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J-GROUPS OF SUSPENSIONS OF STUNTED LENS SPACES MOD 8

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1. Introduction

Let $L^n(q) = S^{2n+1}/\mathbf{Z}_q$ be the $(2n+1)$ -dimensional standard lens space mod q . As defined in [7], we set

$$(1.1) \quad \begin{aligned} L_q^{2n+1} &= L^n(q), \\ L_q^{2n} &= \{[z_0, \dots, z_n] \in L^n(q) \mid z_n \text{ is real and } z_n \geq 0\}. \end{aligned}$$

In the previous paper [10], we determined the J -groups $\tilde{J}(S^j(L_q^m/L_q^n))$ of the suspensions of the stunted lens spaces L_q^m/L_q^n for $q=4$ and for $j \equiv 1 \pmod{2}$. The purpose of this paper is to determine the KO - and J -groups of suspensions of stunted lens spaces mod 8.

This paper is organized as follows. In section 2 we state the main theorems: the structures of $\tilde{KO}(S^j(L_8^m/L_8^n))$ and $\tilde{J}(S^j(L_8^m/L_8^n))$ for $j \equiv 0 \pmod{2}$ are given in Theorems 1 and 2 respectively. In section 3 we prepare some lemmas and recall known results in [8], [9] and [11]. By virtue of the results in [8], the proofs of Theorems 1 and 2 for the case $j \equiv 0 \pmod{4}$ are given in section 4. Applying the method used in the corresponding parts of [10], we prove Theorems 1 and 2 for the case $j \equiv 2 \pmod{4}$ in the final section.

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2. Statement of results

We prepare functions $h_1, h_2, h_3, h_4, a_1, a_2, a_3, a_4, a_5, a_6$ and a_7 defined by

$$(2.1) \quad \begin{cases} h_1(n) = [n/4] + [(n+7)/8] + [(n+4)/8] \\ h_2(n) = [n/4] + [(n+7)/8] + [n/8] + 1 \\ h_3(m, n) = [m/4] - [n/4] \\ h_4(m, n) = [m/8] - [n/8]. \end{cases}$$

$$(2.2) \quad \begin{cases} a_1(m, n) = h_1(m) - [(n+1)/4] - [(n+1)/8] - [(n+6)/8] + 1 \\ a_2(m, n) = h_3(m, n+1) \\ a_3(m, n) = h_4(m-2, n+5) \\ a_4(m, n) = h_4(m, n+7) \\ a_5(m, n) = a_1(m, n) - [(m+4)/8] + [m/8] \\ a_6(m, n) = [(m+4)/8] + [(m-2)/8] - [(n+1)/4] \\ a_7(m, n) = 2[(m+4)/8] - [(n+5)/4]. \end{cases}$$

Let \mathbf{Z}/k denote the cyclic group $\mathbf{Z}/k\mathbf{Z}$ of order k . For an integer n , $G(n)$ denotes the group defined by

$$(2.3) \quad G(n) = \begin{cases} \mathbf{Z}/2 \oplus \mathbf{Z}/2 & (n \equiv 1 \pmod{8}) \\ \mathbf{Z}/2 & (n \equiv 0 \text{ or } 2 \pmod{8}) \\ 0 & (\text{otherwise}). \end{cases}$$

Theorem 1. *Let j , m and n be non-negative integers with $j \equiv 0 \pmod{2}$ and $m > n$.*

(1) *Suppose $j \equiv 0 \pmod{4}$.*

i) *If $n \not\equiv 3 \pmod{4}$ and $m \geq 4[(n+j+15)/8] + 2[(n-j)/4]$, then we have*

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \bigoplus_{i=1}^4 \mathbf{Z}/2^{a_i(m+j, n+j)}.$$

ii) *If $n \equiv 3 \pmod{4}$ and $4[(n+j+15)/8] + 2[(n-j)/4] > m > n$, then we have*

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/2^{a_1(m+j, n+j)} & (m \geq 4[n/4] + 4) \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & (n+j \equiv 0 \pmod{8} \text{ and } n+4 > m \geq n+2) \\ \mathbf{Z}/2 & (h_4(n+j+6, n+j) = [m/2] - [(n+1)/2] = 0) \\ 0 & (\text{otherwise}). \end{cases}$$

iii) *If $n \equiv 3 \pmod{4}$, then we have*

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z} \oplus \widetilde{KO}(S^j(L_8^m/L_8^{n+1})) & (m \geq n+2) \\ \mathbf{Z} & (m = n+1). \end{cases}$$

(2) *Suppose $j \equiv 2 \pmod{4}$.*

i) *If $m \geq 8[(n+j+15)/8] - j + 2$, then we have*

$$\begin{aligned} \widetilde{KO}(S^j(L_8^m/L_8^n)) &\cong \mathbf{Z}/2^{h_3(m+j, n+j-3)} \oplus (\bigoplus_{i=0}^{\frac{1}{2}} \mathbf{Z}/2^{h_4(m+j-4i, n+j-4i+5)}) \\ &\quad \oplus G(m+j) \oplus H(n+j), \end{aligned}$$

where $G(m)$ is the group defined by (2.3) and $H(n)$ is the group defined by

$$H(n) = \begin{cases} \mathbf{Z} & (n \equiv 3 \pmod{4}) \\ G(n) & (\text{otherwise}). \end{cases}$$

ii) If $8[(n+j+15)/8]-j+2 > m \geq 6[(n+j+7)/8]+2[(n+j+1)/8]-j+4$, then we have

$$\begin{aligned} \widetilde{KO}(S^j(L_8^m/L_8^n)) &\cong \mathbf{Z}/2^{h_3(m+j, n+j-3)} \oplus \mathbf{Z}/2^{h_4(m+j-4, n+j+1)} \\ &\oplus G_1(m+j) \oplus H_1(n+j), \end{aligned}$$

where $G_1(m)$ is the group defined by

$$(2.4) \quad G_1(m) = \begin{cases} \mathbf{Z}/4 \oplus \mathbf{Z}/2 & (m \equiv 1 \pmod{8}) \\ \mathbf{Z}/4 & (m \equiv 0 \pmod{8}) \\ G(m) & (\text{otherwise}) \end{cases}$$

and $H_1(n)$ is the group defined by

$$H_1(n) = \begin{cases} \mathbf{Z} & (n \equiv 3 \pmod{4}) \\ G_1(2-n) & (\text{otherwise}). \end{cases}$$

iii) If $6[(n+j+7)/8]+2[(n+j+1)/8]-j+4 > m \geq n+3$, then we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong G_2(m+j) \oplus H_2(n+j),$$

where $G_2(m)$ is the group defined by

$$G_2(m) = \begin{cases} \mathbf{Z}/8 \oplus \mathbf{Z}/2 & (m \equiv 1 \pmod{8}) \\ \mathbf{Z}/8 & (m \equiv 0 \pmod{8}) \\ G(m) & (\text{otherwise}), \end{cases}$$

and $H_2(n)$ is the group defined by

$$H_2(n) = \begin{cases} \mathbf{Z} & (n \equiv 3 \pmod{4}) \\ G_2(2-n) & (\text{otherwise}) \end{cases}$$

iv) If $n+3 > m > n$, then we have $\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \widetilde{KO}(L_8^{m+j}/L_8^{n+j})$.

REMARK. (1) Combining this theorem with [13, Theorem 2], we obtain the complete results for the groups $\widetilde{KO}(S^j(L_8^m/L_8^n))$.

(2) The partial results for the case $n=0$ of this theorem have been obtained in [8].

Let $\nu_p(s)$ denote the exponent of the prime p in the prime power decomposition of s , and $\mathfrak{m}(s)$ the function defined on positive integers as follows (cf. [3]):

$$\nu_p(\mathfrak{m}(s)) = \begin{cases} 0 & (p \neq 2 \text{ and } s \not\equiv 0 \pmod{p-1}) \\ 1 + \nu_p(s) & (p \neq 2 \text{ and } s \equiv 0 \pmod{p-1}) \\ 1 & (p = 2 \text{ and } s \not\equiv 0 \pmod{2}) \\ 2 + \nu_2(s) & (p = 2 \text{ and } s \equiv 0 \pmod{2}). \end{cases}$$

In order to state the next theorem, we set

$$(2.5) \quad \begin{cases} b_1(j, m, n) = \begin{cases} \min \{ \nu + h_4(n+9, n-3), a_1(m+j, n+j) \} & (j \equiv 4 \pmod{8}) \\ \min \{ \nu + h_4(n+9, n-2), a_5(m, n) \} & (\text{otherwise}) \end{cases} \\ b_2(j, m, n) = \begin{cases} \min \{ \nu + h_4(n+3, n-7), a_7(m, n) \} & (j \equiv 4 \pmod{8}) \\ \min \{ \nu + h_4(n+5, n-7), a_6(m, n) \} & (\text{otherwise}) \end{cases} \\ b_3(j, m, n) = \begin{cases} \min \{ \nu + 1, a_3(m+j, n+j) \} & (j \equiv 4 \pmod{8}) \\ \min \{ \nu + 1, a_4(m, n) \} & (\text{otherwise}) \end{cases} \end{cases}$$

where ν is the integer defined by

$$\nu = \begin{cases} \nu_2(j) & (j \neq 0) \\ m & (j = 0). \end{cases}$$

Main result is the following theorem.

Theorem 2. *Let j, m and n be non-negative integers with $j \equiv 0 \pmod{2}$ and $m > n$.*

(1) *Suppose $j \equiv 0 \pmod{4}$ and $n \not\equiv 3 \pmod{4}$.*

i) *If $m \geq 2 \lfloor n/4 \rfloor + 4 \lfloor n/8 \rfloor + 6 + h_4(j, j-4) (2h_4(n-2, n+4) - 4a_4(n, n))$, then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} (\oplus_{i=1}^3 \mathbf{Z}/2^{b_i(j, m, n)}) \oplus \mathbf{Z}/2 & (n \equiv 2 + 2h_4(j+4, j) \pmod{8}) \\ \oplus_{i=1}^3 \mathbf{Z}/2^{b_i(j, m, n)} & (\text{otherwise}). \end{cases}$$

ii) *If $2 \lfloor n/4 \rfloor + 4 \lfloor n/8 \rfloor + 6 + h_4(j, j-4) (2h_4(n-2, n+4) - 4a_4(n, n)) > m > n$, then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/2^{b_1(j, m, n)} \oplus \mathbf{Z}/4 & (h_4(n+j+5, n+j-2) = h_3(m, n+6) = 0) \\ \mathbf{Z}/8 & (h_4(n+j+6, n+j-1) = h_3(m, n+3) = 0) \\ \tilde{KO}(S^j(L_8^m/L_8^n)) & (\text{otherwise}). \end{cases}$$

(2) *Suppose $j \equiv n+1 \equiv 0 \pmod{4}$. Set $M = m((n+j+1)/2)$ and $b_i = b_i(j, m, n)$ ($1 \leq i \leq 3$).*

i) *If $m \geq n + 2h_4(j+4, j) h_4(n+1, n) + 5$, then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/2^{c_1}M \oplus \mathbf{Z}/2^{c_2+i_1} \oplus \mathbf{Z}/2^{b_3} \oplus \mathbf{Z}/2 & (j(n+1) \equiv 4 \pmod{8}) \\ \mathbf{Z}/2^{c_1}M \oplus \mathbf{Z}/2^{c_2+i_1} \oplus \mathbf{Z}/2^{c_3+i_2} \oplus \mathbf{Z}/2^{i_3} & (\text{otherwise}), \end{cases}$$

where i_1, i_2, i_3, c_1, c_2 and c_3 are integers defined by

$$\begin{cases} i_1 = \begin{cases} \min \{ b_1, \nu_2(n+1) - 1 \} & (n+j \equiv 7 \pmod{8}) \\ \min \{ b_1, \nu_2(n+1) \} & (n+j \equiv 3 \pmod{8}) \end{cases} \\ i_2 = \begin{cases} \min \{ b_2, \nu_2(n+1) - 2 \} & (n+j \equiv 7 \pmod{8}) \\ \min \{ b_2, \nu_2(n+1) - 1 \} & (n+j \equiv 3 \pmod{8}) \end{cases} \end{cases}$$

$$(2.6) \quad \begin{cases} i_3 = \begin{cases} \min \{b_3, v_2(n+1)-2\} & (j \equiv n-3 \equiv 4 \pmod{8}) \\ b_3 & (\text{otherwise}) \end{cases} \\ c_1 = \max \{b_k - i_k \mid 1 \leq k \leq 3\} \\ c_3 = \min \{b_k - i_k \mid 1 \leq k \leq 3\} \\ c_2 = (\sum_{k=1}^3 b_k - i_k) - c_1 - c_3. \end{cases}$$

ii) If $n+2h_4(j+4, j) h_4(n+1, n)+5 > m > n$, then we have

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/2^{b-i} M \oplus \mathbf{Z}/2^i & (m \geq n+5) \\ \mathbf{Z}/M \oplus \tilde{KO}(S^j(L_8^m/L_8^{n-1})) & (n+5 > m > n+1) \\ \mathbf{Z}/M & (m = n+1), \end{cases}$$

where $b=b_1$ and $i=\min \{b, v_2(n+1)\}$.

