} = (\bar{\lambda}, 2) : C(\bar{\eta}) \to \Sigma^0 \vee C(\bar{\eta})$ and $\bar{\rho}'_{NP} = (0,1) : C(\bar{\eta}) \to \Sigma^0 \vee C(\bar{\eta})$ when $t = 1$. Here the map $\bar{\lambda} : C(\bar{\eta}) \to \Sigma^0$ satisfies the equalities $\bar{\lambda}\bar{i} = 4$ and $\bar{i}\bar{\lambda} = 4$ (see [13, (1.3)]). Whenever $t \geq 2$, we can regard that the induced homomorphisms $\bar{\rho}'_{NP*}$ and $\lambda'_{NP*} : KU_0C(\bar{\eta}) \to KU_0N'P'_t$ are given by $\bar{\rho}'_{NP}(1) = (1,0,0)$ and $\lambda'_{NP*}(1) = (0,2,1)$ in $KU_0N'P'_t \cong Z \oplus Z \oplus Z/2^{t-1}$ because $\bar{\rho}'_{NP}$ and λ'_{NP} may be replaced by $\bar{\rho}'_{NP} + ki'_{PN}\bar{\lambda}$ and $\lambda'_{NP} + li'_{PN}\bar{\lambda}$ if necessary. By virtue of [13, Lemma 1.5] we obtain that the cofiber of $\bar{\rho}'_{NP}$ is quasi KO_*-equivalent to $\Sigma^4 P'_t$. On the other hand, by use of [13, Lemma 1.2 and Proposition 4.1] (or [9, Theorem 4.2]) we see that the cofiber of λ'_{NP} is quasi KO_*-equivalent to $\Sigma^4 N'_t$. More generally, the cofibers of the maps $2^r\bar{\rho}'_{NP}$ and $2^s\lambda'_{NP}$ are quasi KO_*-equivalent to $\Sigma^4 N'P'_{r,t}$ and $\Sigma^4 P'N'_{s,t}$, respectively, because $N'P'_t$ and $\Sigma^4 N'P'_t$ have the same quasi KO_*-type (see [9, Corollary 4.5]).

Using the maps $f_t^{n,m}$, $\bar{\rho}'_{NP}$ and λ'_{NP} we introduce new small spectra $N'P'F_{r,t}^{n,m}$ and $P'N'F_{s,t}^{n,m}$ constructed as the cofibers of the following maps

(1.7)
$$f_t^{n,m} \vee 2^r\bar{\rho}'_{NP} : \Sigma^0 \vee C(\bar{\eta}) \to N'P'_t,$$
$$f_t^{n,m} \vee 2^s\lambda'_{NP} : \Sigma^0 \vee C(\bar{\eta}) \to N'P'_t,$$

respectively. In particular, $N'P'F_{r,1}^{n,m}$ is equal to $(C(\bar{\eta}) \wedge SZ/2^r) \vee SZ/2^n$ if $n \leq m$, to $(C(\bar{\eta}) \wedge SZ/2^r) \vee SZ/2^{m+2}$ if $n = m+1 > r$, and to $(C(\bar{\eta}) \wedge SZ/2^r) \vee SZ/2^{m+1}$

if $n > m+1 > r$. Moreover it is quasi KO_*-equivalent to $\Sigma^4 V_{r+1} \vee V_{m+1}$, $\Sigma^4 R_{r,m+2}$ or $\Sigma^4 R'_{r,m+1}$ according as $n = m+1 < r$, $n = m+1 = r$ or $n > m+1 \leq r$ (use [6, Proposition 3.1]). On the other hand, $P'N'F^{n,m}_{s,1}$ is just $R'_{n,m+1,s+1}$ introduced in [13]. By a routine computation we can easily show

Proposition 1.6.
i) *The spectrum* $N'P'F^{n,m}_{r,t}$ $(t \geq 2)$ *has the same* $\mathcal{C}$*-type as* $SZ/2^n \vee P'_{m-n+r,t}$ *if* $m \geq n < r$, *and as* $SZ/2^r \vee P'_{m,t}$ *if otherwise.*
ii) *The spectrum* $P'N'F^{n,m}_{s,t}$ $(t \geq 2)$ *has the same* $\mathcal{C}$*-type as* $SZ/2^{n-m+s} \vee P'_{m,t}$ *if* $n > m \leq s$, *and as* $SZ/2^n \vee P'_{s,t}$ *if otherwise.*

Using (1.6) we can easily calculate

Proposition 1.7. *For the spectra* $X = N'P'F^{n,m}_{r,t}$ *and* $P'N'F^{n,m}_{s,t}$ $(t \geq 2)$ *the* KO*-homology groups* KO_iX $(0 \leq i \leq 7)$ *are tabled as follows*:

$i \backslash X$	$N'P'F^{n,m}_{r,t}$	$P'N'F^{n,m}_{s,t}$
0	$Z/2^n \oplus Z/2^{m-n+r+1}$ $(m \geq n \leq r)$ $Z/2^{r+1} \oplus Z/2^m$ (otherwise)	$Z/2^{n-m+s+1} \oplus Z/2^m$ $(n > m \leq s)$ $Z/2^n \oplus Z/2^{s+1}$ (otherwise)
1	$Z/2$	$Z/2$
2	$Z/2^t \oplus Z/2$	$Z/2^t \oplus Z/2$
3	$Z/2$	$Z/2$
4	$Z/2^{n+1} \oplus Z/2^{m-n+r}$ $(m \geq n < r)$ $Z/2^r \oplus Z/2^{m+1}$ (otherwise)	$Z/2^{n-m+s} \oplus Z/2^{m+1}$ $(n > m < s)$ $Z/2^{n+1} \oplus Z/2^s$ (otherwise)
5	$Z/2$	$Z/2$
6	$Z/2^t \oplus Z/2$	$Z/2^t \oplus Z/2$
7	$Z/2$	$Z/2$

2. The same quasi KO_*-type as $SZ/2^r \vee P'_{s,t}$ or $M_r \vee P'_{s,t}$

2.1. Let X be a spectrum having the same $\mathcal{C}$-type as $SZ/2^r \vee P'_{s,t}$. Then its self-conjugate K-homology group KC_iX $(0 \leq i \leq 3)$ is given as follows:

$$KC_iX \cong Z/2^r \oplus Z/2^s \oplus Z/2, Z/2^r \oplus Z/2^{s+1}, Z/2 \oplus Z/2 \oplus Z/2^{t-1}, Z/2 \oplus Z/2^t$$

according as $i = 0, 1, 2, 3$. In addition,

$$KO_1X \oplus KO_5X \cong Z/2 \oplus Z/2 \quad \text{and} \quad KO_3X \oplus KO_7X \cong Z/2 \oplus Z/2.$$

