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## J-GROUPS OF SUSPENSIONS OF STUNTED LENS SPACES MOD 8

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### 1. Introduction

Let  $L^n(q) = S^{2n+1}/\mathbf{Z}_q$  be the  $(2n+1)$ -dimensional standard lens space mod  $q$ . As defined in [7], we set

$$(1.1) \quad \begin{aligned} L_q^{2n+1} &= L^n(q), \\ L_q^{2n} &= \{[z_0, \dots, z_n] \in L^n(q) \mid z_n \text{ is real and } z_n \geqq 0\}. \end{aligned}$$

In the previous paper [10], we determined the  $J$ -groups  $\tilde{J}(S^j(L_q^m/L_q^n))$  of the suspensions of the stunted lens spaces  $L_q^m/L_q^n$  for  $q=4$  and for  $j \equiv 1 \pmod{2}$ . The purpose of this paper is to determine the  $KO$ - and  $J$ -groups of suspensions of stunted lens spaces mod 8.

This paper is organized as follows. In section 2 we state the main theorems: the structures of  $K\widetilde{O}(S^j(L_8^m/L_8^n))$  and  $\tilde{J}(S^j(L_8^m/L_8^n))$  for  $j \equiv 0 \pmod{2}$  are given in Theorems 1 and 2 respectively. In section 3 we prepare some lemmas and recall known results in [8], [9] and [11]. By virtue of the results in [8], the proofs of Theorems 1 and 2 for the case  $j \equiv 0 \pmod{4}$  are given in section 4. Applying the method used in the corresponding parts of [10], we prove Theorems 1 and 2 for the case  $j \equiv 2 \pmod{4}$  in the final section.

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### 2. Statement of results

We prepare functions  $h_1, h_2, h_3, h_4, a_1, a_2, a_3, a_4, a_5, a_6$  and  $a_7$  defined by

$$(2.1) \quad \begin{cases} h_1(n) = [n/4] + [(n+7)/8] + [(n+4)/8] \\ h_2(n) = [n/4] + [(n+7)/8] + [n/8] + 1 \\ h_3(m, n) = [m/4] - [n/4] \\ h_4(m, n) = [m/8] - [n/8]. \end{cases}$$

$$(2.2) \quad \begin{cases} a_1(m, n) = h_1(m) - [(n+1)/4] - [(n+1)/8] - [(n+6)/8] + 1 \\ a_2(m, n) = h_3(m, n+1) \\ a_3(m, n) = h_4(m-2, n+5) \\ a_4(m, n) = h_4(m, n+7) \\ a_5(m, n) = a_1(m, n) - [(m+4)/8] + [m/8] \\ a_6(m, n) = [(m+4)/8] + [(m-2)/8] - [(n+1)/4] \\ a_7(m, n) = 2[(m+4)/8] - [(n+5)/4]. \end{cases}$$

Let  $\mathbf{Z}/k$  denote the cyclic group  $\mathbf{Z}/k\mathbf{Z}$  of order  $k$ . For an integer  $n$ ,  $G(n)$  denotes the group defined by

$$(2.3) \quad G(n) = \begin{cases} \mathbf{Z}/2 \oplus \mathbf{Z}/2 & (n \equiv 1 \pmod{8}) \\ \mathbf{Z}/2 & (n \equiv 0 \text{ or } 2 \pmod{8}) \\ 0 & (\text{otherwise}). \end{cases}$$

**Theorem 1.** Let  $j, m$  and  $n$  be non-negative integers with  $j \equiv 0 \pmod{2}$  and  $m > n$ .

(1) Suppose  $j \equiv 0 \pmod{4}$ .

i) If  $n \equiv 3 \pmod{4}$  and  $m \geq 4[(n+j+15)/8] + 2[(n-j)/4]$ , then we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \bigoplus_{i=1}^{4} \mathbf{Z}/2^{a_i(m+j, n+j)}.$$

ii) If  $n \equiv 3 \pmod{4}$  and  $4[(n+j+15)/8] + 2[(n-j)/4] > m > n$ , then we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/2^{a_1(m+j, n+j)} & (m \geq 4[n/4]+4) \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & (n+j \equiv 0 \pmod{8} \text{ and } n+4 > m \geq n+2) \\ \mathbf{Z}/2 & (h_4(n+j+6, n+j) = [m/2] - [(n+1)/2] = 0) \\ 0 & (\text{otherwise}). \end{cases}$$

iii) If  $n \equiv 3 \pmod{4}$ , then we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z} \oplus \widetilde{KO}(S^j(L_8^m/L_8^{n+1})) & (m \geq n+2) \\ \mathbf{Z} & (m = n+1). \end{cases}$$

(2) Suppose  $j \equiv 2 \pmod{4}$ .

i) If  $m \geq 8[(n+j+15)/8] - j + 2$ , then we have

$$\begin{aligned} \widetilde{KO}(S^j(L_8^m/L_8^n)) &\cong \mathbf{Z}/2^{h_3(m+j, n+j-3)} \oplus \left( \bigoplus_{i=0}^1 \mathbf{Z}/2^{h_4(m+j-4i, n+j-4i+5)} \right) \\ &\quad \oplus G(m+j) \oplus H(n+j), \end{aligned}$$

where  $G(m)$  is the group defined by (2.3) and  $H(n)$  is the group defined by

$$H(n) = \begin{cases} \mathbf{Z} & (n \equiv 3 \pmod{4}) \\ G(n) & (\text{otherwise}). \end{cases}$$

ii) If  $8[(n+j+15)/8]-j+2 > m \geq 6[(n+j+7)/8]+2[(n+j+1)/8]-j+4$ , then we have

$$\begin{aligned} \widetilde{KO}(S^j(L_8^m/L_8^n)) \cong & \mathbf{Z}/2^{h_3(m+j,n+j-3)} \oplus \mathbf{Z}/2^{h_4(m+j-4,n+j+1)} \\ & \oplus G_1(m+j) \oplus H_1(n+j), \end{aligned}$$

where  $G_1(m)$  is the group defined by

$$(2.4) \quad G_1(m) = \begin{cases} \mathbf{Z}/4 \oplus \mathbf{Z}/2 & (m \equiv 1 \pmod{8}) \\ \mathbf{Z}/4 & (m \equiv 0 \pmod{8}) \\ G(m) & (\text{otherwise}) \end{cases}$$

and  $H_1(n)$  is the group defined by

$$H_1(n) = \begin{cases} \mathbf{Z} & (n \equiv 3 \pmod{4}) \\ G_1(2-n) & (\text{otherwise}). \end{cases}$$

iii) If  $6[(n+j+7)/8]+2[(n+j+1)/8]-j+4 > m \geq n+3$ , then we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong G_2(m+j) \oplus H_2(n+j),$$

where  $G_2(m)$  is the group defined by

$$G_2(m) = \begin{cases} \mathbf{Z}/8 \oplus \mathbf{Z}/2 & (m \equiv 1 \pmod{8}) \\ \mathbf{Z}/8 & (m \equiv 0 \pmod{8}) \\ G(m) & (\text{otherwise}), \end{cases}$$

and  $H_2(n)$  is the group defined by

$$H_2(n) = \begin{cases} \mathbf{Z} & (n \equiv 3 \pmod{4}) \\ G_2(2-n) & (\text{otherwise}) \end{cases}$$

iv) If  $n+3 > m > n$ , then we have  $\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \widetilde{KO}(L_8^{m+j}/L_8^{n+j})$ .

REMARK. (1) Combining this theorem with [13, Theorem 2], we obtain the complete results for the groups  $\widetilde{KO}(S^j(L_8^m/L_8^n))$ .

(2) The partial results for the case  $n=0$  of this theorem have been obtained in [8].

Let  $\nu_p(s)$  denote the exponent of the prime  $p$  in the prime power decomposition of  $s$ , and  $m(s)$  the function defined on positive integers as follows (cf. [3]):

$$\nu_p(m(s)) = \begin{cases} 0 & (p \neq 2 \text{ and } s \not\equiv 0 \pmod{(p-1)}) \\ 1 + \nu_p(s) & (p \neq 2 \text{ and } s \equiv 0 \pmod{(p-1)}) \\ 1 & (p = 2 \text{ and } s \not\equiv 0 \pmod{2}) \\ 2 + \nu_2(s) & (p = 2 \text{ and } s \equiv 0 \pmod{2}). \end{cases}$$

In order to state the next theorem, we set

$$(2.5) \quad \begin{cases} b_1(j, m, n) = \begin{cases} \min \{v + h_4(n+9, n-3), a_1(m+j, n+j)\} & (j \equiv 4 \pmod{8}) \\ \min \{v + h_4(n+9, n-2), a_5(m, n)\} & (\text{otherwise}) \end{cases} \\ b_2(j, m, n) = \begin{cases} \min \{v + h_4(n+3, n-7), a_7(m, n)\} & (j \equiv 4 \pmod{8}) \\ \min \{v + h_4(n+5, n-7), a_6(m, n)\} & (\text{otherwise}) \end{cases} \\ b_3(j, m, n) = \begin{cases} \min \{v + 1, a_3(m+j, n+j)\} & (j \equiv 4 \pmod{8}) \\ \min \{v + 1, a_4(m, n)\} & (\text{otherwise}), \end{cases} \end{cases}$$

where  $v$  is the integer defined by

$$v = \begin{cases} v_2(j) & (j \neq 0) \\ m & (j = 0). \end{cases}$$

Main result is the following theorem.

**Theorem 2.** *Let  $j, m$  and  $n$  be non-negative integers with  $j \equiv 0 \pmod{2}$  and  $m > n$ .*

(1) *Suppose  $j \equiv 0 \pmod{4}$  and  $n \not\equiv 3 \pmod{4}$ .*

i) *If  $m \geq 2[n/4] + 4[n/8] + 6 + h_4(j, j-4) (2h_4(n-2, n+4) - 4a_4(n, n))$ , then we have*

$$\tilde{J}(S^j(L_8^n/L_8^m)) \cong \begin{cases} (\bigoplus_{i=1}^3 \mathbf{Z}/2^{b_i(j, m, n)}) \oplus \mathbf{Z}/2 & (n \equiv 2 + 2h_4(j+4, j) \pmod{8}) \\ \bigoplus_{i=1}^3 \mathbf{Z}/2^{b_i(j, m, n)} & (\text{otherwise}). \end{cases}$$

ii) *If  $2[n/4] + 4[n/8] + 6 + h_4(j, j-4) (2h_4(n-2, n+4) - 4a_4(n, n)) > m > n$ , then we have*

$$\tilde{J}(S^j(L_8^n/L_8^m)) \cong \begin{cases} \mathbf{Z}/2^{b_1(j, m, n)} \oplus \mathbf{Z}/4 & (h_4(n+j+5, n+j-2) = h_3(m, n+6) = 0) \\ \mathbf{Z}/8 & (h_4(n+j+6, n+j-1) = h_3(m, n+3) = 0) \\ \widetilde{KO}(S^j(L_8^n/L_8^m)) & (\text{otherwise}). \end{cases}$$

(2) *Suppose  $j \equiv n+1 \equiv 0 \pmod{4}$ . Set  $M = m((n+j+1)/2)$  and  $b_i = b_i(j, m, n)$  ( $1 \leq i \leq 3$ ).*

i) *If  $m \geq n + 2h_4(j+4, j) h_4(n+1, n) + 5$ , then we have*

$$\tilde{J}(S^j(L_8^n/L_8^m)) \cong \begin{cases} \mathbf{Z}/2^{c_1}M \oplus \mathbf{Z}/2^{c_2+i_1} \oplus \mathbf{Z}/2^{b_3} \oplus \mathbf{Z}/2 & (j(n+1) \equiv 4 \pmod{8}) \\ \mathbf{Z}/2^{c_1}M \oplus \mathbf{Z}/2^{c_2+i_1} \oplus \mathbf{Z}/2^{c_3+i_2} \oplus \mathbf{Z}/2^{i_3} & (\text{otherwise}), \end{cases}$$

where  $i_1, i_2, i_3, c_1, c_2$  and  $c_3$  are integers defined by

$$\begin{cases} i_1 = \begin{cases} \min \{b_1, v_2(n+1)-1\} & (n+j \equiv 7 \pmod{8}) \\ \min \{b_1, v_2(n+1)\} & (n+j \equiv 3 \pmod{8}) \end{cases} \\ i_2 = \begin{cases} \min \{b_2, v_2(n+1)-2\} & (n+j \equiv 7 \pmod{8}) \\ \min \{b_2, v_2(n+1)-1\} & (n+j \equiv 3 \pmod{8}) \end{cases} \end{cases}$$

$$(2.6) \quad \begin{cases} i_3 = \begin{cases} \min \{b_3, \nu_2(n+1)-2\} & (j \equiv n-3 \equiv 4 \pmod{8}) \\ b_3 & (\text{otherwise}) \end{cases} \\ c_1 = \max \{b_k - i_k \mid 1 \leq k \leq 3\} \\ c_3 = \min \{b_k - i_k \mid 1 \leq k \leq 3\} \\ c_2 = (\sum_{k=1}^3 b_k - i_k) - c_1 - c_3. \end{cases}$$

ii) If  $n+2h_4(j+4, j) h_4(n+1, n)+5 > m > n$ , then we have

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/2^{b-i} M \oplus \mathbf{Z}/2^i & (m \geq n+5) \\ \mathbf{Z}/M \oplus \widetilde{KO}(S^j(L_8^m/L_8^{n-1})) & (n+5 > m > n+1) \\ \mathbf{Z}/M & (m = n+1), \end{cases}$$

where  $b = b_1$  and  $i = \min \{b, \nu_2(n+1)\}$ .

