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# THE QUASI *KO*,-TYPES OF WEIGHTED MOD 4 LENS SPACES

## Dedicated to Professor Fuichi Uchida on his sixtieth birthday

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## 0. Introduction

Let KU and KO be the complex and the real K-spectrum, respectively. For any CW-spectrum X its KU-homology group  $KU_*X$  is regarded as a (Z/2-graded)abelian group with involution because KU possesses the conjugation  $\psi_C^{-1}$ . Given CW-spectra X and Y we say that X is quasi  $KO_*$ -equivalent to Y if there exists an equivalence  $f: KO \land X \to KO \land Y$  of KO-module spectra (see [8]). If X is quasi  $KO_*$ -equivalent to Y, then  $KO_*X$  is isomorphic to  $KO_*Y$  as a  $KO_*$ -module, and in addition  $KU_*X$  is isomorphic to  $KU_*Y$  as an abelian group with involution. In the latter case we say that X has the same C-type as Y (cf. [2]). In [10] and [11] we have determined the quasi  $KO_*$ -types of the real projective space  $RP^k$  and its stunted projective space  $RP^k/RP^l$ . Moreover in [12] we have determined the quasi  $KO_*$ -types of the mod 4 lens space  $L_4^k$  and its stunted lens space  $L_4^k/L_4^l$  where we simply denote by  $L_4^{2n+1}$  the usual (2n+1)-dimensional mod 4 lens space  $L^n(4)$  and by  $L_4^{2n}$  its 2n-skeleton  $L_0^n(4)$ . In this note we shall generally determine the quasi  $KO_*$ -types of a weighted mod 4 lens space  $L^n(4; q_0, \dots, q_n)$  and its 2n-skeleton  $L_0^n(4; q_0, \dots, q_n)$ along the line of [12].

The weighted mod 4 lens space  $L^n(4; q_0, \dots, q_n)$  is obtained as the fiber of the canonical inclusion  $i : P^n(q_0, \dots, q_n) \to P^{n+1}(4, q_0, \dots, q_n)$  of weighted projective spaces (see [3]). Using the result of Amrani [1, Theorem 3.1] we can calculate the KU-cohomology group  $KU^*L^n(4; q_0, \dots, q_n)$  and the behavior of the conjugation  $\psi_C^{-1}$  on it. Our calculation asserts that  $\Sigma^1 L_0^n(4; q_0, \dots, q_n)$  has the same C-type as one of the small spectra  $\Sigma^2 SZ/2^r \vee P'_{s,t}, SZ/2^r \vee P''_{s,t}$  and  $PP'_{r,s,t}$ , and  $\Sigma^1 L^n(4; q_0, \dots, q_n)$  has the same C-type as one of the small spectra  $\Sigma^2 SZ/2^r \vee P'_{s,t}, SZ/2^r \vee P''_{s,t}$  and  $PP'_{r,s,t}$ , and  $\Sigma^1 L^n(4; q_0, \dots, q_n)$  has the same C-type as one of the small spectra  $\Sigma^2 M_r \vee P''_{s,t}$ ,  $M_r \vee P''_{s,t}, MPP'_{r,s,t}$  and  $\Sigma^{2m} \vee \Sigma^1 L^n_0(4; q_0, \dots, q_n)$  (see Proposition 3.2). Here  $SZ/2^r$  is the Moore spectrum of type  $Z/2^r$  and  $M_r, P'_{s,t}, PP'_{r,s,t}$  and  $MPP'_{r,s,t}$  are the small spectra constructed as the cofibers of the maps  $i\eta : \Sigma^1 \to SZ/2^r$ ,  $i\bar{\eta} : \Sigma^1 SZ/2^t \to SZ/2^s$ ,  $i\bar{\eta} + \bar{\eta}j : \Sigma^1 SZ/2^t \to M_r \vee SZ/2^s$ , respectively, in which  $i : \Sigma^0 \to SZ/2^r$  and  $j : SZ/2^r \to \Sigma^1$  are the bottom cell inclusion and the top cell

projection,  $i_M : SZ/2^r \to M_r$  is the canonical inclusion,  $\eta : \Sigma^1 \to \Sigma^0$  is the stable Hopf map, and  $\bar{\eta} : \Sigma^1 SZ/2^r \to \Sigma^0$  and  $\bar{\eta} : \Sigma^2 \to SZ/2^r$  are its extension and coextension satisfying  $\bar{\eta}i = \eta$  and  $j\tilde{\eta} = \eta$ .

In [12, Proposition 3.1 and Theorem 3.3] we have already characterized the quasi  $KO_*$ -types of spectra having the same C-type as  $SZ/2^r \vee P''_{s,t}$ ,  $M_r \vee P''_{s,t}$ ,  $PP'_{r,s,t}$ or  $MPP'_{r,s,t}$  (see Theorems 1.2 and 1.3). In §1 we introduce some new small spectra X having the same C-type as  $SZ/2^r \vee P'_{s,t}$  or  $M_r \vee P'_{s,t}$ , and calculate their KOhomology groups  $KO_*X$  (Propositions 1.5 and 1.7). In §2 we shall characterize the quasi  $KO_*$ -types of spectra having the same C-type as  $SZ/2^r \vee P'_{s,t}$  or  $M_r \vee P'_{s,t}$  (Theorems 2.3 and 2.4) by using the small spectra introduced in §1. Our discussion developed in  $\S2$  is quite similar to the one done in [6,  $\S4$ ] in order to characterize the quasi  $KO_*$ -types of spectra having the same C-type as  $SZ/2^r \vee SZ/2^s$  (see [6, Theorem 5.3]). In §3 we first calculate the KU-cohomology group  $KU^0L^n(4; q_0, \dots, q_n)$ , and then investigate the behavior of the conjugation  $\psi_C^{-1}$  on it (Proposition 3.1). Dualizing this result we study the C-types of  $L = L^n(4; q_0, \dots, q_n)$  and  $L^n_0(4; q_0, \dots, q_n)$  as is stated above (Proposition 3.2), and moreover calculate the sets  $S(L) = \{2i; KO_{2i}L =$ 0 (0  $\leq i \leq$  3)} (Lemma 3.3). Since  $P_{s,t}'$  and  $\Sigma^2 P_{t-1,s+1}'$  have the same  $\mathcal{C}$ -type we can apply Theorems 1.2, 1.3, 2.3 and 2.4 with the aid of Proposition 3.2 and Lemma 3.3 to determine the quasi  $KO_*$ -types of the weighted mod 4 lens spaces  $L^n(4; q_0, \dots, q_n)$  and  $L^n_0(4; q_0, \dots, q_n)$  as our main results (Theorems 3.5 and 3.6).

## 1. Small spectra having the same C-type as $SZ/2^r \vee P_{s,t}'$ or $M_r \vee P_{s,t}'$

1.1. Let  $SZ/2^m$   $(m \ge 1)$  be the Moore spectrum of type  $Z/2^m$ , and  $i: \Sigma^0 \to SZ/2^m$  and  $j: SZ/2^m \to \Sigma^1$  be the bottom cell inclusion and the top cell projection, respectively. The stable Hopf map  $\eta: \Sigma^1 \to \Sigma^0$  of order 2 admits an extension  $\bar{\eta}: \Sigma^1SZ/2^m \to \Sigma^0$  and a coextension  $\tilde{\eta}: \Sigma^2 \to SZ/2^m$  satisfying  $\bar{\eta}i = \eta$  and  $j\tilde{\eta} = \eta$ . As in [13] (see [8]) we denote by  $M_m$ ,  $N_{m,n}$ ,  $P_{m,n}$ ,  $P'_{m,n}$ ,  $R_{m,n}$ ,  $R'_{m,n}$  and  $K_{m,n}$  the small spectra constructed as the cofibers of the following maps  $i\eta: \Sigma^1 \to SZ/2^m$ ,  $i\eta^2 j$ ,  $\tilde{\eta}j$ ,  $i\bar{\eta}+\tilde{\eta}j: \Sigma^1SZ/2^n \to SZ/2^m$  and  $\tilde{\eta}\eta^2 j$ ,  $i\eta^2 \bar{\eta}, \tilde{\eta}\bar{\eta}: \Sigma^3SZ/2^n \to SZ/2^m$ , respectively. In particular,  $P'_{m-1,1}$  is simply written as  $V_m$ . The spectra  $V_m$  and  $M_m$  are exhibited in the following cofiber sequences:

$$\Sigma^{0} \xrightarrow{2^{m-1}\bar{i}} C(\bar{\eta}) \xrightarrow{\bar{i}_{V}} V_{m} \xrightarrow{\bar{j}_{V}} \Sigma^{1}, \Sigma^{0} \xrightarrow{2^{m}i_{P}} C(\eta) \xrightarrow{h_{M}} M_{m} \xrightarrow{k_{M}} \Sigma^{1}$$

where  $C(\eta)$  and  $C(\bar{\eta})$  are the cofibers of the maps  $\eta : \Sigma^1 \to \Sigma^0$  and  $\bar{\eta} : \Sigma^1 SZ/2 \to \Sigma^0$ , and  $i_P : \Sigma^0 \to C(\eta)$  and  $\bar{i} : \Sigma^0 \to C(\bar{\eta})$  are the bottom cell inclusions. Note that  $C(\bar{\eta})$  is quasi  $KO_*$ -equivalent to  $\Sigma^4$ .