(3) Suppose $j \equiv 2 \pmod{4}$ and $n \equiv 1 \pmod{4}$.

i) If $m \geq 8[(n+17)/8] + h_4(j+4, j) (6h_4(n-2, n+1) + 2h_4(n-6, n+9))$, then we have

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/2^b \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2 \oplus G(n+j) \oplus G(m+j) \\ \quad (h_4(n+5-2h_4(j, j-4), n-2) = 0) \\ \mathbf{Z}/2^b \oplus \mathbf{Z}/8 \oplus G(n+j) \oplus G(m+j) \quad (\text{otherwise}), \end{cases}$$

where $b=b_3(j, m+2, n)$ and $G(m)$ is the group defined by (2.3).

ii) If $8[(n+17)/8] + h_4(j+4, j) (6h_4(n-2, n+1) + 2h_4(n-6, n+9)) > m \geq 8[(n+2)/8] + 10$, then we have

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/4 \oplus \mathbf{Z}/2 \oplus G_1(m+j) & (h_4(n+5-2h_4(j, j-4), n-2) = 0) \\ \mathbf{Z}/8 \oplus G_1(h_4(j+4, j) (2n-1) - n) \oplus G_1(m+j) & (\text{otherwise}), \end{cases}$$

where $G_1(m)$ is the group defined by (2.4).

iii) If $8[(n+2)/8] + 10 > m > n$, then we have

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \tilde{KO}(S^j(L_8^m/L_8^n)).$$

(4) Suppose $j \equiv n+1 \equiv 2 \pmod{4}$. Set $M = m((n+j+1)/2)$.

i) If $m \geq 8[n/8] + 2h_4(j, j-4) + 14$, then we have

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/2^b M \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2 \oplus G(m+j) & (n \equiv 1 \pmod{8}) \\ \mathbf{Z}/4M \oplus \mathbf{Z}/2^b \oplus \mathbf{Z}/2 \oplus G(m+j) & (n \equiv 5 \pmod{8}), \end{cases}$$

where $b=b_3(j, m+2, n)$ and $G(m)$ is the group defined by (2.3).

ii) If $8[n/8] + 14 + 2h_4(j, j-4) > m \geq 4h_4(j+4, j) h_4(n-4, n) + 8[n/8] + 10$, then we have

$$\mathcal{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/2M \oplus \mathbf{Z}/4 \oplus G_1(m+j) & (j \equiv n+1 \equiv 2 \pmod{8}) \\ \mathbf{Z}/2^{c+1}M \oplus \mathbf{Z}/2 \oplus G_1(m+j) & (j \equiv n-3 \equiv 2 \pmod{8}) \\ \mathbf{Z}/2^c M \oplus \mathbf{Z}/4 \oplus G(m+j) & (j \equiv n+5 \equiv 6 \pmod{8}) \\ \mathbf{Z}/2M \oplus \mathbf{Z}/2^d \oplus G_1(m+j) & (j \equiv n+1 \equiv 6 \pmod{8}), \end{cases}$$

where $c = [(m-n-1)/8]$, $d = h_4(m-4, n)$ and $G_1(m)$ is the group defined by (2.4).

iii) If $4h_4(j+4, j)h_4(n-4, n) + 8[n/8] + 10 > m > n$, then we have

$$\mathcal{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/M \oplus \widetilde{KO}(S^j(L_8^m/L_8^{n+1})) & (m \geq 4[n/4] + 6) \\ \mathbf{Z}/M \oplus \mathbf{Z}/2 & (n+j \equiv 7 \pmod{8} \text{ and } n+4 > m \geq n+2) \\ \mathbf{Z}/M & (\text{otherwise}). \end{cases}$$

REMARK. (1) Combining this theorem with [10, Theorem 1], we obtain the complete results for the groups $\mathcal{J}(S^j(L_8^m/L_8^n))$.

(2) The partial results for the case $j=n=0$ of this theorem have been obtained in [9].

3. Preliminaries

In this section we prepare some lemmas and recall known results which are needed to prove Theorems 1 and 2.

Lemma 3.1. *Let j, k, l and s be integers with $j > 0$, $v = v_2(j) \geq s \geq 1$, $l \geq 2$ and $k \equiv \pm 1 \pmod{2^l}$. Then we have*

$$(1) \quad k^j - 1 \equiv (k^{2^v} - 1)(j/2^v) \pmod{2^{2v+2l}}.$$

$$(2) \quad k^j \equiv 1 \pmod{2^{v+l}}.$$

$$(3) \quad k^j - 1 \equiv (k^{2^s} - 1)(j/2^s) \pmod{2^{v+s+2l-1}}.$$

Proof. Since $k^2 \equiv 1 \pmod{2^{l+1}}$, by making use of the method used in the proof of [10, Lemma 3.1] we can show that

$$k^j - 1 \equiv (k^2 - 1)(j/2) \pmod{2^{v+2l}}.$$

This implies (2). In particular, we have

$$k^{2^v} \equiv 1 \pmod{2^{v+l}}.$$

Then, the rest of the proof is similar to that of [10, Lemma 3.1]. q.e.d.

Considering the $\mathbf{Z}/8$ -action on $S^{2n+1} \times \mathbf{C}$ given by

$$\exp(2\pi\sqrt{-1}/8)(z, u) = (z \cdot \exp(2\pi\sqrt{-1}/8), u \cdot \exp(2\pi\sqrt{-1}/8))$$

for $(z, u) \in S^{2n+1} \times \mathbf{C}$, we have a complex line bundle

$$\eta : (S^{2n+1} \times C)/(Z/8) \rightarrow L_8^{2n+1}.$$

Then we have the following elements

$$(3.2) \quad \sigma(i) = \eta^{2^i} - 1 \in \tilde{K}(L_8^{2n+1}) \subset K(L_8^{2n+1}) \quad (0 \leq i \leq 2).$$

We denote the restriction of $\sigma(i)$ in $\tilde{K}(L_8^{2n})$ by the same symbol, and $\sigma(0)$ by σ . The following proposition is well known.

Proposition 3.3 (Mahammed [11]). *The ring $K(L_8^n)$ is isomorphic to the truncated polynomial ring $Z[\sigma]/(\sigma^{[m/2]+1}, (\sigma+1)^8-1)$, where $(\sigma^{[m/2]+1}, (\sigma+1)^8-1)$ means the ideal of $Z[\sigma]$ generated by $\sigma^{[m/2]+1}$ and $(\sigma+1)^8-1$.*

In order to state the next lemma, we set

$$(3.4) \quad (1) \quad \begin{cases} \sigma_{2^i} = \sigma(i) & (0 \leq i \leq 2) \\ \sigma_6 = \sigma_4 \sigma_2 \\ \sigma_{2^{i+1}} = \sigma_{2^i} \sigma_1 & (1 \leq i \leq 3). \end{cases}$$

$$(2) \quad \begin{cases} b(n) = ((-68-48\sqrt{2})^n + (-68+48\sqrt{2})^n)/2 \\ c(n) = ((-68-48\sqrt{2})^n - (-68+48\sqrt{2})^n)/2\sqrt{2}. \end{cases}$$

Then we have

$$(3.5) \quad (1) \quad \begin{cases} b(0) = 1, b(n+1) = -68b(n) - 96c(n), \\ c(0) = 0, c(n+1) = -48b(n) - 68c(n). \end{cases}$$

$$(2) \quad \begin{cases} b(n) \equiv (-4)^n \pmod{2^{2n+4}} \\ c(n) \equiv 0 \pmod{2^{2n+2}}. \end{cases}$$

The following lemma is obtained by Proposition 3.3 and (3.5) (1).

Lemma 3.6. *Let u be a positive integer. Then, in $K(L_8^m)$,*

$$\sigma^u = \sum_{i=1}^7 a_{u,i} \sigma_i,$$

where $a_{u,i}$ ($1 \leq i \leq 7$) are integers defined by $a_{u,1} = (-2)^{u-1}$,

$$a_{u,2} = (1/5)(-4)^{[u/4]+1} + (2/5)(-4)^{[(u-1)/4]} - (1/5)(-4)^{[(u+2)/4]} + (3/5)(-4)^{[(u+1)/4]},$$

$$a_{u,3} = -(-2)^{u-2} - (1/2)a_{u+1,2},$$

$$a_{u,4} = -(1/2)h_4(u, u-1)b([u/8]) + h_4(u-1, u-2)c([u/8]) - h_4(u-2, u-3)c([u/8]) + h_4(u+4, u+3)(b([u/8]) + 2c([u/8])) - h_4(u+3, u+2)(4b([u/8]) + 6c([u/8])) + h_4(u+2, u+1)(10b([u/8]) + 14c([u/8])) - h_4(u+1, u)(20b([u/8]) + 28c([u/8])),$$

$$a_{u,5} = -(-2)^{u-2} - a_{u+1,4} - a_{u+2,4} - (1/2)a_{u+3,4},$$

$$a_{u,6} = (1/2)a_{u,2} - a_{u+1,4} - (1/2)a_{u+2,4}$$

and $a_{u,7} = (-2)^{u-3} - (1/4)a_{u+1,2} - (1/2)a_{u+1,4}$.

Proof. By making use of the relation $(\sigma+1)^8=1$, we obtain equalities $a_{u+1,1} = -2a_{u,1}$, $a_{u+1,2} = a_{u,1} - 2a_{u,3}$, $a_{u+1,3} = a_{u,2} - 2a_{u,3}$, $a_{u+1,4} = a_{u,3} - 2a_{u,7}$, $a_{u+1,5} = a_{u,4} - 2a_{u,5}$, $a_{u+1,6} = a_{u,5} - 2a_{u,7}$ and $a_{u+1,7} = a_{u,6} - 2a_{u,7}$, where $a_{1,1} = 1$ and $a_{1,i} = 0$ ($2 \leq i \leq 7$). Thus the lemma is proved by the induction with respect to u . q.e.d.

In order to state the next proposition, we set

(3.7) (1) Let $F(x)$ denote the free abelian group generated by $\{x_i | 1 \leq i \leq 7\}$. Then X_i and $X_i(n)$ ($7 \geq i \geq 1, n \geq 0$) denote the elements of $F(x)$ defined by $X_1 = 4x_1 + 2x_3 + 2x_5 + x_7$, $X_2 = 2x_2 + x_6$, $X_3 = 2x_3 + x_7$, $X_6 = x_6 + x_7$, $X_i = x_i$ ($i = 4, 5$ or 7), $X_1(n) = 2^{\lfloor n/2 \rfloor} X_1$,

$$\begin{aligned} X_2(n) &= 2^{\lfloor n/4 \rfloor} X_2 - 2^{2\lfloor n/4 \rfloor} X_1, \\ X_3(n) &= 2^{\lfloor (n-2)/4 \rfloor} X_3 + 2^{2\lfloor n/4 \rfloor - 1} h_3(n, n-2) X_1, \\ X_4(n) &= 2^{\lfloor n/8 \rfloor} X_4 + 2^{2\lfloor n/8 \rfloor} h_4(n+4, n) X_2 + 2^{2\lfloor n/8 \rfloor + \lfloor n/4 \rfloor} X_1, \\ X_5(n) &= 2^{\lfloor (n-2)/8 \rfloor} X_5 + 2^{2\lfloor (n-2)/8 \rfloor} h_4(n+2, n-2) X_3 \\ &\quad - 2^{\lfloor (n+2)/4 \rfloor + 2\lfloor (n-2)/8 \rfloor} X_1, \\ X_6(n) &= 2^{\lfloor (n-4)/8 \rfloor} X_6 + 2^{\lfloor n/4 \rfloor - 1} h_4(n, n-4) X_2 \\ &\quad - 2^{\lfloor n/4 \rfloor + 2\lfloor (n-4)/8 \rfloor + 1} X_1 \end{aligned}$$

and

$$\begin{aligned} X_7(n) &= 2^{\lfloor (n-6)/8 \rfloor} X_7 - 2^{\lfloor (n-6)/4 \rfloor} h_4(n+6, n+2) (X_3 - 2X_2) \\ &\quad + 2^{\lfloor (n-2)/4 \rfloor + 2\lfloor (n+2)/8 \rfloor} X_1. \end{aligned}$$

(2) Let $\varphi: F(x) \rightarrow \tilde{K}(L_8^m)$ be the homomorphism defined by setting $\varphi(x_i) = \sigma_i$ ($1 \leq i \leq 7$).

Proposition 3.8 (Kobayashi and Sugawara [9]). *The homomorphism φ is an epimorphism, and the kernel of φ coincides with the subgroup of $F(x)$ generated by $\{X_i(m) | 1 \leq i \leq 7\}$.*

According to [1], we have the following lemma.