(3) Suppose  $j \equiv 2 \pmod{4}$  and  $n \not\equiv 1 \pmod{4}$ .

i) If  $m \geq 8 [(n+17)/8] + h_4(j+4, j) (6h_4(n-2, n+1) + 2h_4(n-6, n+9))$ , then we have

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/2^b \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2 \oplus G(n+j) \oplus G(m+j) \\ \quad (h_4(n+5-2h_4(j, j-4), n-2) = 0) \\ \mathbf{Z}/2^b \oplus \mathbf{Z}/8 \oplus G(n+j) \oplus G(m+j) \quad (\text{otherwise}), \end{cases}$$

where  $b = b_3(j, m+2, n)$  and  $G(m)$  is the group defined by (2.3).

ii) If  $8 [(n+17)/8] + h_4(j+4, j) (6h_4(n-2, n+1) + 2h_4(n-6, n+9)) > m \geq 8[(n+2)/8] + 10$ , then we have

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/4 \oplus \mathbf{Z}/2 \oplus G_1(m+j) & (h_4(n+5-2h_4(j, j-4), n-2) = 0) \\ \mathbf{Z}/8 \oplus G_1(h_4(j+4, j) (2n-1)-n) \oplus G_1(m+j) & (\text{otherwise}), \end{cases}$$

where  $G_1(m)$  is the group defined by (2.4).

iii) If  $8 [(n+2)/8] + 10 > m > n$ , then we have

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \widetilde{KO}(S^j(L_8^m/L_8^n)).$$

(4) Suppose  $j \equiv n+1 \equiv 2 \pmod{4}$ . Set  $M = m((n+j+1)/2)$ .

i) If  $m \geq 8 [n/8] + 2h_4(j, j-4) + 14$ , then we have

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/2^b M \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2 \oplus G(m+j) & (n \equiv 1 \pmod{8}) \\ \mathbf{Z}/4 M \oplus \mathbf{Z}/2^b \oplus \mathbf{Z}/2 \oplus G(m+j) & (n \equiv 5 \pmod{8}), \end{cases}$$

where  $b = b_3(j, m+2, n)$  and  $G(m)$  is the group defined by (2.3).

ii) If  $8 [n/8] + 14 + 2h_4(j, j-4) > m \geq 4h_4(j+4, j) h_4(n-4, n) + 8 [n/8] + 10$ , then we have

$$\tilde{J}(S^j(L_8^n/L_8^s)) \cong \begin{cases} \mathbf{Z}/2M \oplus \mathbf{Z}/4 \oplus G_1(m+j) & (j \equiv n+1 \equiv 2 \pmod{8}) \\ \mathbf{Z}/2^{c+1}M \oplus \mathbf{Z}/2 \oplus G_1(m+j) & (j \equiv n-3 \equiv 2 \pmod{8}) \\ \mathbf{Z}/2^cM \oplus \mathbf{Z}/4 \oplus G(m+j) & (j \equiv n+5 \equiv 6 \pmod{8}) \\ \mathbf{Z}/2M \oplus \mathbf{Z}/2^d \oplus G_1(m+j) & (j \equiv n+1 \equiv 6 \pmod{8}), \end{cases}$$

where  $c = [(m-n-1)/8]$ ,  $d = h_4(m-4, n)$  and  $G_1(m)$  is the group defined by (2.4).

iii) If  $4h_4(j+4, j)h_4(n-4, n)+8[n/8]+10 > m > n$ , then we have

$$\tilde{J}(S^j(L_8^n/L_8^s)) \cong \begin{cases} \mathbf{Z}/M \oplus \widetilde{KO}(S^j(L_8^n/L_8^{n+1})) & (m \geq 4[n/4]+6) \\ \mathbf{Z}/M \oplus \mathbf{Z}/2 & (n+j \equiv 7 \pmod{8} \text{ and } n+4 > m \geq n+2) \\ \mathbf{Z}/M & (\text{otherwise}). \end{cases}$$

**REMARK.** (1) Combining this theorem with [10, Theorem 1], we obtain the complete results for the groups  $\tilde{J}(S^j(L_8^n/L_8^s))$ .  
(2) The partial results for the case  $j=n=0$  of this theorem have been obtained in [9].

### 3. Preliminaries

In this section we prepare some lemmas and recall known results which are needed to prove Theorems 1 and 2.

**Lemma 3.1.** *Let  $j, k, l$  and  $s$  be integers with  $j > 0$ ,  $v=v_2(j) \geq s \geq 1$ ,  $l \geq 2$  and  $k \equiv \pm 1 \pmod{2^l}$ . Then we have*

- (1)  $k^j - 1 \equiv (k^{2^v} - 1)(j/2^v) \pmod{2^{v+2l}}$ .
- (2)  $k^j \equiv 1 \pmod{2^{v+l}}$ .
- (3)  $k^j - 1 \equiv (k^{2^s} - 1)(j/2^s) \pmod{2^{v+s+2l-1}}$ .

**Proof.** Since  $k^2 \equiv 1 \pmod{2^{l+1}}$ , by making use of the method used in the proof of [10, Lemma 3.1] we can show that

$$k^j - 1 \equiv (k^2 - 1)(j/2) \pmod{2^{v+2l}}.$$

This implies (2). In particular, we have

$$k^{2^v} \equiv 1 \pmod{2^{v+l}}.$$

Then, the rest of the proof is similar to that of [10, Lemma 3.1]. q.e.d.

Considering the  $\mathbf{Z}/8$ -action on  $S^{2n+1} \times \mathbf{C}$  given by

$$\exp(2\pi\sqrt{-1}/8)(z, u) = (z \cdot \exp(2\pi\sqrt{-1}/8), u \cdot \exp(2\pi\sqrt{-1}/8))$$

for  $(z, u) \in S^{2n+1} \times \mathbf{C}$ , we have a complex line bundle

$$\eta: (S^{2n+1} \times \mathbf{C}) / (\mathbf{Z}/8) \rightarrow L_8^{2n+1}.$$

Then we have the following elements

$$(3.2) \quad \sigma(i) = \eta^{2^i} - 1 \in \tilde{K}(L_8^{2n+1}) \subset K(L_8^{2n+1}) \quad (0 \leq i \leq 2).$$

We denote the restriction of  $\sigma(i)$  in  $\tilde{K}(L_8^{2n})$  by the same symbol, and  $\sigma(0)$  by  $\sigma$ . The following proposition is well known.

**Proposition 3.3** (Mahammed [11]). *The ring  $K(L_8^n)$  is isomorphic to the truncated polynomial ring  $\mathbf{Z}[\sigma]/(\sigma^{\lceil m/2 \rceil + 1}, (\sigma + 1)^8 - 1)$ , where  $(\sigma^{\lceil m/2 \rceil + 1}, (\sigma + 1)^8 - 1)$  means the ideal of  $\mathbf{Z}[\sigma]$  generated by  $\sigma^{\lceil m/2 \rceil + 1}$  and  $(\sigma + 1)^8 - 1$ .*

In order to state the next lemma, we set

$$(3.4) \quad (1) \quad \begin{cases} \sigma_{2^i} = \sigma(i) & (0 \leq i \leq 2) \\ \sigma_6 = \sigma_4 \sigma_2 \\ \sigma_{2i+1} = \sigma_{2i} \sigma_1 & (1 \leq i \leq 3). \end{cases}$$

$$(2) \quad \begin{cases} b(n) = ((-68 - 48\sqrt{2})^n + (-68 + 48\sqrt{2})^n)/2 \\ c(n) = ((-68 - 48\sqrt{2})^n - (-68 + 48\sqrt{2})^n)/2\sqrt{2}. \end{cases}$$

Then we have

$$(3.5) \quad (1) \quad \begin{cases} b(0) = 1, b(n+1) = -68b(n) - 96c(n), \\ c(0) = 0, c(n+1) = -48b(n) - 68c(n). \end{cases}$$

$$(2) \quad \begin{cases} b(n) \equiv (-4)^n \pmod{2^{2n+4}} \\ c(n) = 0 \pmod{2^{2n+2}}. \end{cases}$$

The following lemma is obtained by Proposition 3.3 and (3.5) (1).

**Lemma 3.6.** *Let  $u$  be a positive integer. Then, in  $K(L_8^n)$ ,*

$$\sigma^u = \sum_{i=1}^7 a_{u,i} \sigma_i,$$

$$\begin{aligned} \text{where } a_{u,i} \quad (1 \leq i \leq 7) \text{ are integers defined by } a_{u,1} &= (-2)^{u-1}, \\ a_{u,2} &= (1/5)(-4)^{\lfloor u/4 \rfloor + 1} + (2/5)(-4)^{\lfloor (u-1)/4 \rfloor} - (1/5)(-4)^{\lfloor (u+2)/4 \rfloor} \\ &\quad + (3/5)(-4)^{\lfloor (u+1)/4 \rfloor}, \\ a_{u,3} &= -(-2)^{u-2} - (1/2)a_{u+1,2}, \\ a_{u,4} &= -(1/2)h_4(u, u-1)b([u/8]) + h_4(u-1, u-2)c([u/8]) \\ &\quad - h_4(u-2, u-3)c([u/8]) + h_4(u+4, u+3)(b([u/8]) + 2c([u/8])) \\ &\quad - h_4(u+3, u+2)(4b([u/8]) + 6c([u/8])) \\ &\quad + h_4(u+2, u+1)(10b([u/8]) + 14c([u/8])) \\ &\quad - h_4(u+1, u)(20b([u/8]) + 28c([u/8])), \\ a_{u,5} &= -(-2)^{u-2} - a_{u+1,4} - a_{u+2,4} - (1/2)a_{u+3,4}, \\ a_{u,6} &= (1/2)a_{u,2} - a_{u+1,4} - (1/2)a_{u+2,4} \end{aligned}$$

and  $a_{u,7} = (-2)^{u-3} - (1/4) a_{u+1,2} - (1/2) a_{u+1,4}$ .

Proof. By making use of the relation  $(\sigma+1)^8=1$ , we obtain equalities  $a_{u+1,1}=-2a_{u,1}$ ,  $a_{u+1,2}=a_{u,1}-2a_{u,3}$ ,  $a_{u+1,3}=a_{u,2}-2a_{u,3}$ ,  $a_{u+1,4}=a_{u,3}-2a_{u,7}$ ,  $a_{u+1,5}=a_{u,4}-2a_{u,5}$ ,  $a_{u+1,6}=a_{u,5}-2a_{u,7}$  and  $a_{u+1,7}=a_{u,6}-2a_{u,7}$ , where  $a_{1,1}=1$  and  $a_{1,i}=0$  ( $2 \leq i \leq 7$ ). Thus the lemma is proved by the induction with respect to  $u$ . q.e.d.

In order to state the next proposition, we set

(3.7) (1) Let  $F(x)$  denote the free abelian group generated by  $\{x_i \mid 1 \leq i \leq 7\}$ . Then  $X_i$  and  $X_i(n)$  ( $7 \geq i \geq 1$ ,  $n \geq 0$ ) denote the elements of  $F(x)$  defined by  $X_1=4x_1+2x_3+2x_5+x_7$ ,  $X_2=2x_2+x_6$ ,  $X_3=2x_3+x_7$ ,  $X_6=x_6+x_7$ ,  $X_i=x_i$  ( $i=4, 5$  or  $7$ ),  $X_1(n)=2^{\lfloor n/2 \rfloor} X_1$ ,

$$\begin{aligned} X_2(n) &= 2^{\lfloor n/4 \rfloor} X_2 - 2^{2\lfloor n/4 \rfloor} X_1, \\ X_3(n) &= 2^{\lfloor (n-2)/4 \rfloor} X_3 + 2^{2\lfloor n/4 \rfloor-1} h_3(n, n-2) X_1, \\ X_4(n) &= 2^{\lfloor n/8 \rfloor} X_4 + 2^{2\lfloor n/8 \rfloor} h_4(n+4, n) X_2 + 2^{2\lfloor n/8 \rfloor+\lfloor n/4 \rfloor} X_1, \\ X_5(n) &= 2^{\lfloor (n-2)/8 \rfloor} X_5 + 2^{2\lfloor (n-2)/8 \rfloor} h_4(n+2, n-2) X_3 \\ &\quad - 2^{\lfloor (n+2)/4 \rfloor+2\lfloor (n-2)/8 \rfloor} X_1, \\ X_6(n) &= 2^{\lfloor (n-4)/8 \rfloor} X_6 + 2^{\lfloor n/4 \rfloor-1} h_4(n, n-4) X_2 \\ &\quad - 2^{\lfloor n/4 \rfloor+2\lfloor (n-4)/8 \rfloor+1} X_1 \end{aligned}$$

and

$$\begin{aligned} X_7(n) &= 2^{\lfloor (n-6)/8 \rfloor} X_7 - 2^{\lfloor (n-6)/4 \rfloor} h_4(n+6, n+2) (X_3-2X_2) \\ &\quad + 2^{\lfloor (n-2)/4 \rfloor+2\lfloor (n+2)/8 \rfloor} X_1. \end{aligned}$$

(2) Let  $\varphi: F(x) \rightarrow \tilde{K}(L_8^m)$  be the homomorphism defined by setting  $\varphi(x_i)=\sigma_i$  ( $1 \leq i \leq 7$ ).

**Proposition 3.8** (Kobayashi and Sugawara [9]). *The homomorphism  $\varphi$  is an epimorphism, and the kernel of  $\varphi$  coincides with the subgroup of  $F(x)$  generated by  $\{X_i(m) \mid 1 \leq i \leq 7\}$ .*

According to [1], we have the following lemma.

**Lemma 3.9.** *The Adams operations are given by the following formulae, where  $s_i=\varphi(X_i)$  ( $1 \leq i \leq 7$ ).*

$$(1) \quad \psi^k(s_1) = \begin{cases} s_1 & (k \equiv 1 \pmod{2}) \\ 2s_2 & (k \equiv 2 \pmod{4}) \\ 4s_4 & (k \equiv 4 \pmod{8}) \\ 0 & (k \equiv 0 \pmod{8}) \end{cases}$$

$$(2) \quad \psi^k(s_2) = \begin{cases} s_2 & (k \equiv 1 \pmod{2}) \\ 2s_4 & (k \equiv 2 \pmod{4}) \\ 0 & (k \equiv 0 \pmod{4}) \end{cases}$$

$$(3) \quad \psi^k(s_3) = \begin{cases} s_3 - 2h_3(k+1, k)(s_2 + s_3) & (k \equiv 1 \pmod{2}) \\ 2s_6 - 2s_7 & (k \equiv 2 \pmod{8}) \\ -4s_4 - 2s_6 + 2s_7 & (k \equiv 6 \pmod{8}) \\ 0 & (k \equiv 0 \pmod{4}) \end{cases}$$

$$(4) \quad \psi^k(s_4) = \begin{cases} s_4 & (k \equiv 1 \pmod{2}) \\ 0 & (k \equiv 0 \pmod{2}) \end{cases}$$

$$(5) \quad \psi^k(s_5) = \begin{cases} s_5 - 2h_4(k+4, k)(s_4 + s_5) & (k \equiv 1 \pmod{4}) \\ s_5 + s_6 - 2h_4(k+4, k)(s_4 + s_6 + s_5) & (k \equiv 3 \pmod{4}) \\ 0 & (k \equiv 0 \pmod{2}) \end{cases}$$

$$(6) \quad \psi^k(s_6) = \begin{cases} (1 - 2h_4(k+5, k+1))s_6 & (k \equiv 1 \pmod{2}) \\ 0 & (k \equiv 0 \pmod{2}) \end{cases}$$

$$(7) \quad \psi^k(s_7) = \begin{cases} s_7 - 2h_4(k+4, k)s_6 & (k \equiv 1 \pmod{4}) \\ 2s_4 - s_7 + 2h_4(k+4, k)s_6 & (k \equiv 3 \pmod{4}) \\ 0 & (k \equiv 0 \pmod{2}) \end{cases}$$

For each integer  $n$  with  $0 \leq n < m$ , we denote the inclusion map of  $L_8^n$  into  $L_8^m$  by  $i_n^m$ , and denote the kernel of the homomorphism

$$(i_n^m)! : \tilde{K}(L_8^n) \rightarrow \tilde{K}(L_8^m)$$

by  $V_n$ . Then by Proposition 3.8, Lemma 3.6 and (3.5) (2), we obtain the following lemma.