Moreover we denote by  $_{V}P_{m,n}$ ,  $P_{m,n}^{V}$ ,  $_{V}R_{m,n}$ ,  $R_{m,n}^{V}$ ,  $VR_{m,n}$ ,  $MP_{m,n}$ ,  $PM_{m,n}$ ,  $MR_{m,n}$  and  $RM_{m,n}$  the small spectra constructed as the cofibers of the following maps:

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$$i_{V}\tilde{\eta}j: \Sigma^{1}SZ/2^{n} \to V_{m}, \qquad \tilde{\eta}\bar{j}_{V}: \Sigma^{1}V_{n} \to SZ/2^{m},$$

$$i_{V}\tilde{\eta}\eta^{2}j: \Sigma^{3}SZ/2^{n} \to V_{m}, \qquad \tilde{\eta}\eta^{2}\bar{j}_{V}: \Sigma^{3}V_{n} \to SZ/2^{m},$$

$$(1.1) \qquad \qquad \xi_{V}\eta j: \Sigma^{5}SZ/2^{n} \to V_{m},$$

$$i_{M}\tilde{\eta}j: \Sigma^{1}SZ/2^{n} \to M_{m}, \qquad \tilde{\eta}k_{M}: \Sigma^{1}M_{n} \to SZ/2^{m},$$

$$i_{M}\tilde{\eta}\eta^{2}j: \Sigma^{3}SZ/2^{n} \to M_{m}, \qquad \tilde{\eta}\eta^{2}k_{M}: \Sigma^{3}M_{n} \to SZ/2^{m},$$

respectively, where  $i_V: SZ/2^{m-1} \to V_m$  and  $i_M: \Sigma^0 \to M_m$  are the canonical inclusions, and  $\xi_V: \Sigma^5 \to V_m$  is the map satisfying  $j_V \xi_V = \tilde{\eta} \eta$  for the canonical projection  $j_V: V_m \to \Sigma^2 SZ/2$ . Here we understand  $i_V \tilde{\eta} = i: \Sigma^0 \to SZ/2$  and  $\xi_V = \tilde{\eta}\eta:$  $\Sigma^3 \rightarrow SZ/2$  when m = 1. According to [6, Proposition 3.2] and its dual the spectra  $_V P_{m,n}$ ,  $P_{m,n}^V$ ,  $_V R_{m,n}$  and  $R_{m,n}^V$   $(m \ge 2)$  are quasi  $KO_*$ -equivalent to  $\Sigma^2 P_{n+1,m-1}$ ,  $\Sigma^6 P_{n+1,m-1}$ ,  $\Sigma^2 V' N_{m,n}$  and  $\Sigma^6 V' N_{m,n}$ , respectively. Here the spectrum  $V' N_{m,n}$  is constructed as the cofiber of the map  $\tilde{\eta}j \vee i\eta^2 j$ :  $\Sigma^1 SZ/2^{m-1} \vee \Sigma^1 SZ/2^n \to SZ/2$ , and it is quasi  $KO_*$ -equivalent to  $\Sigma^6 V_m \vee \Sigma^2 SZ/2^n$  if  $m \ge n$ . The S-dual spectrum  $NV_{n,m}$  of  $V'N_{m,n}$  and the spectrum  $VR_{m,n}$  have been introduced in [13, Proposition 3.1], and the spectra  $MP_{m,n}$  and  $PM_{m,n}$  were written as  $MV'_{m,n}$  and  $V'M_{m,n}$ , respectively, in [12, Propositions 2.3 and 2.4]. On the other hand, the spectra  $MR_{m,n}$ and  $RM_{n,m}$  have the same C-type as  $M_m \vee SZ/2^n$ . Note that  $MR_{m,n}$  is quasi  $KO_*$ equivalent to  $M_m \vee \Sigma^4 SZ/2^n$  if  $m \geq n$ , and  $RM_{m,n}$  is quasi  $KO_*$ -equivalent to  $SZ/2^m \vee \Sigma^4 M_n$  if m > n. By a routine computation we obtain the KO-homology groups  $KO_iX$   $(0 \le i \le 7)$  of  $X = MR_{m,n}$  (m < n) and  $RM_{m,n}$   $(m \le n)$  as follows:

where  $(*)_1 \cong Z/4$  and  $(*)_k \cong Z/2 \oplus Z/2$  if  $k \ge 2$ .

For any maps  $f: \Sigma^i SZ/2^t \to Z_r$  and  $g: \Sigma^i Z_r \to SZ/2^s$  whose cofibers are denoted by  $X_{r,t}$  and  $Y_{s,r}$ , we introduce new small spectra  $XP'_{r,s,t}$  and  $P'Y_{s,t,r}$  constructed as the cofibers of the following maps

(1.3) 
$$(f, i\bar{\eta}) : \Sigma^i SZ/2^t \to Z_r \vee \Sigma^{i-1} SZ/2^s,$$
$$i\bar{\eta} \vee g : \Sigma^1 SZ/2^t \vee \Sigma^i Z_r \to SZ/2^s,$$

respectively. In particular, the spectra  $NP'_{r,s,1}$  and  $PP'_{r,s,1}$  are written as  $NV_{r,s+1}$ and  $PV_{r,s+1}$  in [13, Proposition 3.1], respectively, and  $RP'_{r,s,1} = SZ/2^r \vee \Sigma^2 V_{s+1}$ and  $_V RP'_{r,s,1} = V_r \vee \Sigma^2 V_{s+1}$ . By virtue of [6, Propositions 3.2 and 3.3] the spectra  $_V PP'_{r,s,1}$ ,  $P'P_{s,1,r}$ ,  $P'P^V_{s,1,r}$  and  $P'R_{s,1,r}$  are quasi  $KO_*$ -equivalent to  $\Sigma^4 K_{r,s+1}$ ,  $\Sigma^2 P_{r+1,s}$ ,  $\Sigma^4 P_{s+1,r}$  and  $\Sigma^2 V' N_{s+1,r}$ , respectively. On the other hand, the spectrum  $V RP'_{r,s,1}$  is quasi  $KO_*$ -equivalent to  $R'_{r,s+1}$ ,  $R'R_{r,s+1}$  or  $V_r \vee \Sigma^4 V_{s+1}$  according as r > s + 1, r = s + 1 or  $r \leq s$ , and the spectrum  $P'R^V_{s,1,r}$  is quasi  $KO_*$ -equivalent to  $\Sigma^4 R_{s+1,r}$ ,  $R'R_{s+1,r}$  or  $V_{s+1} \vee \Sigma^4 V_r$  according as r > s+1, r = s+1 or  $r \le s$ . Here the spectrum  $R'R_{m,n}$  has been introduced in [13, Proposition 3.3]. The spectra  $PP'_{r,s,t}, VPP'_{r,s,t}, P'P_{s,t,r}, MPP'_{r,s,t}$  and  $P'PM_{s,t,r}$  were written as  $U_{s,r,t}, V_{s,r,t}, U'_{s,t,r}, MU_{s,r,t}$  and  $U'M_{s,t,r}$  in [12], respectively, and their KU-homology groups with the conjugation  $\psi_C^{-1}$  and their KO-homology groups have been obtained in [12, Propositions 2.1, 2.2, 2.3 and 2.4].

i) "The 
$$X = PP'_{r,s,t}$$
 or  $_VPP'_{r,s,t}$  case"

	<i>r</i> :	> t > s		$r \ge t \le$	s	
$KU_0X \in$	$\cong Z/2^r \oplus$	$Z/2^t \oplus Z/2^t$	$Z/2^r \in$	$\partial Z/2^{t-1}$	$\oplus Z/2$	s+1
$\psi_C^{-1}$ =	$= \begin{pmatrix} 1\\0\\0 \end{pmatrix}$	$\begin{pmatrix} 2^{r-t} & 0 \\ -1 & 0 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1\\ 0\\ 0 \end{pmatrix}$	$2^{r-t+1}$ -1 0	$-2^{r-1}$ 1 1	
		( + > -				
	$r \leq$	$\leq \iota \geq s$		$r \leq t \leq$	5	
$KU_0X \in$	$\stackrel{r}{:}{\stackrel{\scriptstyle <}{_{\scriptstyle 2}}} \cong Z/2^{r-1} \oplus$	$\leq \iota \geq s$ $Z/2^{t+1} \oplus Z_{s}$	$/2^{s} Z/2^{r-1}$	$r \leq t \leq 1$ $1 \oplus Z/2^t$	$\oplus Z/2$	s+1

ii) "The  $X = MPP'_{r,s,t}$  case"

iii) Their KO-homology groups 
$$KO_iX$$
  $(0 \le i \le 7)$  are tabled as follows:

$X \setminus i$	0	1	2	3	4	5	6	7
$PP'_{r,s,t}$	$Z/2^r\oplus Z/2^s$	Z/2	$(*)_{t-1,r}\oplus Z/2$	Z/2	$\mathbb{Z}/2^{r-1}\oplus\mathbb{Z}/2^{s+1}$	0	$Z/2^t$	0
$VPP'_{r,s,t}$	$Z/2^{r-1} \oplus Z/2^s$	0	$Z/2^t\oplus Z/2$	Z/2	$Z/2^r \oplus Z/2^{s+1}$	Z/2	$(*)_{t-1,r}$	0
$MPP'_{r,s,t}$	$Z/2^r\oplus Z/2^s$	0	$Z\oplus Z/2^t\oplus Z/2$	Z/2	$Z/2^r \oplus Z/2^{s+1}$	0	$Z\oplus Z/2^t$	0

where  $(*)_{k,1} \cong Z/2^{k+2}$  and  $(*)_{k,l} \cong Z/2^{k+1} \oplus Z/2$  if  $l \ge 2$ .

For any spectrum X having the same C-type as  $PP'_{r,s,t}$  or  $MPP'_{r,s,t}$  we have already determined its quasi  $KO_*$ -type in [12, Theorem 3.3].

### Theorem 1.2.

- i) If a spectrum X has the same C-type as  $PP'_{r,s,t}$ , then it is quasi  $KO_*$ -equivalent to one of the following small spectra  $PP'_{r,s,t}$ ,  $\Sigma^4 PP'_{r,s,t}$ ,  $_VPP'_{r,s,t}$  and  $\Sigma^4_V PP'_{r,s,t}$ .
- ii) If a spectrum X has the same C-type as  $MPP'_{r,s,t}$ , then it is quasi  $KO_*$ -equivalent to either of the small spectra  $MPP'_{r,s,t}$  and  $\Sigma^4 MPP'_{r,s,t}$ .

Applying Theorem 1.2 we see that

(1.4) the spectra  $P'P_{s,t,r}$ ,  $P'P_{s,t,r}^V$  and  $P'PM_{s,t,r}$  are quasi  $KO_*$ -equivalent to  $\Sigma^2 PP'_{r+1,t-1,s}$ ,  $\Sigma^2 VPP'_{r+1,t-1,s}$  and  $\Sigma^2 MPP'_{r+1,t-1,s}$ , respectively (see [12, Corollary 3.4]).

We can also show the following result (see [12, Proposition 3.1]).

### Theorem 1.3.

- i) If a spectrum X has the same C-type as  $SZ/2^r \vee P_{s,t}''$ , then it is quasi  $KO_*$ -equivalent to one of the following wedge sums  $SZ/2^r \vee P_{s,t}''$ ,  $\Sigma^4 SZ/2^r \vee P_{s,t}''$ ,  $V_r \vee P_{s,t}''$  and  $\Sigma^4 V_r \vee P_{s,t}''$ .
- ii) If a spectrum X has the same C-type as  $M_r \vee P_{s,t}''$ , then it is quasi  $KO_*$ equivalent to either of the following wedge sums  $M_r \vee P_{s,t}''$  and  $\Sigma^4 M_r \vee P_{s,t}''$ .