Lemma 3.9. *The Adams operations are given by the following formulae, where $s_i = \varphi(X_i)$ ($1 \leq i \leq 7$).*

$$\begin{aligned} (1) \quad \psi^k(s_1) &= \begin{cases} s_1 & (k \equiv 1 \pmod{2}) \\ 2s_2 & (k \equiv 2 \pmod{4}) \\ 4s_4 & (k \equiv 4 \pmod{8}) \\ 0 & (k \equiv 0 \pmod{8}). \end{cases} \\ (2) \quad \psi^k(s_2) &= \begin{cases} s_2 & (k \equiv 1 \pmod{2}) \\ 2s_4 & (k \equiv 2 \pmod{4}) \\ 0 & (k \equiv 0 \pmod{4}). \end{cases} \end{aligned}$$

$$\begin{aligned}
 (3) \quad \psi^k(s_3) &= \begin{cases} s_3 - 2h_3(k+1, k) (s_2 + s_3) & (k \equiv 1 \pmod{2}) \\ 2s_6 - 2s_7 & (k \equiv 2 \pmod{8}) \\ -4s_4 - 2s_6 + 2s_7 & (k \equiv 6 \pmod{8}) \\ 0 & (k \equiv 0 \pmod{4}). \end{cases} \\
 (4) \quad \psi^k(s_4) &= \begin{cases} s_4 & (k \equiv 1 \pmod{2}) \\ 0 & (k \equiv 0 \pmod{2}). \end{cases} \\
 (5) \quad \psi^k(s_5) &= \begin{cases} s_5 - 2h_4(k+4, k) (s_4 + s_5) & (k \equiv 1 \pmod{4}) \\ s_5 + s_6 - 2h_4(k+4, k) (s_4 + s_6 + s_5) & (k \equiv 3 \pmod{4}) \\ 0 & (k \equiv 0 \pmod{2}). \end{cases} \\
 (6) \quad \psi^k(s_6) &= \begin{cases} (1 - 2h_4(k+5, k+1)) s_6 & (k \equiv 1 \pmod{2}) \\ 0 & (k \equiv 0 \pmod{2}). \end{cases} \\
 (7) \quad \psi^k(s_7) &= \begin{cases} s_7 - 2h_4(k+4, k) s_6 & (k \equiv 1 \pmod{4}) \\ 2s_4 - s_7 + 2h_4(k+4, k) s_6 & (k \equiv 3 \pmod{4}) \\ 0 & (k \equiv 0 \pmod{2}). \end{cases}
 \end{aligned}$$

For each integer n with $0 \leq n < m$, we denote the inclusion map of L_8^n into L_8^m by i_n^m , and denote the kernel of the homomorphism

$$(i_n^m)!: \tilde{K}(L_8^m) \rightarrow \tilde{K}(L_8^n)$$

by V_n . Then by Proposition 3.8, Lemma 3.6 and (3.5) (2), we obtain the following lemma.

Lemma 3.10. (1) *The group V_n is the subgroup of $\tilde{K}(L_8^m)$ generated by $\{\varphi(X_i(n)) \mid 1 \leq i \leq 7\}$.*
 (2) *Let u be a positive integer with $2u < m$. Then we have*

$$\sigma^u \equiv \begin{cases} -\varphi(X_4(2u-2)) & (u \equiv 0 \pmod{4}) \\ -\varphi(X_5(2u-2)) & (u \equiv 1 \pmod{4}) \\ -\varphi(X_6(2u-2) + X_1(2u-2)) & (u \equiv 2 \pmod{4}) \\ \varphi(X_7(2u-2)) & (u \equiv 3 \pmod{4}) \end{cases}$$

modulo the subgroup V_{2u} .

Considering the $\mathbf{Z}/8$ -action on $S^{2n+1} \times \mathbf{R}$ given by

$$\exp(2\pi\sqrt{-1}/8) (z, v) = (z \cdot \exp(2\pi\sqrt{-1}/8), -v)$$

for $(z, v) \in S^{2n+1} \times \mathbf{R}$, we have a real line bundle

$$\nu: (S^{2n+1} \times \mathbf{R})/(\mathbf{Z}/8) \rightarrow L_8^{2n+1}.$$

We set $\kappa = \nu - 1 \in \tilde{K}\tilde{O}(L_8^{2n+1})$. It is easy to see that

$$(3.11) \quad \begin{cases} c(\kappa) = \sigma(2) \\ r(\sigma(2)) = 2\kappa, \end{cases}$$

where $c: KO \rightarrow K$ is the complexification and $r: K \rightarrow KO$ is the real restriction.

Let $I: \tilde{K}(X) \rightarrow \tilde{K}(S^2X)$ (resp. $I_R: \tilde{KO}(X) \rightarrow \tilde{KO}(S^2X)$) be the Bott periodicity isomorphisms for K (resp. KO)-theory. In order to state the next proposition, we set

(3.12) Let j be a non-negative integer with $j \equiv 0 \pmod{4}$.

$$(1) \quad \begin{cases} \tau_i = r(I^{j/2}(\sigma_i)) & (1 \leq i \leq 2) \\ \tau_3 = r(I^{j/2}(\sigma_5)) \\ \tau_4 = \begin{cases} I_K^{j/8}(\kappa) & (j \equiv 0 \pmod{8}) \\ r(I^{j/2}(\sigma_4)) & (j \equiv 4 \pmod{8}). \end{cases} \end{cases}$$

(2) Let $F(y)$ denote the free abelian group generated by y_1, y_2, y_3 and y_4 . Then $X_i^j, Y_i^j, X_i^j(n)$ and $Y_i^j(n)$ ($1 \leq i \leq 4, n \geq 0$) denote the elements of $F(y)$ defined by $Y_4^j = y_4$,

$$\begin{aligned} Y_1^j &= (-1)^{(j/4)} X_1^j = h_4(j+12, j) (2y_1 - y_2 + y_3) + y_4, \\ Y_2^j &= X_2^j = h_4(j+12, j) y_2 - y_4, \\ Y_3^j &= -y_3 - h_4(j, j-12) y_4, \\ X_i^j &= Y_{i+h_4(j+4, j)(7-2i)}^j \quad (3 \leq i \leq 4), \\ X_1^j(n) &= (-1)^{(j/4)} Y_1^j(n) = 2^{h_4(n+j) - h_4(j)} X_1^j, \\ X_2^j(n) &= Y_2^j(n) = 2^{[n/4] - h_4(j+4, j)} (X_2^j - (-2)^{[n/4]} h_4(n+j+7, n+j) Y_1^j), \\ X_3^j(n) &= 2^{h_4(n+j-2, j+4)} X_3^j + 2^{[n/4] - h_4(j+12, j)} h_4(n+j, n+j-2) X_2^j \\ &\quad - 2^{2h_4(n+j+4, j+4) + [n/4] - 1} h_4(n+j+4, n+j-2) X_1^j, \\ X_4^j(n) &= 2^{h_4(n+j, j+4)} X_4^j + 2^{2[n-(n-4)/8]} h_4(n+j+4, n+j) X_2^j \\ &\quad + 2^{2h_4(n+j, j+4) + [n/4]} h_4(n+j+7, n+j) X_1^j \end{aligned}$$

and $Y_i^j(n) = X_{i+h_4(j+4, j)(7-2i)}^j(n)$ ($3 \leq i \leq 4$).

(3) Let $\mu_j: F(y) \rightarrow \tilde{KO}(S^j L_8^n)$ be the homomorphism defined by setting $\mu_j(y_i) = \tau_i$ ($1 \leq i \leq 4$).

Proposition 3.13 (Kobayashi [8]). *Let j be a non-negative integer with $j \equiv 0 \pmod{4}$. Then the homomorphism μ_j is an epimorphism, and the kernel of μ_j coincides with the subgroup of $F(y)$ generated by $\{Y_i^j(m) \mid 1 \leq i \leq 4\}$.*

According to [1] and [4], we have the following lemma.

Lemma 3.14. *Let j be a non-negative integer with $j \equiv 0 \pmod{4}$. Then the Adams operations are given by the following formulae, where $t_i = \mu_j(Y_i^j)$*

$(1 \leq i \leq 4)$ and $k \equiv 1 \pmod{2}$.

- (1) $\psi^k(t_i) = k^{i/2} t_i \quad (i = 1, 2 \text{ or } 4).$
- (2) $\psi^k(t_3) = k^{i/2} (1 - 2h_4(k+5, k+1)) t_3.$

By Lemma 3.9 and (3.11), we have the following Lemma.

Lemma 3.15. *Let j be a non-negative integer with $j \equiv 0 \pmod{4}$. Then homomorphisms $c: KO \rightarrow K$ and $r: K \rightarrow KO$ are given by the following formulae, where $s_i = I^{i/2}(\varphi(X_i))$ ($1 \leq i \leq 7$) and $t_i = \mu_j(Y_i^j)$ ($1 \leq i \leq 4$).*

- (1) $r(s_i) = h_4(j, j-12) t_i \quad (i = 1, 2 \text{ or } 4).$
- (2) $r(s_3) = -r(s_2).$
- (3) $r(s_5) = -t_3 - h_4(j, j-12) t_4.$
- (4) $r(s_6) = 2t_3.$
- (5) $r(s_7) = 2t_3 + h_4(j, j-12) t_4.$
- (6) $c(t_i) = h_4(j+12, j) s_i \quad (i = 1, 2 \text{ or } 4).$
- (7) $c(t_3) = s_6.$

4. Proof for the case $j \equiv 0 \pmod{4}$

In this section we prove the parts of the case $j \equiv 0 \pmod{4}$ of Theorems 1 and 2. Throughout this section, j denotes a non-negative integer with $j \equiv 0 \pmod{4}$, and ν the integer defined by

$$(4.1) \quad \nu = \begin{cases} \nu_2(j) & (j > 0) \\ m & (j = 0). \end{cases}$$

For each integer n with $0 \leq n < m$, we denote the kernel of the homomorphism

$$(i_n^m)!: \widetilde{KO}(S^j L_8^m) \rightarrow \widetilde{KO}(S^j L_8^n)$$

by $VO_{m,n}^j$. It follows from Proposition 3.13 that we have

(4.2) *The group $VO_{m,n}^j$ is the subgroup of $\widetilde{KO}(S^j L_8^m)$ generated by*

$$\{\mu_j(Y_i^j(n)) \mid 1 \leq i \leq 4\},$$

where $\mu_j: F(y) \rightarrow \widetilde{KO}(S^j L_8^m)$ is the homomorphism defined in (3.12).

By Lemma 3.14, we have the following lemma.

Lemma 4.3. *The Adams operations are given by the following formulae, where $T_i = \mu_j(Y_i^j(n))$ ($1 \leq i \leq 4$) and $k \equiv 1 \pmod{2}$.*

- (1) $\psi^k(T_i) = k^{i/2} T_i \quad (i = 1, 2 \text{ or } 4).$

(2) $\psi^k(T_3) = k^{j/2}(T_3 + h_4(k+5, k+1)(-2T_3 + \alpha(0, j, n) T_2 - \alpha(1, j, n) T_1))$,
 where $\alpha(l, j, n) = h_4(n + h_4(j, j-28l), n + (-2)^{l-1} h_4(j+12, j+16l))$ ($0 \leq l \leq 1$).

We set

$$(4.4) \quad UO_{m,n}^j = \sum_k (\cap_k k^e (\psi^k - 1) VO_{m,n}^j),$$

where the intersection runs over all non-negative integers e . Since the order of $VO_{m,n}^j$ is equal to a power of 2, we have

$$UO_{m,n}^j = \sum_{k: \text{odd}} (\psi^k - 1) VO_{m,n}^j.$$

It follows from Lemmas 3.1 and 4.3 that we have

(4.5) *The group $UO_{m,n}^j$ is the subgroup of $VO_{m,n}^j$ generated by*

$$\{2^{v+1} T_i \mid i = 1, 2 \text{ or } 4\} \cup \{R\},$$

where $T_i = \mu_j(Y_i^j(n))$ ($1 \leq i \leq 4$),

$$R = 2(2^v - 1) T_3 + \alpha(0, j, n) T_2 - \alpha(1, j, n) T_1$$

and $\alpha(l, j, n)$ is the integer defined in Lemma 4.3 ($0 \leq l \leq 1$).

4.1. Proof for the case $n \equiv 3 \pmod{4}$. Suppose that $n \equiv 3 \pmod{4}$. According to [13], we have the exact sequence

$$0 \rightarrow \widetilde{KO}(S^j(L_8^m/L_8^n)) \rightarrow \widetilde{KO}(S^j L_8^m) \xrightarrow{\binom{i_m}{n}!} \widetilde{KO}(S^j L_8^n) \rightarrow 0.$$

Hence we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong VO_{m,n}^j.$$

Since the order of $VO_{m,n}^j$ is finite, we have

$$\widetilde{J}(S^j(L_8^m/L_8^n)) \cong VO_{m,n}^j/UO_{m,n}^j.$$

It follows from Proposition 3.13 that we have

$$VO_{m,n}^j \cong \langle \{w_i \mid 1 \leq i \leq 4\} \rangle / \langle \{R_i \mid 1 \leq i \leq 4\} \rangle,$$

where $w_i = X_i^j(n)$ ($1 \leq i \leq 4$), $R_1 = 2^{h_1(m+j) - h_1(n+j)} w_1$,

$$R_2 = 2^{h_3(m,n) + a_4(n+j, n+j)} (\widetilde{w}_2 + 2^{h_3(m,n)} w_1),$$

$$R_3 = 2^{a_3(m+j, n+j)} (\widetilde{w}_3 + 2^{a_4(m+j+4, n+j)} (\widetilde{w}_2 - 2^{a_7(m+j, n+j)+1} w_1)),$$

$$R_4 = 2^{a_4(m+j, n+j)} (\widetilde{w}_4 + 2^{h_4(m+j, n+j+4)} (\widetilde{w}_2 + 2^{a_7(m+j-4, n+j)+2} w_1)),$$

$$\widetilde{w}_2 = h_4(n+j+15, n+j) w_2 - (-1)^{\lfloor (n+j)/4 \rfloor} a_4(n+j, n+j) w_1,$$

$$\widetilde{w}_3 = h_4(n+j+12, n+j-2) w_3 - h_4(n+j, n+j-2) w_2 + a_4(n+j+12, n+j) w_1$$

and $\widetilde{w}_4 = h_4(n+j+15, n+j) w_4 - h_4(n+j+4, n+j) w_2 + a_4(n+j+4, n+j) w_1$.