**Lemma 3.10.** (1) *The group  $V_n$  is the subgroup of  $\tilde{K}(L_8^m)$  generated by  $\{\varphi(X_i(n)) \mid 1 \leq i \leq 7\}$ .*  
(2) *Let  $u$  be a positive integer with  $2u < m$ . Then we have*

$$\sigma^u \equiv \begin{cases} -\varphi(X_4(2u-2)) & (u \equiv 0 \pmod{4}) \\ -\varphi(X_5(2u-2)) & (u \equiv 1 \pmod{4}) \\ -\varphi(X_6(2u-2) + X_1(2u-2)) & (u \equiv 2 \pmod{4}) \\ \varphi(X_7(2u-2)) & (u \equiv 3 \pmod{4}) \end{cases}$$

modulo the subgroup  $V_{2u}$ .

Considering the  $\mathbb{Z}/8$ -action on  $S^{2n+1} \times \mathbf{R}$  given by

$$\exp(2\pi\sqrt{-1}/8)(z, v) = (z \cdot \exp(2\pi\sqrt{-1}/8), -v)$$

for  $(z, v) \in S^{2n+1} \times \mathbf{R}$ , we have a real line bundle

$$\nu : (S^{2n+1} \times \mathbf{R})/(\mathbb{Z}/8) \rightarrow L_8^{2n+1}.$$

We set  $\kappa = \nu - 1 \in \widetilde{KO}(L_8^{2n+1})$ . It is easy to see that

$$(3.11) \quad \begin{cases} c(\kappa) = \sigma(2) \\ r(\sigma(2)) = 2\kappa, \end{cases}$$

where  $c: KO \rightarrow K$  is the complexification and  $r: K \rightarrow KO$  is the real restriction.

Let  $I: \widetilde{K}(X) \rightarrow \widetilde{K}(S^2X)$  (resp.  $I_R: \widetilde{KO}(X) \rightarrow \widetilde{KO}(S^8X)$ ) be the Bott periodicity isomorphisms for  $K$ (resp.  $KO$ )-theory. In order to state the next proposition, we set

(3.12) Let  $j$  be a non-negative integer with  $j \equiv 0 \pmod{4}$ .

$$(1) \quad \begin{cases} \tau_i = r(I^{j/2}(\sigma_i)) & (1 \leq i \leq 2) \\ \tau_3 = r(I^{j/2}(\sigma_5)) \\ \tau_4 = \begin{cases} I_R^{j/8}(\kappa) & (j \equiv 0 \pmod{8}) \\ r(I^{j/2}(\sigma_4)) & (j \equiv 4 \pmod{8}) \end{cases} \end{cases}$$

(2) Let  $F(y)$  denote the free abelian group generated by  $y_1, y_2, y_3$  and  $y_4$ . Then  $X_i^j, Y_i^j, X_i^j(n)$  and  $Y_i^j(n)$  ( $1 \leq i \leq 4, n \geq 0$ ) denote the elements of  $F(y)$  defined by  $Y_4^j = y_4$ ,

$$\begin{aligned} Y_1^j &= (-1)^{(j/4)} X_1^j = h_4(j+12, j)(2y_1 - y_2 + y_3) + y_4, \\ Y_2^j &= X_2^j = h_4(j+12, j)y_2 - y_4, \\ Y_3^j &= -y_3 - h_4(j, j-12)y_4, \\ X_1^j &= Y_{i+h_4(j+4, j)(7-2i)}^j \quad (3 \leq i \leq 4), \\ X_1^j(n) &= (-1)^{(j/4)} Y_1^j(n) = 2^{h_1(n+j) - h_1(j)} X_1^j, \\ X_2^j(n) &= Y_2^j(n) = 2^{[n/4] - h_4(j+4, j)} (X_2^j - (-2)^{[n/4]} h_4(n+j+7, n+j) Y_1^j), \\ X_3^j(n) &= 2^{h_4(n+j-2, j+4)} X_3^j + 2^{[n/4] - h_4(j+12, j)} h_4(n+j, n+j-2) X_2^j \\ &\quad - 2^{2h_4(n+j+4, j+4) + [n/4] - 1} h_4(n+j+4, n+j-2) X_1^j, \\ X_4^j(n) &= 2^{h_4(n+j, j+4)} X_4^j + 2^{2[n-(n-4)/8]} h_4(n+j+4, n+j) X_2^j \\ &\quad + 2^{2h_4(n+j, j+4) + [n/4]} h_4(n+j+7, n+j) X_1^j \end{aligned}$$

and  $Y_i^j(n) = X_{i+h_4(j+4, j)(7-2i)}^j(n) \quad (3 \leq i \leq 4)$ .

(3) Let  $\mu_j: F(y) \rightarrow \widetilde{KO}(S^j L_8^m)$  be the homomorphism defined by setting  $\mu_j(y_i) = \tau_i$  ( $1 \leq i \leq 4$ ).

**Proposition 3.13** (Kobayashi [8]). *Let  $j$  be a non-negative integer with  $j \equiv 0 \pmod{4}$ . Then the homomorphism  $\mu_j$  is an epimorphism, and the kernel of  $\mu_j$  coincides with the subgroup of  $F(y)$  generated by  $\{Y_i^j(m) \mid 1 \leq i \leq 4\}$ .*

According to [1] and [4], we have the following lemma.

**Lemma 3.14.** *Let  $j$  be a non-negative integer with  $j \equiv 0 \pmod{4}$ . Then the Adams operations are given by the following formulae, where  $t_i = \mu_j(Y_i^j)$*

( $1 \leq i \leq 4$ ) and  $k \equiv 1 \pmod{2}$ .

- (1)  $\psi^k(t_i) = k^{j/2} t_i$  ( $i = 1, 2$  or  $4$ ).
- (2)  $\psi^k(t_3) = k^{j/2} (1 - 2h_4(k+5, k+1)) t_3$ .

By Lemma 3.9 and (3.11), we have the following Lemma.

**Lemma 3.15.** *Let  $j$  be a non-negative integer with  $j \equiv 0 \pmod{4}$ . Then homomorphisms  $c: KO \rightarrow K$  and  $r: K \rightarrow KO$  are given by the following formulae, where  $s_i = I^{j/2}(\varphi(X_i))$  ( $1 \leq i \leq 7$ ) and  $t_i = \mu_j(Y_i)$  ( $1 \leq i \leq 4$ ).*

- (1)  $r(s_i) = h_4(j, j-12) t_i$  ( $i = 1, 2$  or  $4$ ).
- (2)  $r(s_3) = -r(s_2)$ .
- (3)  $r(s_5) = -t_3 - h_4(j, j-12) t_4$ .
- (4)  $r(s_6) = 2t_3$ .
- (5)  $r(s_7) = 2t_3 + h_4(j, j-12) t_4$ .
- (6)  $c(t_i) = h_4(j+12, j) s_i$  ( $i = 1, 2$  or  $4$ ).
- (7)  $c(t_3) = s_6$ .

#### 4. Proof for the case $j \equiv 0 \pmod{4}$

In this section we prove the parts of the case  $j \equiv 0 \pmod{4}$  of Theorems 1 and 2. Throughout this section,  $j$  denotes a non-negative integer with  $j \equiv 0 \pmod{4}$ , and  $\nu$  the integer defined by

$$(4.1) \quad \nu = \begin{cases} \nu_2(j) & (j > 0) \\ m & (j = 0). \end{cases}$$

For each integer  $n$  with  $0 \leq n < m$ , we denote the kernel of the homomorphism

$$(i_n^m)^!: \widetilde{KO}(S^j L_8^n) \rightarrow \widetilde{KO}(S^j L_8^m)$$

by  $VO_{m,n}^j$ . It follows from Proposition 3.13 that we have

(4.2) *The group  $VO_{m,n}^j$  is the subgroup of  $\widetilde{KO}(S^j L_8^m)$  generated by*

$$\{\mu_j(Y_i^j(n)) \mid 1 \leq i \leq 4\},$$

where  $\mu_j: F(y) \rightarrow \widetilde{KO}(S^j L_8^m)$  is the homomorphism defined in (3.12).

By Lemma 3.14, we have the following lemma.

**Lemma 4.3.** *The Adams operations are given by the following formulae, where  $T_i = \mu_j(Y_i^j(n))$  ( $1 \leq i \leq 4$ ) and  $k \equiv 1 \pmod{2}$ .*

- (1)  $\psi^k(T_i) = k^{j/2} T_i$  ( $i = 1, 2$  or  $4$ ).

$$(2) \quad \psi^k(T_3) = k^{j/2} (T_3 + h_4(k+5, k+1) (-2T_3 + \alpha(0, j, n) T_2 - \alpha(1, j, n) T_1)),$$

where  $\alpha(l, j, n) = h_4(n+h_4(j, j-28l), n+(-2)^{1-l} h_4(j+12, j+16l))$  ( $0 \leq l \leq 1$ ).

We set

$$(4.4) \quad UO_{m,n}^j = \sum_k (\bigcap_e k^e (\psi^k - 1) VO_{m,n}^j),$$

where the intersection runs over all non-negative integers  $e$ . Since the order of  $VO_{m,n}^j$  is equal to a power of 2, we have

$$UO_{m,n}^j = \sum_{k: \text{odd}} (\psi^k - 1) VO_{m,n}^j.$$

It follows from Lemmas 3.1 and 4.3 that we have

(4.5) *The group  $UO_{m,n}^j$  is the subgroup of  $VO_{m,n}^j$  generated by*

$$\{2^{v+1} T_i \mid i = 1, 2 \text{ or } 4\} \cup \{R\},$$

where  $T_i = \mu_j(Y_i^j(n))$  ( $1 \leq i \leq 4$ ),

$$R = 2(2^v - 1) T_3 + \alpha(0, j, n) T_2 - \alpha(1, j, n) T_1$$

and  $\alpha(l, j, n)$  is the integer defined in Lemma 4.3 ( $0 \leq l \leq 1$ ).

**4.1. Proof for the case  $n \not\equiv 3 \pmod{4}$ .** Suppose that  $n \not\equiv 3 \pmod{4}$ . According to [13], we have the exact sequence

$$0 \rightarrow \widetilde{KO}(S^j(L_8^n/L_8^n)) \rightarrow \widetilde{KO}(S^j L_8^n) \xrightarrow{(i_n^m)^!} \widetilde{KO}(S^j L_8^n) \rightarrow 0.$$

Hence we have

$$\widetilde{KO}(S^j(L_8^n/L_8^n)) \cong VO_{m,n}^j.$$

Since the order of  $VO_{m,n}^j$  is finite, we have

$$\widetilde{J}(S^j(L_8^n/L_8^n)) \cong VO_{m,n}^j / UO_{m,n}^j.$$

It follows from Proposition 3.13 that we have

$$VO_{m,n}^j \cong \langle \{w_i \mid 1 \leq i \leq 4\} \rangle / \langle \{R_i \mid 1 \leq i \leq 4\} \rangle,$$

where  $w_i = X_i^j(n)$  ( $1 \leq i \leq 4$ ),  $R_1 = 2^{h_1(m+j)-h_1(n+j)} w_1$ ,

$$R_2 = 2^{h_3(m,n)+a_4(n+j,n+j)} (\tilde{w}_2 + 2^{h_3(m,n)} w_1),$$

$$R_3 = 2^{a_3(m+j,n+j)} (\tilde{w}_3 + 2^{a_4(m+j+4,n+j)} (\tilde{w}_2 - 2^{a_7(m+j,n+j)+1} w_1)),$$

$$R_4 = 2^{a_4(m+j,n+j)} (\tilde{w}_4 + 2^{h_4(m+j,n+j+4)} (\tilde{w}_2 + 2^{a_7(m+j-4,n+j)+2} w_1)),$$

$$\tilde{w}_2 = h_4(n+j+15, n+j) w_2 - (-1)^{\lceil (n+j)/4 \rceil} a_4(n+j, n+j) w_1,$$

$$\tilde{w}_3 = h_4(n+j+12, n+j-2) w_3 - h_4(n+j, n+j-2) w_2 + a_4(n+j+12, n+j) w_1,$$

and  $\tilde{w}_4 = h_4(n+j+15, n+j) w_4 - h_4(n+j+4, n+j) w_2 + a_4(n+j+4, n+j) w_1$ .