Hence $KO_{2i+1}X$ $(0 \leq i \leq 3)$ are divided into the nine cases (A,D) with A=a, b, c and D=d, e, f as follows:

(2.1)
$$\begin{aligned}&\text{(a) } KO_1X \cong KO_5X \cong Z/2 \quad \text{(b) } KO_5X = 0 \quad \text{(c) } KO_1X = 0\\ &\text{(d) } KO_3X \cong KO_7X \cong Z/2 \quad \text{(e) } KO_7X = 0 \quad \text{(f) } KO_3X = 0.\end{aligned}$$

The induced homomorphisms $(-\tau, \tau\pi_C)_* : KC_iX \to KO_{i+1}X \oplus KO_{i+5}X$ $(i = 0, 2)$ are represented by the following matrices

$$\Phi_0 = \varphi_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} : Z/2^r \oplus Z/2^s \oplus Z/2 \to Z/2 \oplus Z/2$$
$$\Phi_2 = \varphi_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : Z/2 \oplus Z/2 \oplus Z/2^{t-1} \to Z/2 \oplus Z/2,$$

respectively, where φ_0, $\varphi_2 : Z/2 \oplus Z/2 \to Z/2 \oplus Z/2$ is one of the following matrices:

(2.2)
$$\overset{(1)}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \overset{(2)}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \overset{(3)}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} \overset{(4)}{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} \overset{(5)}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} \overset{(6)}{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}.$$

Evidently it is sufficient to take as φ_0 or φ_2 only the matrix (1) in case of (b), (c), (e) or (f). By using a suitable transformation of KU_0X similarly to [6, 4.1] we can verify that in case of (a) the matrix (1) as φ_0 is replaced by (5), and in case of (d) the matrix (1) as φ_2 is replaced by (5) if $r \leq s$, and by (3) if $r > s$. Therefore it is sufficient to take as φ_0 the matrices (1), (2) and (3) in case of (a), and as φ_2 the matrices (1), (2) and (3) in case of (d) and $r \leq s$, and (1), (2) and (6) in case of (d) and $r > s$.

Let X be a spectrum having the same $\mathcal{C}$-type as $M_r \vee P'_{s,t}$. Then its self-conjugate K-homology group KC_iX $(0 \leq i \leq 3)$ is given as follows:

$$KC_iX \cong Z/2^r \oplus Z/2^s \oplus Z/2, Z/2^{r+1} \oplus Z/2^{s+1},$$
$$Z \oplus Z/2 \oplus Z/2 \oplus Z/2^{t-1}, Z \oplus Z/2^t$$

according as $i = 0$, 1, 2, 3. In addition,

$$KO_1X \oplus KO_5X \cong Z/2 \quad \text{and} \quad KO_3X \oplus KO_7X \cong Z/2 \oplus Z/2.$$

Hence $KO_{2i+1}X$ $(0 \leq i \leq 3)$ are divided into the six cases (A, D) with A = b, c and D = d, e, f given in (2.1). The induced homomorphisms $(-\tau, \tau\pi_C)_* : KC_iX \to KO_{i+1}X \oplus KO_{i+5}X$ $(i = 0, 2)$ are represented by the following matrices

$$\Phi_0 = (0, 0, 1) : Z/2^r \oplus Z/2^s \oplus Z/2 \to Z/2$$
$$\Phi_2 = \varphi_2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} : Z \oplus Z/2 \oplus Z/2 \oplus Z/2^{t-1} \to Z/2 \oplus Z/2,$$

where $\varphi_2 : Z/2 \oplus Z/2 \to Z/2 \oplus Z/2$ is one of the matrices given in (2.2). Evidently it is sufficient to take as φ_2 only the matrix (1) in case of (e) or (f). On the other hand, it is sufficient to take as φ_2 the matrices (1), (2) and (3) in case of (d) and $r \le s$, and (1), (2) and (6) in case of (d) and $r > s$.

Given a spectrum X having the same $\mathcal{C}$-type as $SZ/2^r \vee P'_{s,t}$ or $M_r \vee P'_{s,t}$ we define its φ-type (A, D, i, j) where A = a, b, c, D = d, e, f and $1 \le i, j \le 6$, using the above notations as in [6, §4].

Lemma 2.1.

i) *Let X be a spectrum having the same $\mathcal{C}$-type as $SZ/2^r \vee P'_{s,t}$. Then its φ-type is one of the following* 25 *types*: *(a, d, i, j), (a, e, i, 1), (a, f, i, 1), (b, d, 1, j), (c, d, 1, j), (b, e, 1, 1), (b, f, 1, 1), (c, e, 1, 1) and (c, f, 1, 1) where i = 1, 2, 3, and j = 1, 2, 3 if $r \le s$ and j = 1, 2, 6 if $r > s$.*

ii) *Let X be a spectrum having the same $\mathcal{C}$-type as $M_r \vee P'_{s,t}$. Then its φ-type is one of the following* 10 *types*: *(b, d, 1, j), (c, d, 1, j), (b, e, 1, 1), (b, f, 1, 1), (c, e, 1, 1) and (c, f, 1, 1) where j = 1, 2, 3 if $r \le s$ and j = 1, 2, 6 if $r > s$.*

2.2. Using [6, Lemmas 4.2 and 4.3] we can easily determine the φ-types of the small spectra appearing in Propositions 1.3 and 1.5.

Proposition 2.2.

i) *The spectra $NP'_{r,s,t}$, $VRP'_{r,s,t}$ ($t \ge 2$), $RP'_{r,t-1,s+1}$, $_VRP'_{r,t-1,s+1}$, $P'R_{s,t,r}$ and $P'R^V_{s,t,r}$ ($t \ge 2$) have the following φ-types (a, e, 3, 1), (a, d, 4, 1), (a, d, 1, 3), (c, d, 1, 3), (c, d, 1, 6) and (a, d, 1, 6), respectively.*

ii) *The spectra $MRP'_{r,t-1,s+1}$ and $P'RM_{s,t,r}$ have the following φ-types (c, d, 1, 3) and (c, d, 1, 6), respectively.*

iii) *The spectrum $N'P'F^{n,m}_{r,t}$ ($t \ge 2$) has the following φ-type (a, d, 4, 1), (a, d, 4, 3) or (a, d, 4, 2) according as $m \ge n < r$, $m \ge n = r$ or otherwise, and the spectrum $P'N'F^{n,m}_{s,t}$ ($t \ge 2$) has the following φ-type (a, d, 4, 2), (a, d, 4, 6) or (a, d, 4, 1) according as $n > m < s$, $n > m = s$ or otherwise.*

Let X be a spectrum having the same $\mathcal{C}$-type as $SZ/2^r \vee P'_{s,t}$ or $M_r \vee P'_{s,t}$. If a spectrum Y has the same $\mathcal{C}$-type as X, then we can choose a quasi KU_*-equivalence $f : Y \to KU \wedge X$ with $(\psi_C^{-1} \wedge 1)f = f$. If there exists a map $h : Y \to KO \wedge X$ satisfying $(\epsilon_U \wedge 1)h = f$ for the complexification map $\epsilon_U : KO \to KU$, then h becomes a quasi KO_*-equivalence (see [8, Proposition 1.1]). After choosing a suitable small spectrum Y having the same φ-type as X we can prove the following theorems by applying the same method developed in [6], [8] or [9].