Suppose that $m \geq 4 \lfloor (n+j+7)/8 \rfloor + 4 \lfloor (n-j+12)/8 \rfloor$, and set $A_i = R_i$ ($3 \leq i \leq 4$),

$$A_1 = R_1 - (-1)^{\lfloor (n+j)/4 \rfloor} 2^{h_1(m+j)-2\lfloor (n+j)/4 \rfloor - h_3(m,n)} R_2 \\ - h_4(n+j, n+j-2) 2^{a_1(m+j, n+j)} (2^{h_4(n+j, m+j+6)} R_3 - 2^{h_4(n+j, m+j)-1} R_2) \\ + h_4(n+j+4, n+j) 2^{a_1(m+j, n+j)} (2^{h_4(n+j, m+j)} R_4 - 2^{h_4(n+j, m+j+4)} R_2),$$

$$A_2 = R_2 + h_4(n+j-1, n+j-4) 2^{h_4(m+j+4, n+j)} R_4 \\ + h_4(n+j+4, n+j) 2^{h_3(m+j, n+j) - h_4(m+j-2, n+j)} R_3,$$

$$u_1 = -(-1)^{\lfloor (n+j)/4 \rfloor} 2^{h_3(m,n) + a_4(n+j, n+j) - 1} w_1 \\ - h_4(n+j-2, n+j-4) (w_2 + 2^{h_3(m,n)-2} w_1) \\ - h_4(n+j, n+j-2) (w_3 - 2^{a_4(m+j-4, n+j)} (2^{h_3(m,n)} + 2^{a_7(m+j, n+j)+1}) w_1) \\ + h_4(n+j+4, n+j) (w_4 - 2^{h_4(m+j, n+j)-2} (2^{h_3(m,n)} - 2^{2h_4(m+j, n+j)}) w_1),$$

$$u_2 = w_2 + 2^{h_3(m,n) + a_4(n+j, n+j)} w_1 \\ + h_4(n+j-1, n+j-4) (w_4 + 2^{a_4(m+j, n+j)} (\tilde{w}_2 + 2^{2h_4(m+j, n+j)+1} w_1)) \\ + h_4(n+j+4, n+j) (w_3 + 2^{a_4(m+j-4, n+j)} (\tilde{w}_2 - 2^{a_7(m+j, n+j)+1} w_1)),$$

$$u_3 = \tilde{w}_3 + 2^{a_4(m+j+4, n+j)} (\tilde{w}_2 - 2^{a_7(m+j, n+j)+1} w_1)$$

and $u_4 = \tilde{w}_4 + 2^{h_4(m+j, n+j+4)} (\tilde{w}_2 + 2^{a_7(m+j-4, n+j)+2} w_1)$. Then we have

$$A_i = 2^{a_i(m+j, n+j)} u_i \quad (1 \leq i \leq 4),$$

$$w_1 = h_4(n+j, n+j-2) h_4(n+j+15, n+j) (2u_1 + u_2 + u_3 - 2^{h_4(m+j+4, n+j)} u_2) \\ + h_4(n+j-1, n+j-2) (2^{h_4(m+j+4, n+j)} - 1) u_4 \\ + h_4(n+j-2, n+j-4) (2u_1 + 2u_2 - u_4) \\ + h_4(n+j+4, n+j) (4u_1 - 2u_2 + u_3 - 2u_4 + 2^{h_4(m+j, n+j)} (2u_2 - u_3))$$

and

$$w_2 = -2^{h_3(m,n) + a_4(n+j, n+j)} w_1 - h_4(n+j-2, n+j-4) u_1 + h_4(n+j, n+j-1) u_2 \\ - h_4(n+j-1, n+j-2) (2u_1 + u_3 - 2^{h_4(m+j-4, n+j)} (2u_2 - u_4)) \\ + h_4(n+j+4, n+j) (2u_1 - u_4 + 2^{a_4(m+j, n+j)} (2u_2 - u_3)).$$

This implies that $\langle \{w_i \mid 1 \leq i \leq 4\} \rangle = \langle \{u_i \mid 1 \leq i \leq 4\} \rangle$ and

$$VO_{m,n}^j \cong \langle \{u_i \mid 1 \leq i \leq 4\} \rangle / \langle \{A_i \mid 1 \leq i \leq 4\} \rangle \cong \bigoplus_{i=1}^4 \mathbf{Z} / 2^{a_i(m+j, n+j)}.$$

Suppose that $4 \lfloor (n+j+7)/8 \rfloor + 4 \lfloor (n-j+12)/8 \rfloor > m \geq h_1(n+j) + 2 \lfloor (n-j+4)/8 \rfloor + 2 \lfloor (n-j+6)/8 \rfloor + 1$, and set $A_i = R_i$ ($2 \leq i \leq 4$), $u_{2i} = R_{2i}$ ($1 \leq i \leq 2$),

$$A_1 = \begin{cases} 2^{h_3(m+3, n)}(2R_4 - R_2) - R_1 & (h_4(n+j, n+j-4) = 0) \\ R_1 - 4R_2 + 8R_3 & (n+j \equiv 1 \pmod{8} \text{ and } m \geq n+3) \\ R_1 + 4R_2 & (n+j \equiv 2 \pmod{8}) \\ R_1 & (h_3(m, n) = 0), \end{cases}$$

$$u_1 = \begin{cases} w_4 + w_1 & (h_4(n+j, n+j-4) = 0) \\ w_3 - 2w_1 & (n+j \equiv 1 \pmod{8} \text{ and } m \geq n+3) \\ w_2 + 2w_1 & (n+j \equiv 2 \pmod{8}) \\ w_1 & (h_3(m, n) = 0) \end{cases}$$

and

$$u_3 = \begin{cases} w_3 + w_2 - w_1 & (h_4(n+j, n+j-4) = 0) \\ 2w_3 + w_2 - 3w_1 & (n+j \equiv 1 \pmod{8} \text{ and } m \geq n+3) \\ w_3 + 2w_2 - 3w_1 & (n+j \equiv 2 \pmod{8}) \\ w_3 & (h_3(m, n) = 0). \end{cases}$$

Then we have $A_i = u_i$ ($i=2$ or 4),

$$A_1 = \begin{cases} 2^{a_1(m+j, n+j)} u_1 & (m \geq 4 \lfloor n/4 \rfloor + 4) \\ 2u_1 & (n+j \equiv 0 \pmod{8}) \\ u_1 & (n+j \equiv 1 \pmod{8} \text{ and } n+3 > m > n), \end{cases}$$

$$A_3 = \begin{cases} 2u_3 & (h_4(n+j-2, n+j-4) = \lfloor (m+j)/2 \rfloor - 4 \lfloor (n+j+4)/8 \rfloor - 1 = 0) \\ u_3 & (\text{otherwise}), \end{cases}$$

$$w_1 = \begin{cases} 2u_4 - u_2 - 4u_1 & (h_4(n+j, n+j-4) = 0) \\ 4u_1 + u_2 - 2u_3 & (n+j \equiv 1 \pmod{8} \text{ and } m \geq n+3) \\ 2u_1 - u_2 & (n+j \equiv 2 \pmod{8}) \\ u_1 & (h_3(m, n) = 0) \end{cases}$$

and

$$w_2 = \begin{cases} 2u_1 + u_2 - u_4 & (h_4(n+j, n+j-4) = 0) \\ -6u_1 - u_2 + 3u_3 & (n+j \equiv 1 \pmod{8} \text{ and } m \geq n+3) \\ 2u_2 - 3u_1 & (n+j \equiv 2 \pmod{8}) \\ u_2 - u_1 & (h_3(m, n) = 0). \end{cases}$$

This implies that $\langle \{w_i \mid 1 \leq i \leq 4\} \rangle = \langle \{u_i \mid 1 \leq i \leq 4\} \rangle$ and

$$VO_{m,n}^j \cong \begin{cases} \mathbf{Z}/16 \oplus \mathbf{Z}/2 & (h_4(n+j, n+j-4) = \lfloor m/2 \rfloor - 2 \lfloor n/4 \rfloor - 3 = 0) \\ \mathbf{Z}/2^{a_1(m+j, n+j)} & (4 \lfloor (n+j+15)/8 \rfloor + 2 \lfloor (n-j)/4 \rfloor > m \geq 4 \lfloor n/4 \rfloor + 4) \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & (n+j \equiv 0 \pmod{8} \text{ and } \lfloor m/2 \rfloor = \lfloor n/2 \rfloor + 1) \\ \mathbf{Z}/2 & (h_4(n+j+6, n+j) = \lfloor m/2 \rfloor - \lfloor (n+1)/2 \rfloor = 0). \end{cases}$$

If $h_1(n+j) + 2 \lfloor (n-j+4)/8 \rfloor + 2 \lfloor (n-j+6)/8 \rfloor \geq m > n$, then we have $VO_{m,n}^j \cong 0$. Thus the proof for the case $j \equiv 0 \pmod{4}$ and $n \not\equiv 3 \pmod{4}$ of Theorem 1 is completed.

Consider the case $j \equiv 0 \pmod{8}$. It follows from (4.5) that we have

$$\mu_j^{-1}(UO_{m,n}^j) = \langle \{2^{\nu+h_3(i+5,i)} w_i \mid 1 \leq i \leq 4\} \cup \{R_i \mid 0 \leq i \leq 4\} \rangle,$$

where ν is the integer defined by (4.1) and

$$R_0 = 2(2^\nu - 1)w_3 + h_4(n, n-2)w_2 - h_4(n+4, n-1)w_1.$$

Suppose that $m \geq 4 \lfloor (n+15)/8 \rfloor + 2 \lfloor (n+6)/8 \rfloor + 2 \lfloor n/8 \rfloor$, and set

$$\begin{aligned}
 v_1 &= h_4(n-2, n-4) w_2 + h_4(n, n-2) w_3 + h_4(n+4, n) w_4, \\
 v_2 &= h_4(n-1, n-2) (w_4 + 2w_3 + 2^{h_3(m,n)+1} (1 - 2^{h_4(m+6,n)}) w_3 + 2^v w_3) \\
 &\quad + h_4(n-2, n-4) (w_4 + w_2 + 2^{h_3(m,n)} w_2) + h_4(n+4, n) (w_3 + 2w_4 - 2^{h_3(m,n)+1} w_4) \\
 &\quad + h_4(n, n-1) (w_2 + 2^{h_3(m,n)+1} w_3 + 2^v w_3), \\
 v_3 &= h_4(n, n-1) (w_4 + 2^{h_4(m,n)} w_2 - 2^{3h_4(m,n)+2} w_3) \\
 &\quad + h_4(n-1, n-2) (2w_4 - w_1 + 2^{h_4(m,n)+1} (w_4 + 2w_3) - 2^{3h_4(m,n)+3} w_3) \\
 &\quad + h_4(n-2, n-4) (2w_4 - w_1 + 2^{h_4(m,n)+1} (w_4 + w_2) - 2^{3h_4(m,n)+2} w_2) \\
 &\quad + h_4(n+4, n) (2w_4 - w_2 + 2^{h_4(m,n)} (w_3 + 2w_4) + 2^{3h_4(m,n)+1} w_4)
 \end{aligned}$$

and $v_4 = R_0 + h_4(n-2, n-3) (w_3 - R_0)$. Then we have $\langle \{w_i \mid 1 \leq i \leq 4\} \rangle = \langle \{v_i \mid 1 \leq i \leq 4\} \rangle$ and

$$VO_{m,n}^j / UO_{m,n}^j \cong \begin{cases} (\oplus_{i=1}^3 \mathbf{Z}/2^{b_i(j,m,n)}) \oplus \mathbf{Z}/2 & (n \equiv 2 \pmod{8}) \\ \oplus_{i=1}^3 \mathbf{Z}/2^{b_i(j,m,n)} & (\text{otherwise}). \end{cases}$$

Suppose that $4 \lfloor (n+15)/8 \rfloor + 2 \lfloor (n+6)/8 \rfloor + 2 \lfloor n/8 \rfloor > m \geq 4 \lfloor (n+14)/8 \rfloor + 4 \lfloor n/8 \rfloor$, and set

$$\begin{aligned}
 v_1 &= h_4(n-2, n-3) w_2 + h_4(n-1, n-2) w_3 + h_4(n+4, n) w_4, \\
 v_2 &= h_4(n-2, n-3) (w_4 + 5w_2) + h_4(n-1, n-2) (w_4 - 2w_3) \\
 &\quad + h_4(n+4, n) (w_3 - 2w_4), \\
 v_3 &= h_4(n-2, n-3) (2w_4 - w_1 - 16w_2) + h_4(n-1, n-2) (w_1 - 4w_3) \\
 &\quad + h_4(n+4, n) (w_2 + 6w_4)
 \end{aligned}$$

and $v_4 = R_0 + h_4(n-2, n-3) (w_3 + 2w_2 - 3w_1 - R_0)$. Then we have $\langle \{w_i \mid 1 \leq i \leq 4\} \rangle = \langle \{v_i \mid 1 \leq i \leq 4\} \rangle$ and

$$VO_{m,n}^j / UO_{m,n}^j \cong \begin{cases} \mathbf{Z}/2^{b_1(j,m,n)} \oplus \mathbf{Z}/4 & (n \equiv 2 \pmod{8}) \\ \mathbf{Z}/8 & (n \equiv 1 \pmod{8}) \\ VO_{m,n}^j & (h_4(n, n-4) = 0). \end{cases}$$

If $4 \lfloor (n+14)/8 \rfloor + 4 \lfloor n/8 \rfloor > m > n$, then we have $UO_{m,n}^j \cong 0$. Thus the proof for the case $j \equiv 0 \pmod{8}$ and $n \equiv 3 \pmod{4}$ of Theorem 2 is completed.