Suppose that  $m \geq 4 [(n+j+7)/8] + 4 [(n-j+12)/8]$ , and set  $A_i = R_i$  ( $3 \leq i \leq 4$ ),

$$\begin{aligned} A_1 &= R_1 - (-1)^{[(n+j)/4]} 2^{h_1(m+j)-2[(n+j)/4]-h_3(m,n)} R_2 \\ &\quad - h_4(n+j, n+j-2) 2^{a_1(m+j, n+j)} (2^{h_4(n+j, m+j+6)} R_3 - 2^{h_4(n+j, m+j)-1} R_2) \\ &\quad + h_4(n+j+4, n+j) 2^{a_1(m+j, n+j)} (2^{h_4(n+j, m+j)} R_4 - 2^{h_4(n+j, m+j+4)} R_2), \end{aligned}$$

$$\begin{aligned} A_2 &= R_2 + h_4(n+j-1, n+j-4) 2^{h_4(m+j+4, n+j)} R_4 \\ &\quad + h_4(n+j+4, n+j) 2^{h_3(m+j, n+j)-h_4(m+j-2, n+j)} R_3, \end{aligned}$$

$$\begin{aligned} u_1 &= -(-1)^{[(n+j)/4]} 2^{h_3(m,n)+a_4(n+j, n+j)-1} w_1 \\ &\quad - h_4(n+j-2, n+j-4) (w_2 + 2^{h_3(m,n)-2} w_1) \\ &\quad - h_4(n+j, n+j-2) (w_3 - 2^{a_4(m+j-4, n+j)} (2^{h_3(m,n)} + 2^{a_7(m+j, n+j)+1}) w_1) \\ &\quad + h_4(n+j+4, n+j) (w_4 - 2^{h_4(m+j, n+j)-2} (2^{h_3(m,n)} - 2^{2h_4(m+j, n+j)}) w_1), \end{aligned}$$

$$\begin{aligned} u_2 &= w_2 + 2^{h_3(m,n)+a_4(n+j, n+j)} w_1 \\ &\quad + h_4(n+j-1, n+j-4) (w_4 + 2^{a_4(m+j, n+j)} (\tilde{w}_2 + 2^{2h_4(m+j, n+j)+1} w_1)) \\ &\quad + h_4(n+j+4, n+j) (w_3 + 2^{a_4(m+j-4, n+j)} (\tilde{w}_2 - 2^{a_7(m+j, n+j)+1} w_1)), \end{aligned}$$

$$u_3 = \tilde{w}_3 + 2^{a_4(m+j+4, n+j)} (\tilde{w}_2 - 2^{a_7(m+j, n+j)+1} w_1)$$

and  $u_4 = \tilde{w}_4 + 2^{h_4(m+j, n+j+4)} (\tilde{w}_2 + 2^{a_7(m+j-4, n+j)+2} w_1)$ . Then we have

$$A_i = 2^{a_i(m+j, n+j)} u_i \quad (1 \leq i \leq 4),$$

$$\begin{aligned} w_1 &= h_4(n+j, n+j-2) h_4(n+j+15, n+j) (2u_1 + u_2 + u_3 - 2^{h_4(m+j+4, n+j)} u_2) \\ &\quad + h_4(n+j-1, n+j-2) (2^{h_4(m+j+4, n+j)} - 1) u_4 \\ &\quad + h_4(n+j-2, n+j-4) (2u_1 + 2u_2 - u_4) \\ &\quad + h_4(n+j+4, n+j) (4u_1 - 2u_2 + u_3 - 2u_4 + 2^{h_4(m+j, n+j)} (2u_2 - u_3)) \end{aligned}$$

and

$$\begin{aligned} w_2 &= -2^{h_3(m,n)+a_4(n+j, n+j)} w_1 - h_4(n+j-2, n+j-4) u_1 + h_4(n+j, n+j-1) u_2 \\ &\quad - h_4(n+j-1, n+j-2) (2u_1 + u_3 - 2^{h_4(m+j-4, n+j)} (2u_2 - u_4)) \\ &\quad + h_4(n+j+4, n+j) (2u_1 - u_4 + 2^{a_4(m+j, n+j)} (2u_2 - u_3)). \end{aligned}$$

This implies that  $\langle \{w_i \mid 1 \leq i \leq 4\} \rangle = \langle \{u_i \mid 1 \leq i \leq 4\} \rangle$  and

$$VO_{m,n}^j \cong \langle \{u_i \mid 1 \leq i \leq 4\} \rangle / \langle \{A_i \mid 1 \leq i \leq 4\} \rangle \cong \bigoplus_{i=1}^4 \mathbb{Z}/2^{a_i(m+j, n+j)}.$$

Suppose that  $4[(n+j+7)/8] + 4[(n-j+12)/8] > m \geq h_1(n+j) + 2[(n-j+4)/8] + 2[(n-j+6)/8] + 1$ , and set  $A_i = R_i$  ( $2 \leq i \leq 4$ ),  $u_{2i} = R_{2i}$  ( $1 \leq i \leq 2$ ),

$$\begin{aligned} A_1 &= \begin{cases} 2^{h_3(m+3,n)}(2R_4-R_2)-R_1 & (h_4(n+j, n+j-4) = 0) \\ R_1-4R_2+8R_3 & (n+j \equiv 1 \pmod{8} \text{ and } m \geq n+3) \\ R_1+4R_2 & (n+j \equiv 2 \pmod{8}) \\ R_1 & (h_3(m, n) = 0), \end{cases} \\ u_1 &= \begin{cases} w_4+w_1 & (h_4(n+j, n+j-4) = 0) \\ w_3-2w_1 & (n+j \equiv 1 \pmod{8} \text{ and } m \geq n+3) \\ w_2+2w_1 & (n+j \equiv 2 \pmod{8}) \\ w_1 & (h_3(m, n) = 0) \end{cases} \end{aligned}$$

and

$$u_3 = \begin{cases} w_3 + w_2 - w_1 & (h_4(n+j, n+j-4) = 0) \\ 2w_3 + w_2 - 3w_1 & (n+j \equiv 1 \pmod{8} \text{ and } m \geq n+3) \\ w_3 + 2w_2 - 3w_1 & (n+j \equiv 2 \pmod{8}) \\ w_3 & (h_3(m, n) = 0) \end{cases}$$

Then we have  $A_i = u_i$  ( $i=2$  or  $4$ ),

$$\begin{aligned} A_1 &= \begin{cases} 2^{a_1(m+j, n+j)} u_1 & (m \geq 4[n/4]+4) \\ 2u_1 & (n+j \equiv 0 \pmod{8}) \\ u_1 & (n+j \equiv 1 \pmod{8} \text{ and } n+3 > m > n) \end{cases}, \\ A_3 &= \begin{cases} 2u_3 & (h_4(n+j-2, n+j-4) = [(m+j)/2] - 4[(n+j+4)/8] - 1 = 0) \\ u_3 & (\text{otherwise}), \end{cases} \\ w_1 &= \begin{cases} 2u_4 - u_2 - 4u_1 & (h_4(n+j, n+j-4) = 0) \\ 4u_1 + u_2 - 2u_3 & (n+j \equiv 1 \pmod{8} \text{ and } m \geq n+3) \\ 2u_1 - u_2 & (n+j \equiv 2 \pmod{8}) \\ u_1 & (h_3(m, n) = 0) \end{cases} \end{aligned}$$

and

$$w_2 = \begin{cases} 2u_1 + u_2 - u_4 & (h_4(n+j, n+j-4) = 0) \\ -6u_1 - u_2 + 3u_3 & (n+j \equiv 1 \pmod{8} \text{ and } m \geq n+3) \\ 2u_2 - 3u_1 & (n+j \equiv 2 \pmod{8}) \\ u_2 - u_1 & (h_3(m, n) = 0) \end{cases}$$

This implies that  $\langle \{w_i \mid 1 \leq i \leq 4\} \rangle = \langle \{u_i \mid 1 \leq i \leq 4\} \rangle$  and

$$VO_{m,n}^j \cong \begin{cases} \mathbf{Z}/16 \oplus \mathbf{Z}/2 & (h_4(n+j, n+j-4) = [m/2] - 2[n/4] - 3 = 0) \\ \mathbf{Z}/2^{a_1(m+j, n+j)} & (4[(n+j+15)/8] + 2[(n-j)/4] > m \geq 4[n/4]+4) \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & (n+j \equiv 0 \pmod{8} \text{ and } [m/2] = (n/2)+1) \\ \mathbf{Z}/2 & (h_4(n+j+6, n+j) = [m/2] - [(n+1)/2] = 0). \end{cases}$$

If  $h_4(n+j) + 2[(n-j+4)/8] + 2[(n-j+6)/8] \geq m > n$ , then we have  $VO_{m,n}^j \cong 0$ . Thus the proof for the case  $j \equiv 0 \pmod{4}$  and  $n \not\equiv 3 \pmod{4}$  of Theorem 1 is completed.

Consider the case  $j \equiv 0 \pmod{8}$ . It follows from (4.5) that we have

$$\mu_j^{-1}(VO_{m,n}^j) = \langle \{2^{\nu+h_3(i+5,i)} w_i \mid 1 \leq i \leq 4\} \cup \{R_i \mid 0 \leq i \leq 4\} \rangle,$$

where  $\nu$  is the integer defined by (4.1) and

$$R_0 = 2(2^\nu - 1) w_3 + h_4(n, n-2) w_2 - h_4(n+4, n-1) w_1.$$

Suppose that  $m \geq 4[(n+15)/8] + 2[(n+6)/8] + 2[n/8]$ , and set

$$\begin{aligned}
v_1 &= h_4(n-2, n-4) w_2 + h_4(n, n-2) w_3 + h_4(n+4, n) w_4, \\
v_2 &= h_4(n-1, n-2) (w_4 + 2w_3 + 2^{h_3(m, n)+1} (1 - 2^{h_4(m+6, n)}) w_3 + 2^v w_3) \\
&\quad + h_4(n-2, n-4) (w_4 + w_2 + 2^{h_3(m, n)} w_2) + h_4(n+4, n) (w_3 + 2w_4 - 2^{h_3(m, n)+1} w_4) \\
&\quad + h_4(n, n-1) (w_2 + 2^{h_3(m, n)+1} w_3 + 2^v w_3), \\
v_3 &= h_4(n, n-1) (w_4 + 2^{h_4(m, n)} w_2 - 2^{3h_4(m, n)+2} w_3) \\
&\quad + h_4(n-1, n-2) (2w_4 - w_1 + 2^{h_4(m, n)+1} (w_4 + 2w_3) - 2^{3h_4(m, n)+3} w_3) \\
&\quad + h_4(n-2, n-4) (2w_4 - w_1 + 2^{h_4(m, n)+1} (w_4 + w_2) - 2^{3h_4(m, n)+2} w_2) \\
&\quad + h_4(n+4, n) (2w_4 - w_2 + 2^{h_4(m, n)} (w_3 + 2w_4) + 2^{3h_4(m, n)+1} w_4)
\end{aligned}$$

and  $v_4 = R_0 + h_4(n-2, n-3) (w_3 - R_0)$ . Then we have  $\langle \{w_i \mid 1 \leq i \leq 4\} \rangle = \langle \{v_i \mid 1 \leq i \leq 4\} \rangle$  and

$$VO_{m,n}^j / UO_{m,n}^j \cong \begin{cases} (\bigoplus_{i=1}^3 \mathbf{Z}/2^{b_i(j, m, n)}) \oplus \mathbf{Z}/2 & (n \equiv 2 \pmod{8}) \\ \bigoplus_{i=1}^3 \mathbf{Z}/2^{b_i(j, m, n)} & (\text{otherwise}). \end{cases}$$

Suppose that  $4[(n+15)/8] + 2[(n+6)/8] + 2[n/8] > m \geq 4[(n+14)/8] + 4[n/8]$ , and set

$$\begin{aligned}
v_1 &= h_4(n-2, n-3) w_2 + h_4(n-1, n-2) w_3 + h_4(n+4, n) w_4, \\
v_2 &= h_4(n-2, n-3) (w_4 + 5w_2) + h_4(n-1, n-2) (w_4 - 2w_3) \\
&\quad + h_4(n+4, n) (w_3 - 2w_4), \\
v_3 &= h_4(n-2, n-3) (2w_4 - w_1 - 16w_2) + h_4(n-1, n-2) (w_1 - 4w_3) \\
&\quad + h_4(n+4, n) (w_2 + 6w_4)
\end{aligned}$$

and  $v_4 = R_0 + h_4(n-2, n-3) (w_3 + 2w_2 - 3w_1 - R_0)$ . Then we have  $\langle \{w_i \mid 1 \leq i \leq 4\} \rangle = \langle \{v_i \mid 1 \leq i \leq 4\} \rangle$  and

$$VO_{m,n}^j / UO_{m,n}^j \cong \begin{cases} \mathbf{Z}/2^{b_1(j, m, n)} \oplus \mathbf{Z}/4 & (n \equiv 2 \pmod{8}) \\ \mathbf{Z}/8 & (n \equiv 1 \pmod{8}) \\ VO_{m,n}^j & (h_4(n, n-4) = 0). \end{cases}$$

If  $4[(n+14)/8] + 4[n/8] > m > n$ , then we have  $UO_{m,n}^j \cong 0$ . Thus the proof for the case  $j \equiv 0 \pmod{8}$  and  $n \not\equiv 3 \pmod{4}$  of Theorem 2 is completed.

Consider the case  $j \equiv 4 \pmod{8}$ . It follows from (4.5) that we have

$$\mu_j^{-1}(UO_{m,n}^j) = \langle \{2^{3+h_3(i, i-1)} w_i \mid 1 \leq i \leq 4\} \cup \{R_i \mid 0 \leq i \leq 4\} \rangle,$$

where  $R_0 = 2w_4 - h_4(n+3, n) (8w_4 + w_1) - h_4(n, n-4) (8w_4 + w_2)$ . Suppose that  $m \geq 2[n/4] + 4[(n+2)/8] + 6$ , and set

$$\begin{aligned}
v_1 &= h_4(n, n-4) w_4 + h_4(n+4, n+2) w_3 + h_4(n+2, n) w_2, \\
v_2 &= h_4(n, n-4) (w_3 - 2w_4) + h_4(n+4, n+3) (w_2 + w_4 - 2^{h_3(m, n)+1} w_3) \\
&\quad + h_4(n+3, n+2) (w_4 - 2w_3) + h_4(n+2, n) (w_4 + w_2),
\end{aligned}$$

$$\begin{aligned}
v_3 = & h_4(n, n-4) (2w_3 + w_1 + 2^{h_4(m,n)+1}(w_3 - 2w_4)) \\
& + h_4(n+4, n+3) (2w_3 - w_2 + w_1 + 2^{h_4(m,n)} w_2) \\
& + h_4(n+3, n+2) (2w_3 - w_2 + 2^{h_4(m,n)}(w_4 - 2w_3)) \\
& + h_4(n+2, n) (w_3 + 2^{h_4(m,n)}(w_4 + w_2))
\end{aligned}$$

and  $v_4 = R_0 + h_4(n+4, n+3)(w_4 - R_0)$ . Then we have  $\langle \{w_i \mid 1 \leq i \leq 4\} \rangle = \langle \{v_i \mid 1 \leq i \leq 4\} \rangle$  and

$$VO_{m,n}^j / UO_{m,n}^j \cong \begin{cases} (\bigoplus_{i=1}^3 \mathbf{Z}/2^{b_i(j,m,n)}) \oplus \mathbf{Z}/2 & (n \equiv 4 \pmod{8}) \\ \bigoplus_{i=1}^3 \mathbf{Z}/2^{b_i(j,m,n)} & (\text{otherwise}). \end{cases}$$

If  $2[n/4] + 4[(n+2)/8] + 6 > m > n$ , then we have  $UO_{m,n}^j \cong 0$ . Thus the proof for the case  $j \equiv 4 \pmod{8}$  and  $n \not\equiv 3 \pmod{4}$  of Theorem 2 is completed.