1.2. Since  $P'_{s,t}$  and  $\Sigma^2 P'_{t-1,s+1}$  have the same C-type a routine computation shows

### **Proposition 1.4.**

- i) The spectra  $NP'_{r,s,t}$ ,  $VRP'_{r,s,t}$ ,  $RP'_{r,t-1,s+1}$ ,  $VRP'_{r,t-1,s+1}$ ,  $P'R_{s,t,r}$  and  $P'R^V_{s,t,r}$ have the same C-type as the wedge sum  $SZ/2^r \vee P'_{s,t}$ .
- ii) The spectra  $MRP'_{r,t-1,s+1}$  and  $P'RM_{s,t,r}$  have the same C-type as the wedge sum  $M_r \vee P'_{s,t}$ .

Note that if  $r \geq t$  the spectra  $RP'_{r,s,t}$ ,  $_VRP'_{r,s,t}$  and  $MRP'_{r,s,t}$  are quasi  $KO_*$ equivalent to  $SZ/2^r \vee \Sigma^2 P'_{s,t}$ ,  $V_r \vee \Sigma^2 P'_{s,t}$  and  $M_r \vee \Sigma^2 P'_{s,t}$ , respectively, and if  $r \leq s$ the spectra  $P'R_{s,t,r}$ ,  $P'R^V_{s,t,r}$  and  $P'RM_{s,t,r}$  are quasi  $KO_*$ -equivalent to  $\Sigma^4 SZ/2^r \vee P'_{s,t}$ ,  $\Sigma^4 V_r \vee P'_{s,t}$  and  $\Sigma^4 M_r \vee P'_{s,t}$ , respectively. By use of [13, Propositions 2.2 and 3.1] and (1.2) we can easily calculate

**Proposition 1.5.** For the small spectra X listed in Proposition 1.4 the KOhomology groups  $KO_iX$  ( $0 \le i \le 7$ ) are tabled as follows:

$i \setminus X$	$NP'_{r,s,t}$	$RP_{r,s,t}^{\prime}$	$_V RP'_{r,s,t}$	$VRP'_{r,s,t}$
	$(t \ge 2)$	(r < t)	(r < t)	$(t \ge 2)$
0	$Z/2^r \oplus Z/2^s$	$Z/2^r\oplus Z/2^t$	$Z/2^{r-1} \oplus Z/2^t$	$Z/2^r \oplus Z/2^{s+1}$
1	Z/2	Z/2	0	Z/2
2	$Z/2^t\oplus Z/2\oplus Z/2$	$Z/2^s \oplus (*)_r$	$Z/2^s\oplus Z/2$	$Z/2^t\oplus Z/2$
3	$Z/2\oplus Z/2$	Z/2	Z/2	Z/2
4	$Z/2^{r+1} \oplus Z/2^{s+1}$	$Z/2^{r-1}\oplus Z/2^t\oplus Z/2$	$Z/2^r\oplus Z/2^t\oplus Z/2$	$\mathbb{Z}/2^{r+1}\oplus\mathbb{Z}/2^s$
5	Z/2	Z/2	$Z/2\oplus Z/2$	Z/2
6	$Z/2^t$	$Z/2^{s+1}\oplus Z/2$	$Z/2^{s+1} \oplus (*)_r$	$Z/2^t\oplus Z/2$
7	0	Z/2	Z/2	Z/2

$i \setminus X$	$P'R_{s,t,r}$	$P'R_{s,t,r}^V$	$MRP'_{r,s,t}$	$P'RM_{s,t,r}$
ļ	$(s < r, t \geq 2)$	$(s < r, t \geq 2)$	(r < t)	(s < r)
0	$Z/2^s\oplus Z/2^r$	$Z/2^s \oplus Z/2^{r+1}$	$Z/2^r \oplus Z/2^t$	$Z/2^s \oplus Z/2^{r+1}$
1	0	Z/2	0	0
2	$Z/2^{t-1}\oplus Z/2$	$Z/2^{t-1}\oplus Z/2\oplus Z/2$	$Z\oplus Z/2^s\oplus Z/2$	$Z\oplus Z/2^{t-1}\oplus Z/2$
3	Z/2	Z/2	Z/2	Z/2
4	$(*)_{s-1,t} \oplus \mathbb{Z}/2^{r+1}$	$(*)_{s-1,t}\oplus Z/2^r$	$Z/2^r\oplus Z/2^s\oplus Z/2$	$(*)_{s-1,t} \oplus \mathbb{Z}/2^{r+1}$
5	$Z/2\oplus Z/2$	Z/2	Z/2	Z/2
6	$Z/2^t\oplus Z/2\oplus Z/2$	$Z/2^t\oplus Z/2$	$Z\oplus Z/2^{s+1}\oplus Z/2$	$Z\oplus Z/2^t\oplus Z/2$
7	Z/2	Z/2	Z/2	Z/2

where  $(*)_{k,1} \cong Z/2^{k+2}$  and  $(*)_{k,l} \cong Z/2^{k+1} \oplus Z/2$  if  $l \ge 2$ , and  $(*)_{0,l}$  is abbreviated as  $(*)_{l}$ .

Let  $N'_t$ ,  $P'_t$  and  $R'_t$  denote the small spectra constructed as the cofibers of the following maps  $\eta^2 j$ ,  $\bar{\eta} : \Sigma^1 SZ/2^t \to \Sigma^0$  and  $\eta^2 \bar{\eta} : \Sigma^3 SZ/2^t \to \Sigma^0$ , respectively. Consider the small spectrum  $N'P'_t$  constructed as the cofiber of the map  $(\eta^2 j, \bar{\eta}) : \Sigma^1 SZ/2^t \to \Sigma^0 \vee \Sigma^0$ . Then we have two maps  $i'_{NP} : \Sigma^0 \to N'P'_t$  and  $\rho'_{NP} : \Sigma^0 \to N'P'_t$  whose cofibers are  $N'_t$  and  $P'_t$ , respectively. These two maps are related by the equality  $i'_{NP}\bar{\eta} = \rho'_{NP}\eta^2 j : \Sigma^1 SZ/2^t \to N'P'_t$ . In particular,  $i'_{NP} = (2, \bar{i}) : \Sigma^0 \to \Sigma^0 \vee C(\bar{\eta})$  and  $\rho'_{NP} = (1,0) : \Sigma^0 \to \Sigma^0 \vee C(\bar{\eta})$  when t = 1. We denote by  $N'P'_{r,t}$ ,  $P'N'_{s,t}$  and  $F^{n,m}_t$  the spectra constructed as the cofibers of the following maps  $2^r \rho'_{NP}$ ,  $2^{si'_{NP}}$  and  $f^{n,m}_t = 2^n \rho'_{NP} + 2^m i'_{NP} : \Sigma^0 \to N'P'_t$ , respectively. In particular,  $N'P'_{r,1} = C(\bar{\eta}) \vee SZ/2^r$  and  $P'N'_{s,1}$  is quasi  $KO_*$ -equivalent to  $\Sigma^4 R'_{s+1}$ . On the other hand,  $F^{n,m}_1 = C(\bar{\eta}) \vee SZ/2^n$  if  $n \leq m, F^{n,m}_1 = C(\bar{\eta}) \vee V_{m+1}$  if n = m + 1, and it is quasi  $KO_*$ -equivalent to  $\Sigma^4 R'_{m+1}$  if n > m + 1. Whenever  $t \geq 2$  we can regard that the induced homomorphisms  $\rho'_{NP_*}$  and  $i'_{NP_*} : KU_0\Sigma^0 \to KU_0N'P'_t$  are

given by  $\rho'_{NP*}(1) = (1,0,0)$  and  $i'_{NP*}(1) = (0,2,1)$  in  $KU_0N'P'_t \cong Z \oplus Z \oplus Z/2^{t-1}$ because  $i'_{NP}$  may be replaced by  $i'_{NP} + 2q\rho'_{NP}$  if necessary. Hence it is easily shown that

- (1.5) i) the spectra  $N'P'_{r,t}$  and  $P'N'_{s,t}$  have the same C-type as  $SZ/2^r \vee P'_t$  and  $\Sigma^0 \vee P'_{s,t}$ , respectively, and
  - ii) the spectrum  $F_t^{n,m}$  has the same C-type as  $SZ/2^n \vee P'_t$  when  $n \leq m$ , and as  $\Sigma^0 \vee P'_{m,t}$  when n > m.

By use of [8, Proposition 4.2] and [9, Proposition 2.4] we can easily calculate the KO-homology groups  $KO_iX \ (0 \le i \le 7)$  of  $X = N'P'_{r,t}$ ,  $P'N'_{s,t}$  and  $F_t^{n,m} \ (t \ge 2)$  as follows:

	$X \setminus i$	0	1	2	3	4	5	6	7
(1.6)	$N'P_{r,t}'$	$Z\oplus Z/2^r$	Z/2	$Z/2^t\oplus Z/2$	Z/2	$Z\oplus Z/2^{r+1}$	Z/2	$Z/2^t$	0
(1.0)	$P'N'_{s,t}$	$Z\oplus Z/2^s$	Z/2	$Z/2^t\oplus Z/2$	Z/2	$Z\oplus Z/2^{s+1}$	Z/2	$Z/2^t$	0
	$F_t^{n,m}$	$Z\oplus Z/2^l$	Z/2	$Z/2^t\oplus Z/2$	Z/2	$Z\oplus Z/2^{l+1}$	Z/2	$Z/2^t$	0

where  $l = \min\{n, m\}$ .