Consider the case $j \equiv 4 \pmod{8}$. It follows from (4.5) that we have

$$\mu_j^{-1}(UO_{m,n}^j) = \langle \{2^{3+h_3(i,i-1)} w_i \mid 1 \leq i \leq 4\} \cup \{R_i \mid 0 \leq i \leq 4\} \rangle,$$

where $R_0 = 2w_4 - h_4(n+3, n) (8w_4 + w_1) - h_4(n, n-4) (8w_4 + w_2)$. Suppose that $m \geq 2 \lfloor n/4 \rfloor + 4 \lfloor (n+2)/8 \rfloor + 6$, and set

$$\begin{aligned}
 v_1 &= h_4(n, n-4) w_4 + h_4(n+4, n+2) w_3 + h_4(n+2, n) w_2, \\
 v_2 &= h_4(n, n-4) (w_3 - 2w_4) + h_4(n+4, n+3) (w_2 + w_4 - 2^{h_3(m,n)+1} w_3) \\
 &\quad + h_4(n+3, n+2) (w_4 - 2w_3) + h_4(n+2, n) (w_4 + w_2),
 \end{aligned}$$

$$\begin{aligned}
 v_3 = & h_4(n, n-4) (2w_3 + w_1 + 2^{h_4(m,n)+1}(w_3 - 2w_4)) \\
 & + h_4(n+4, n+3) (2w_3 - w_2 + w_1 + 2^{h_4(m,n)} w_2) \\
 & + h_4(n+3, n+2) (2w_3 - w_2 + 2^{h_4(m,n)}(w_4 - 2w_3)) \\
 & + h_4(n+2, n) (w_3 + 2^{h_4(m,n)}(w_4 + w_2))
 \end{aligned}$$

and $v_4 = R_0 + h_4(n+4, n+3) (w_4 - R_0)$. Then we have $\langle \{w_i \mid 1 \leq i \leq 4\} \rangle = \langle \{v_i \mid 1 \leq i \leq 4\} \rangle$ and

$$VO_{m,n}^j / UO_{m,n}^j \cong \begin{cases} (\oplus_{i=1}^3 \mathbf{Z} / 2^{b_i(j,m,n)}) \oplus \mathbf{Z} / 2 & (n \equiv 4 \pmod{8}) \\ \oplus_{i=1}^3 \mathbf{Z} / 2^{b_i(j,m,n)} & (\text{otherwise}). \end{cases}$$

If $2 \lfloor n/4 \rfloor + 4 \lfloor (n+2)/8 \rfloor + 6 > m > n$, then we have $UO_{m,n}^j \cong 0$. Thus the proof for the case $j \equiv 4 \pmod{8}$ and $n \not\equiv 3 \pmod{4}$ of Theorem 2 is completed.

4.2. Proof for the case $n \equiv 3 \pmod{4}$. Now, we turn to the case $n \equiv 3 \pmod{4}$. It follows from [13] that we have the following commutative diagram, in which rows are exact.

$$(4.6) \quad \begin{array}{ccccccc}
 0 & \rightarrow & VO_{m,n+1}^j & \xrightarrow{f_1} & \widetilde{KO}(S^j(L_8^m/L_8^n)) & \xrightarrow{f_2} & \widetilde{KO}(S^{j+n+1}) \rightarrow 0 \\
 & & \parallel & & \downarrow f_3 & & \downarrow \\
 0 & \rightarrow & VO_{m,n+1}^j & \rightarrow & \widetilde{KO}(S^j L_8^m) & \xrightarrow{f_4} & \widetilde{KO}(S^j L_8^{n+1}) \rightarrow 0.
 \end{array}$$

Since $\widetilde{KO}(S^{j+n+1})$ is isomorphic to \mathbf{Z} , the upper row of (4.6) splits. Choose $y \in \widetilde{KO}(S^j(L_8^m/L_8^n))$ such that $\beta = f_2(y)$ generates the group $\widetilde{KO}(S^{j+n+1})$. Then we have an isomorphism

$$f: VO_{m,n+1}^j \oplus \widetilde{KO}(S^{j+n+1}) \rightarrow \widetilde{KO}(S^j(L_8^m/L_8^n))$$

defined by $f(x, k\beta) = f_1(x) + ky$ for every $(x, k) \in VO_{m,n+1}^j \oplus \mathbf{Z}$. This proves the case $j \equiv n+1 \equiv 0 \pmod{4}$ of Theorem 1.

Lemma 4.7. *If $j \equiv n+1 \equiv 0 \pmod{4}$, then there is an element $y \in \widetilde{KO}(S^j(L_8^m/L_8^n))$ which satisfies the following conditions:*

- (1) $\beta = f_2(y)$ generates the group $\widetilde{KO}(S^{j+n+1})$,
- (2) $f_3(y) = \begin{cases} \mu_j(Y_4^j(n)) & (j \equiv n+1 \equiv 0 \pmod{8}) \\ \mu_j(Y_3^j(n) + Y_1^j(n)) & (j \equiv n+1 \equiv 4 \pmod{8}) \\ \mu_j(Y_3^j(n) + Y_2^j(n) + 2Y_1^j(n)) & (j \equiv n-3 \equiv 0 \pmod{8}) \\ \mu_j(Y_4^j(n) + Y_2^j(n)) & (j \equiv n-3 \equiv 4 \pmod{8}). \end{cases}$

Proof. Consider the following commutative diagram, in which rows are exact:

$$\begin{array}{ccccccc}
 0 & \rightarrow & V_{m,n+1}^j & \xrightarrow{f_{c,1}} & \tilde{K}(S^j(L_8^m/L_8^n)) & \xrightarrow{f_{c,2}} & \tilde{K}(S^{n+j+1}) \rightarrow 0 \\
 & & \parallel & & \downarrow f_{c,3} & & \downarrow \\
 0 & \rightarrow & V_{m,n+1}^i & \longrightarrow & \tilde{K}(S^j L_8^m) & \xrightarrow{f_{c,4}} & \tilde{K}(S^j L_8^{n+1}) \rightarrow 0.
 \end{array}$$

According to Lemma 3.10, there is an element $x \in \tilde{K}(S^j(L_8^m/L_8^n))$ such that $f_{c,2}(x)$ generates the group $\tilde{K}(S^{n+j+1})$ and

$$f_{c,3}(x) = \begin{cases} I^{j/2}(\varphi(X_4(n))) & (n \equiv 7 \pmod{8}) \\ I^{j/2}(\varphi(X_6(n)+X_1(n))) & (n \equiv 3 \pmod{8}). \end{cases}$$

If $n+j \equiv 3 \pmod{8}$, then $r: \tilde{K}(S^{n+j+1}) \rightarrow \tilde{K}\tilde{O}(S^{n+j+1})$ is an isomorphism. It follows from Lemma 3.15 that $y=r(x)$ satisfies the conditions (1) and (2). If $n+j \equiv 7 \pmod{8}$, then $c: \tilde{K}\tilde{O}(S^{n+j+1}) \rightarrow \tilde{K}(S^{n+j+1})$ is an isomorphism and $c: \tilde{K}\tilde{O}(S^j L_8^{n+1}) \rightarrow \tilde{K}(S^j L_8^{n+1})$ is a monomorphism. There is an element $\mathfrak{y} \in \tilde{K}\tilde{O}(S^j(L_8^m/L_8^n))$ such that $f_2(\mathfrak{y})$ generates the group $\tilde{K}\tilde{O}(S^{n+j+1})$ and $f_{c,4}(c(f_3(\mathfrak{y})))=f_{c,4}(f_{c,3}(x))$. It follows from Lemma 3.15 that we have $f_{c,3}(x)=c(Y)$, where

$$Y = \begin{cases} \mu_j(Y_4^j(n)) & (n+1 \equiv j \equiv 0 \pmod{8}) \\ \mu_j(Y_3^j(n)+Y_1^j(n)) & (n+1 \equiv j \equiv 4 \pmod{8}). \end{cases}$$

This implies that $f_4(f_3(\mathfrak{y}))=f_4(Y)$ and $y=\mathfrak{y}+f_1(Y-f_3(\mathfrak{y}))$ satisfies the conditions (1) and (2). q.e.d.

In the rest of this section, we fix an element $y \in \tilde{K}\tilde{O}(S^j(L_8^m/L_8^n))$ which satisfies the conditions of Lemma 4.7, and set

$$(4.8) \quad T_i = \mu_j(Y_i^j(n+1)) \quad (1 \leq i \leq 4).$$

Lemma 4.9. *If k is an odd integer, then the Adams operation ψ^k is given by*

$$\psi^k(y) = k^{(n+j+1)/2} y + ((k^{j/2} - k^{(n+j+1)/2})/8) f_1(8f_3(y)) + k^{j/2} f_1(w),$$

where $w = h_4(k+5, k+1)(W - 8f_3(y))$ and

$$(4.10) \quad W = 8f_3(y) - h_4(n, n-4)(T_3 + h_4(j, j-4)(T_3 + T_1)).$$

Proof. We necessarily have

$$\psi^k(y) = uy + f_1(x)$$

for some integer u and an element $x \in VO_{m,n+1}^j$. By using the ψ -map f_2 , we see that $u = k^{(n+j+1)/2}$. Under $f_3, f_1(x)$ maps into x and y maps into $f_3(y)$, and we see that

$$\psi^k(f_3(y)) = k^{(n+j+1)/2} f_3(y) + x.$$

It follows from Lemma 4.3 that

$$k^{j/2}(f_3(y) + w) = k^{(n+j+1)/2} f_3(y) + x.$$

This implies that

$$x = ((k^{j/2} - k^{(n+j+1)/2})/8)(8f_3(y)) + k^{j/2} w. \quad \text{q.e.d.}$$

We now recall some definition in [3]. Set $Y = \widetilde{KO}(S^j(L_8^m/L_8^n))$ and let f be a function which assigns to each integer k a non-negative integer $f(k)$. Given such a function f , we define Y_f to be the subgroup of Y generated by

$$\{k^{f(k)}(\psi^k - 1)(y) \mid k \in \mathbf{Z}, y \in Y\};$$

that is,

$$Y_f = \langle \{k^{f(k)}(\psi^k - 1)(y) \mid k \in \mathbf{Z}, y \in Y\} \rangle.$$

Then the kernel of the homomorphism $J'' : Y \rightarrow J''(Y)$ coincides with $\bigcap_f Y_f$, where the intersection runs over all functions f .