Theorem 2.3. *Let X be a spectrum having the same $\mathcal{C}$-type as $SZ/2^r \vee P'_{s,t}$ ($t \ge$ 2). Then it is quasi KO_*-equivalent to one of the following small spectra (cf.* [6, Theorem 5.3]):

i) *The case of* $r \le s: Y_r \vee Y_{s,t}$, $NP'_{r,s,t}$, $\Sigma^4 NP'_{r,s,t}$, $vRP'_{r,t-1,s+1}$, $\Sigma^4 vRP'_{r,t-1,s+1}$, $VRP'_{r,s,t}$, $\Sigma^4 VRP'_{r,s,t}$, $RP'_{r,t-1,s+1}$, $\Sigma^4 RP'_{r,t-1,s+1}$, $N'P'F^{r,s}_{r,t}$.

ii) *The case of* $r > s: Y_r \vee Y_{s,t}$, $NP'_{r,s,t}$, $\Sigma^4 NP'_{r,s,t}$, $P'R_{s,t,r}$, $\Sigma^4 P'R_{s,t,r}$, $VRP'_{r,s,t}$, $\Sigma^4 VRP'_{r,s,t}$, $P'R^V_{s,t,r}$, $\Sigma^4 P'R^V_{s,t,r}$, $P'N'F^{r,s}_{s,t}$.

Here $Y_r = SZ/2^r$, $\Sigma^4 SZ/2^r$, V_r *or* $\Sigma^4 V_r$, *and* $Y_{s,t} = P'_{s,t}$, $\Sigma^4 P'_{s,t}$, $\Sigma^2 P'_{t-1,s+1}$ *or* $\Sigma^6 P'_{t-1,s+1}$.

Theorem 2.4. *Let* X *be a spectrum having the same* $\mathcal{C}$*-type as* $M_r \vee P'_{s,t}$. *Then it is quasi* KO_**-equivalent to one of the following small spectra*:

i) *The case of* $r \le s: Y_r \vee Y_{s,t}$, $MRP'_{r,t-1,s+1}$, $\Sigma^4 MRP'_{r,t-1,s+1}$.

ii) *The case of* $r > s: Y_r \vee Y_{s,t}$, $P'RM_{s,t,r}$, $\Sigma^4 P'RM_{s,t,r}$.

Here $Y_r = M_r$ *or* $\Sigma^4 M_r$, *and* $Y_{s,t} = P'_{s,t}$, $\Sigma^4 P'_{s,t}$, $\Sigma^2 P'_{t-1,s+1}$ *or* $\Sigma^6 P'_{t-1,s+1}$.

Combining Theorem 2.3 with Proposition 2.2 iii) we get

Corollary 2.5.

i) *The spectrum* $N'P'F^{n,m}_{r,t}$ $(t \ge 2)$ *is quasi* KO_**-equivalent to* $VRP'_{n,m-n+r,t}$ *if* $m \ge n < r$, *and to* $\Sigma^4 VRP'_{r,m,t}$ *if* $m \ge n > r$ *or* $m < n$.

ii) *The spectrum* $P'N'F^{n,m}_{s,t}$ $(t \ge 2)$ *is quasi* KO_**-equivalent to* $\Sigma^4 VRP'_{n-m+s,m,t}$ *if* $n > m < s$, *and to* $VRP'_{n,s,t}$ *if* $n > m > s$ *or* $n \le m$.

3. Weighted mod 4 lens spaces

3.1. Let $S^{2n+1}(q_0, \cdots, q_n)$ denote the unit sphere $S^{2n+1} \subset C^{n+1}$ with S^1-action defined by $\lambda \cdot (x_0, \cdots, x_n) = (\lambda^{q_0} x_0, \cdots, \lambda^{q_n} x_n) \in C^{n+1}$ for any $\lambda \in S^1 \subset C$. Then we set

$$P^n(q_0, \cdots, q_n) = S^{2n+1}(q_0, \cdots, q_n)/S^1$$

$$L^n(q; q_0, \cdots, q_n) = S^{2n+1}(q_0, \cdots, q_n)/(Z/q)$$

where Z/q is the q-th roots of the unity in $S^1 \subset C$. Denote by $L^n_0(q; q_0, \cdots, q_n)$ the subspace of $L^n(q; q_0, \cdots, q_n)$ defined by

$$L^n_0(q; q_0, \cdots, q_n) = \{[x_0, \cdots, x_n] \in L^n(q; q_0, \cdots, q_n) | x_n \text{ is real } \ge 0\}.$$

Of course, $P^n(1, \cdots, 1)$, $L^n(q; 1, \cdots, 1)$ and $L^n_0(q; 1, \cdots, 1)$ are the usual complex projective space CP^n, the usual mod q lens space $L^n(q)$ and its $2n$-skeleton $L^n_0(q)$, respectively. For a weighted mod 4 lens space $L^n(4; q_0, \cdots, q_n)$ we may assume that $q_0 = \cdots = q_{r-1} = 4$, $q_r = \cdots = q_{r+s-1} = 2$ and $q_{r+s} = \cdots = q_n = 1$ where $0 \le r \le r+s \le n$. For such a tuple $(q_0, \cdots, q_n)$ we simply set $P(r,s,t) = P^n(q_0, \cdots, q_n)$, $L(r,s,t) = L^n(4; q_0, \cdots, q_n)$ and $L_0(r,s,t) = L^n_0(4; q_0, \cdots, q_n)$ with $n = r+s+t$.

Moreover we shall omit the "r" as $P(s,t)$, $L(s,t)$ or $L_0(s,t)$ when $r=0$. Notice that $L(r,s,t)=\Sigma^{2r}L(s,t)$ and $L_0(r,s,t)=\Sigma^{2r}L_0(s,t)$.