Choose two maps  $h'_N$  :  $\Sigma^2 \to N'_t$  and  $\bar{\rho}'_N$  :  $C(\bar{\eta}) \to N'_t$  whose cofibers coincide with  $C(\eta^2)$  and  $V'_t$ , respectively, where  $C(\eta^2)$  is the cofiber of the map  $\eta^2: \Sigma^2 \to \Sigma^0$  and  $V'_t = P_{1,t-1}$  which is quasi  $KO_*$ -equivalent to  $\Sigma^6 V_t$  (see [13]). Then there exist two maps  $\lambda'_{NP}$  :  $C(\bar{\eta}) \rightarrow N'P'_t$  and  $\bar{\rho}'_{NP}$  :  $C(\bar{\eta}) \rightarrow N'P'_t$  satisfying  $j'_{NP}\lambda'_{NP} = h'_N\eta j\bar{j}$  and  $j'_{NP}\bar{\rho}'_{NP} = \bar{\rho}'_N$  for the canonical projection  $j'_{NP}$ :  $N'P'_t \to N'_t$ . In particular, we may choose as  $\lambda'_{NP} = (\bar{\lambda}, 2) : C(\bar{\eta}) \to \Sigma^0 \vee C(\bar{\eta})$ and  $\bar{\rho}'_{NP} = (0,1) : C(\bar{\eta}) \to \Sigma^0 \vee C(\bar{\eta})$  when t = 1. Here the map  $\bar{\lambda} : C(\bar{\eta}) \to \Sigma^0$ satisfies the equalities  $\overline{\lambda}\overline{i} = 4$  and  $\overline{i}\overline{\lambda} = 4$  (see [13, (1.3)]). Whenever  $t \ge 2$ , we can regard that the induced homomorphisms  $\bar{\rho}'_{NP*}$  and  $\lambda'_{NP*}: KU_0C(\bar{\eta}) \to KU_0N'P'_t$ are given by  $\bar{\rho}'_{NP}(1) = (1,0,0)$  and  $\lambda'_{NP*}(1) = (0,2,1)$  in  $KU_0N'P'_t \cong Z \oplus Z \oplus$  $Z/2^{t-1}$  because  $\bar{\rho}'_{NP}$  and  $\lambda'_{NP}$  may be replaced by  $\bar{\rho}'_{NP} + k i'_{PN} \bar{\lambda}$  and  $\lambda'_{NP} + l i'_{PN} \bar{\lambda}$ if necessary. By virtue of [13, Lemma 1.5] we obtain that the cofiber of  $\bar{\rho}'_{NP}$  is quasi  $KO_*$ -equivalent to  $\Sigma^4 P'_t$ . On the other hand, by use of [13, Lemma 1.2 and Proposition 4.1] (or [9, Theorem 4.2]) we see that the cofiber of  $\lambda'_{NP}$  is quasi  $KO_*$ equivalent to  $\Sigma^4 N'_t$ . More generally, the cofibers of the maps  $2^r \bar{\rho}'_{NP}$  and  $2^s \lambda'_{NP}$ are quasi  $KO_*$ -equivalent to  $\Sigma^4 N'P'_{r,t}$  and  $\Sigma^4 P'N'_{s,t}$ , respectively, because  $N'P'_t$  and  $\Sigma^4 N' P'_t$  have the same quasi  $KO_*$ -type (see [9, Corollary 4.5]).

Using the maps  $f_t^{n,m}$ ,  $\bar{\rho}'_{NP}$  and  $\lambda'_{NP}$  we introduce new small spectra  $N'P'F_{r,t}^{n,m}$ and  $P'N'F_{s,t}^{n,m}$  constructed as the cofibers of the following maps

(1.7) 
$$\begin{aligned} f_t^{n,m} \vee 2^r \bar{\rho}'_{NP} &: \Sigma^0 \vee C(\bar{\eta}) \to N' P'_t, \\ f_t^{n,m} \vee 2^s \lambda'_{NP} &: \Sigma^0 \vee C(\bar{\eta}) \to N' P'_t, \end{aligned}$$

respectively. In particular,  $N'P'F_{r,1}^{n,m}$  is equal to  $(C(\bar{\eta}) \wedge SZ/2^r) \vee SZ/2^n$  if  $n \leq m$ , to  $(C(\bar{\eta}) \wedge SZ/2^r) \vee SZ/2^{m+2}$  if n = m+1 > r, and to  $(C(\bar{\eta}) \wedge SZ/2^r) \vee SZ/2^{m+1}$ 

if n > m+1 > r. Moreover it is quasi  $KO_*$ -equivalent to  $\Sigma^4 V_{r+1} \vee V_{m+1}$ ,  $\Sigma^4 R_{r,m+2}$ or  $\Sigma^4 R'_{r,m+1}$  according as n = m+1 < r, n = m+1 = r or  $n > m+1 \le r$  (use [6, Proposition 3.1]). On the other hand,  $P'N'F^{n,m}_{s,1}$  is just  $R'_{n,m+1,s+1}$  introduced in [13]. By a routine computation we can easily show

### **Proposition 1.6.**

- i) The spectrum  $N'P'F_{r,t}^{n,m}$   $(t \ge 2)$  has the same C-type as  $SZ/2^n \lor P'_{m-n+r,t}$  if  $m \ge n < r$ , and as  $SZ/2^r \lor P'_{m,t}$  if otherwise.
- ii) The spectrum  $P'N'F_{s,t}^{n,m}$   $(t \ge 2)$  has the same C-type as  $SZ/2^{n-m+s} \lor P'_{m,t}$  if  $n > m \le s$ , and as  $SZ/2^n \lor P'_{s,t}$  if otherwise.

Using (1.6) we can easily calculate

**Proposition 1.7.** For the spectra  $X = N'P'F_{r,t}^{n,m}$  and  $P'N'F_{s,t}^{n,m}$   $(t \ge 2)$  the KO-homology groups  $KO_iX$   $(0 \le i \le 7)$  are tabled as follows:

$i \setminus X$	$N'P'F_{r,t}^{n,m}$	$P'N'F^{n,m}_{s,t}$
0	$\begin{cases} Z/2^n \oplus Z/2^{m-n+r+1} & (m \ge n \le r) \\ Z/2^{r+1} \oplus Z/2^m & (\text{otherwise}) \end{cases}$	$\begin{cases} Z/2^{n-m+s+1} \oplus Z/2^m & (n > m \le s) \\ Z/2^n \oplus Z/2^{s+1} & (\text{otherwise}) \end{cases}$
1	Z/2	Z/2
2	$Z/2^t\oplus Z/2$	$Z/2^t\oplus Z/2$
3	Z/2	Z/2
4	$\begin{cases} Z/2^{n+1} \oplus Z/2^{m-n+r} & (m \ge n < r) \\ Z/2^r \oplus Z/2^{m+1} & (\text{otherwise}) \end{cases}$	$\begin{cases} Z/2^{n-m+s} \oplus Z/2^{m+1} \ (n > m < s) \\ Z/2^{n+1} \oplus Z/2^s \ (\text{otherwise}) \end{cases}$
5	Z/2	Z/2
6	$Z/2^t\oplus Z/2$	$Z/2^t\oplus Z/2$
7	Z/2	Z/2

## 2. The same quasi $KO_*$ -type as $SZ/2^r \vee P'_{s,t}$ or $M_r \vee P'_{s,t}$

**2.1.** Let X be a spectrum having the same C-type as  $SZ/2^r \vee P'_{s,t}$ . Then its self-conjugate K-homology group  $KC_iX$   $(0 \le i \le 3)$  is given as follows:

 $KC_i X \cong Z/2^r \oplus Z/2^s \oplus Z/2, Z/2^r \oplus Z/2^{s+1}, Z/2 \oplus Z/2 \oplus Z/2^{t-1}, Z/2 \oplus Z/2^t$ 

according as i = 0, 1, 2, 3. In addition,

 $KO_1X \oplus KO_5X \cong Z/2 \oplus Z/2$  and  $KO_3X \oplus KO_7X \cong Z/2 \oplus Z/2$ .

Hence  $KO_{2i+1}X$  ( $0 \le i \le 3$ ) are divided into the nine cases (A,D) with A=a, b, c

and D=d, e, f as follows:

(2.1)   
(a) 
$$KO_1X \cong KO_5X \cong Z/2$$
 (b)  $KO_5X = 0$  (c)  $KO_1X = 0$   
(d)  $KO_3X \cong KO_7X \cong Z/2$  (e)  $KO_7X = 0$  (f)  $KO_3X = 0$ .

The induced homomorphisms  $(-\tau, \tau \pi_C)_* : KC_i X \to KO_{i+1} X \oplus KO_{i+5} X$  (i = 0, 2) are represented by the following matrices

$$\Phi_0 = \varphi_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} : Z/2^r \oplus Z/2^s \oplus Z/2 \to Z/2 \oplus Z/2$$

$$\Phi_2 = \varphi_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : Z/2 \oplus Z/2 \oplus Z/2^{t-1} \to Z/2 \oplus Z/2,$$

respectively, where  $\varphi_0, \varphi_2: Z/2 \oplus Z/2 \to Z/2 \oplus Z/2$  is one of the following matrices:

(2.2) 
$$\begin{pmatrix} (1) & (2) & (3) & (4) & (5) & (6) \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Evidently it is sufficient to take as  $\varphi_0$  or  $\varphi_2$  only the matrix (1) in case of (b), (c), (e) or (f). By using a suitable transformation of  $KU_0X$  similarly to [6, 4.1] we can verify that in case of (a) the matrix (1) as  $\varphi_0$  is replaced by (5), and in case of (d) the matrix (1) as  $\varphi_2$  is replaced by (5) if  $r \leq s$ , and by (3) if r > s. Therefore it is sufficient to take as  $\varphi_0$  the matrices (1), (2) and (3) in case of (a), and as  $\varphi_2$  the matrices (1), (2) and (3) in case of (d) and  $r \leq s$ , and (1), (2) and (6) in case of (d) and r > s.

Let X be a spectrum having the same C-type as  $M_r \vee P'_{s,t}$ . Then its self-conjugate K-homology group  $KC_iX$   $(0 \le i \le 3)$  is given as follows:

$$KC_i X \cong Z/2^r \oplus Z/2^s \oplus Z/2, Z/2^{r+1} \oplus Z/2^{s+1},$$
$$Z \oplus Z/2 \oplus Z/2 \oplus Z/2^{t-1}, Z \oplus Z/2^t$$

according as i = 0, 1, 2, 3. In addition,

$$KO_1X \oplus KO_5X \cong Z/2$$
 and  $KO_3X \oplus KO_7X \cong Z/2 \oplus Z/2$ .

Hence  $KO_{2i+1}X$  ( $0 \le i \le 3$ ) are divided into the six cases (A, D) with A = b, c and D = d, e, f given in (2.1). The induced homomorphisms  $(-\tau, \tau\pi_C)_* : KC_iX \to KO_{i+1}X \oplus KO_{i+5}X$  (i = 0, 2) are represented by the following matrices

$$\begin{split} \Phi_0 &= (0,0,1) \, : \, Z/2^r \oplus Z/2^s \oplus Z/2 \to Z/2 \\ \Phi_2 &= \varphi_2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \, : \, Z \oplus Z/2 \oplus Z/2 \oplus Z/2^{t-1} \to Z/2 \oplus Z/2, \end{split}$$

where  $\varphi_2: Z/2 \oplus Z/2 \to Z/2 \oplus Z/2$  is one of the matrices given in (2.2). Evidently it is sufficient to take as  $\varphi_2$  only the matrix (1) in case of (e) or (f). On the other hand, it is sufficient to take as  $\varphi_2$  the matrices (1), (2) and (3) in case of (d) and  $r \leq s$ , and (1), (2) and (6) in case of (d) and r > s.

Given a spectrum X having the same C-type as  $SZ/2^r \vee P'_{s,t}$  or  $M_r \vee P'_{s,t}$  we define its  $\varphi$ -type (A, D, *i*, *j*) where A = a, b, c, D = d, e, f and  $1 \le i, j \le 6$ , using the above notations as in [6, §4].