Suppose that f satisfies

$$(4.11) \quad f(k) \geq m + \max \{v_p(\mathfrak{m}((n+j+1)/2)) \mid p \text{ is a prime divisor of } k\}$$

for every $k \in \mathbf{Z}$. For each odd integer i , $N(i)$ denotes the integer chosen to satisfy the property

$$(4.12) \quad iN(i) \equiv 1 \pmod{2^m}.$$

In the following calculation we put $(n+j+1)/2 = u$ for the sake of simplicity. From Lemma 3.1, (4.5) and Lemma 4.9, we have

$$\begin{aligned} & k^{f(k)}(\psi^k - 1)(y) \\ &= k^{f(k)}(k^u - 1)y + (k^{f(k)}(k^{j/2} - k^u)/8)f_1(8f_3(y)) + k^{f(k)+(j/2)}f_1(w) \\ &\equiv k^{f(k)}(k^u - 1)y + (k^{f(k)}(k^{j/2} - k^u)/8)f_1(W) \pmod{f_1(UO_{m,n+1}^j)} \\ &= k^{f(k)}(k^u - 1)y + (k^{f(k)}N(u/2^{v_2(u)})(u(k^{j/2} - 1) - u(k^u - 1))/2^{v_2(8u)})f_1(W) \\ &\equiv k^{f(k)}(k^u - 1)y \\ &\quad + (k^{f(k)}N(u/2^{v_2(u)})((j/2)(k^u - 1) - u(k^u - 1))/2^{v_2(8u)})f_1(W) \pmod{f_1(UO_{m,n+1}^j)} \\ &= (k^{f(k)}(k^u - 1)/2^{v_2(4u)})(2^{v_2(4u)}y - N(u/2^{v_2(u)}((n+1)/4)f_1(W)(2^v + 1)). \end{aligned}$$

By virtue of [3; II, Theorem (2.7) and Lemma (2.12)], we have

$$\begin{aligned} Y_f &= \langle f_1(UO_{m,n+1}^j) \cup \{k^{f(k)}(\psi^k - 1)(y) \mid k \in \mathbf{Z}\} \rangle \\ &= \langle f_1(UO_{m,n+1}^j) \cup \{\mathfrak{m}(u)y - Mf_1(W)\} \rangle, \end{aligned}$$

where $M = (\mathfrak{m}(u)/2^{\nu_2(u)+2}) N(u/2^{\nu_2(u)}) ((n+1)/4) (2^\nu + 1)$. Since this is true for every function f which satisfies (4.11), we have

$$(4.13) \quad J''(Y) \cong Y / \langle f_1(UO_{m,n+1}^j) \cup \{\mathfrak{m}((n+j+1)/2)y - Mf_1(W)\} \rangle,$$

where $\nu_2(M) = \nu_2(n+1) - 2$ and

$$W = \begin{cases} 4T_4 + 2T_2 + T_1 & (j \equiv n+1 \equiv 0 \pmod{8}) \\ 8T_4 + 4T_2 - T_1 & (j \equiv n-3 \equiv 4 \pmod{8}) \\ 6T_3 + 4T_2 + 4T_1 & (j \equiv n-3 \equiv 0 \pmod{8}) \\ 3T_3 + 2T_2 + 3T_1 & (j \equiv n+1 \equiv 4 \pmod{8}). \end{cases}$$

Suppose that $m \geq n+5+2h_4(j+4, j)h_4(n+1, n)$. It follows from the proof for the case $n \not\equiv 3 \pmod{4}$ that we have

$$W \equiv \sum_{i=1}^3 m_i z_i \pmod{UO_{m,n+1}^j},$$

where $z_i = \mu_j(v_i)$ ($1 \leq i \leq 3$), $\nu_2(m_i) = 2 + h_4(n+j, n+j-4) - i$ ($1 \leq i \leq 2$) and $\nu_2(m_3) = 2h_4(j, j-4)$. Therefore

$$J''(Y) \cong F(v) / \langle \{\sum_{i=0}^3 M_i v_i\} \cup \{B_i | 1 \leq i \leq 4\} \rangle,$$

where $F(v)$ is the free abelian group generated by $\{v_i | 0 \leq i \leq 4\}$, $M_0 = \mathfrak{m}((n+j+1)/2)$,

$$\begin{aligned} B_i &= 2^{b_i(j, m, n)} v_i \quad (1 \leq i \leq 3), \\ B_4 &= (h_4(j+4, j)h_4(n+5, n+1) + 1) v_4 \end{aligned}$$

and $M_i = -m_i M$ ($1 \leq i \leq 3$). Set

$$(4.14) \quad i_k = \min \{b_k(j, m, n), \nu_2(n+1) + \nu_2(m_k) - 2\} \quad (1 \leq k \leq 3).$$

For the sake of simplicity, we put $b_k = b_k(j, m, n)$ ($1 \leq k \leq 3$) in the following calculation. Since $\nu_2(M) = \nu_2(n+1) - 2$, the greatest common divisor of M_k and 2^{b_k} is equal to 2^{i_k} ($1 \leq k \leq 3$). Choose integers $e_{11}, e_{12}, e_{21}, e_{22}, e_{31}$ and e_{32} with $e_{k1} 2^{b_k} + e_{k2} M_k = 2^{i_k}$ ($1 \leq k \leq 3$). If $b_1 - i_1 > b_2 - i_2$ and $i_2 \leq \nu_2(M_3)$, then we have

$$A_1 \begin{pmatrix} \sum_{i=0}^3 M_i v_i \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 2^{b_1-i_1} M_0 v_0 \\ 2^{b_2-i_2} (e_{12} M_0 v_0 + 2^{i_1} v_1) \\ e_{22} (M_0 v_0 + M_1 v_1 + M_3 v_3) + 2^{i_2} v_2 \\ B_3 \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} 2^{b_1-i_1} & -M_1/2^{i_1} & -2^{b_1-b_2-i_1} M_2 & -2^{b_1-b_3-i_1} M_3 \\ e_{12} 2^{b_2-i_2} & e_{11} 2^{b_2-i_2} & -e_{12} M_2/2^{i_2} & -2^{b_2-b_3-i_2} e_{12} M_3 \\ e_{22} & 0 & e_{21} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\det A_1=1$. This implies that

$$J''(Y) \cong \begin{cases} \mathbf{Z}/2^{b_1-1}M_0 \oplus \mathbf{Z}/2^{b_2+1} \oplus \mathbf{Z}/2^{b_3} \oplus \mathbf{Z}/2 & (j \equiv n+1 \equiv 4 \pmod{8}) \\ \mathbf{Z}/2^{b_1-i_1}M_0 \oplus \mathbf{Z}/2^{b_2-i_2+i_1} \oplus \mathbf{Z}/2^{i_2} \oplus \mathbf{Z}/2^{b_3} & (\text{otherwise}). \end{cases}$$

If $b_2-i_2 \geq b_1-i_1$ and $\nu_2(M_3) \geq i_2$, then we have

$$A_2 \begin{pmatrix} \sum_{i=0}^3 M_i v_i \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 2^{b_2-i_2} M_0 v_0 \\ B_1 \\ e_{22}(M_0 v_0 + M_1 v_1 + M_3 v_3) + 2^{i_2} v_2 \\ B_3 \end{pmatrix},$$

where

$$A_2 = \begin{pmatrix} 2^{b_2-i_2} & -M_1 2^{b_2-b_1-i_2} & -M_2/2^{i_2} & -2^{b_2-b_3-i_2} M_3 \\ 0 & 1 & 0 & 0 \\ e_{22} & 0 & e_{21} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\det A_2=1$. This implies that

$$J''(Y) \cong \begin{cases} \mathbf{Z}/2^{b_2}M_0 \oplus \mathbf{Z}/2^{b_1} \oplus \mathbf{Z}/2^{b_3} \oplus \mathbf{Z}/2 & (j \equiv n+1 \equiv 4 \pmod{8}) \\ \mathbf{Z}/2^{b_2-i_2}M_0 \oplus \mathbf{Z}/2^{b_1} \oplus \mathbf{Z}/2^{i_2} \oplus \mathbf{Z}/2^{b_3} & (\text{otherwise}). \end{cases}$$

If $i_2 > \nu_2(M_3)$, then we necessarily have $j \equiv n-3 \equiv 4 \pmod{8}$, $b_1=4$, $i_1 \geq 3$, $b_2=3$ and $i_2=i_1-1=\nu_2(M_3)+1$. If $i_2 > \nu_2(M_3)$ and $b_2-i_2 > b_3-i_3$, then we have $i_2=2$, $b_3=i_3$ and

$$A_3 \begin{pmatrix} \sum_{i=0}^3 M_i v_i \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 2M_0 v_0 \\ B_1 \\ e_{22}(M_0 v_0 + M_1 v_1) + 4v_2 \\ B_3 \end{pmatrix},$$

where

$$A_3 = \begin{pmatrix} 2 & -M_1/8 & -M_2/4 & -2M_3/2^{b_3} \\ 0 & 1 & 0 & 0 \\ e_{22} & 0 & e_{21} & -e_{22} M_3/2^{b_3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\det A_3=1$. This implies that

$$J''(Y) \cong \mathbf{Z}/2M_0 \oplus \mathbf{Z}/16 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2^{b_3}.$$

If $i_2 > \nu_2(M_3)$ and $b_3-i_3 \geq b_2-i_2$, then we have

$$A_4 \begin{pmatrix} \sum_{i=0}^3 M_i v_i \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 2^{b_3-i_3} M_0 v_0 \\ B_1 \\ B_2 \\ e_{32}(M_0 v_0 + M_1 v_1 + M_2 v_2) + 2^{i_3} v_3 \end{pmatrix},$$

where

$$A_4 = \begin{pmatrix} 2^{b_3-i_3} & -M_1 2^{b_3-i_3-4} & -M_2 2^{b_3-i_3-3} & -M_3/2^{i_3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ e_{32} & 0 & 0 & e_{31} \end{pmatrix}$$

and $\det A_4=1$. This implies that

$$J''(Y) \cong \mathbf{Z}/2^{b_3-i_3}M_0 \oplus \mathbf{Z}/16 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/2^{i_3}.$$

If $n+7 > m \geq n+5$ and $j \equiv n-3 \equiv 4 \pmod{8}$, then we have $W \equiv m_1 T_3 \pmod{UO_{m,n+1}^i}$, where $v_2(m_1)=2$. Therefore

$$J''(Y) \cong F(v) / \langle \{M_0 v_0 + M_1 v_1, B_1\} \rangle,$$

where $F(v)$ is the free abelian group generated by $\{v_i | 0 \leq i \leq 1\}$, $M_0 = m((n+j+1)/2)$, $B_1 = 2^b v_1$, $b = b_1(j, m, n)$ and $M_1 = -m_1 M$. Set $i = \min\{b, v_2(n+1)\}$. Choose integers e_1 and e_2 with $e_1 2^b + e_2 M_1 = 2^i$. Then we have

$$\begin{pmatrix} 2^{b-i} & -M_1/2^i \\ e_2 & e_1 \end{pmatrix} \begin{pmatrix} M_0 v_0 + M_1 v_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} 2^{b-i} M_0 v_0 \\ e_2 M_0 v_0 + 2^i v_1 \end{pmatrix}.$$

This implies that $J''(Y) \cong \mathbf{Z}/2^{b-i}M_0 \oplus \mathbf{Z}/2^i$.

If $n+5 > m > n$, then we have $Mf_1(W) \equiv 0 \pmod{f_1(UO_{m,n}^i)}$ and

$$J''(Y) \cong \mathbf{Z}/m((n+j+1)/2) \oplus (VO_{m,n+1}^i / UO_{m,n+1}^i).$$

According to [2], [3] and [12], we have $\tilde{J}(S^j(L_8^m/L_8^n)) \cong J''(Y)$. Thus, the proof for the parts of the case $n+1 \equiv j \equiv 0 \pmod{8}$ of Theorem 2 is completed.

5. Proof for the case $j \equiv 2 \pmod{4}$

In this section we prove the parts of the case $j \equiv 2 \pmod{4}$ of Theorems 1 and 2. Throughout this section j denotes a positive integer with $j \equiv 2 \pmod{4}$. Consider the elements S_i ($1 \leq i \leq 7$) of $\tilde{K}(S^j L_8^m)$ defined by

$$(5.1) \quad S_i = I^{j/2}(\varphi(X_i(n+1))) \quad (1 \leq i \leq 7),$$

where $\varphi: F(x) \rightarrow \tilde{K}(L_8^m)$ is the homomorphism defined in (3.7). For each integer n with $0 \leq n \leq m$, we denote the kernel of the homomorphism

$$(i_n^m)^!: \tilde{K}(S^j L_8^m) \rightarrow \tilde{K}(S^j L_8^n)$$

by $V_{m,n}^j$. By Proposition 3.8, we have

$$(5.2) \quad V_{m,2[(n+1)/2]}^j = \langle \{S_i \mid 1 \leq i \leq 7\} \rangle .$$

Consider the Bott exact sequence (cf. [5] and [6, (12.2)])

$$(5.3) \quad \rightarrow \tilde{K}\tilde{O}(S^{j+2}X) \xrightarrow{c} \tilde{K}(S^{j+2}X) \xrightarrow{r \circ I^{-1}} \tilde{K}\tilde{O}(S^jX) \xrightarrow{\partial} \tilde{K}\tilde{O}(S^{j-1}X) \rightarrow$$

for $X=L_8^m/L_8^n$, where ∂ is the homomorphism defined by the exterior product with the generator of $\tilde{K}\tilde{O}(S^1)$. Using the isomorphisms

$$VO_{m,2[(n+1)/2]}^{j+2} \cong \tilde{K}\tilde{O}(S^{j+2}(L_8^m/L_8^{2[(n+1)/2]}))$$

and

$$V_{m,2[(n+1)/2]}^j \cong \tilde{K}(S^j(L_8^m/L_8^{2[(n+1)/2]})) ,$$

we obtain the exact sequence

$$(5.4) \quad \rightarrow VO_{m,2u}^{j+2} \xrightarrow{I^{-1} \circ c} V_{m,2u}^j \xrightarrow{r_1} \tilde{K}\tilde{O}(S^j(L_8^m/L_8^{2u})) \xrightarrow{\partial} G \rightarrow 0 ,$$

where $u=[(n+1)/2]$ and

$$G = \begin{cases} \tilde{K}\tilde{O}(S^{j+1}(L_8^m/L_8^{2u})) & (m+j \equiv 0, 1 \text{ or } 2 \pmod{8}) \\ 0 & (\text{otherwise}) . \end{cases}$$

Using Lemma 3.15, we obtain the following lemma.