Denote by γ the canonical line bundle over CP^n and set $a=[\gamma]-1\in KU^0CP^n$. Then it is well known that the (reduced) KU-cohomology group $KU^*CP^n_+\cong Z[a]/(a^{n+1})$ where CP^n_+ denotes the disjoint union of CP^n and a point. According to [1, Theorem 3.1] the map $\varphi: CP^n\to P(r,s,t)$ defined by $\varphi[x_0,\cdots,x_n]=[x_0^{q_0},\cdots,x_n^{q_n}]$ with $n=r+s+t$ induces a monomorphism $\varphi^*: KU^*P(r,s,t)\to KU^*CP^n$ and the free abelian group $KU^*P(r,s,t)$ has the following basis $\{T_1,\cdots,T_n\}$ such that $\varphi^*T_l=a(2)^l$ for $1\le l\le r$, $\varphi^*T_{r+k}=a(2)^r a(1)^k$ for $1\le k\le s$ and $\varphi^*T_{r+s+h}=a(2)^r a(1)^s a^h$ for $1\le h\le t$, where $a(1)=(a+1)^2-1$ and $a(2)=(a+1)^4-1$.

In order to calculate the KU-cohomology group $KU^*L(s,t)$ we use the following cofiber sequence

$$L(s,t)\xrightarrow{\theta}P(s,t)\xrightarrow{i}P(1,s,t) \tag{3.1}$$

where θ is the natural surjection and i is the canonical inclusion (cf. [3, Assertion 1]). Since $a(2)=2a(1)+a(1)^2=2a(1)+2a(1)a+a(1)a^2=4a+6a^2+4a^3+a^4$, the induced homomorphism $i^*: KU^*P(1,s,t)\to KU^*P(s,t)$ is given as follows: $i^*T_k=2T_k+T_{k+1}$ for $1\le k\le s-1$, $i^*T_s=2T_s+2T_{s+1}+T_{s+2}$, $i^*T_{s+h}=4T_{s+h}+6T_{s+h+1}+4T_{s+h+2}+T_{s+h+3}$ for $1\le h\le t$ and $i^*T_{s+t+1}=0$. Using the (n,n)-matrix $E_k=(e_k,\cdots,e_n,0,\cdots,0)$ we here introduce the two (n,n)-matrices $A_n=2E_1+E_2$ and $B_n=4E_1+6E_2+4E_3+E_4$, where e_j is the unit column vector entried "1" only in the j-th component. Moreover we set

$$C_{s,t}=\begin{pmatrix}A_s & 0\\ \xi & B_t\end{pmatrix}\quad\text{where } \xi=(0,\cdots,0,2e_1+e_2).$$

Then the induced homomorphism $i^*: KU^0P(1,s,t)\to KU^0P(s,t)$ is expressed as $(C_{s,t},0): \bigoplus_{s+t+1}Z\to\bigoplus_{s+t}Z$. Therefore $KU^0L(s,t)\cong \mathrm{Coker}C_{s,t}$ and $KU^1L(s,t)\cong Z$. In particular, $KU^0L^n(2)\cong KU^0L(n,0)\cong\mathrm{Coker}A_n$ and $KU^0L^n(4)\cong\mathrm{Coker}B_n$.

Recall that the KU-cohomology groups $KU^0L^n(2)\cong Z[\sigma]/(\sigma^{n+1},\sigma(1))$ and $KU^0L^n(4)\cong Z[\sigma]/(\sigma^{n+1},\sigma(2))$ are given as follows (see [4, 5]):

i) $KU^0L^n(2)\cong Z/2^n$ with generator σ,

ii) $KU^0L^{2m}(4)\cong Z/2^{2m+1}\oplus Z/2^m\oplus Z/2^{m-1}$ with generators σ, $\sigma(1)$ and $\sigma(1)\sigma$, $KU^0L^{2m+1}(4)\cong Z/2^{2m+2}\oplus Z/2^m\oplus Z/2^m$ with generators σ, $\sigma(1)+2^{m+1}\sigma$ and $\sigma(1)\sigma$, where $\sigma=\theta^*a$ and $\sigma(i)=\theta^*a(i)$.

Therefore the induced homomorphism $\theta^*: KU^0CP^n\to KU^0L^n(2)$ is given by the following row:

$$\alpha_n=(-1)^{n-1}(1,-2,\cdots,(-2)^{n-1}):\bigoplus_n Z\to Z/2^n. \tag{3.2}$$

On the other hand, the induced homomorphism $\theta^* : KU^0CP^n \to KU^0L^n(4)$ is represented by the following $(3,n)$-matrix β_n:

$$\text{(3.3)}\quad \beta_{2m} = \begin{pmatrix} 1 & -2 & 4-2^{m+1} & * \\ 0 & 1 & -2 & * \\ 0 & 0 & 1 & * \end{pmatrix},\ \beta_{2m+1} = \begin{pmatrix} 1 & -2-2^{m+1} & 4+2^{m+2} & * \\ 0 & 1 & -2 & * \\ 0 & 0 & 1 & * \end{pmatrix}.$$

Notice that $KU^0L(s,t)$ is isomorphic to the cokernel of

$$\begin{pmatrix} A_s \\ \beta_t\xi \end{pmatrix} : \bigoplus_s Z \to (\bigoplus_s Z) \oplus \mathrm{Coker} B_t.$$

Since $\beta_{2m}\xi = (0,\cdots,0,e_2)$ and $\beta_{2m+1}\xi = (0,\cdots,0,-2^{m+1}e_1+e_2)$, we can easily calculate the KU-cohomology group $KU^0L(s,t)$ for $t \geq 1$ as follows:

$$\text{(3.4)}\quad \begin{aligned} KU^0L(s,2m) &\cong Z/2^{s+m} \oplus Z/2^{2m+1} \oplus Z/2^{m-1} \\ KU^0L(s,2m+1) &\cong \begin{cases} Z/2^{s+m} \oplus Z/2^{2m+2} \oplus Z/2^{m} & (s \leq m) \\ Z/2^{s+m+1} \oplus Z/2^{2m+1} \oplus Z/2^{m} & (s > m). \end{cases} \end{aligned}$$

Moreover we see that the quotient morphism $\delta_{s,t} : (\bigoplus_s Z) \oplus \mathrm{Coker} B_t \to KU^0L(s,t)$ is represented by the following matrix:

$$\underset{t=2m}{\begin{pmatrix} \alpha_s & 0 & -2^s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \underset{t=2m+1>2s}{\begin{pmatrix} \alpha_s & 0 & -2^s & 0 \\ 2^{m-s+1}\alpha_s & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \underset{t=2m+1<2s}{\begin{pmatrix} \alpha_s & 2^{s-m-1} & 0 & 0 \\ 0 & 1 & 2^{m+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}.$$