### Lemma 2.1.

- i) Let X be a spectrum having the same C-type as  $SZ/2^r \vee P'_{s,t}$ . Then its  $\varphi$ -type is one of the following 25 types: (a, d, i, j), (a, e, i, 1), (a, f, i, 1), (b, d, 1, j), (c, d, 1, j), (b, e, 1, 1), (b, f, 1, 1), (c, e, 1, 1) and (c, f, 1, 1) where i = 1, 2, 3, and j = 1, 2, 3 if  $r \leq s$  and j = 1, 2, 6 if r > s.
- ii) Let X be a spectrum having the same C-type as  $M_r \vee P'_{s,t}$ . Then its  $\varphi$ -type is one of the following 10 types: (b, d, 1, j), (c, d, 1, j), (b, e, 1, 1), (b, f, 1, 1), (c, e, 1, 1) and (c, f, 1, 1) where j = 1, 2, 3 if  $r \leq s$  and j = 1, 2, 6 if r > s.

**2.2.** Using [6, Lemmas 4.2 and 4.3] we can easily determine the  $\varphi$ -types of the small spectra appearing in Propositions 1.3 and 1.5.

### **Proposition 2.2.**

- i) The spectra  $NP'_{r,s,t}$ ,  $VRP'_{r,s,t}$   $(t \ge 2)$ ,  $RP'_{r,t-1,s+1}$ ,  $_VRP'_{r,t-1,s+1}$ ,  $P'R_{s,t,r}$ and  $P'R^V_{s,t,r}$   $(t \ge 2)$  have the following  $\varphi$ -types (a, e, 3, 1), (a, d, 4, 1), (a, d, 1, 3), (c, d, 1, 3), (c, d, 1, 6) and (a, d, 1, 6), respectively.
- ii) The spectra  $MRP'_{r,t-1,s+1}$  and  $P'RM_{s,t,r}$  have the following  $\varphi$ -types (c, d, 1, 3) and (c, d, 1, 6), respectively.
- iii) The spectrum  $N'P'F_{r,t}^{n,m}$   $(t \ge 2)$  has the following  $\varphi$ -type (a, d, 4, 1), (a, d, 4, 3) or (a, d, 4, 2) according as  $m \ge n < r$ ,  $m \ge n = r$  or otherwise, and the spectrum  $P'N'F_{s,t}^{n,m}$   $(t \ge 2)$  has the following  $\varphi$ -type (a, d, 4, 2), (a, d, 4, 6) or (a, d, 4, 1) according as n > m < s, n > m = s or otherwise.

Let X be a spectrum having the same C-type as  $SZ/2^r \vee P'_{s,t}$  or  $M_r \vee P'_{s,t}$ . If a spectrum Y has the same C-type as X, then we can choose a quasi  $KU_*$ -equivalence  $f: Y \to KU \wedge X$  with  $(\psi_C^{-1} \wedge 1)f = f$ . If there exists a map  $h: Y \to KO \wedge X$  satisfying  $(\epsilon_U \wedge 1)h = f$  for the complexification map  $\epsilon_U : KO \to KU$ , then h becomes a quasi  $KO_*$ -equivalence (see [8, Proposition 1.1]). After choosing a suitable small spectrum Y having the same  $\varphi$ -type as X we can prove the following theorems by applying the same method developed in [6], [8] or [9].

**Theorem 2.3.** Let X be a spectrum having the same C-type as  $SZ/2^r \vee P'_{s,t}$   $(t \ge 2)$ . Then it is quasi  $KO_*$ -equivalent to one of the following small spectra (cf. [6, The-

orem 5.3]):

- i) The case of  $r \leq s: Y_r \vee Y_{s,t}, NP'_{r,s,t}, \Sigma^4 NP'_{r,s,t}, VRP'_{r,t-1,s+1}, \Sigma^4 VRP'_{r,t-1,s+1}, VRP'_{r,s,t}, \Sigma^4 VRP'_{r,s,t}, RP'_{r,t-1,s+1}, \Sigma^4 RP'_{r,t-1,s+1}, N'P'F^{r,s}_{r,t}.$
- ii) The case of  $r > s : Y_r \lor Y_{s,t}$ ,  $NP'_{r,s,t}$ ,  $\Sigma^4 NP'_{r,s,t}$ ,  $P'R_{s,t,r}$ ,  $\Sigma^4 P'R_{s,t,r}$ ,  $VRP'_{r,s,t}$ ,  $\Sigma^4 VRP'_{r,s,t}$ ,  $P'R^V_{s,t,r}$ ,  $\Sigma^4 P'R^V_{s,t,r}$ ,  $P'N'F^{r,s}_{s,t}$ . Here  $Y_r = SZ/2^r$ ,  $\Sigma^4 SZ/2^r$ ,  $V_r$  or  $\Sigma^4 V_r$ , and  $Y_{s,t} = P'_{s,t}$ ,  $\Sigma^4 P'_{s,t}$ ,  $\Sigma^2 P'_{t-1,s+1}$ or  $\Sigma^6 P'_{t-1,s+1}$ .

**Theorem 2.4.** Let X be a spectrum having the same C-type as  $M_r \vee P'_{s,t}$ . Then it is quasi  $KO_*$ -equivalent to one of the following small spectra:

- i) The case of  $r \leq s : Y_r \lor Y_{s,t}$ ,  $MRP'_{r,t-1,s+1}$ ,  $\Sigma^4 MRP'_{r,t-1,s+1}$ .
- ii) The case of  $r > s : Y_r \lor Y_{s,t}$ ,  $P'RM_{s,t,r}$ ,  $\Sigma^4 P'RM_{s,t,r}$ . Here  $Y_r = M_r$  or  $\Sigma^4 M_r$ , and  $Y_{s,t} = P'_{s,t}$ ,  $\Sigma^4 P'_{s,t}$ ,  $\Sigma^2 P'_{t-1,s+1}$  or  $\Sigma^6 P'_{t-1,s+1}$ .

Combining Theorem 2.3 with Proposition 2.2 iii) we get

## Corollary 2.5.

- i) The spectrum  $N'P'F_{r,t}^{n,m}$   $(t \ge 2)$  is quasi  $KO_*$ -equivalent to  $VRP'_{n,m-n+r,t}$  if  $m \ge n < r$ , and to  $\Sigma^4 VRP'_{r,m,t}$  if  $m \ge n > r$  or m < n.
- ii) The spectrum  $P'N'F_{s,t}^{n,m}$   $(t \ge 2)$  is quasi  $KO_*$ -equivalent to  $\Sigma^4 VRP'_{n-m+s,m,t}$ if n > m < s, and to  $VRP'_{n,s,t}$  if n > m > s or  $n \le m$ .

### 3. Weighted mod 4 lens spaces

**3.1.** Let  $S^{2n+1}(q_0, \dots, q_n)$  denote the unit sphere  $S^{2n+1} \subset C^{n+1}$  with  $S^1$ -action defined by  $\lambda \cdot (x_0, \dots, x_n) = (\lambda^{q_0} x_0, \dots, \lambda^{q_n} x_n) \in C^{n+1}$  for any  $\lambda \in S^1 \subset C$ . Then we set

$$P^{n}(q_{0}, \dots, q_{n}) = S^{2n+1}(q_{0}, \dots, q_{n})/S^{1}$$
$$L^{n}(q; q_{0}, \dots, q_{n}) = S^{2n+1}(q_{0}, \dots, q_{n})/(Z/q)$$

where Z/q is the q-th roots of the unity in  $S^1 \subset C$ . Denote by  $L_0^n(q;q_0,\dots,q_n)$  the subspace of  $L^n(q;q_0,\dots,q_n)$  defined by

$$L_0^n(q; q_0, \dots, q_n) = \{ [x_0, \dots, x_n] \in L^n(q; q_0, \dots, q_n) | x_n \text{ is real } \ge 0 \}.$$

Of course,  $P^n(1, \dots, 1)$ ,  $L^n(q; 1, \dots, 1)$  and  $L_0^n(q; 1, \dots, 1)$  are the usual complex projective space  $CP^n$ , the usual mod q lens space  $L^n(q)$  and its 2n-skeleton  $L_0^n(q)$ , respectively. For a weighted mod 4 lens space  $L^n(4; q_0, \dots, q_n)$  we may assume that  $q_0 = \dots = q_{r-1} = 4$ ,  $q_r = \dots = q_{r+s-1} = 2$  and  $q_{r+s} = \dots = q_n = 1$  where  $0 \le r \le$  $r+s \le n$ . For such a tuple  $(q_0, \dots, q_n)$  we simply set  $P(r, s, t) = P^n(q_0, \dots, q_n)$ ,  $L(r, s, t) = L^n(4; q_0, \dots, q_n)$  and  $L_0(r, s, t) = L_0^n(4; q_0, \dots, q_n)$  with n = r + s + t. Moreover we shall omit the "r" as P(s,t), L(s,t) or  $L_0(s,t)$  when r = 0. Notice that  $L(r,s,t) = \Sigma^{2r}L(s,t)$  and  $L_0(r,s,t) = \Sigma^{2r}L_0(s,t)$ .

Denote by  $\gamma$  the canonical line bundle over  $CP^n$  and set  $a = [\gamma] - 1 \in KU^0 CP^n$ . Then it is well known that the (reduced) KU-cohomology group  $KU^*CP_+^n \cong Z[a]/(a^{n+1})$  where  $CP_+^n$  denotes the disjoint union of  $CP^n$  and a point. According to [1, Theorem 3.1] the map  $\varphi : CP^n \to P(r, s, t)$  defined by  $\varphi[x_0, \dots, x_n] = [x_0^{q_0}, \dots, x_n^{q_n}]$  with n = r + s + t induces a monomorphism  $\varphi^* : KU^*P(r, s, t) \to KU^*CP^n$  and the free abelian group  $KU^*P(r, s, t)$  has the following basis  $\{T_1, \dots, T_n\}$  such that  $\varphi^*T_l = a(2)^l$  for  $1 \leq l \leq r$ ,  $\varphi^*T_{r+k} = a(2)^r a(1)^k$  for  $1 \leq k \leq s$  and  $\varphi^*T_{r+s+h} = a(2)^r a(1)^s a^h$  for  $1 \leq h \leq t$ , where  $a(1) = (a+1)^2 - 1$  and  $a(2) = (a+1)^4 - 1$ .