**4.2. Proof for the case  $n \equiv 3 \pmod{4}$ .** Now, we turn to the case  $n \equiv 3 \pmod{4}$ . It follows from [13] that we have the following commutative diagram, in which rows are exact.

$$(4.6) \quad \begin{array}{ccccccc} 0 & \rightarrow & VO_{m,n+1}^j & \xrightarrow{f_1} & \widetilde{KO}(S^j(L_8^m/L_8^n)) & \xrightarrow{f_2} & \widetilde{KO}(S^{j+n+1}) \rightarrow 0 \\ & & \parallel & & \downarrow f_3 & & \downarrow \\ 0 & \rightarrow & VO_{m,n+1}^j & \rightarrow & \widetilde{KO}(S^j L_8^m) & \xrightarrow{f_4} & \widetilde{KO}(S^j L_8^{n+1}) \rightarrow 0. \end{array}$$

Since  $\widetilde{KO}(S^{j+n+1})$  is isomorphic to  $\mathbf{Z}$ , the upper row of (4.6) splits. Choose  $y \in \widetilde{KO}(S^j(L_8^m/L_8^n))$  such that  $\beta = f_2(y)$  generates the group  $\widetilde{KO}(S^{j+n+1})$ . Then we have an isomorphism

$$f: VO_{m,n+1}^j \oplus \widetilde{KO}(S^{j+n+1}) \rightarrow \widetilde{KO}(S^j(L_8^m/L_8^n))$$

defined by  $f(x, k\beta) = f_1(x) + ky$  for every  $(x, k) \in VO_{m,n+1}^j \oplus \mathbf{Z}$ . This proves the case  $j \equiv n+1 \equiv 0 \pmod{4}$  of Theorem 1.

**Lemma 4.7.** *If  $j \equiv n+1 \equiv 0 \pmod{4}$ , then there is an element  $y \in \widetilde{KO}(S^j(L_8^m/L_8^n))$  which satisfies the following conditions:*

(1)  $\beta = f_2(y)$  generates the group  $\widetilde{KO}(S^{j+n+1})$ ,

$$(2) \quad f_3(y) = \begin{cases} \mu_j(Y_4^j(n)) & (j \equiv n+1 \equiv 0 \pmod{8}) \\ \mu_j(Y_3^j(n) + Y_1^j(n)) & (j \equiv n+1 \equiv 4 \pmod{8}) \\ \mu_j(Y_3^j(n) + Y_2^j(n) + 2Y_1^j(n)) & (j \equiv n-3 \equiv 0 \pmod{8}) \\ \mu_j(Y_4^j(n) + Y_2^j(n)) & (j \equiv n-3 \equiv 4 \pmod{8}) \end{cases}$$

Proof. Consider the following commutative diagram, in which rows are exact:

$$\begin{array}{ccccccc}
0 \rightarrow & V_{m,n+1}^j & \xrightarrow{f_{C,1}} & \tilde{K}(S^j(L_8^m/L_8^n)) & \xrightarrow{f_{C,2}} & \tilde{K}(S^{n+j+1}) & \rightarrow 0 \\
& \parallel & & \downarrow f_{C,3} & & \downarrow & \\
0 \rightarrow & V_{m,n+1}^j & \longrightarrow & \tilde{K}(S^jL_8^m) & \xrightarrow{f_{C,4}} & \tilde{K}(S^jL_8^{n+1}) & \rightarrow 0.
\end{array}$$

According to Lemma 3.10, there is an element  $x \in \tilde{K}(S^j(L_8^m/L_8^n))$  such that  $f_{C,2}(x)$  generates the group  $\tilde{K}(S^{n+j+1})$  and

$$f_{C,3}(x) = \begin{cases} I^{1/2}(\varphi(X_4(n))) & (n \equiv 7 \pmod{8}) \\ I^{1/2}(\varphi(X_6(n)+X_1(n))) & (n \equiv 3 \pmod{8}) \end{cases}.$$

If  $n+j \equiv 3 \pmod{8}$ , then  $r: \tilde{K}(S^{n+j+1}) \rightarrow \widetilde{KO}(S^{n+j+1})$  is an isomorphism. It follows from Lemma 3.15 that  $y=r(x)$  satisfies the conditions (1) and (2). If  $n+j \equiv 7 \pmod{8}$ , then  $c: \widetilde{KO}(S^{n+j+1}) \rightarrow \tilde{K}(S^{n+j+1})$  is an isomorphism and  $c: \widetilde{KO}(S^jL_8^{n+1}) \rightarrow \tilde{K}(S^jL_8^{n+1})$  is a monomorphism. There is an element  $\mathfrak{y} \in \widetilde{KO}(S^j(L_8^m/L_8^n))$  such that  $f_2(\mathfrak{y})$  generates the group  $\widetilde{KO}(S^{n+j+1})$  and  $f_{C,4}(c(f_3(\mathfrak{y})))=f_{C,4}(f_{C,3}(x))$ . It follows from Lemma 3.15 that we have  $f_{C,3}(x)=c(Y)$ , where

$$Y = \begin{cases} \mu_j(Y_4^i(n)) & (n+1 \equiv j \equiv 0 \pmod{8}) \\ \mu_j(Y_3^i(n)+Y_1^i(n)) & (n+1 \equiv j \equiv 4 \pmod{8}) \end{cases}.$$

This implies that  $f_4(f_3(\mathfrak{y}))=f_4(Y)$  and  $y=\mathfrak{y}+f_1(Y-f_3(\mathfrak{y}))$  satisfies the conditions (1) and (2). q.e.d.

In the rest of this section, we fix an element  $y \in \widetilde{KO}(S^j(L_8^m/L_8^n))$  which satisfies the conditions of Lemma 4.7, and set

$$(4.8) \quad T_i = \mu_j(Y_i^j(n+1)) \quad (1 \leq i \leq 4).$$

**Lemma 4.9.** *If  $k$  is an odd integer, then the Adams operation  $\psi^k$  is given by*

$$\psi^k(y) = k^{(n+j+1)/2} y + ((k^{1/2} - k^{(n+j+1)/2})/8) f_1(8f_3(y)) + k^{j/2} f_1(w),$$

where  $w=h_4(k+5, k+1)(W-8f_3(y))$  and

$$(4.10) \quad W = 8f_3(y) - h_4(n, n-4)(T_3 + h_4(j, j-4)(T_3 + T_1)).$$

Proof. We necessarily have

$$\psi^k(y) = uy + f_1(x)$$

for some integer  $u$  and an element  $x \in VO_{m,n+1}^j$ . By using the  $\psi$ -map  $f_2$ , we see that  $u=k^{(n+j+1)/2}$ . Under  $f_3, f_1(x)$  maps into  $x$  and  $y$  maps into  $f_3(y)$ , and we see that

$$\psi^k(f_3(y)) = k^{(n+j+1)/2} f_3(y) + x.$$

It follows from Lemma 4.3 that

$$k^{j/2}(f_3(y) + w) = k^{(n+j+1)/2} f_3(y) + x.$$

This implies that

$$x = ((k^{j/2} - k^{(n+j+1)/2})/8)(8f_3(y)) + k^{j/2} w. \quad \text{q.e.d.}$$

We now recall some definition in [3]. Set  $Y = \widetilde{KO}(S^j(L_8^n/L_8^m))$  and let  $f$  be a function which assigns to each integer  $k$  a non-negative integer  $f(k)$ . Given such a function  $f$ , we define  $Y_f$  to be the subgroup of  $Y$  generated by

$$\{k^{f(k)}(\psi^k - 1)(y) \mid k \in \mathbf{Z}, y \in Y\};$$

that is,

$$Y_f = \langle \{k^{f(k)}(\psi^k - 1)(y) \mid k \in \mathbf{Z}, y \in Y\} \rangle.$$

Then the kernel of the homomorphism  $J'': Y \rightarrow J''(Y)$  coincides with  $\bigcap_f Y_f$ , where the intersection runs over all functions  $f$ .

Suppose that  $f$  satisfies

$$(4.11) \quad f(k) \geq m + \max \{v_p(m((n+j+1)/2)) \mid p \text{ is a prime divisor of } k\}$$

for every  $k \in \mathbf{Z}$ . For each odd integer  $i$ ,  $N(i)$  denotes the integer chosen to satisfy the property

$$(4.12) \quad iN(i) \equiv 1 \pmod{2^m}.$$

In the following calculation we put  $(n+j+1)/2 = u$  for the sake of simplicity. From Lemma 3.1, (4.5) and Lemma 4.9, we have

$$\begin{aligned} & k^{f(k)}(\psi^k - 1)(y) \\ &= k^{f(k)}(k^u - 1)y + (k^{f(k)}(k^{j/2} - k^u)/8)f_1(8f_3(y)) + k^{f(k)+(j/2)}f_1(w) \\ &\equiv k^{f(k)}(k^u - 1)y + (k^{f(k)}(k^{j/2} - k^u)/8)f_1(W) \pmod{f_1(UO_{m,n+1}^j)} \\ &= k^{f(k)}(k^u - 1)y + (k^{f(k)} N(u/2^{v_2(u)}) (u(k^{j/2} - 1) - u(k^u - 1))/2^{v_2(8u)})f_1(W) \\ &\equiv k^{f(k)}(k^u - 1)y \\ &\quad + (k^{f(k)} N(u/2^{v_2(u)}) ((j/2)(k^u - 1) - u(k^u - 1))/2^{v_2(8u)})f_1(W)(2^v + 1) \\ &\qquad \qquad \qquad \pmod{f_1(UO_{m,n+1}^j)} \\ &= (k^{f(k)}(k^u - 1)/2^{v_2(4u)}) (2^{v_2(4u)}y - N(u/2^{v_2(u)}) ((n+1)/4)f_1(W)(2^v + 1)). \end{aligned}$$

By virtue of [3; II, Theorem (2.7) and Lemma (2.12)], we have

$$\begin{aligned} Y_f &= \langle f_1(UO_{m,n+1}^j) \cup \{k^{f(k)}(\psi^k - 1)(y) \mid k \in \mathbf{Z}\} \rangle \\ &= \langle f_1(UO_{m,n+1}^j) \cup \{m(u)y - Mf_1(W)\} \rangle, \end{aligned}$$

where  $M = (\mathbf{m}(u)/2^{v_2(u)+2}) N(u/2^{v_2(u)}) ((n+1)/4) (2^v+1)$ . Since this is true for every function  $f$  which satisfies (4.11), we have

$$(4.13) \quad J''(Y) \cong Y / \langle f_1(UO_{m,n+1}^j) \cup \{\mathbf{m}((n+j+1)/2) y - Mf_1(W)\} \rangle,$$

where  $v_2(M) = v_2(n+1) - 2$  and

$$W = \begin{cases} 4T_4 + 2T_2 + T_1 & (j \equiv n+1 \equiv 0 \pmod{8}) \\ 8T_4 + 4T_2 - T_1 & (j \equiv n-3 \equiv 4 \pmod{8}) \\ 6T_3 + 4T_2 + 4T_1 & (j \equiv n-3 \equiv 0 \pmod{8}) \\ 3T_3 + 2T_2 + 3T_1 & (j \equiv n+1 \equiv 4 \pmod{8}) \end{cases}.$$

Suppose that  $m \geq n+5 + 2h_4(j+4, j) h_4(n+1, n)$ . It follows from the proof for the case  $n \not\equiv 3 \pmod{4}$  that we have

$$W \equiv \sum_{i=1}^3 m_i z_i \pmod{UO_{m,n+1}^j},$$

where  $z_i = \mu_j(v_i)$  ( $1 \leq i \leq 3$ ),  $v_2(m_i) = 2 + h_4(n+j, n+j-4) - i$  ( $1 \leq i \leq 2$ ) and  $v_2(m_3) = 2h_4(j, j-4)$ . Therefore

$$J''(Y) \cong F(v) / \langle \{\sum_{i=0}^3 M_i v_i\} \cup \{B_i \mid 1 \leq i \leq 4\} \rangle,$$

where  $F(v)$  is the free abelian group generated by  $\{v_i \mid 0 \leq i \leq 4\}$ ,  $M_0 = \mathbf{m}((n+j+1)/2)$ ,

$$\begin{aligned} B_i &= 2^{b_i(j,m,n)} v_i \quad (1 \leq i \leq 3), \\ B_4 &= (h_4(j+4, j) h_4(n+5, n+1) + 1) v_4 \end{aligned}$$

and  $M_i = -m_i M$  ( $1 \leq i \leq 3$ ). Set

$$(4.14) \quad i_k = \min \{b_k(j, m, n), v_2(n+1) + v_2(m_k) - 2\} \quad (1 \leq k \leq 3).$$

For the sake of simplicity, we put  $b_k = b_k(j, m, n)$  ( $1 \leq k \leq 3$ ) in the following calculation. Since  $v_2(M) = v_2(n+1) - 2$ , the greatest common divisor of  $M_k$  and  $2^{b_k}$  is equal to  $2^{i_k}$  ( $1 \leq k \leq 3$ ). Choose integers  $e_{11}, e_{12}, e_{21}, e_{22}, e_{31}$  and  $e_{32}$  with  $e_{k1} 2^{b_k} + e_{k2} M_k = 2^{i_k}$  ( $1 \leq k \leq 3$ ). If  $b_1 - i_1 > b_2 - i_2$  and  $i_2 \leq v_2(M_3)$ , then we have

$$A_1 \begin{pmatrix} \sum_{i=0}^3 M_i v_i \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 2^{b_1-i_1} M_0 v_0 \\ 2^{b_2-i_2} (e_{12} M_0 v_0 + 2^{i_1} v_1) \\ e_{22} (M_0 v_0 + M_1 v_1 + M_3 v_3) + 2^{i_2} v_2 \\ B_3 \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} 2^{b_1-i_1} & -M_1/2^{i_1} & -2^{b_1-b_2-i_1} M_2 & -2^{b_1-b_3-i_1} M_3 \\ e_{12} 2^{b_2-i_2} & e_{11} 2^{b_2-i_2} & -e_{12} M_2/2^{i_2} & -2^{b_2-b_3-i_2} e_{12} M_3 \\ e_{22} & 0 & e_{21} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and  $\det A_1=1$ . This implies that

$$J''(Y) \cong \begin{cases} \mathbf{Z}/2^{b_1-1}M_0 \oplus \mathbf{Z}/2^{b_2+1} \oplus \mathbf{Z}/2^{b_3} \oplus \mathbf{Z}/2 & (j \equiv n+1 \equiv 4 \pmod{8}) \\ \mathbf{Z}/2^{b_1-i_1}M_0 \oplus \mathbf{Z}/2^{b_2-i_2+i_1} \oplus \mathbf{Z}/2^{i_2} \oplus \mathbf{Z}/2^{b_3} & (\text{otherwise}). \end{cases}$$