Since the induced homomorphism $\theta^* : KU^0P(s,t) \to KU^0L(s,t)$ is expressed as the composition $\delta_{s,t}(1 \oplus \beta_t)$, we can immediately give a basis of $KU^0L(s,t)$ $(s,t \geq 1)$ as follows:

$$\text{(3.5)}\quad (\sigma(1), \sigma(s,1), \sigma(s,3))B'_{s,t}$$

where $\sigma(1) = \theta^*T_1$, $\sigma(s,i) = \theta^*T_{s+i}$ and $B'_{s,t}$ $(s,t \geq 1)$ is the matrix tabled below:

$$\text{(3.6)}\quad B'_{s,2m} = \begin{pmatrix} (-1)^{s-1} & 0 & (-1)^s 2^{s+1} \\ 0 & 1 & 2^{m+1}-4 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B'_{s,2m+1} = \underset{s \leq m}{\begin{pmatrix} (-1)^{s-1} & 0 & (-1)^s 2^{s+1} \\ -2^{m-s+1} & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}} \underset{s>m}{\begin{pmatrix} (-1)^{s-1} & (-1)^s 2^{s-m-1} & (-1)^s 2^{s+1} \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}}.$$

3.2. Next we shall investigate the behavior of the conjugation ψ_C^{-1} on $KU^0L(s,t)$ $(s,t \geq 1)$. Note that $\psi_C^{-1}a^h = (-1)^h a^h(1+a)^{-h}$ and $\psi_C^{-1}a(1)^k = (-1)^k a(1)^k(1+a)^{-2k}$ in KU^0CP^n. Since $a(2) = (1+a(1))^2 - 1$ and $a(1)^s a(2) = a(1)^s\{(a+1)^4 - 1\}$ it follows immediately that

$$\psi_C^{-1}a(1) \equiv a(1) \bmod a(2)$$

$$\psi_C^{-1}a(1)^s a \equiv \begin{cases} a(1)^s(a^3+3a^2+3a) \\ a(1)^s(a^2+a) \end{cases} \bmod \ a(1)^s a(2) \quad \begin{matrix} s: \text{even} \\ s: \text{odd} \end{matrix}$$

$$\psi_C^{-1}a(1)^s a^3 \equiv \begin{cases} a(1)^s(3a^3+6a^2+4a) \\ -a(1)^s(a^3+2a^2+4a) \end{cases} \bmod \ a(1)^s a(2) \quad \begin{matrix} s: \text{even} \\ s: \text{odd.} \end{matrix}$$

Since $a(1)^s a^2 \equiv (-1)^s 2^s a(1) - 2a(1)^s a \bmod a(2)$, the conjugation ψ_C^{-1} on $KU^0L(s,t)$ behaves as

$$\psi_C^{-1}(\sigma(1), \sigma(s,1), \sigma(s,3)) = (\sigma(1), \sigma(s,1), \sigma(s,3))P_s$$

for the following matrix P_s:

$$(3.7) \quad P_{2n} = \begin{pmatrix} 1 & 3\cdot 2^n & 3\cdot 2^{2n+1} \\ 0 & -3 & -8 \\ 0 & 1 & 3 \end{pmatrix}, \quad P_{2n+1} = \begin{pmatrix} 1 & -2^{2n+1} & 2^{2n+2} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Consider the following matrix $C_{s,t}$ $(s,t \geq 1)$ representing an automorphism on $KU^0L(s,t)$:

$$(3.8) \quad C_{s,2m} = \underset{s=2n\leq m,\, s=2n+1}{\begin{pmatrix} 1 & 2^{s-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \qquad \underset{s=2n>m}{\begin{pmatrix} 1 & 2^{s-m-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$C_{s,2m+1} = \underset{s=2n\leq m}{\begin{pmatrix} 1+2^m & 0 & -2^s \\ 0 & 1 & 0 \\ -2^{m-s} & 0 & 1 \end{pmatrix}} \quad \underset{s=2n+1\leq m}{\begin{pmatrix} 1 & 2^{s-1}(1-2^m) & -2^s(1-2^m) \\ 2^{2m-s+1} & 1+2^{2s}(1-2^m) & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\underset{s=2n>m\geq 0}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{s=2n+1>m\geq 1}{\begin{pmatrix} 1 & 2^{s-m}+2^{s-1} & 2^{s+m} \\ 0 & 1 & 2^{m+1} \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{s=2n+1>m=0}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}.$$

In order to express the conjugation ψ_C^{-1} on $KU^0L(s,t)$ plainly we here change the basis of $KU^0L(s,t)$ given in (3.5) slightly as follows:

$$(3.9) \qquad (\sigma(1), \sigma(s,1), \sigma(s,3))B_{s,t} \text{ where } B_{s,t} = B'_{s,t}C_{s,t}.$$

Then the conjugation ψ_C^{-1} on $KU^0L(s,t)$ is represented by the composition $B_{s,t}^{-1}P_sB_{s,t}$. Therefore a routine computation shows

Proposition 3.1. *On the KU-cohomology group $KU^0L(s,t)$ with basis $(\sigma(1), \sigma(s,1), \sigma(s,3))B_{s,t}$ $(s,t \geq 1)$ the conjugation ψ_C^{-1} behaves as follows:*
i) *On $KU^0L(s,2m) \cong Z/2^{s+m} \oplus Z/2^{2m+1} \oplus Z/2^{m-1}$,*

$$\psi_C^{-1} = \underset{s=2n\leq m}{\begin{pmatrix} 1 & -2^s & 2^{s+1} \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}} \underset{s=2n>m}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-2^{m+1} & 2^{m+2} \\ 0 & 1 & -1 \end{pmatrix}} \underset{s=2n+1}{\begin{pmatrix} 1 & 0 & 2^{s+1} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}.$$

ii) *On $KU^0L(s,2m+1) \cong Z/2^{s+m} \oplus Z/2^{2m+2} \oplus Z/2^m$ $(s \leq m)$,*

$$\psi_C^{-1} = \underset{s=2n\leq m}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-2^{m+1} & 2^{m+2} \\ 0 & 1 & -1 \end{pmatrix}} \underset{s=2n+1\leq m}{\begin{pmatrix} 1 & 0 & 0 \\ 2^{m-s+2} & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}.$$

iii) *On $KU^0L(s,2m+1) \cong Z/2^{s+m+1} \oplus Z/2^{2m+1} \oplus Z/2^m$ $(s > m)$,*

$$\psi_C^{-1} = \underset{s=2n>m\geq 0}{\begin{pmatrix} 1 & -2^s & 2^{s+1} \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}} \underset{s=2n+1>m\geq 1}{\begin{pmatrix} 1 & 2^{s-m} & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}} \underset{s=2n+1>m=0}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}.$$

REMARK. When $t = 0$, the conjugation $\psi_C^{-1} = 1$ on $KU^0L(s,0) \cong Z/2^s$ with basis $\sigma(1)$.