Lemma 5.5. *For the homomorphism r_1 in the exact sequence (5.4), we have*

$$\text{Ker } r_1 = \begin{cases} \langle \{S_1, S_2, S_4, S_6\} \rangle & (2u+j \equiv 4 \text{ or } 6 \pmod{8}) \\ \langle \{2S_1, S_2, S_4, S_6\} \rangle & (2u+j \equiv 2 \pmod{8}) \\ \langle \{S_1, S_2, S_4, 2S_6\} \rangle & (2u \equiv j-4 \equiv 2 \pmod{8}) \\ \langle \{S_1, S_2, 2S_4, S_6\} \rangle & (2u \equiv j-4 \equiv 6 \pmod{8}) , \end{cases}$$

where $u=[(n+1)/2]$.

It follows from Lemmas 3.9 and 5.5 that we have

Lemma 5.6. *The Adams operations are given by the following formulae, where $u=[(n+1)/2]$, $T_i=r_i(S_i)$ ($1 \leq i \leq 7$) and $k \equiv 1 \pmod{2}$.*

$$(1) \quad \psi^k(T_3) = (-1)^{h_3(k+2,k)} k^{j/2} T_3 .$$

$$(2) \quad \psi^k(T_5) = (-1)^{h_4(k+4,k)} k^{j/2} \tilde{T}_5 ,$$

where $\tilde{T}_5 = T_5 - h_4(k+5, k+1) (h_3(u+1, u-1) T_3 + h_4(j+4, j) h_3(u+2, u+1) T_1)$.

$$(3) \quad \psi^k(T_7) = (-1)^{h_3(k+2,k)} k^{j/2} \tilde{T}_7,$$

where $\tilde{T}_7 = T_7 - h_4(k+5, k+1) h_4(j+4, j) h_3(u+2, u+1) T_1$.

$$(4) \quad \psi^k(T_i) = T_i \quad (i = 1, 2, 4 \text{ or } 6).$$

We set

$$(5.7) \quad U_{m,2u}^j = \sum_{k: \text{ odd}} (\psi^k - 1) r_1(V_{m,2u}^j).$$

It follows from Lemma 5.6 that we have

$$(5.8) \quad U_{m,2u}^j = \langle \{4T_3, R_1, R_2\} \rangle,$$

where $u = [(n+1)/2]$,

$$R_1 = 2T_5 + h_3(u+1, u-1) T_3 - h_4(j+4, j) h_3(u+2, u+1) T_1,$$

$$R_2 = 4T_7 - h_4(j+4, j) h_3(u+2, u+1) T_1$$

and $T_i = r_1(S_i) (1 \leq i \leq 7)$.

5.1. Proof for the case $n \equiv 0 \pmod{2}$. Suppose $n \equiv 0 \pmod{2}$. If $m \geq n+2$, then by Proposition 3.8 and Lemma 5.5, we have

$$r_1(V_{m,n}^j) \cong \langle \{w_i \mid 0 \leq i \leq 3\} \rangle / \langle \{R_i \mid 4 \leq i \leq 7\} \rangle,$$

where $w_0 = h_3(n, n-2) X_1(n) + h_4(n+6, n+4) X_6(n) + h_4(n+2, n) X_4(n)$,

$$w_1 = (1 - 2h_4(n+6, n+2)) X_3(n),$$

$$w_2 = h_4(n+2, n-2) X_5(n) + h_4(n+6, n+2) X_7(n),$$

$$w_3 = h_4(n+6, n+2) X_5(n) + h_4(n+2, n-2) X_7(n),$$

$$R_4 = h_4(n+j, n+j-12) w_0,$$

$$R_5 = 2^{h_3(m+j, n+j)-1} \tilde{w}_1,$$

$$R_6 = 2^{a_8(m+j, n+j)} (\tilde{w}_2 + 2^{a_8(m+j, n+j)} \tilde{w}_1),$$

$$R_7 = 2^{a_9(m+j, n+j)-1} (\tilde{w}_3 - 2^{a_9(m+j, n+j)} \tilde{w}_1),$$

$$\tilde{w}_1 = (1 - 2h_4(n+6, n+2)) (2X_3(n) - h_4(n+j+6, n+j+4) w_0),$$

$$\tilde{w}_2 = 2w_2 - w_1 + h_4(j, j-4) h_4(n, n-2) w_0,$$

$$\tilde{w}_3 = 2w_3 - h_4(n+j+6, n+j+4) w_0,$$

$$a_8(m+j, n+j) = \sum_{i=0}^1 h_4(n+j+4i, n+j+4i-4) h_4(m+j+4i-4, n+j+4i)$$

and $a_9(m+j, n+j) = a_8(m+j+8, n+j+4)$. If $m \geq n+2+12h_4(n+j+6, n+j+4) + 2h_3(n+2, n)$, then we have

$$r_1(V_{m,n}^j) \cong \langle \{u_i \mid 0 \leq i \leq 3\} \rangle / \langle \{A_i \mid 0 \leq i \leq 3\} \rangle,$$

where $u_0 = w_0, u_1 = w_2, A_0 = R_4, A_1 = 2^{h_3(m+j, n+j)+1} u_1$,

$$A_i = 2^{a_i + 6(m+j, n+j)} u_i \quad (2 \leq i \leq 3),$$

$$u_2 = 2w_2 - w_1 + 2^{a_8(m+j, n+j)+2} w_2$$

and $u_3 = w_3 - 2^{a_9(m+j, n+j)+1} w_2$. If $n+2+12h_4(n+j+6, n+j+4)+2h_3(n+2, n) > m \geq n+2+2h_4(n+j+4, n+j+2)$, then we have

$$r_1(V_{m,n}^j) \cong \langle \{u_i | 0 \leq i \leq 3\} \rangle / \langle \{A_i | 0 \leq i \leq 3\} \rangle,$$

where $u_1 = w_2$,

$$u_0 = \begin{cases} 2w_3 - 4w_1 - w_0 & (n+j \equiv 2 \pmod{8} \text{ and } n+14 > m \geq n+6) \\ 2w_1 - w_0 & (n+j \equiv 2 \pmod{8} \text{ and } n+6 > m \geq n+2) \\ w_0 & (n+j \equiv 0 \pmod{8} \text{ and } n+4 > m \geq n+2), \end{cases}$$

$$u_2 = \begin{cases} 2w_2 - w_1 & (n+j \equiv 2 \pmod{8} \text{ and } n+14 > m \geq n+10) \\ 2w_2 + w_1 - h_4(n+4, n) w_0 & (n+j \equiv 2 \pmod{8} \text{ and } n+10 > m \geq n+2) \\ w_1 & (n+j \equiv 0 \pmod{8} \text{ and } n+4 > m \geq n+2), \end{cases}$$

$$u_3 = \begin{cases} w_3 - w_1 - h_4(n+4, n) w_0 & (n+j \equiv 2 \pmod{8} \text{ and } n+6 > m \geq n+2) \\ w_3 - 2w_1 & (\text{otherwise}), \end{cases}$$

$$A_0 = \begin{cases} 2u_0 & (n+j \equiv 0 \pmod{8} \text{ and } n+4 > m \geq n+2) \\ u_0 & (\text{otherwise}), \end{cases}$$

$$A_1 = \begin{cases} 2^{h_4(m+j+20, n+j)} u_1 & (n+j \equiv 2 \pmod{8}) \\ u_1 & (n+j \equiv 0 \pmod{8}), \end{cases}$$

$$A_2 = \begin{cases} 2u_2 & (n+j \equiv 2 \pmod{8} \text{ and } n+14 > m \geq n+10) \\ u_2 & (\text{otherwise}) \end{cases}$$

and

$$A_3 = \begin{cases} 4u_3 & (n+j \equiv 2 \pmod{8} \text{ and } n+14 > m \geq n+6) \\ u_3 & (\text{otherwise}). \end{cases}$$

If $n+2+2h_4(n+j+4, n+j+2) > m > n$, then we have $r_1(V_{m,n}^j) \cong 0$.

Suppose $j \equiv 2 \pmod{8}$. If $m \geq 8[(n+2)/8] + 10 + 4h_4(n, n-2)$, then we have

$$J''(r_1(V_{m,n}^j)) \cong \langle \{v_i | 0 \leq i \leq 3\} \rangle / \langle \{B_i | 0 \leq i \leq 3\} \rangle,$$

where $v_0 = w_0$, $v_1 = w_2$, $v_2 = 2w_2 + w_1$, $v_3 = w_3$, $B_0 = R_4$,

$$B_1 = 4h_4(n+10, n-2) v_1,$$

$$B_2 = 2^{h_4(n+6, n+2)b_3(j, m+2, n)} v_2$$

and $B_3 = 2^{h_4(n+6, n+2)+h_4(n+2, n-2)b_3(j, m+2, n)} v_3$. If $8[(n+2)/8] + 10 + 4h_4(n, n-2) > m \geq 8[(n+2)/8] + 10$, then we have

$$J''(r_1(V_{m,n}^j)) \cong \langle \{v_i | 0 \leq i \leq 3\} \rangle / \langle \{B_i | 0 \leq i \leq 3\} \rangle,$$

where $v_0=2w_3-w_0, v_1=w_2, v_2=2w_2+w_1, v_3=w_3, B_0=v_0, B_1=8v_1, B_2=v_2$ and $B_3=4v_3$. If $8[(n+2)/8]+10 > m < n$, then we have $U_{m,n}^j \cong 0$.

Suppose $j \equiv 6 \pmod{8}$. If $m \geq 8[(n+2)/8]+10$, then we have

$$J''(r_1(V_{m,n}^j)) \cong \langle \{v_i \mid 0 \leq i \leq 3\} \rangle / \langle \{B_i \mid 0 \leq i \leq 3\} \rangle,$$

where $v_0=w_0-4h_4(n+4, n+2)w_2, v_1=w_2, v_2=2w_2+(1-2h_4(n+4, n+2))w_1,$

$$v_3 = w_3 - 2h_4(n+4, n+2)w_2,$$

$$B_0 = h_4(n+14, n+4)v_0,$$

$$B_1 = 4h_4(n+12, n-2)v_1,$$

$$B_2 = 2^{h_4(n+6, n+2)b_3(j, m+2, n)}v_2$$

and $B_3=2^{h_4(n+6, n+2)+h_4(n+2, n-2)b_3(j, m+2, n)}v_3$. If $8[(n+2)/8]+10 > m < n$, then we have $U_{m,n}^j \cong 0$. Thus we obtain

(5.9) Suppose that $n \equiv 0 \pmod{2}$.

(1) If $m \geq n+2+12h_4(n+j+6, n+j+4)+2h_3(n+2, n)$, then we have

$$r_1(V_{m,n}^j) \cong \mathbf{Z}/2^{h_3(m+j, n+j)+1} \oplus (\oplus_{i=0}^1 \mathbf{Z}/2^{h_4(m+j-4i, n+j-4i+4)}) \oplus G(n+j),$$

where $G(n+j)$ is the group defined by (2.3).

(2) If $n+2+12h_4(n+j+6, n+j+4)+2h_3(n+2, n) > m > n$, then we have

$$r_1(V_{m,n}^j) \cong \begin{cases} \mathbf{Z}/2^{h_3(m+j, n+j)+1} \oplus \mathbf{Z}/2^{h_4(m+j-4, n+j)} \oplus \mathbf{Z}/4 & (m \geq n+6) \\ \mathbf{Z}/8 & (n+j \equiv 2 \pmod{8} \text{ and } n+6 > m \geq n+2) \\ \mathbf{Z}/2 & (n+j \equiv 0 \pmod{8} \text{ and } n+4 > m \geq n+2) \\ 0 & (\text{otherwise}). \end{cases}$$

(3) If $m \geq 8[(n+2)/8]+10+4h_4(n, n-2)h_4(j, j-4)$, then we have

$$J''(r_1(V_{m,n}^j)) \cong \begin{cases} \mathbf{Z}/8 \oplus \mathbf{Z}/2^b \oplus G(n+j) & (h_4(n+6, n+2)h_4(n+j+4, n+j-2) = 0) \\ \mathbf{Z}/4 \oplus \mathbf{Z}/2^b \oplus \mathbf{Z}/2 \oplus G(n+j) & (\text{otherwise}), \end{cases}$$

where $b=b_3(j, m+2, n)$ and $G(n+j)$ is the group defined by (2.3).

(4) If $j \equiv n+2 \equiv 2 \pmod{8}$ and $n+14 > m \geq n+10$, then we have

$$J''(r_1(V_{m,n}^j)) \cong \mathbf{Z}/8 \oplus \mathbf{Z}/4.$$

(5) If $8[(n+2)/8]+10 > m > n$, then we have $J''(r_1(V_{m,n}^j)) \cong r_1(V_{m,n}^j)$.

By (5.4) and (5.9), we obtain the results for the cases $j \equiv 2 \pmod{4}, n \equiv 0 \pmod{2}$ and $m+j \equiv 3, 4, 5, 6$ or $7 \pmod{8}$.