In order to calculate the KU-cohomology group  $KU^*L(s,t)$  we use the following cofiber sequence

(3.1) 
$$L(s,t) \xrightarrow{\theta} P(s,t) \xrightarrow{i} P(1,s,t)$$

where  $\theta$  is the natural surjection and *i* is the canonical inclusion (cf. [3, Assertion 1]). Since  $a(2) = 2a(1) + a(1)^2 = 2a(1) + 2a(1)a + a(1)a^2 = 4a + 6a^2 + 4a^3 + a^4$ , the induced homomorphism  $i^* : KU^*P(1,s,t) \to KU^*P(s,t)$  is given as follows:  $i^*T_k = 2T_k + T_{k+1}$  for  $1 \le k \le s - 1$ ,  $i^*T_s = 2T_s + 2T_{s+1} + T_{s+2}$ ,  $i^*T_{s+h} = 4T_{s+h} + 6T_{s+h+1} + 4T_{s+h+2} + T_{s+h+3}$  for  $1 \le h \le t$  and  $i^*T_{s+t+1} = 0$ . Using the (n, n)-matrix  $E_k = (e_k, \dots, e_n, 0, \dots, 0)$  we here introduce the two (n, n)-matrices  $A_n = 2E_1 + E_2$  and  $B_n = 4E_1 + 6E_2 + 4E_3 + E_4$ , where  $e_j$  is the unit column vector entried "1" only in the *j*-th component. Moreover we set

$$C_{s,t} = \begin{pmatrix} A_s & 0\\ \xi & B_t \end{pmatrix} \text{ where } \xi = (0, \cdots, 0, 2e_1 + e_2).$$

Then the induced homomorphism  $i^* : KU^0P(1,s,t) \to KU^0P(s,t)$  is expressed as  $(C_{s,t},0) : \bigoplus_{s+t+1} Z \to \bigoplus_{s+t} Z$ . Therefore  $KU^0L(s,t) \cong \operatorname{Coker} C_{s,t}$  and  $KU^1L(s,t) \cong Z$ . In particular,  $KU^0L^n(2) \cong KU^0L(n,0) \cong \operatorname{Coker} A_n$  and  $KU^0L^n(4) \cong \operatorname{Coker} B_n$ .

Recall that the KU-cohomology groups  $KU^0L^n(2) \cong Z[\sigma]/(\sigma^{n+1}, \sigma(1))$  and  $KU^0L^n(4) \cong Z[\sigma]/(\sigma^{n+1}, \sigma(2))$  are given as follows (see [4, 5]):

i)  $KU^0L^n(2) \cong Z/2^n$  with generator  $\sigma$ ,

ii)  $KU^0L^{2m}(4) \cong Z/2^{2m+1} \oplus Z/2^m \oplus Z/2^{m-1}$  with generators  $\sigma$ ,  $\sigma(1)$  and  $\sigma(1)\sigma$ ,  $KU^0L^{2m+1}(4) \cong Z/2^{2m+2} \oplus Z/2^m \oplus Z/2^m$  with generators  $\sigma$ ,  $\sigma(1) + 2^{m+1}\sigma$ and  $\sigma(1)\sigma$ , where  $\sigma = \theta^*a$  and  $\sigma(i) = \theta^*a(i)$ .

Therefore the induced homomorphism  $\theta^* : KU^0 CP^n \to KU^0 L^n(2)$  is given by the following row:

(3.2) 
$$\alpha_n = (-1)^{n-1} (1, -2, \cdots, (-2)^{n-1}) : \bigoplus_n Z \to Z/2^n.$$

On the other hand, the induced homomorphism  $\theta^* : KU^0 CP^n \to KU^0 L^n(4)$  is represented by the following (3, n)-matrix  $\beta_n$ :

$$(3.3) \ \beta_{2m} = \begin{pmatrix} 1 & -2 & 4 - 2^{m+1} & * \\ 0 & 1 & -2 & * \\ 0 & 0 & 1 & * \end{pmatrix}, \ \beta_{2m+1} = \begin{pmatrix} 1 & -2 - 2^{m+1} & 4 + 2^{m+2} & * \\ 0 & 1 & -2 & * \\ 0 & 0 & 1 & * \end{pmatrix}.$$

Notice that  $KU^0L(s,t)$  is isomorphic to the cokernel of

$$\begin{pmatrix} A_s \\ \beta_t \xi \end{pmatrix} : \bigoplus_s Z \to (\bigoplus_s Z) \oplus \operatorname{Coker} B_t.$$

Since  $\beta_{2m}\xi = (0, \dots, 0, e_2)$  and  $\beta_{2m+1}\xi = (0, \dots, 0, -2^{m+1}e_1 + e_2)$ , we can easily calculate the KU-cohomology group  $KU^0L(s,t)$  for  $t \ge 1$  as follows:

(3.4)  

$$KU^{0}L(s,2m) \cong Z/2^{s+m} \oplus Z/2^{2m+1} \oplus Z/2^{m-1}$$

$$KU^{0}L(s,2m+1) \cong \begin{cases} Z/2^{s+m} \oplus Z/2^{2m+2} \oplus Z/2^{m} & (s \le m) \\ Z/2^{s+m+1} \oplus Z/2^{2m+1} \oplus Z/2^{m} & (s > m). \end{cases}$$

Moreover we see that the quotient morphism  $\delta_{s,t} : (\bigoplus_s Z) \oplus \operatorname{Coker} B_t \to KU^0L(s,t)$  is represented by the following matrix:

$$\begin{array}{ccccc} t=2m & t=2m+1>2s & t=2m+1<2s \\ \begin{pmatrix} \alpha_s & 0 & -2^s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} \alpha_s & 0 & -2^s & 0 \\ 2^{m-s+1}\alpha_s & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} \alpha_s & 2^{s-m-1} & 0 & 0 \\ 0 & 1 & 2^{m+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the induced homomorphism  $\theta^* : KU^0P(s,t) \to KU^0L(s,t)$  is expressed as the composition  $\delta_{s,t}(1 \oplus \beta_t)$ , we can immediately give a basis of  $KU^0L(s,t)$   $(s,t \ge 1)$  as follows:

$$(3.5) \qquad \qquad (\sigma(1), \sigma(s, 1), \sigma(s, 3))B'_{s,t}$$

where  $\sigma(1) = \theta^* T_1$ ,  $\sigma(s, i) = \theta^* T_{s+i}$  and  $B'_{s,t}$   $(s, t \ge 1)$  is the matrix tabled below:

(3.6) 
$$B'_{s,2m} = \begin{pmatrix} (-1)^{s-1} & 0 & (-1)^{s}2^{s+1} \\ 0 & 1 & 2^{m+1}-4 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B'_{s,2m+1} = \begin{pmatrix} s \leq m & s > m \\ (-1)^{s-1} & 0 & (-1)^{s}2^{s+1} \\ -2^{m-s+1} & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} (-1)^{s-1} & (-1)^{s}2^{s-m-1} & (-1)^{s}2^{s+1} \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}.$$

**3.2.** Next we shall investigate the behavior of the conjugation  $\psi_C^{-1}$  on  $KU^0L(s,t)$   $(s,t \ge 1)$ . Note that  $\psi_C^{-1}a^h = (-1)^ha^h(1+a)^{-h}$  and  $\psi_C^{-1}a(1)^k = (-1)^ka(1)^k(1+a)^{-2k}$  in  $KU^0CP^n$ . Since  $a(2) = (1+a(1))^2 - 1$  and  $a(1)^sa(2) = a(1)^s\{(a+1)^4 - 1\}$  it follows immediately that

$$\begin{split} \psi_C^{-1} a(1) &\equiv a(1) \bmod a(2) \\ \psi_C^{-1} a(1)^s a &\equiv \begin{cases} a(1)^s (a^3 + 3a^2 + 3a) \\ a(1)^s (a^2 + a) \\ \psi_C^{-1} a(1)^s a^3 &\equiv \begin{cases} a(1)^s (3a^3 + 6a^2 + 4a) \\ -a(1)^s (a^3 + 2a^2 + 4a) \\ -a(1)^s (a^3 + 2a^2 + 4a) \end{cases} & \text{mod} \quad a(1)^s a(2) \\ s : \text{odd} \\ s : \text{odd}. \end{split}$$

Since  $a(1)^s a^2 \equiv (-1)^s 2^s a(1) - 2a(1)^s a \mod a(2)$ , the conjugation  $\psi_C^{-1}$  on  $KU^0L(s,t)$  behaves as

$$\psi_C^{-1}(\sigma(1), \sigma(s, 1), \sigma(s, 3)) = (\sigma(1), \sigma(s, 1), \sigma(s, 3))P_s$$

for the following matrix  $P_s$ :

$$(3.7) \quad P_{2n} = \begin{pmatrix} 1 & 3 \cdot 2^n & 3 \cdot 2^{2n+1} \\ 0 & -3 & -8 \\ 0 & 1 & 3 \end{pmatrix}, \quad P_{2n+1} = \begin{pmatrix} 1 & -2^{2n+1} & 2^{2n+2} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Consider the following matrix  $C_{s,t}$   $(s,t \ge 1)$  representing an automorphism on  $KU^0L(s,t)$ :

$$(3.8) \qquad s = 2n \le m, s = 2n + 1 \qquad s = 2n > m \\ \begin{pmatrix} 1 & 2^{s-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2^{s-m-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ s = 2n \le m \qquad s = 2n + 1 \le m \\ \begin{pmatrix} 1 + 2^m & 0 & -2^s \\ 0 & 1 & 0 \\ -2^{m-s} & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2^{s-1}(1-2^m) & -2^s(1-2^m) \\ 2^{2m-s+1} & 1+2^{2s}(1-2^m) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ s = 2n > m \ge 0 \qquad s = 2n + 1 > m \ge 1 \\ s = 2n > m \ge 0 \qquad s = 2n + 1 > m \ge 1 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2^{s-m} + 2^{s-1} & 2^{s+m} \\ 0 & 1 & 2^{s+m} \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In order to express the conjugation  $\psi_C^{-1}$  on  $KU^0L(s,t)$  plainly we here change the basis of  $KU^0L(s,t)$  given in (3.5) slightly as follows:

(3.9) 
$$(\sigma(1), \sigma(s, 1), \sigma(s, 3))B_{s,t}$$
 where  $B_{s,t} = B'_{s,t}C_{s,t}$ .

Then the conjugation  $\psi_C^{-1}$  on  $KU^0L(s,t)$  is represented by the composition  $B_{s,t}^{-1}P_sB_{s,t}$ . Therefore a routine computation shows

**Proposition 3.1.** On the KU-cohomology group  $KU^0L(s,t)$  with basis  $(\sigma(1), \sigma(s,1), \sigma(s,3))B_{s,t}$   $(s,t \ge 1)$  the conjugation  $\psi_C^{-1}$  behaves as follows: i) On  $KU^0L(s,2m) \cong Z/2^{s+m} \oplus Z/2^{2m+1} \oplus Z/2^{m-1}$ ,

$$\psi_C^{-1} = \begin{pmatrix} s = 2n \le m & s = 2n > m & s = 2n + 1 \\ 1 & -2^s & 2^{s+1} \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - 2^{m+1} & 2^{m+2} \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2^{s+1} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

ii) On  $KU^0L(s, 2m+1) \cong Z/2^{s+m} \oplus Z/2^{2m+2} \oplus Z/2^m \ (s \le m),$ 

$$\psi_C^{-1} = \begin{pmatrix} s = 2n \le m & s = 2n + 1 \le m \\ 1 & 0 & 0 \\ 0 & 1 - 2^{m+1} & 2^{m+2} \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} s = 2n + 1 \le m \\ 1 & 0 & 0 \\ 2^{m-s+2} & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

iii) On  $KU^0L(s, 2m+1) \cong Z/2^{s+m+1} \oplus Z/2^{2m+1} \oplus Z/2^m \ (s > m)$ ,

$$\psi_C^{-1} = \begin{pmatrix} s = 2n > m \ge 0 & s = 2n + 1 > m \ge 1 \ s = 2n + 1 > m = 0 \\ \begin{pmatrix} 1 & -2^s & 2^{s+1} \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2^{s-m} & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

REMARK. When t = 0, the conjugation  $\psi_C^{-1} = 1$  on  $KU^0L(s,0) \cong Z/2^s$  with basis  $\sigma(1)$ .