We shall use the dual of Proposition 3.1 to study the behavior of the conjugation ψ_C^{-1} on $KU_*L_0(s,t)$ and $KU_*L(s,t)$.

Proposition 3.2. *The weighted mod 4 lens spaces $\Sigma^1L_0(s,t)$ and $\Sigma^1L(s,t)$ $(s \geq 1, t \geq 0)$ have the same $\mathcal{C}$-types as the small spectra tabled below, respectively (cf. [12, Proposition 5.1]):*

	$\Sigma^1L_0(s,2m)$	$\Sigma^1L(s,2m)$	$\Sigma^1L_0(s,2m+1)$
$s=2n\leq m$	$PP'_{2m+1,s+m-1,m}$	$MPP'_{2m+1,s+m-1,m}$	$SZ/2^{s+m} \vee P''_{2m+1,m+1}$
$s=2n>m$	$SZ/2^{s+m} \vee P''_{2m,m}$	$M_{s+m} \vee P''_{2m,m}$	$PP'_{2m+1,s+m,m+1}$
$s=2n+1, m\geq 1$	$\Sigma^2SZ/2^{2m+1} \vee P'_{s+m-1,m}$	$\Sigma^2M_{2m+1} \vee P'_{s+m-1,m}$	$\Sigma^2SZ/2^m \vee P'_{s+m,2m+2}$
$s=2n+1, m=0$	$SZ/2^s$	$\Sigma^0 \vee SZ/2^s$	$SZ/2 \vee SZ/2^{s+1}$

Moreover $\Sigma^1 L(s, 2m+1)$ *has the same* $\mathcal{C}$*-type as the wedge sum* $\Sigma^{2s} \vee \Sigma^1 L_0(s, 2m+1)$.

Proof. By dualizing Proposition 3.1 we can immediately determine the $\mathcal{C}$-type of $\Sigma^1 L_0(s,t)$ because $KU_{-1}L_0(s,t) \cong KU^0 L_0(s,t)$ and $KU_0 L_0(s,t) = 0$. On the other hand, Proposition 3.4 below implies that $\Sigma^1 L(s, 2m+1)$ has the same $\mathcal{C}$-type as $\Sigma^{2s} \vee \Sigma^1 L_0(s, 2m+1)$. We shall now investigate the $\mathcal{C}$-type of $\Sigma^1 L(s, 2m)$ in case of $s = 2n \leq m$. Note that $KU_{-1}L(s,t) \cong KU_{-1}\Sigma^{2s+2t+1} \oplus KU_{-1}L_0(s,t)$ and $KU_0 L(s,t) = 0$. According to the dual of Proposition 3.1 the conjugations ψ_C^{-1} on $KU_{-1}L(s,2m) \cong Z \oplus Z/2^{s+m} \oplus Z/2^{2m+1} \oplus Z/2^{m-1}$ and $KU_{-1}L_0(s, 2m+1) \cong Z/2^{s+m} \oplus Z/2^{2m+2} \oplus Z/2^m$ are represented by the following matrices

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & -2^{m+1} & 1 & 2^{m+2} \\ c & 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-2^{m+1} & 2^{m+2} \\ 0 & 1 & -1 \end{pmatrix}$$

for some integers a, b and c, respectively. As is easily verified, we may regard that $a = c = 0$ and $b = 0$ or -1 after changing the direct sum decomposition of $KU_{-1}L(s, 2m)$ suitably if necessary. Consider the canonical inclusion map $i_{L_0} : L(s,t) \to L_0(s, t+1)$. By virtue of (3.9) the induced homomorphism $i^*_{L_0} : KU^0 L_0(s, t+1) \to KU^0 L(s,t)$ is actually represented by the matrix $F_{s,t} = B^{-1}_{s,t} B_{s,t+1}$. Since a routine computation shows that

$$F_{s,2m} = \begin{pmatrix} 1+2^m & 0 & -2^s \\ -2^{m-s+2}(1+2^{m-1}) & 1 & 2^{m+1} \\ -2^{m-s+1} & 0 & 1 \end{pmatrix},$$

the induced homomorphism $i_{L_0 *} : KU_{-1}L(s,2m) \to KU_{-1}L_0(s, 2m+1)$ is expressed as the following matrix

$$\begin{pmatrix} x & 1+2^m & -2(1+2^{m-1}) & -2^{m+2} \\ y & 0 & 2 & 0 \\ z & -1 & 1 & 2 \end{pmatrix}$$

for some integers x, y and z. Here y must be odd because $i_{L_0 *}$ is an epimorphism. Using the equality $\psi_C^{-1} i_{L_0 *} = i_{L_0 *} \psi_C^{-1}$ we get immediately that $b \equiv y \bmod 2^m$, thus $b = -1$. Therefore $\Sigma^1 L(s, 2m)$ has the same $\mathcal{C}$-type as $MPP'_{2m+1, s+m-1, m}$ when $s = 2n \leq m$. In the other three cases the $\mathcal{C}$-types of $\Sigma^1 L(s, 2m)$ are similarly obtained. □

3.3. Using Proposition 3.2 we can immediately calculate $KO_i X \oplus KO_{i+4} X$ ($i = 0, 2$) for $X = L_0(s,t)$ and $L(s,t)$ ($s \geq 1, t \geq 0$) as tabled below:

$X=$	$L_0(2n,2m)$	$L(2n,2m)$	$L_0(2n,2m+1)$	$L(2n,2m+1)$
$KO_0X\oplus KO_4X\cong$	$Z/2$	0	$Z/2$	$Z/2\oplus Z/2$
$KO_2X\oplus KO_6X\cong$	$Z/2$	$Z/2$	$Z/2$	$Z/2$
$X=$	$L_0(2n+1,2m)$	$L(2n+1,2m)$	$L_0(2n+1,2m+1)$	$L(2n+1,2m+1)$
$KO_0X\oplus KO_4X\cong$	$(**)_m$	$Z/2\oplus Z/2$	$Z/2\oplus Z/2$	$Z/2\oplus Z/2$
$KO_2X\oplus KO_6X\cong$	$(**)_m$	$Z/2$	$Z/2\oplus Z/2$	$Z/2\oplus Z/2\oplus Z/2$

where $(**)_0\cong Z/2$ and $(**)_m\cong Z/2\oplus Z/2$ if $m\geq 1$.