If  $b_2-i_2 \geq b_1-i_1$  and  $\nu_2(M_3) \geq i_2$ , then we have

$$A_2 \begin{pmatrix} \sum_{i=0}^3 M_i v_i \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 2^{b_2-i_2} M_0 v_0 \\ B_1 \\ e_{22}(M_0 v_0 + M_1 v_1 + M_3 v_3) + 2^{i_2} v_2 \\ B_3 \end{pmatrix},$$

where

$$A_2 = \begin{pmatrix} 2^{b_2-i_2} & -M_1 2^{b_2-b_1-i_2} & -M_2/2^{i_2} & -2^{b_2-b_3-i_2} M_3 \\ 0 & 1 & 0 & 0 \\ e_{22} & 0 & e_{21} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and  $\det A_2=1$ . This implies that

$$J''(Y) \cong \begin{cases} \mathbf{Z}/2^{b_2}M_0 \oplus \mathbf{Z}/2^{b_1} \oplus \mathbf{Z}/2^{b_3} \oplus \mathbf{Z}/2 & (j \equiv n+1 \equiv 4 \pmod{8}) \\ \mathbf{Z}/2^{b_2-i_2}M_0 \oplus \mathbf{Z}/2^{b_1} \oplus \mathbf{Z}/2^{i_2} \oplus \mathbf{Z}/2^{b_3} & (\text{otherwise}). \end{cases}$$

If  $i_2 > \nu_2(M_3)$ , then we necessarily have  $j \equiv n-3 \equiv 4 \pmod{8}$ ,  $b_1=4$ ,  $i_1 \geq 3$ ,  $b_2=3$  and  $i_2=i_1-1=\nu_2(M_3)+1$ . If  $i_2 > \nu_2(M_3)$  and  $b_2-i_2 > b_3-i_3$ , then we have  $i_2=2$ ,  $b_3=i_3$  and

$$A_3 \begin{pmatrix} \sum_{i=0}^3 M_i v_i \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 2M_0 v_0 \\ B_1 \\ e_{22}(M_0 v_0 + M_1 v_1) + 4v_2 \\ B_3 \end{pmatrix},$$

where

$$A_3 = \begin{pmatrix} 2 & -M_1/8 & -M_2/4 & -2M_3/2^{b_3} \\ 0 & 1 & 0 & 0 \\ e_{22} & 0 & e_{21} & -e_{22} M_3/2^{b_3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and  $\det A_3=1$ . This implies that

$$J''(Y) \cong \mathbf{Z}/2M_0 \oplus \mathbf{Z}/16 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2^{b_3}.$$

If  $i_2 > \nu_2(M_3)$  and  $b_3-i_3 \geq b_2-i_2$ , then we have

$$A_4 \begin{pmatrix} \sum_{i=0}^3 M_i v_i \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 2^{b_3-i_3} M_0 v_0 \\ B_1 \\ B_2 \\ e_{32}(M_0 v_0 + M_1 v_1 + M_2 v_2) + 2^{i_3} v_3 \end{pmatrix},$$

where

$$A_4 = \begin{pmatrix} 2^{b_3-i_3} & -M_1 2^{b_3-i_3-4} & -M_2 2^{b_3-i_3-3} & -M_3/2^{i_3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ e_{32} & 0 & 0 & e_{31} \end{pmatrix}$$

and  $\det A_4 = 1$ . This implies that

$$J''(Y) \cong \mathbf{Z}/2^{b_3-i_3}M_0 \oplus \mathbf{Z}/16 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/2^{i_3}.$$

If  $n+7 > m \geq n+5$  and  $j \equiv n-3 \pmod{8}$ , then we have  $W \equiv m_1 T_3 \pmod{UO_{m,n+1}^j}$ , where  $\nu_2(m_1) = 2$ . Therefore

$$J''(Y) \cong F(v)/\langle \{M_0 v_0 + M_1 v_1, B_1\} \rangle,$$

where  $F(v)$  is the free abelian group generated by  $\{v_i \mid 0 \leq i \leq 1\}$ ,  $M_0 = \mathfrak{m}((n+j+1)/2)$ ,  $B_1 = 2^b v_1$ ,  $b = b_1(j, m, n)$  and  $M_1 = -m_1 M$ . Set  $i = \min\{b, \nu_2(n+1)\}$ . Choose integers  $e_1$  and  $e_2$  with  $e_1 2^b + e_2 M_1 = 2^i$ . Then we have

$$\begin{pmatrix} 2^{b-i} & -M_1/2^i \\ e_2 & e_1 \end{pmatrix} \begin{pmatrix} M_0 v_0 + M_1 v_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} 2^{b-i} M_0 v_0 \\ e_2 M_0 v_0 + 2^i v_1 \end{pmatrix}.$$

This implies that  $J''(Y) \cong \mathbf{Z}/2^{b-i}M_0 \oplus \mathbf{Z}/2^i$ .

If  $n+5 > m > n$ , then we have  $Mf_1(W) \equiv 0 \pmod{f_1(UO_{m,n}^j)}$  and

$$J''(Y) \cong \mathbf{Z}/\mathfrak{m}((n+j+1)/2) \oplus (VO_{m,n+1}^j/UO_{m,n+1}^j).$$

According to [2], [3] and [12], we have  $\tilde{J}(S^j(L_8^n/L_8^n)) \cong J''(Y)$ . Thus, the proof for the parts of the case  $n+1 \equiv j \pmod{8}$  of Theorem 2 is completed.

## 5. Proof for the case $j \equiv 2 \pmod{4}$

In this section we prove the parts of the case  $j \equiv 2 \pmod{4}$  of Theorems 1 and 2. Throughout this section  $j$  denotes a positive integer with  $j \equiv 2 \pmod{4}$ . Consider the elements  $S_i$  ( $1 \leq i \leq 7$ ) of  $\tilde{K}(S^j L_8^n)$  defined by

$$(5.1) \quad S_i = I^{j/2}(\varphi(X_i(n+1))) \quad (1 \leq i \leq 7),$$

where  $\varphi: F(x) \rightarrow \tilde{K}(L_8^n)$  is the homomorphism defined in (3.7). For each integer  $n$  with  $0 \leq n \leq m$ , we denote the kernel of the homomorphism

$$(i_n^m)^! : \tilde{K}(S^j L_8^m) \rightarrow \tilde{K}(S^j L_8^n)$$

by  $V_{m,n}^j$ . By Proposition 3.8, we have

$$(5.2) \quad V_{m,2[(n+1)/2]}^j = \langle \{S_i \mid 1 \leq i \leq 7\} \rangle.$$

Consider the Bott exact sequence (cf. [5] and [6, (12.2)])

$$(5.3) \quad \rightarrow \widetilde{KO}(S^{j+2}X) \xrightarrow{c} \widetilde{K}(S^{j+2}X) \xrightarrow{r \circ I^{-1}} \widetilde{KO}(S^j X) \xrightarrow{\partial} \widetilde{KO}(S^{j-1}X) \rightarrow$$

for  $X = L_8^m / L_8^n$ , where  $\partial$  is the homomorphism defined by the exterior product with the generator of  $\widetilde{KO}(S^1)$ . Using the isomorphisms

$$VO_{m,2[(n+1)/2]}^{j+2} \cong \widetilde{KO}(S^{j+2}(L_8^m / L_8^{2[(n+1)/2]}))$$

and

$$V_{m,2[(n+1)/2]}^j \cong \widetilde{K}(S^j(L_8^m / L_8^{2[(n+1)/2]})),$$

we obtain the exact sequence

$$(5.4) \quad \rightarrow VO_{m,2u}^{j+2} \xrightarrow{I^{-1} \circ c} V_{m,2u}^j \xrightarrow{r_1} \widetilde{KO}(S^j(L_8^m / L_8^{2u})) \xrightarrow{\partial} G \rightarrow 0,$$

where  $u = [(n+1)/2]$  and

$$G = \begin{cases} \widetilde{KO}(S^{j+1}(L_8^m / L_8^{2u})) & (m+j \equiv 0, 1 \text{ or } 2 \pmod{8}) \\ 0 & (\text{otherwise}). \end{cases}$$

Using Lemma 3.15, we obtain the following lemma.

**Lemma 5.5.** *For the homomorphism  $r_1$  in the exact sequence (5.4), we have*

$$\text{Ker } r_1 = \begin{cases} \langle \{S_1, S_2, S_4, S_6\} \rangle & (2u+j \equiv 4 \text{ or } 6 \pmod{8}) \\ \langle \{2S_1, S_2, S_4, S_6\} \rangle & (2u+j \equiv 2 \pmod{8}) \\ \langle \{S_1, S_2, S_4, 2S_6\} \rangle & (2u \equiv j-4 \equiv 2 \pmod{8}) \\ \langle \{S_1, S_2, 2S_4, S_6\} \rangle & (2u \equiv j-4 \equiv 6 \pmod{8}), \end{cases}$$

where  $u = [(n+1)/2]$ .

It follows from Lemmas 3.9 and 5.5 that we have

**Lemma 5.6.** *The Adams operations are given by the following formulae, where  $u = [(n+1)/2]$ ,  $T_i = r_1(S_i)$  ( $1 \leq i \leq 7$ ) and  $k \equiv 1 \pmod{2}$ .*

$$(1) \quad \psi^k(T_3) = (-1)^{h_3(k+2,k)} k^{j/2} T_3.$$

$$(2) \quad \psi^k(T_5) = (-1)^{h_4(k+4,k)} k^{j/2} \tilde{T}_5,$$

where  $\tilde{T}_5 = T_5 - h_4(k+5, k+1) (h_3(u+1, u-1) T_3 + h_4(j+4, j) h_3(u+2, u+1) T_1)$ .

$$(3) \quad \psi^k(T_7) = (-1)^{h_3(k+2,k)} k^{j/2} \tilde{T}_7,$$

where  $\tilde{T}_7 = T_7 - h_4(k+5, k+1) h_4(j+4, j) h_3(u+2, u+1) T_1$ .

$$(4) \quad \psi^k(T_i) = T_i \quad (i = 1, 2, 4 \text{ or } 6).$$

We set

$$(5.7) \quad U_{m,2u}^j = \sum_{k: \text{odd}} (\psi^k - 1) r_1(V_{m,2u}^j).$$

It follows from Lemma 5.6 that we have

$$(5.8) \quad U_{m,2u}^j = \langle \{4T_3, R_1, R_2\} \rangle,$$

where  $u = [(n+1)/2]$ ,

$$\begin{aligned} R_1 &= 2T_5 + h_3(u+1, u-1) T_3 - h_4(j+4, j) h_3(u+2, u+1) T_1, \\ R_2 &= 4T_7 - h_4(j+4, j) h_3(u+2, u+1) T_1 \end{aligned}$$

and  $T_i = r_1(S_i)$  ( $1 \leq i \leq 7$ ).

**5.1. Proof for the case  $n \equiv 0 \pmod{2}$ .** Suppose  $n \equiv 0 \pmod{2}$ . If  $m \geq n+2$ , then by Proposition 3.8 and Lemma 5.5, we have

$$r_1(V_{m,n}^j) \cong \langle \{w_i \mid 0 \leq i \leq 3\} \rangle / \langle \{R_i \mid 4 \leq i \leq 7\} \rangle,$$

where  $w_0 = h_3(n, n-2) X_1(n) + h_4(n+6, n+4) X_6(n) + h_4(n+2, n) X_4(n)$ ,

$$w_1 = (1 - 2h_4(n+6, n+2)) X_3(n),$$

$$w_2 = h_4(n+2, n-2) X_5(n) + h_4(n+6, n+2) X_7(n),$$

$$w_3 = h_4(n+6, n+2) X_5(n) + h_4(n+2, n-2) X_7(n),$$

$$R_4 = h_4(n+j, n+j-12) w_0,$$

$$R_5 = 2^{h_3(m+j, n+j)-1} \tilde{w}_1,$$

$$R_6 = 2^{a_8(m+j, n+j)} (\tilde{w}_2 + 2^{a_8(m+j, n+j)} \tilde{w}_1),$$

$$R_7 = 2^{a_9(m+j, n+j)-1} (\tilde{w}_3 - 2^{a_9(m+j, n+j)} \tilde{w}_1),$$

$$\tilde{w}_1 = (1 - 2h_4(n+6, n+2)) (2X_3(n) - h_4(n+j+6, n+j+4) w_0),$$

$$\tilde{w}_2 = 2w_2 - w_1 + h_4(j, j-4) h_4(n, n-2) w_0,$$

$$\tilde{w}_3 = 2w_3 - h_4(n+j+6, n+j+4) w_0,$$

$$a_8(m+j, n+j) = \sum_{i=0}^1 h_4(n+j+4i, n+j+4i-4) h_4(m+j+4i-4, n+j+4i)$$

and  $a_9(m+j, n+j) = a_8(m+j+8, n+j+4)$ . If  $m \geq n+2+12h_4(n+j+6, n+j+4)+2h_3(n+2, n)$ , then we have

$$r_1(V_{m,n}^j) \cong \langle \{u_i \mid 0 \leq i \leq 3\} \rangle / \langle \{A_i \mid 0 \leq i \leq 3\} \rangle,$$

where  $u_0 = w_0$ ,  $u_1 = w_2$ ,  $A_0 = R_4$ ,  $A_1 = 2^{h_3(m+j, n+j)+1} u_1$ ,

$$\begin{aligned} A_i &= 2^{a_{i+6}(m+j, n+j)} u_i \quad (2 \leq i \leq 3), \\ u_2 &= 2w_2 - w_1 + 2^{a_6(m+j, n+j)+2} w_2 \end{aligned}$$

and  $u_3 = w_3 - 2^{a_9(m+j, n+j)+1} w_2$ . If  $n+2+12h_4(n+j+6, n+j+4)+2h_3(n+2, n) > m \geq n+2+2h_4(n+j+4, n+j+2)$ , then we have

$$r_1(V_{m,n}^j) \cong \langle \{u_i \mid 0 \leq i \leq 3\} \rangle / \langle \{A_i \mid 0 \leq i \leq 3\} \rangle,$$

where  $u_1 = w_2$ ,

$$\begin{aligned} u_0 &= \begin{cases} 2w_3 - 4w_1 - w_0 & (n+j \equiv 2 \pmod{8} \text{ and } n+14 > m \geq n+6) \\ 2w_1 - w_0 & (n+j \equiv 2 \pmod{8} \text{ and } n+6 > m \geq n+2) \\ w_0 & (n+j \equiv 0 \pmod{8} \text{ and } n+4 > m \geq n+2), \end{cases} \\ u_2 &= \begin{cases} 2w_2 - w_1 & (n+j \equiv 2 \pmod{8} \text{ and } n+14 > m \geq n+10) \\ 2w_2 + w_1 - h_4(n+4, n) w_0 & (n+j \equiv 2 \pmod{8} \text{ and } n+10 > m \geq n+2) \\ w_1 & (n+j \equiv 0 \pmod{8} \text{ and } n+4 > m \geq n+2), \end{cases} \\ u_3 &= \begin{cases} w_3 - w_1 - h_4(n+4, n) w_0 & (n+j \equiv 2 \pmod{8} \text{ and } n+6 > m \geq n+2) \\ w_3 - 2w_1 & (\text{otherwise}), \end{cases} \\ A_0 &= \begin{cases} 2u_0 & (n+j \equiv 0 \pmod{8} \text{ and } n+4 > m \geq n+2) \\ u_0 & (\text{otherwise}), \end{cases} \\ A_1 &= \begin{cases} 2^{h_4(m+j+20, n+j)} u_1 & (n+j \equiv 2 \pmod{8}) \\ u_1 & (n+j \equiv 0 \pmod{8}), \end{cases} \\ A_2 &= \begin{cases} 2u_2 & (n+j \equiv 2 \pmod{8} \text{ and } n+14 > m \geq n+10) \\ u_2 & (\text{otherwise}) \end{cases} \end{aligned}$$

and

$$A_3 = \begin{cases} 4u_3 & (n+j \equiv 2 \pmod{8} \text{ and } n+14 > m \geq n+6) \\ u_3 & (\text{otherwise}). \end{cases}$$

If  $n+2+2h_4(n+j+4, n+j+2) > m > n$ , then we have  $r_1(V_{m,n}^j) \cong 0$ .