We shall use the dual of Proposition 3.1 to study the behavior of the conjugation  $\psi_C^{-1}$  on  $KU_*L_0(s,t)$  and  $KU_*L(s,t)$ .

**Proposition 3.2.** The weighted mod 4 lens spaces  $\Sigma^1 L_0(s,t)$  and  $\Sigma^1 L(s,t)$   $(s \ge 1, t \ge 0)$  have the same C-types as the small spectra tabled below, respectively (cf. [12, Proposition 5.1]):

	$\Sigma^1 L_0(s,2m)$	$\Sigma^1 L(s,2m)$	$\Sigma^1 L_0(s, 2m+1)$
$s=2n\leq m$	$PP'_{2m+1,s+m-1,m}$	$MPP'_{2m+1,s+m-1,m}$	$SZ/2^{s+m} \vee P_{2m+1,m+1}^{\prime\prime}$
s = 2n > m	$SZ/2^{s+m} \vee P_{2m,m}^{\prime\prime}$	$M_{s+m} \vee P_{2m,m}''$	$PP_{2m+1,s+m,m+1}'$
$s=2n+1,m\geq 1$	$\Sigma^2 SZ/2^{2m+1} \vee P'_{s+m-1,m}$	$\Sigma^2 M_{2m+1} \vee P'_{s+m-1,m}$	$\Sigma^2 SZ/2^m \vee P'_{s+m,2m+2}$
s=2n+1, m=0	$SZ/2^s$	$\Sigma^0 \vee SZ/2^s$	$SZ/2 \vee SZ/2^{s+1}$

Moreover  $\Sigma^1 L(s, 2m+1)$  has the same C-type as the wedge sum  $\Sigma^{2s} \vee \Sigma^1 L_0(s, 2m+1)$ .

Proof. By dualizing Proposition 3.1 we can immediately determine the C-type of  $\Sigma^1 L_0(s,t)$  because  $KU_{-1}L_0(s,t) \cong KU^0 L_0(s,t)$  and  $KU_0 L_0(s,t) = 0$ . On the other hand, Proposition 3.4 below implies that  $\Sigma^1 L(s, 2m + 1)$  has the same C-type as  $\Sigma^{2s} \vee \Sigma^1 L_0(s, 2m + 1)$ . We shall now investigate the C-type of  $\Sigma^1 L(s, 2m)$  in case of  $s = 2n \leq m$ . Note that  $KU_{-1}L(s,t) \cong KU_{-1}\Sigma^{2s+2t+1} \oplus KU_{-1}L_0(s,t)$  and  $KU_0L(s,t) = 0$ . According to the dual of Proposition 3.1 the conjugations  $\psi_C^{-1}$  on  $KU_{-1}L(s, 2m) \cong Z \oplus Z/2^{s+m} \oplus Z/2^{2m+1} \oplus Z/2^{2m-1}$  and  $KU_{-1}L_0(s, 2m + 1) \cong Z/2^{s+m} \oplus Z/2^{2m+2} \oplus Z/2^m$  are represented by the following matrices

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & -2^{m+1} & 1 & 2^{m+2} \\ c & 1 & 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - 2^{m+1} & 2^{m+2} \\ 0 & 1 & -1 \end{pmatrix}$$

for some integers a, b and c, respectively. As is easily verified, we may regard that a = c = 0 and b = 0 or -1 after changing the direct sum decomposition of  $KU_{-1}L(s, 2m)$  suitably if necessary. Consider the canonical inclusion map  $i_{L_0} : L(s,t) \to L_0(s,t+1)$ . By virtue of (3.9) the induced homomorphism  $i_{L_0}^* :$  $KU^0L_0(s,t+1) \to KU^0L(s,t)$  is actually represented by the matrix  $F_{s,t} = B_{s,t}^{-1}B_{s,t+1}$ . Since a routine computation shows that

$$F_{s,2m} = \begin{pmatrix} 1+2^m & 0 & -2^s \\ -2^{m-s+2}(1+2^{m-1}) & 1 & 2^{m+1} \\ -2^{m-s+1} & 0 & 1 \end{pmatrix},$$

the induced homomorphism  $i_{L_0*}: KU_{-1}L(s, 2m) \to KU_{-1}L_0(s, 2m+1)$  is expressed as the following matrix

$$\begin{pmatrix} x & 1+2^m & -2(1+2^{m-1}) & -2^{m+2} \\ y & 0 & 2 & 0 \\ z & -1 & 1 & 2 \end{pmatrix}$$

for some integers x, y and z. Here y must be odd because  $i_{L_0*}$  is an epimorphism. Using the equality  $\psi_C^{-1}i_{L_0*} = i_{L_0*}\psi_C^{-1}$  we get immediately that  $b \equiv y \mod 2^m$ , thus b = -1. Therefore  $\Sigma^1 L(s, 2m)$  has the same C-type as  $MPP'_{2m+1,s+m-1,m}$  when  $s = 2n \leq m$ . In the other three cases the C-types of  $\Sigma^1 L(s, 2m)$  are similarly obtained.

**3.3.** Using Proposition 3.2 we can immediately calculate  $KO_i X \oplus KO_{i+4}X$  (i = 0, 2) for  $X = L_0(s, t)$  and L(s, t)  $(s \ge 1, t \ge 0)$  as tabled below:

X =	$L_0(2n,2m)$	L(2n,2m)	$L_0(2n,2m+1)$	L(2n, 2m + 1)
$KO_0X \oplus KO_4X \cong$	Z/2	0	Z/2	$Z/2\oplus Z/2$
$KO_2X \oplus KO_6X \cong$	Z/2	Z/2	Z/2	Z/2
X =	$L_0(2n+1,2m)$	L(2n+1,2m)	$L_0(2n+1, 2m+1)$	L(2n+1,2m+1)
$KO_0X \oplus KO_4X \cong$	$(**)_m$	$Z/2\oplus Z/2$	$Z/2\oplus Z/2$	$Z/2\oplus Z/2$
$KO_2X \oplus KO_6X \cong$	(**) <i>m</i>	Z/2	$Z/2\oplus Z/2$	$Z/2 \oplus Z/2 \oplus Z/2$

where  $(**)_0 \cong Z/2$  and  $(**)_m \cong Z/2 \oplus Z/2$  if  $m \ge 1$ .

**Lemma 3.3.** For  $X = L_0(s,t)$  and L(s,t)  $(s \ge 1, t \ge 0)$  the sets  $S(X) = \{2i; KO_{2i}X = 0 \ (0 \le i \le 3)\}$  are given as follows:

$$\begin{array}{rcl} (\mathbf{i}) & X = & L_0(2n,2m) \ L(2n,2m) \ L_0(2n,2m+1) \ L(2n,2m+1) \\ \\ S(X) = & \{4,6\} & \{0,4,6\} & \{0,6\} & \{0,6\} & n+m: \mathrm{even} \\ & \{0,6\} & \{0,4,6\} & \{4,6\} & \{4,6\} & n+m: \mathrm{odd} \end{array}$$

$$\begin{array}{rcl} (\mathbf{ii}) & X = & L_0(2n+1,2m) \ L(2n+1,2m) \ L_0(2n+1,2m+1) \ L(2n+1,2m+1) \\ \\ S(X) = & \{0,6\} & \{0,6\} & \{0\} & \{0\} & n,m: \mathrm{even} \\ & \{0\} & \{0,6\} & \{0,6\} & \{0,6\} & \{0,6\} & n,m+1: \mathrm{even} \\ \\ & \{4,(6)\}_m & \{4,6\} & \{4,6\} & \{4,6\} & n,m+1: \mathrm{odd} \end{array}$$

**{4}** 

**{4}** 

n, m: odd

where  $\{4, (6)\}_0 = \{4, 6\}$  and  $\{4, (6)\}_m = \{4\}$  if  $m \ge 1$ .

 $\{4, 6\}$ 

 $\{4, 6\}$ 

Proof. Consider the following (homotopy) commutative diagram

$$\begin{array}{ccc} L_0(s,t) \xrightarrow{\theta_0} P(s,t) \xrightarrow{i_0} P(1,s,t-1) \\ i_L \downarrow & \parallel & \downarrow \tilde{i} \\ L(s,t) \xrightarrow{\theta} P(s,t) \xrightarrow{i} P(1,s,t) \end{array}$$

with two cofiber sequences, where the maps  $i_L$ , i and  $\tilde{i}$  are the canonical inclusions, and the map  $i_0$  is defined by  $i_0[x_0, \dots, x_{s+t}] = [x_{s+t}^4, x_0, \dots, x_{s+t}]$ . According to [7, Theorem 2.4] the weighted projective space P(s,t) is quasi  $KO_*$ -equivalent to the wedge sum  $\vee_{n+m}C(\eta)$ ,  $\Sigma^{4n+4m+4} \vee (\vee_{n+m}C(\eta))$ ,  $\Sigma^{4n+2} \vee (\vee_{n+m}C(\eta))$  or  $\Sigma^{4n+2} \vee \Sigma^{4n+4m+4} \vee (\vee_{n+m}C(\eta))$  according as (s,t) = (2n,2m), (2n,2m+1), (2n+1,2m) or (2n+1,2m+1). In addition, P(1,s,t) is quasi  $KO_*$ -equivalent to the wedge sum  $\Sigma^2 \vee \Sigma^2 P(s,t)$ . Using the above commutative diagram we can immediately obtain our result.

**Proposition 3.4.** The weighted mod 4 lens space L(s, 2m + 1) is quasi  $KO_*$ -equivalent to the wedge sum  $\Sigma^{2s+4m+3} \vee L_0(s, 2m + 1)$ .