Lemma 3.3. *For $X=L_0(s,t)$ and $L(s,t)$ $(s\geq 1, t\geq 0)$ the sets $S(X)=\{2i; KO_{2i}X=0\ (0\leq i\leq 3)\}$ are given as follows*:

(i)

$X=$	$L_0(2n,2m)$	$L(2n,2m)$	$L_0(2n,2m+1)$	$L(2n,2m+1)$	
$S(X)=$	$\{4,6\}$	$\{0,4,6\}$	$\{0,6\}$	$\{0,6\}$	$n+m$: even
	$\{0,6\}$	$\{0,4,6\}$	$\{4,6\}$	$\{4,6\}$	$n+m$: odd

(ii)

$X=$	$L_0(2n+1,2m)$	$L(2n+1,2m)$	$L_0(2n+1,2m+1)$	$L(2n+1,2m+1)$	
$S(X)=$	$\{0,6\}$	$\{0,6\}$	$\{0\}$	$\{0\}$	n,m : even
	$\{0\}$	$\{0,6\}$	$\{0,6\}$	$\{0,6\}$	$n,m+1$: even
	$\{4,(6)\}_m$	$\{4,6\}$	$\{4,6\}$	$\{4,6\}$	$n,m+1$: odd
	$\{4,6\}$	$\{4,6\}$	$\{4\}$	$\{4\}$	n,m : odd

where $\{4,(6)\}_0=\{4,6\}$ *and* $\{4,(6)\}_m=\{4\}$ *if* $m\geq 1$.

Proof. Consider the following (homotopy) commutative diagram

$$\begin{array}{ccccc} L_0(s,t) & \xrightarrow{\theta_0} & P(s,t) & \xrightarrow{i_0} & P(1,s,t-1) \\ i_L\downarrow & & \| & & \downarrow\tilde{i} \\ L(s,t) & \xrightarrow{\theta} & P(s,t) & \xrightarrow{i} & P(1,s,t) \end{array}$$

with two cofiber sequences, where the maps i_L, i and $\tilde{i}$ are the canonical inclusions, and the map i_0 is defined by $i_0[x_0,\cdots,x_{s+t}]=[x_{s+t}^4,x_0,\cdots,x_{s+t}]$. According to [7, Theorem 2.4] the weighted projective space $P(s,t)$ is quasi KO_*-equivalent to the wedge sum $\vee_{n+m}C(\eta)$, $\Sigma^{4n+4m+4}\vee(\vee_{n+m}C(\eta))$, $\Sigma^{4n+2}\vee(\vee_{n+m}C(\eta))$ or $\Sigma^{4n+2}\vee\Sigma^{4n+4m+4}\vee(\vee_{n+m}C(\eta))$ according as $(s,t)=(2n,2m)$, $(2n,2m+1)$, $(2n+1,2m)$ or $(2n+1,2m+1)$. In addition, $P(1,s,t)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^2\vee\Sigma^2P(s,t)$. Using the above commutative diagram we can immediately obtain our result. □

Proposition 3.4. *The weighted mod 4 lens space $L(s,2m+1)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^{2s+4m+3}\vee L_0(s,2m+1)$.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} \Sigma^{2s+4m+3} & \xrightarrow{\tilde{\alpha}} & P(1,s,2m) & \xrightarrow{\bar{i}} & P(1,s,2m+1) \\ \| & & \downarrow & & \downarrow \\ \Sigma^{2s+4m+3} & \xrightarrow{\alpha} & \Sigma^1 L_0(s,2m+1) & \xrightarrow{i_L} & \Sigma^1 L(s,2m+1) \end{array}$$

with two cofiber sequences. Since the quasi KO_*-type of $P(1,s,t)$ is given as in the proof of Lemma 3.3 we see that the map $1 \wedge \tilde{\alpha} : \Sigma^{2s+4m+3}KO \to KO \wedge P(1,s,2m)$ is trivial. Hence our result is immediate. □

Applying Theorems 1.2 and 1.3 and Proposition 3.4 with the aid of Proposition 3.2 and Lemma 3.3 we can immediately obtain

Theorem 3.5. *The weighted mod* 4 *lens spaces* $\Sigma^1 L_0(2n,t)$ *and* $\Sigma^1 L(2n,t)$ *for* $n \geq 1$ *are quasi* KO_**-equivalent to the small spectra tabled below, respectively (cf.* [12, Theorem 3]):

		$\Sigma^1 L_0(2n,2m)$	$\Sigma^1 L(2n,2m)$	$\Sigma^1 L_0(2n,2m+1)$
i)	$n+m$: even	$PP'_{2m+1,2n+m-1,m}$	$MPP'_{2m+1,2n+m-1,m}$	$V_{2n+m} \vee P''_{2m+1,m+1}$
	$n+m$: odd	$vPP'_{2m+1,2n+m-1,m}$	$MPP'_{2m+1,2n+m-1,m}$	$SZ/2^{2n+m} \vee P''_{2m+1,m+1}$
ii)	$n+m$: even	$SZ/2^{2n+m} \vee P''_{2m,m}$	$M_{2n+m} \vee P''_{2m,m}$	$vPP'_{2m+1,2n+m,m+1}$
	$n+m$: odd	$V_{2n+m} \vee P''_{2m,m}$	$M_{2n+m} \vee P''_{2m,m}$	$PP'_{2m+1,2n+m,m+1}$

in cases when i) $2n \leq m$ *and* ii) $2n > m$. *Moreover* $\Sigma^1 L(2n,2m+1)$ *is quasi* KO_**-equivalent to* $\Sigma^{4n+4m+4} \vee \Sigma^1 L_0(2n,2m+1)$.

Applying Theorems 2.3 and 2.4 in place of Theorems 1.2 and 1.3 we show

Theorem 3.6. *The weighted mod* 4 *lens spaces* $\Sigma^1 L_0(2n+1,t)$ *and* $\Sigma^1 L(2n+1,t)$ *are quasi* KO_**-equivalent to the small spectra tabled below, respectively*:

	$\Sigma^1 L_0(2n+1,2m)$	$\Sigma^1 L(2n+1,2m)$	$\Sigma^1 L_0(2n+1,2m+1)$
i)	V_{2n+1}	$\Sigma^4 \vee V_{2n+1}$	$\Sigma^4 SZ/2 \vee V_{2n+2}$
ii)	$\Sigma^2 SZ/2^{2m+1} \vee P'_{2n+m,m}$	$\Sigma^2 M_{2m+1} \vee P'_{2n+m,m}$	$\Sigma^2 V_m \vee P'_{2n+m+1,2m+2}$
iii)	$\Sigma^2 V_{2m+1} \vee P'_{2n+m,m}$	$\Sigma^2 M_{2m+1} \vee P'_{2n+m,m}$	$\Sigma^2 SZ/2^m \vee P'_{2n+m+1,2m+2}$
iv)	$SZ/2^{2n+1}$	$\Sigma^0 \vee SZ/2^{2n+1}$	$SZ/2 \vee SZ/2^{2n+2}$
v)	$\Sigma^6 SZ/2^{2m+1} \vee \Sigma^6 P'_{m-1,2n+m+1}$	$\Sigma^6 M_{2m+1} \vee \Sigma^6 P'_{m-1,2n+m+1}$	$\Sigma^6 V_m \vee \Sigma^6 P'_{2m+1,2n+m+2}$
vi)	$\Sigma^6 V_{2m+1} \vee \Sigma^6 P'_{m-1,2n+m+1}$	$\Sigma^6 M_{2m+1} \vee \Sigma^6 P'_{m-1,2n+m+1}$	$\Sigma^6 SZ/2^m \vee \Sigma^6 P'_{2m+1,2n+m+2}$

in cases when i) n *is even and* $m = 0$, ii) n *and* $m \geq 2$ *are even,* iii) n *is even and* m *is odd,* iv) n *is odd and* $m = 0$, v) n *is odd and* $m \geq 2$ *is even, and* vi) n *and* m *are odd. Moreover* $\Sigma^1 L(2n+1,2m+1)$ *is quasi* KO_**-equivalent to* $\Sigma^{4n+4m+6} \vee \Sigma^1 L_0(2n+1,2m+1)$.

Proof. By a quite similar argument to the case of the real projective space RP^k (cf. [10, Theorem 5]) we can easily determine the quasi KO_*-types of $\Sigma^1 L_0(2n+1,0)$ and $\Sigma^1 L(2n+1,0)$. The quasi KO_*-type of $\Sigma^1 L(2n+1,2m)$ for $m \geq 1$ is immediately determined by applying Theorem 2.4 ii) with the aid of Proposition 3.2 and Lemma 3.3. On the other hand, the quasi KO_*-types of $\Sigma^1 L_0(2n+1,2m)$ in cases of ii) and vi) and those of $\Sigma^1 L_0(2n+1,2m+1)$ in cases of iii), iv) and v) are also determined by applying Theorem 2.3 and [6, Theorem 5.3] in place of Theorem 2.4 ii).

We shall now investigate the quasi KO_*-types of $\Sigma^1 L_0(2n+1,2m-1)$ and $\Sigma^1 L_0(2n+1,2m)$ in case when n is even and m is odd. Consider the following two cofiber sequences

$$\begin{array}{l} \Sigma^{4n+4m} \xrightarrow{\alpha_0} \Sigma^1 L(2n+1,2m-2) \xrightarrow{i_{L_0}} \Sigma^1 L_0(2n+1,2m-1) \\ \Sigma^{4n+4m+2} \xrightarrow{\alpha_0} \Sigma^1 L(2n+1,2m-1) \xrightarrow{i_{L_0}} \Sigma^1 L_0(2n+1,2m) \end{array}$$

where $\Sigma^1 L(2n+1,2m-1)$ is quasi KO_*-equivalent to $\Sigma^{4n+4m+2} \vee \Sigma^1 L_0(2n+1,2m-1)$ according to Proposition 3.4. Note that $\Sigma^1 L(2n+1,0)$ is quasi KO_*-equivalent to $\Sigma^4 \vee V_{2n+1}$. Since $\Sigma^1 L_0(2n+1,1)$ has the same $\mathcal{C}$-type as $SZ/2 \vee SZ/2^{2n+2}$ by Proposition 3.2, [6, Proposition 3.2] asserts that it must be quasi KO_*-equivalent to $\Sigma^4 SZ/2 \vee V_{2n+2}$. Hence it is easily calculated that $KO_3 L_0(2n+1,2) \cong Z/2 \oplus Z/2^{2n+3}$ and $KO_7 L_0(2n+1,2)$ is isomorphic to the cokernel of $\alpha_{0*} : Z/2 \to Z/2 \oplus Z/2 \oplus Z/2^{2n+1}$. From Lemma 3.3 we recall that the set $S(X)$ consists of only 0 for $X = L_0(2n+1,2m-1)$ or $L_0(2n+1,2m)$ under our assumption on n and m. Applying Theorem 2.3 i) and ii) combined with Proposition 3.2 we see that $\Sigma^1 L_0(2n+1,2m-1)$ is quasi KO_*-equivalent to one of the three spectra $\Sigma^2 V_{m-1} \vee P'_{2n+m,2m}$, $\Sigma^2 SZ/2^{m-1} \vee \Sigma^2 P'_{2m-1,2n+m+1}$ and $\Sigma^2 NP'_{m-1,2m-1,2n+m+1}$ when $m \geq 3$, and $\Sigma^1 L_0(2n+1,2m)$ is quasi KO_*-equivalent to one of the three spectra $\Sigma^2 V_{2m+1} \vee P'_{2n+m,m}$, $\Sigma^2 SZ/2^{2m+1} \vee \Sigma^2 P'_{m-1,2n+m+1}$ and $\Sigma^2 NP'_{2m+1,m-1,2n+m+1}$ when $m \geq 1$. Since $\Sigma^1 L(2n+1,2m-2)$ is quasi KO_*-equivalent to $\Sigma^2 M_{2m-1} \vee P'_{2n+m-1,m-1}$ when $m \geq 3$, it is immediate that $KO_1 L_0(2n+1,2m-1) \cong Z/2^{2m-1} \oplus Z/2^{m-2} \oplus Z/2$. Therefore $\Sigma^1 L_0(2n+1,2m-1)$ must be quasi KO_*-equivalent to $\Sigma^2 V_{m-1} \vee P'_{2n+m,2m}$ when $m \geq 3$. Hence it is easily calculated that $KO_3 L_0(2n+1,2m) \cong Z/2 \oplus Z/2^{2n+m+1} \oplus Z/2$ and $KO_7 L_0(2n+1,2m)$ is isomorphic to the cokernel of $\alpha_{0*} : Z/2 \to Z/2 \oplus Z/2 \oplus Z/2^{2n+m}$. Therefore $\Sigma^1 L_0(2n+1,2m)$ must be quasi KO_*-equivalent to $\Sigma^2 V_{2m+1} \vee P'_{2n+m,m}$ when $m \geq 3$ as well as $m = 1$.

In case when n is odd and $m \geq 2$ is even the quasi KO_*-types of $\Sigma^1 L_0(2n+1,2m-1)$ and $\Sigma^1 L_0(2n+1,2m)$ are determined by a parallel argument. □

REMARK. According to Theorems 3.5 and 3.6, $L_0(s,0)$ and $L(s,0)$ are quasi KO_*-equivalent to the real projective spaces RP^{2s} and RP^{2s+1}, respectively.

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