Suppose  $j \equiv 2 \pmod{8}$ . If  $m \geq 8[(n+2)/8] + 10 + 4h_4(n, n-2)$ , then we have

$$J''(r_1(V_{m,n}^j)) \cong \langle \{v_i \mid 0 \leq i \leq 3\} \rangle / \langle \{B_i \mid 0 \leq i \leq 3\} \rangle,$$

where  $v_0 = w_0$ ,  $v_1 = w_2$ ,  $v_2 = 2w_2 + w_1$ ,  $v_3 = w_3$ ,  $B_0 = R_4$ ,

$$\begin{aligned} B_1 &= 4h_4(n+10, n-2) v_1, \\ B_2 &= 2^{h_4(n+6, n+2) + b_3(j, m+2, n)} v_2 \end{aligned}$$

and  $B_3 = 2^{h_4(n+6, n+2) + h_4(n+2, n-2) + b_3(j, m+2, n)} v_3$ . If  $8[(n+2)/8] + 10 + 4h_4(n, n-2) > m \geq 8[(n+2)/8] + 10$ , then we have

$$J''(r_1(V_{m,n}^j)) \cong \langle \{v_i \mid 0 \leq i \leq 3\} \rangle / \langle \{B_i \mid 0 \leq i \leq 3\} \rangle,$$

where  $v_0=2w_3-w_0$ ,  $v_1=w_2$ ,  $v_2=2w_2+w_1$ ,  $v_3=w_3$ ,  $B_0=v_0$ ,  $B_1=8v_1$ ,  $B_2=v_2$  and  $B_3=4v_3$ . If  $8[(n+2)/8]+10>m< n$ , then we have  $U_{m,n}^j\simeq 0$ .

Suppose  $j\equiv 6 \pmod{8}$ . If  $m\geq 8[(n+2)/8]+10$ , then we have

$$J''(r_1(V_{m,n}^j))\simeq \langle\{v_i|0\leq i\leq 3\}\rangle/\langle\{B_i|0\leq i\leq 3\}\rangle,$$

where  $v_0=w_0-4h_4(n+4, n+2)w_2$ ,  $v_1=w_2$ ,  $v_2=2w_2+(1-2h_4(n+4, n+2))w_1$ ,

$$\begin{aligned} v_3 &= w_3-2h_4(n+4, n+2)w_2, \\ B_0 &= h_4(n+14, n+4)v_0, \\ B_1 &= 4h_4(n+12, n-2)v_1, \\ B_2 &= 2^{h_4(n+6, n+2)b_3(j, m+2, n)}v_2 \end{aligned}$$

and  $B_3=2^{h_4(n+6, n+2)+h_4(n+2, n-2)b_3(j, m+2, n)}v_3$ . If  $8[(n+2)/8]+10>m< n$ , then we have  $U_{m,n}^j\simeq 0$ . Thus we obtain

(5.9) Suppose that  $n\equiv 0 \pmod{2}$ .

(1) If  $m\geq n+2+12h_4(n+j+6, n+j+4)+2h_3(n+2, n)$ , then we have

$$r_1(V_{m,n}^j)\simeq \mathbf{Z}/2^{h_3(m+j, n+j)+1}\oplus(\bigoplus_{i=0}^1 \mathbf{Z}/2^{h_4(m+j-4i, n+j-4i+4)})\oplus G(n+j),$$

where  $G(n+j)$  is the group defined by (2.3).

(2) If  $n+2+12h_4(n+j+6, n+j+4)+2h_3(n+2, n)>m>n$ , then we have

$$r_1(V_{m,n}^j)\simeq \begin{cases} \mathbf{Z}/2^{h_3(m+j, n+j)+1}\oplus \mathbf{Z}/2^{h_4(m+j-4, n+j)}\oplus \mathbf{Z}/4 & (m\geq n+6) \\ \mathbf{Z}/8 & (n+j\equiv 2 \pmod{8} \text{ and } n+6>m\geq n+2) \\ \mathbf{Z}/2 & (n+j\equiv 0 \pmod{8} \text{ and } n+4>m\geq n+2) \\ 0 & (\text{otherwise}) \end{cases}$$

(3) If  $m\geq 8[(n+2)/8]+10+4h_4(n, n-2)h_4(j, j-4)$ , then we have

$$J''(r_1(V_{m,n}^j))\simeq \begin{cases} \mathbf{Z}/8\oplus \mathbf{Z}/2^b\oplus G(n+j) & (h_4(n+6, n+2)h_4(n+j+4, n+j-2)=0) \\ \mathbf{Z}/4\oplus \mathbf{Z}/2^b\oplus \mathbf{Z}/2\oplus G(n+j) & (\text{otherwise}), \end{cases}$$

where  $b=b_3(j, m+2, n)$  and  $G(n+j)$  is the group defined by (2.3).

(4) If  $j\equiv n+2\equiv 2 \pmod{8}$  and  $n+14>m\geq n+10$ , then we have

$$J''(r_1(V_{m,n}^j))\simeq \mathbf{Z}/8\oplus \mathbf{Z}/4.$$

(5) If  $8[(n+2)/8]+10>m>n$ , then we have  $J''(r_1(V_{m,n}^j))\simeq r_1(V_{m,n}^j)$ .

By (5.4) and (5.9), we obtain the results for the cases  $j\equiv 2 \pmod{4}$ ,  $n\equiv 0 \pmod{2}$  and  $m+j\equiv 3, 4, 5, 6$  or  $7 \pmod{8}$ .

We now turn to the case  $m+j\equiv 0 \pmod{8}$ . Consider the commutative diagram

$$\begin{array}{ccccccc}
& 0 & & 0 & & & \\
& \uparrow & & \uparrow & & & \\
V_{m-1,n}^j & \xrightarrow{r_1} & \widetilde{KO}(S^j(L_8^{m-1}/L_8^n)) & \longrightarrow & 0 & & \\
& \uparrow & & \uparrow & & \uparrow & \\
V_{m,n}^j & \xrightarrow{r_1} & \widetilde{KO}(S^j(L_8^m/L_8^n)) & \xrightarrow{\partial} & \widetilde{KO}(S^{j+1}(L_8^m/L_8^n)) & \longrightarrow 0 & \\
& f \uparrow & g \uparrow & & & \uparrow & \\
0 \rightarrow \widetilde{K}(S^{m+j}) & \xrightarrow{r} & \widetilde{KO}(S^{m+j}) & \xrightarrow{\partial_1} & \widetilde{KO}(S^{m+j+1}) & \longrightarrow 0 & \\
& & & & \uparrow & & \\
& & & & 0 & & 
\end{array}$$

of exact sequences. Since  $\widetilde{KO}(S^{m+j+1}) \cong \mathbb{Z}/2$ , Lemma 3.10 implies that  $\widetilde{K}(S^{m+j}) \cong \mathbb{Z}$  has a generator  $\gamma$  with

$$f(\gamma) = \begin{cases} I^{j/2}(\varphi(X_5(m-2))) & (j \equiv 6 \pmod{8}) \\ I^{j/2}(\varphi(X_7(m-2))) & (j \equiv 2 \pmod{8}) \end{cases}$$

and  $r(\gamma) = 2\beta$ , where  $\beta$  is a generator of the group  $\widetilde{KO}(S^{m+j}) \cong \mathbb{Z}$ . It follows from Lemma 5.5 that we have

$$2g(\beta) = r_1(f(\gamma)) = 2^{h_4(m+j-16, n+j)} W_2 + 2^{h_3(m+j-12, n+j)} W_1,$$

where  $W_1 = (1 - 2h_4(n+j, n+j-4)) r_1(I^{j/2}(\varphi(\tilde{w}_1)))$  and

$$W_2 = \begin{cases} r_1(I^{j/2}(\varphi(\tilde{w}_2))) & (h_4(n+j, n+j-4) = 0) \\ r_1(I^{j/2}(\varphi(\tilde{w}_3))) & (\text{otherwise}). \end{cases}$$

If  $m > n + 12$ , we set

$$\alpha = g(\beta) - \begin{cases} 2^{[(m-n-14)/8]} W_3 - 2^{[(m-n-10)/4]} W_1 & (h_4(n+j+4, n+j) = 0) \\ 2^{[(m-n-14)/8]} (2W_2 - W_1) + 2^{[(m-n-10)/4]} W_1 & (\text{otherwise}), \end{cases}$$

where  $W_i = r_1(I^{j/2}(\varphi(w_i)))$  ( $1 \leq i \leq 3$ ). Then we have  $\partial(\alpha) \neq 0$  and  $2\alpha = 0^{m-n-14}W_0$ , where  $W_0 = r_1(I^{j/2}(\varphi(w_0)))$ . By Lemma 3.9, Lemma 5.5, (5.8) and the fact  $8g(\beta) = 0$ , we have

$$(5.10) \quad \psi^k(\alpha) \equiv (k - 2[k/2]) \alpha \pmod{U_{m,n}^j}.$$

According to [3, II], the Adams operations on  $\widetilde{KO}(S^{m+j+1})$  are given by  $\psi^k = k - 2[k/2]$ . If  $m > n + 14$ , then the short exact sequence

$$0 \rightarrow r_1(V_{m,n}^j) \rightarrow \widetilde{KO}(S^j(L_8^m/L_8^n)) \xrightarrow{\partial} \widetilde{KO}(S^{j+1}(L_8^m/L_8^n)) \rightarrow 0$$

splits. Hence we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong r_1(V_{m,n}^j) \oplus \mathbb{Z}/2.$$

It follows from (5.10) that we have

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong J''(r_1(V_{m,n}^j)) \oplus \mathbb{Z}/2.$$

If  $m=n+14$ , then we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) = \langle r_1(V_{m,n}^j) \cup \{\alpha\} \rangle = \langle \{W_1, W_2, W_3, \alpha\} \rangle.$$

Since  $\text{ord } \widetilde{KO}(S^j(L_8^m/L_8^n))=1024$  by [13],  $\text{ord } \langle \{W_1, W_2, W_3\} \rangle=256$  and  $\text{ord } \langle \alpha \rangle=4$ , we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \mathbb{Z}/32 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2$$

and

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 & (j \equiv 2 \pmod{8}) \\ \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & (j \equiv 6 \pmod{8}) \end{cases}.$$

If  $n+12 \geq m \geq n+2$ , we set

$$\alpha = g(\beta) - ([\!(m-n-2)/8\!]+[\!(m-n+2)/8\!]) W_2.$$

Then we have

$$2\alpha = W_2 + [\!(m-n+2)/8\!] (W_3 - W_2) + [\!(m-n-2)/8\!] (W_1 - 2W_2 - W_3).$$

Hence  $\widetilde{KO}(S^j(L_8^m/L_8^n)) = \langle r_1(V_{m,n}^j) \cup \{\alpha\} \rangle$

$$= \begin{cases} \langle \{W_2, W_3, \alpha\} \rangle & (m \geq n+10) \\ \langle \{W_0, W_2, \alpha\} \rangle & (m = n+8) \\ \langle \{W_2, \alpha\} \rangle & (m = n+6) \\ \langle \alpha \rangle & (n+6 > m \geq n+2), \end{cases}$$

$$\text{ord } \langle \{W_2, W_3\} \rangle = 32 \quad (m \geq n+10),$$

$$\text{ord } \langle W_2 \rangle = 8 \quad (n+10 > m \geq n+6)$$

and

$$\text{ord } \langle \alpha \rangle = 2^{[(n-m+31)/8]}.$$

Since  $\text{ord } \widetilde{KO}(S^j(L_8^m/L_8^n)) = 2^{2h_3(m+j, n+j) + h_4(n+j, n+j-4)+1}$  by [13], we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbb{Z}/16 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 & (m \geq n+10) \\ \mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 & (m = n+8) \\ \mathbb{Z}/8 \oplus \mathbb{Z}/8 & (m = n+6) \\ \mathbb{Z}/8 & (n+6 > m \geq n+2) \end{cases}$$

and

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & (m \geq n+10 \text{ and } j \equiv 6 \pmod{8}) \\ \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 & (m \geq n+8 \text{ and } h_4(n+2, n-2) = 0) \\ \mathbb{Z}/8 \oplus \mathbb{Z}/4 & (m = n+6 \text{ and } j \equiv 6 \pmod{8}) \\ \widetilde{KO}(S^j(L_8^m/L_8^n)) & (\text{otherwise}). \end{cases}$$

Thus we obtain the results for the case  $m+j \equiv 0 \pmod{8}$ .

Modifying the proof above, we obtain the results for the case  $m+j \equiv 1 \pmod{8}$  (cf. [10]).