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Proof. Consider the following commutative diagram

$$\begin{array}{cccc} \Sigma^{2s+4m+3} & \xrightarrow{\tilde{\alpha}} & P(1,s,2m) & \xrightarrow{\tilde{i}} & P(1,s,2m+1) \\ & & & \downarrow & & \downarrow \\ \Sigma^{2s+4m+3} & \xrightarrow{\alpha} & \Sigma^1 L_0(s,2m+1) & \xrightarrow{i_L} & \Sigma^1 L(s,2m+1) \end{array}$$

with two cofiber sequences. Since the quasi  $KO_*$ -type of P(1, s, t) is given as in the proof of Lemma 3.3 we see that the map  $1 \wedge \tilde{\alpha} : \Sigma^{2s+4m+3}KO \to KO \wedge P(1, s, 2m)$  is trivial. Hence our result is immediate.

Applying Theorems 1.2 and 1.3 and Proposition 3.4 with the aid of Proposition 3.2 and Lemma 3.3 we can immediately obtain

**Theorem 3.5.** The weighted mod 4 lens spaces  $\Sigma^1 L_0(2n, t)$  and  $\Sigma^1 L(2n, t)$  for  $n \ge 1$  are quasi KO<sub>\*</sub>-equivalent to the small spectra tabled below, respectively (cf. [12, Theorem 3]):

		$\Sigma^1 L_0(2n,2m)$	$\Sigma^1 L(2n,2m)$	$\Sigma^1 L_0(2n,2m+1)$
;)	n+m: even	$PP_{2m+1,2n+m-1,m}'$	$MPP'_{2m+1,2n+m-1,m}$	$V_{2n+m} \vee P_{2m+1,m+1}^{\prime\prime}$
1)	$n+m: \mathrm{odd}$	$_{V}PP'_{2m+1,2n+m-1,m}$	$MPP'_{2m+1,2n+m-1,m}$	$SZ/2^{2n+m} \vee P_{2m+1,m+1}''$
;;)	n+m:even	$SZ/2^{2n+m}\vee P_{2m,m}^{\prime\prime}$	$M_{2n+m} \vee P_{2m,m}^{\prime\prime}$	$_VPP'_{2m+1,2n+m,m+1}$
,	n+m: odd	$V_{2n+m} \vee P_{2m,m}^{\prime\prime}$	$M_{2n+m} \vee P_{2m,m}^{\prime\prime}$	$PP_{2m+1,2n+m,m+1}$

in cases when i)  $2n \leq m$  and ii) 2n > m. Moreover  $\Sigma^1 L(2n, 2m + 1)$  is quasi  $KO_*$ -equivalent to  $\Sigma^{4n+4m+4} \vee \Sigma^1 L_0(2n, 2m + 1)$ .

Applying Theorems 2.3 and 2.4 in place of Theorems 1.2 and 1.3 we show

**Theorem 3.6.** The weighted mod 4 lens spaces  $\Sigma^1 L_0(2n + 1, t)$  and  $\Sigma^1 L(2n + 1, t)$  are quasi KO<sub>\*</sub>-equivalent to the small spectra tabled below, respectively:

	$\sum^{1} L_0(2n+1,2m)$	$\Sigma^1 L(2n+1,2m)$	$\Sigma^1 L_0(2n+1, 2m+1)$
i)	$V_{2n+1}$	$\Sigma^4 \vee V_{2n+1}$	$\Sigma^4 SZ/2 \vee V_{2n+2}$
ii)	$\Sigma^2 SZ/2^{2m+1} \vee P'_{2n+m,m}$	$\Sigma^2 M_{2m+1} \vee P'_{2n+m,m}$	$\Sigma^2 V_m \vee P'_{2n+m+1,2m+2}$
iii)	$\Sigma^2 V_{2m+1} \vee P'_{2n+m,m}$	$\Sigma^2 M_{2m+1} \vee P'_{2n+m,m}$	$\Sigma^2 SZ/2^m \vee P'_{2n+m+1,2m+2}$
iv)	$SZ/2^{2n+1}$	$\Sigma^0 \vee SZ/2^{2n+1}$	$SZ/2 \vee SZ/2^{2n+2}$
v)	$\Sigma^{6}SZ/2^{2m+1} \vee \Sigma^{6}P'_{m-1,2n+m+1}$	$\Sigma^6 M_{2m+1} \vee \Sigma^6 P'_{m-1,2n+m+1}$	$\Sigma^6 V_m \vee \Sigma^6 P'_{2m+1,2n+m+2}$
vi)	$\Sigma^6 V_{2m+1} \vee \Sigma^6 P'_{m-1,2n+m+1}$	$\Sigma^6 M_{2m+1} \vee \Sigma^6 P'_{m-1,2n+m+1}$	$\Sigma^6 SZ/2^m \vee \Sigma^6 P'_{2m+1,2n+m+2}$

in cases when i) n is even and m = 0, ii) n and  $m \ge 2$  are even, iii) n is even and m is odd, iv) n is odd and m = 0, v) n is odd and  $m \ge 2$  is even, and vi) n and m are odd. Moreover  $\Sigma^1 L(2n+1, 2m+1)$  is quasi  $KO_*$ -equivalent to  $\Sigma^{4n+4m+6} \lor \Sigma^1 L_0(2n+1, 2m+1)$ .

Proof. By a quite similar argument to the case of the real projective space  $RP^k$ (cf. [10, Theorem 5]) we can easily determine the quasi  $KO_*$ -types of  $\Sigma^1 L_0(2n + 1, 0)$  and  $\Sigma^1 L(2n + 1, 0)$ . The quasi  $KO_*$ -type of  $\Sigma^1 L(2n + 1, 2m)$  for  $m \ge 1$  is immediately determined by applying Theorem 2.4 ii) with the aid of Proposition 3.2 and Lemma 3.3. On the other hand, the quasi  $KO_*$ -types of  $\Sigma^1 L_0(2n + 1, 2m)$  in cases of ii) and vi) and those of  $\Sigma^1 L_0(2n + 1, 2m + 1)$  in cases of iii), iv) and v) are also determined by applying Theorem 2.3 and [6, Theorem 5.3] in place of Theorem 2.4 ii).

We shall now investigate the quasi  $KO_*$ -types of  $\Sigma^1 L_0(2n + 1, 2m - 1)$  and  $\Sigma^1 L_0(2n + 1, 2m)$  in case when n is even and m is odd. Consider the following two cofiber sequences

$$\Sigma^{4n+4m} \xrightarrow{\alpha_0} \Sigma^1 L(2n+1,2m-2) \xrightarrow{i_{L_0}} \Sigma^1 L_0(2n+1,2m-1)$$

$$\Sigma^{4n+4m+2} \xrightarrow{\alpha_0} \Sigma^1 L(2n+1,2m-1) \xrightarrow{i_{L_0}} \Sigma^1 L_0(2n+1,2m)$$

where  $\Sigma^1 L(2n+1, 2m-1)$  is quasi  $KO_*$ -equivalent to  $\Sigma^{4n+4m+2} \vee \Sigma^1 L_0(2n+1, 2m-1)$ 1) according to Proposition 3.4. Note that  $\Sigma^1 L(2n+1,0)$  is quasi  $KO_*$ -equivalent to  $\Sigma^4 \vee V_{2n+1}$ . Since  $\Sigma^1 L_0(2n+1,1)$  has the same C-type as  $SZ/2 \vee SZ/2^{2n+2}$  by Proposition 3.2, [6, Proposition 3.2] asserts that it must be quasi KO<sub>\*</sub>-equivalent to  $\Sigma^4 SZ/2 \lor V_{2n+2}$ . Hence it is easily calculated that  $KO_3L_0(2n+1,2) \cong Z/2 \oplus Z/2^{2n+3}$ and  $KO_7L_0(2n+1,2)$  is isomorphic to the cokernel of  $\alpha_{0*}: Z/2 \to Z/2 \oplus Z/2 \oplus$  $Z/2^{2n+1}$ . From Lemma 3.3 we recall that the set S(X) consists of only 0 for X = $L_0(2n+1,2m-1)$  or  $L_0(2n+1,2m)$  under our assumption on n and m. Applying Theorem 2.3 i) and ii) combined with Proposition 3.2 we see that  $\Sigma^1 L_0(2n +$ 1, 2m-1) is quasi  $KO_*$ -equivalent to one of the three spectra  $\Sigma^2 V_{m-1} \vee P'_{2n+m,2m}$ ,  $\Sigma^2 SZ/2^{m-1}$   $\vee$   $\Sigma^2 P'_{2m-1,2n+m+1}$  and  $\Sigma^2 NP'_{m-1,2m-1,2n+m+1}$  when  $m \geq 3$ , and  $\Sigma^1 L_0(2n+1,2m)$  is quasi  $KO_*$ -equivalent to one of the three spectra  $\Sigma^2 V_{2m+1} \vee$  $\begin{array}{l} P_{2n+m,m}', \ \Sigma^2 SZ/2^{2m+1} \vee \Sigma^2 P_{m-1,2n+m+1}' \ \text{and} \ \Sigma^2 NP_{2m+1,m-1,2n+m+1}' \ \text{when} \ m \geq \\ 1. \ \text{Since} \ \Sigma^1 L(2n+1,2m-2) \ \text{is quasi} \ KO_* \text{-equivalent to} \ \Sigma^2 M_{2m-1} \vee P_{2n+m-1,m-1}' \end{array}$ when  $m \geq 3$ , it is immediate that  $KO_1L_0(2n+1, 2m-1) \cong Z/2^{2m-1} \oplus Z/2^{m-2} \oplus Z/2$ . Therefore  $\Sigma^1 L_0(2n+1, 2m-1)$  must be quasi  $KO_*$ -equivalent to  $\Sigma^2 V_{m-1} \vee P'_{2n+m,2m}$ when  $m \geq 3$ . Hence it is easily calculated that  $KO_3L_0(2n+1,2m) \cong Z/2 \oplus$  $Z/2^{2n+m+1}\oplus Z/2$  and  $KO_7L_0(2n+1,2m)$  is isomorphic to the cokernel of  $\alpha_{0*}$  :  $Z/2 \rightarrow Z/2 \oplus Z/2 \oplus Z/2^{2n+m}$ . Therefore  $\Sigma^1 L_0(2n+1,2m)$  must be quasi  $KO_*$ equivalent to  $\Sigma^2 V_{2m+1} \vee P'_{2n+m,m}$  when  $m \geq 3$  as well as m = 1.

In case when n is odd and  $m \ge 2$  is even the quasi  $KO_*$ -types of  $\Sigma^1 L_0(2n + 1, 2m - 1)$  and  $\Sigma^1 L_0(2n + 1, 2m)$  are determined by a parallel argument.

REMARK. According to Theorems 3.5 and 3.6,  $L_0(s,0)$  and L(s,0) are quasi  $KO_*$ -equivalent to the real projective spaces  $RP^{2s}$  and  $RP^{2s+1}$ , respectively.

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