Finally we consider the case  $m+j \equiv 2 \pmod{8}$ . Since

$$\widetilde{KO}(S^j(L_8^m/L_8^{m-2})) \cong \widetilde{KO}(S^{j+m-2}L_8^2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

by Proposition 3.13, and the Adams operations on  $\widetilde{KO}(S^j(L_8^m/L_8^{m-2}))$  are given by  $\psi^k = k - 2 [k/2]$ , the proof for this case is similar to the corresponding proof of [10].

Thus the proof for the case  $j \equiv 2 \pmod{4}$  and  $n \equiv 0 \pmod{2}$  is completed.

**5.2. Proof for the case  $n \equiv 1 \pmod{2}$ .** Consider the following commutative diagram, in which the row is exact.

$$\begin{array}{ccccccc} 0 & \rightarrow & V_{m,n+1}^j & \xrightarrow{f_1} & \tilde{K}(S^j(L_8^m/L_8^n)) & \xrightarrow{f_2} & \tilde{K}(S^{n+j+1}) \rightarrow 0 \\ & & \parallel & & \downarrow f_3 & & \\ & & V_{m,n+1}^j & \hookrightarrow & \tilde{K}(S^j L_8^m) & & \end{array}$$

By Lemma 3.10, we can choose an element  $x \in \tilde{K}(S^j(L_8^m/L_8^n))$  such that  $f_2(x)$  generates the group  $\tilde{K}(S^{n+j+1}) \cong \mathbb{Z}$  and

$$f_3(x) = \begin{cases} I^{j/2}(\varphi(X_5(n))) & (n \equiv 1 \pmod{8}) \\ I^{j/2}(\varphi(X_6(n) + X_1(n))) & (n \equiv 3 \pmod{8}) \\ I^{j/2}(\varphi(X_7(n))) & (n \equiv 5 \pmod{8}) \\ I^{j/2}(\varphi(X_4(n))) & (n \equiv 7 \pmod{8}) \end{cases}$$

Inspect the following commutative diagram

$$\begin{array}{ccccccc} 0 & & 0 & & \widetilde{KO}(S^{n+j+2}) & & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ VO_{m,n+1}^{j+2} & \xrightarrow{I^{-1} \circ c} & V_{m,n+1}^j & \xrightarrow{r_1} & \widetilde{KO}(S^j(L_8^m/L_8^{n+1})) & \xrightarrow{\partial} & \widetilde{KO}(S^{j+1}(L_8^m/L_8^{n+1})) \\ \downarrow & & f_1 \downarrow & & g_1 \downarrow & & h_1 \downarrow \\ \widetilde{KO}(S^{j+2}(L_8^m/L_8^n)) & \rightarrow & \tilde{K}(S^j(L_8^m/L_8^n)) & \xrightarrow{r_2} & \widetilde{KO}(S^j(L_8^m/L_8^n)) & \xrightarrow{\partial} & \widetilde{KO}(S^{j+1}(L_8^m/L_8^n)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \widetilde{KO}(S^{n+j+3}) & \xrightarrow{I^{-1} \circ c} & \tilde{K}(S^{n+j+1}) & \xrightarrow{r} & \widetilde{KO}(S^{n+j+1}) & \xrightarrow{\partial} & \widetilde{KO}(S^{n+j+2}) \rightarrow 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & \widetilde{KO}(S^{j-1}(L_8^m/L_8^{n+1})) & & \end{array}$$

of exact sequences. By Lemma 3.9, Proposition 3.13 and Lemma 3.15, we obtain

$$(5.11) \quad \text{Ker } r_2 = \begin{cases} \langle \{f_1(S_i) \mid i = 1, 2, 4 \text{ or } 6\} \rangle & (n \equiv 1 \pmod{4}) \\ \langle f_1(\text{Ker } r_1) \cup \{2x\} \rangle & (n+j \equiv 1 \pmod{8}) \\ \langle f_1(\text{Ker } r_1) \cup \{x\} \rangle & (n+j = 5 \pmod{8}) \end{cases}$$

Suppose  $m \geq n+3$ . Then we have

$$\text{Coker } g_2 \cong \widetilde{KO}(S^{n+j+2}) \cong \begin{cases} \mathbb{Z}/2 & (n+j \equiv 7 \pmod{8}) \\ 0 & (\text{otherwise}), \end{cases}$$

and hence

$$r(\tilde{K}(S^{n+j+1})) = g_2(\widetilde{KO}(S^j(L_8^n/L_8^{n+1}))) = \begin{cases} 2\widetilde{KO}(S^{n+j+1}) & (n+j \equiv 7 \pmod{8}) \\ \widetilde{KO}(S^{n+j+1}) & (\text{otherwise}). \end{cases}$$

Since  $h_1$  is a monomorphism, we have  $\text{Ker } g_1 \subset r_1(V_{m,n+1}^j)$ . Thus we obtain a split short exact sequence

$$0 \rightarrow \widetilde{KO}(S^j(L_8^n/L_8^{n+1}))/\text{Ker } g_1 \xrightarrow{\tilde{g}_1} \widetilde{KO}(S^j(L_8^n/L_8^{n+1})) \xrightarrow{\tilde{g}_2} H \rightarrow 0,$$

where  $\text{Ker } g_1 = \langle \{r_1(S_4), r_1(S_6)\} \rangle$  and

$$H = \begin{cases} 2\widetilde{KO}(S^{n+j+1}) & (n+j \equiv 7 \pmod{8}) \\ \widetilde{KO}(S^{n+j+1}) & (\text{otherwise}). \end{cases}$$

Applying the method used in the proof of Lemma 4.11 to  $x$ , we obtain the following result by Lemma 3.9 and (5.11).

(5.12) (1) If  $n \equiv 3 \pmod{4}$ , then the Adams operations are given by

$$\psi^k(r_2(x)) = (k - 2[k/2])r_2(x).$$

(2) Suppose  $n \equiv 1 \pmod{4}$  and  $k \equiv \varepsilon \pmod{8}$ , where  $\varepsilon$  is an odd integer with  $-3 \leq \varepsilon \leq 3$ . Then the Adams operation  $\psi^k$  is given by

$$\psi^k(r_2(x)) = k''r_2(x) + ((\varepsilon k^{j/2} - k'')/8)r_2(f_1(8f_3(x))) + k^{j/2}w,$$

where  $u = (n+j+1)/2$ ,  $w = -(\varepsilon/3)h_4(\varepsilon+5, \varepsilon+1)W$  and

$$W = \begin{cases} r_2(f_1(S_3 + S_5)) & (n \equiv 1 \pmod{8}) \\ r_2(f_1(2S_7 - S_3)) & (n \equiv 5 \pmod{8}) \end{cases}.$$

Suppose  $n \equiv 3 \pmod{4}$ . Then using (5.11) and (5.12) (1), we see that the short exact sequence

$$0 \rightarrow \widetilde{KO}(S^j(L_8^n/L_8^{n+1})) \rightarrow \widetilde{KO}(S^j(L_8^n/L_8^{n+1})) \rightarrow \widetilde{KO}(S^{n+j+1}) \rightarrow 0$$

of  $\psi$ -maps splits. This implies that

$$\widetilde{KO}(S^j(L_8^n/L_8^{n+1})) \cong \widetilde{KO}(S^j(L_8^n/L_8^{n+1})) \oplus \widetilde{KO}(S^{n+j+1})$$

and

$$\tilde{J}(S^j(L_8^n/L_8^{n+1})) \cong \tilde{J}(S^j(L_8^n/L_8^{n+1})) \oplus \tilde{J}(S^{n+j+1}).$$

Thus, results of the case  $j \equiv 2 \pmod{4}$ ,  $n \equiv 3 \pmod{4}$  and  $m \geq n+3$  follow from those of the case  $j \equiv 2 \pmod{4}$  and  $n \equiv 0 \pmod{4}$ .

Now, we turn to the case  $n \equiv 1 \pmod{4}$ . If  $m \geq n+5$ , then we have

$$\text{Im } r_2 \cong \mathbb{Z} \oplus \mathbb{Z}/2^{h_3(m+j, n+j)} \oplus (\bigoplus_{i=0}^1 \mathbb{Z}/2^{h_4(m+j-4i, n+j-4i+5)}) .$$

If  $n+5 > m \geq n+3$ , then we have  $\text{Im } r_2 \cong \mathbb{Z}$ . Under the assumption of (5.12) (2), we have

$$\begin{aligned} ((\varepsilon k^{j/2} - k^u)/8) r_2(f_1(8f_3(x))) + k^{j/2} w \\ \equiv ((n+1)/2) N(u/2^{v_2(u)}) ((k^u - 1)/2^{v_2(u)+2}) W \pmod{g_1(U_{m,n+1}^j)}, \end{aligned}$$

where  $u = (n+j+1)/2$ . Thus we have  $\tilde{J}(S^j(L_8^m/L_8^n)) \cong \widetilde{KO}(S^j(L_8^m/L_8^n))/U_1$ , where  $U_1$  is the subgroup of  $\widetilde{KO}(S^j(L_8^m/L_8^n))$  generated by

$$g_1(U_{m,n+1}^j) \cup \{m((n+j+1)/2) r_2(x) + MW\}$$

with  $M \equiv 1 \pmod{2}$ . By (5.8), we have

$$(5.13) \quad U_1 = \begin{cases} \langle \{4v_1, 2v_2, 4v_3, M_0 v_0 + M(v_1 + v_2)\} \rangle & (n \equiv 1 \pmod{8}) \\ \langle \{8v_2, 2v_2 + v_1, 4v_3, M_0 v_0 + 2M(v_2 + v_3)\} \rangle & (n \equiv 5 \pmod{8}), \end{cases}$$

where  $M_0 = m((n+j+1)/2)$ ,  $v_2(M) = 0$ ,  $v_i = g_1(r_1(S_{2i+1}))$  ( $1 \leq i \leq 3$ ) and  $v_0 = r_2(x)$ .

This implies that we have

$$J''(\text{Im } r_2) \cong F(v)/\langle \{B_i \mid 0 \leq i \leq 3\} \rangle,$$

where  $F(v)$  is the free abelian group generated by  $\{v_i \mid 0 \leq i \leq 3\}$ ,

$$\begin{aligned} B_0 &= \begin{cases} 4M_0 v_0 & (m \geq n+9+12h_4(n, n-4)) \\ 2M_0 v_0 & (n+9+12h_4(n, n-4) > m \geq n+5+4h_4(n, n-4)) \\ M_0 v_0 & (n+5+4h_4(n, n-4) > m), \end{cases} \\ B_1 &= \begin{cases} v_1 + 2v_2 & (n \equiv 5 \pmod{8}) \\ 2M_0 v_0 + 2v_1 & (n \equiv 1 \pmod{8}) \text{ and } m \geq n+13 \\ v_1 - 2v_3 & (n \equiv 1 \pmod{8}) \text{ and } n+13 > m, \end{cases} \\ B_2 &= \begin{cases} 2M_0 v_0 + 4v_2 & (n \equiv 5 \pmod{8}) \text{ and } m \geq n+17 \\ MM_0 v_0 + 2v_2 & (n \equiv 5 \pmod{8}) \text{ and } n+17 > m \geq n+5 \\ M_0 v_0 + Mv_1 + v_2 & (n \equiv 1 \pmod{8}) \text{ and } m \geq n+9 \\ v_2 + 2h_4(n, n-4) v_3 & (n+5+4h_4(n, n-4) > m) \end{cases} \end{aligned}$$

and

$$B_3 = \begin{cases} M_0 v_0 + 2Mv_0 + 2v_3 & (n \equiv 5 \pmod{8}) \text{ and } m \geq n+9 \\ 4v_3 & (n \equiv 1 \pmod{8}) \text{ and } m \geq n+5 \\ v_3 + 2h_4(n+4, n) v_2 & (n+5+4h_4(n+4, n) > m). \end{cases}$$

By the proof for the case  $n \equiv 2 \pmod{4}$ , we obtain

(5.14) Suppose  $j-1 \equiv n \equiv 1 \pmod{4}$  and  $m \geq n+3$ .

(1) If  $m+j \equiv 0 \pmod{8}$ , then we have

$$\widetilde{KO}(S^j(L_8^n/L_8^n)) \cong \begin{cases} (\text{Im } r_2) \oplus \mathbb{Z}/2 & (m \geq n+17) \\ \mathbb{Z} \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 & (m = n+13) \\ \mathbb{Z} \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/4 & (m = n+9) \\ \mathbb{Z} \oplus \mathbb{Z}/8 & (m = n+5) \end{cases}$$

and

$$\widetilde{J}(S^j(L_8^n/L_8^n)) \cong \begin{cases} J''(\text{Im } r_2) \oplus \mathbb{Z}/2 & (m \geq n+17) \\ \mathbb{Z}/2M_0 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 & (h_4(n+4, n) = m-n-13 = 0) \\ \mathbb{Z}/4M_0 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & (h_4(n, n-4) = m-n-13 = 0) \\ \mathbb{Z}/2M_0 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 & (h_4(n+4, n) = m-n-9 = 0) \\ \mathbb{Z}/4M_0 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 & (h_4(n, n-4) = m-n-9 = 0) \\ \mathbb{Z}/M_0 \oplus \mathbb{Z}/8 & (h_4(n+4, n) = m-n-5 = 0) \\ \mathbb{Z}/2M_0 \oplus \mathbb{Z}/4 & (h_4(n, n-4) = m-n-5 = 0), \end{cases}$$

where  $M_0 = \min((n+j+1)/2)$ .

(2) If  $m+j \equiv 1 \pmod{8}$ , then we have

$$\widetilde{KO}(S^j(L_8^n/L_8^n)) \cong \widetilde{KO}(S^j(L_8^{n-1}/L_8^n)) \oplus \mathbb{Z}/2$$

and

$$\widetilde{J}(S^j(L_8^n/L_8^n)) \cong \widetilde{J}(S^j(L_8^{n-1}/L_8^n)) \oplus \mathbb{Z}/2.$$

(3) If  $m+j \equiv 2 \pmod{8}$ , then we have

$$\widetilde{KO}(S^j(L_8^n/L_8^n)) \cong (\text{Im } r_2) \oplus \mathbb{Z}/2$$

and

$$\widetilde{J}(S^j(L_8^n/L_8^n)) \cong J''(\text{Im } r_2) \oplus \mathbb{Z}/2.$$

Noting the fact that we have  $S^j(L_8^{n+1}/L_8^n) \approx S^{n+j+1}$  and

$$S^j(L_8^{n+2}/L_8^n) \cong \begin{cases} S^{n+j+2} \vee S^{n+j+1} & (n \equiv 1 \pmod{2}) \\ S^{n+j} L_8^2 & (n \equiv 0 \pmod{2}), \end{cases}$$

we obtain the result for the case  $n \equiv 1 \pmod{2}$ .

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