We now turn to the case $m+j \equiv 0 \pmod{8}$. Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & V_{m-1,n}^j & \xrightarrow{r_1} & \widetilde{KO}(S^j(L_8^{m-1}/L_8^n)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & V_{m,n}^j & \xrightarrow{r_1} & \widetilde{KO}(S^j(L_8^m/L_8^n)) & \xrightarrow{\partial} & \widetilde{KO}(S^{j+1}(L_8^m/L_8^n)) \longrightarrow 0 \\
 & & f \uparrow & & g \uparrow & & \uparrow \\
 0 \rightarrow & \widetilde{K}(S^{m+j}) & \xrightarrow{r} & \widetilde{KO}(S^{m+j}) & \xrightarrow{\partial_1} & \widetilde{KO}(S^{m+j+1}) & \longrightarrow 0 \\
 & & & & & \uparrow & \\
 & & & & & 0 &
 \end{array}$$

of exact sequences. Since $\widetilde{KO}(S^{m+j+1}) \cong \mathbf{Z}/2$, Lemma 3.10 implies that $\widetilde{K}(S^{m+j}) \cong \mathbf{Z}$ has a generator γ with

$$f(\gamma) = \begin{cases} I^{j/2}(\varphi(X_5(m-2))) & (j \equiv 6 \pmod{8}) \\ I^{j/2}(\varphi(X_7(m-2))) & (j \equiv 2 \pmod{8}) \end{cases}$$

and $r(\gamma) = 2\beta$, where β is a generator of the group $\widetilde{KO}(S^{m+j}) \cong \mathbf{Z}$. It follows from Lemma 5.5 that we have

$$2g(\beta) = r_1(f(\gamma)) = 2^{h_4(m+j-16, n+j)} W_2 + 2^{h_3(m+j-12, n+j)} W_1,$$

where $W_1 = (1 - 2h_4(n+j, n+j-4)) r_1(I^{j/2}(\varphi(\tilde{w}_1)))$ and

$$W_2 = \begin{cases} r_1(I^{j/2}(\varphi(\tilde{w}_2))) & (h_4(n+j, n+j-4) = 0) \\ r_1(I^{j/2}(\varphi(\tilde{w}_3))) & (\text{otherwise}). \end{cases}$$

If $m > n + 12$, we set

$$\alpha = g(\beta) - \begin{cases} 2^{\lfloor (m-n-14)/8 \rfloor} W_3 - 2^{\lfloor (m-n-10)/4 \rfloor} W_1 & (h_4(n+j+4, n+j) = 0) \\ 2^{\lfloor (m-n-14)/8 \rfloor} (2W_2 - W_1) + 2^{\lfloor (m-n-10)/4 \rfloor} W_1 & (\text{otherwise}), \end{cases}$$

where $W_i = r_1(I^{j/2}(\varphi(w_i)))$ ($1 \leq i \leq 3$). Then we have $\partial(\alpha) \neq 0$ and $2\alpha = 0^{m-n-14} W_0$, where $W_0 = r_1(I^{j/2}(\varphi(w_0)))$. By Lemma 3.9, Lemma 5.5, (5.8) and the fact $8g(\beta) = 0$, we have

$$(5.10) \quad \psi^k(\alpha) \equiv (k - 2 \lfloor k/2 \rfloor) \alpha \pmod{U_{m,n}^j}.$$

According to [3, II], the Adams operations on $\widetilde{KO}(S^{m+j+1})$ are given by $\psi^k = k - 2 \lfloor k/2 \rfloor$. If $m > n + 14$, then the short exact sequence

$$0 \rightarrow r_1(V_{m,n}^j) \rightarrow \widetilde{KO}(S^j(L_8^m/L_8^n)) \xrightarrow{\partial} \widetilde{KO}(S^{j+1}(L_8^m/L_8^n)) \rightarrow 0$$

splits. Hence we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong r_1(V_{m,n}^j) \oplus \mathbf{Z}/2.$$

It follows from (5.10) that we have

$$J(S^j(L_8^m/L_8^n)) \cong J''(r_1(V_{m,n}^j)) \oplus \mathbf{Z}/2.$$

If $m=n+14$, then we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) = \langle r_1(V_{m,n}^j) \cup \{\alpha\} \rangle = \langle \{W_1, W_2, W_3, \alpha\} \rangle.$$

Since $\text{ord } \widetilde{KO}(S^j(L_8^m/L_8^n))=1024$ by [13], $\text{ord } \langle \{W_1, W_2, W_3\} \rangle=256$ and $\text{ord } \langle \alpha \rangle=4$, we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \mathbf{Z}/32 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2$$

and

$$J(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/8 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/4 & (j \equiv 2 \pmod{8}) \\ \mathbf{Z}/8 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 & (j \equiv 6 \pmod{8}). \end{cases}$$

If $n+12 \geq m \geq n+2$, we set

$$\alpha = g(\beta) - ([(m-n-2)/8] + [(m-n+2)/8]) W_2.$$

Then we have

$$2\alpha = W_2 + [(m-n+2)/8] (W_3 - W_2) + [(m-n-2)/8] (W_1 - 2W_2 - W_3).$$

Hence $\widetilde{KO}(S^j(L_8^m/L_8^n)) = \langle r_1(V_{m,n}^j) \cup \{\alpha\} \rangle$

$$= \begin{cases} \langle \{W_2, W_3, \alpha\} \rangle & (m \geq n+10) \\ \langle \{W_0, W_2, \alpha\} \rangle & (m = n+8) \\ \langle \{W_2, \alpha\} \rangle & (m = n+6) \\ \langle \alpha \rangle & (n+6 > m \geq n+2), \end{cases}$$

$$\text{ord } \langle \{W_2, W_3\} \rangle = 32 \quad (m \geq n+10),$$

$$\text{ord } \langle W_2 \rangle = 8 \quad (n+10 > m \geq n+6)$$

and

$$\text{ord } \langle \alpha \rangle = 2^{[(n-m+31)/8]}.$$

Since $\text{ord } \widetilde{KO}(S^j(L_8^m/L_8^n))=2^{2h_3(m+j, n+j)+h_4(n+j, n+j-4)+1}$ by [13], we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/16 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2 & (m \geq n+10) \\ \mathbf{Z}/8 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2 & (m = n+8) \\ \mathbf{Z}/8 \oplus \mathbf{Z}/8 & (m = n+6) \\ \mathbf{Z}/8 & (n+6 > m \geq n+2) \end{cases}$$

and

$$J(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/8 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 & (m \geq n+10 \text{ and } j \equiv 6 \pmod{8}) \\ \mathbf{Z}/4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2 & (m \geq n+8 \text{ and } h_4(n+2, n-2) = 0) \\ \mathbf{Z}/8 \oplus \mathbf{Z}/4 & (m = n+6 \text{ and } j \equiv 6 \pmod{8}) \\ \widetilde{KO}(S^j(L_8^m/L_8^n)) & (\text{otherwise}). \end{cases}$$

Thus we obtain the results for the case $m+j \equiv 0 \pmod{8}$.

Modifying the proof above, we obtain the results for the case $m+j \equiv 1 \pmod{8}$ (cf. [10]).

Finally we consider the case $m+j \equiv 2 \pmod{8}$. Since

$$\widetilde{KO}(S^j(L_8^m/L_8^{m-2})) \cong \widetilde{KO}(S^{j+m-2}L_8^2) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$$

by Proposition 3.13, and the Adams operations on $\widetilde{KO}(S^j(L_8^m/L_8^{m-2}))$ are given by $\psi^k = k - 2 [k/2]$, the proof for this case is similar to the corresponding proof of [10].

Thus the proof for the case $j \equiv 2 \pmod{4}$ and $n \equiv 0 \pmod{2}$ is completed.

5.2. Proof for the case $n \equiv 1 \pmod{2}$. Consider the following commutative diagram, in which the row is exact.

$$\begin{array}{ccccccc} 0 & \rightarrow & V_{m,n+1}^j & \xrightarrow{f_1} & \widetilde{K}(S^j(L_8^m/L_8^n)) & \xrightarrow{f_2} & \widetilde{K}(S^{n+j+1}) \rightarrow 0 \\ & & \parallel & & \downarrow f_3 & & \\ & & V_{m,n+1}^j & \hookrightarrow & \widetilde{K}(S^j L_8^m) & & \end{array}$$

By Lemma 3.10, we can choose an element $x \in \widetilde{K}(S^j(L_8^m/L_8^n))$ such that $f_2(x)$ generates the group $\widetilde{K}(S^{n+j+1}) \cong \mathbf{Z}$ and

$$f_3(x) = \begin{cases} I^{j/2}(\varphi(X_5(n))) & (n \equiv 1 \pmod{8}) \\ I^{j/2}(\varphi(X_6(n) + X_1(n))) & (n \equiv 3 \pmod{8}) \\ I^{j/2}(\varphi(X_7(n))) & (n \equiv 5 \pmod{8}) \\ I^{j/2}(\varphi(X_4(n))) & (n \equiv 7 \pmod{8}) \end{cases}$$

Inspect the following commutative diagram

$$\begin{array}{ccccccc} 0 & & 0 & & \widetilde{KO}(S^{n+j+2}) & & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ VO_{m,n+1}^{j+2} & \xrightarrow{I^{-1} \circ c} & V_{m,n+1}^j & \xrightarrow{r_1} & \widetilde{KO}(S^j(L_8^m/L_8^{n+1})) & \xrightarrow{\partial} & \widetilde{KO}(S^{j+1}(L_8^m/L_8^{n+1})) \\ \downarrow & & f_1 \downarrow & & g_1 \downarrow & & h_1 \downarrow \\ \widetilde{KO}(S^{j+2}(L_8^m/L_8^n)) & \rightarrow & \widetilde{K}(S^j(L_8^m/L_8^n)) & \xrightarrow{r_2} & \widetilde{KO}(S^j(L_8^m/L_8^n)) & \xrightarrow{\partial} & \widetilde{KO}(S^{j+1}(L_8^m/L_8^n)) \\ \downarrow & & f_2 \downarrow & & g_2 \downarrow & & h_2 \downarrow \\ \widetilde{KO}(S^{n+j+3}) & \xrightarrow{I^{-1} \circ c} & \widetilde{K}(S^{n+j+1}) & \xrightarrow{r} & \widetilde{KO}(S^{n+j+1}) & \xrightarrow{\partial} & \widetilde{KO}(S^{n+j+2}) \rightarrow 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & \widetilde{KO}(S^{j-1}(L_8^m/L_8^{n+1})) & & \end{array}$$

of exact sequences. By Lemma 3.9, Proposition 3.13 and Lemma 3.15, we obtain

$$(5.11) \quad \text{Ker } r_2 = \begin{cases} \langle \{f_1(S_i) \mid i = 1, 2, 4 \text{ or } 6\} \rangle & (n \equiv 1 \pmod{4}) \\ \langle f_1(\text{Ker } r_1) \cup \{2x\} \rangle & (n+j \equiv 1 \pmod{8}) \\ \langle f_1(\text{Ker } r_1) \cup \{x\} \rangle & (n+j \equiv 5 \pmod{8}) \end{cases}$$

Suppose $m \geq n + 3$. Then we have

$$\text{Coker } g_2 \cong \widetilde{KO}(S^{n+j+2}) \cong \begin{cases} \mathbf{Z}/2 & (n+j \equiv 7 \pmod{8}) \\ 0 & (\text{otherwise}), \end{cases}$$

and hence

$$r(\widetilde{K}(S^{n+j+1})) = g_2(\widetilde{KO}(S^j(L_8^m/L_8^n))) = \begin{cases} 2\widetilde{KO}(S^{n+j+1}) & (n+j \equiv 7 \pmod{8}) \\ \widetilde{KO}(S^{n+j+1}) & (\text{otherwise}). \end{cases}$$

Since h_1 is a monomorphism, we have $\text{Ker } g_1 \subset r_1(V_{m,n+1}^j)$. Thus we obtain a split short exact sequence

$$0 \rightarrow \widetilde{KO}(S^j(L_8^m/L_8^{n+1}))/\text{Ker } g_1 \xrightarrow{\bar{g}_1} \widetilde{KO}(S^j(L_8^m/L_8^n)) \xrightarrow{\bar{g}_2} H \rightarrow 0,$$

where $\text{Ker } g_1 = \langle \{r_1(S_4), r_1(S_6)\} \rangle$ and

$$H = \begin{cases} 2\widetilde{KO}(S^{n+j+1}) & (n+j \equiv 7 \pmod{8}) \\ \widetilde{KO}(S^{n+j+1}) & (\text{otherwise}). \end{cases}$$

Applying the method used in the proof of Lemma 4.11 to x , we obtain the following result by Lemma 3.9 and (5.11).

(5.12) (1) *If $n \equiv 3 \pmod{4}$, then the Adams operations are given by*

$$\psi^k(r_2(x)) = (k - 2[k/2])r_2(x).$$

(2) *Suppose $n \equiv 1 \pmod{4}$ and $k \equiv \varepsilon \pmod{8}$, where ε is an odd integer with $-3 \leq \varepsilon \leq 3$. Then the Adams operation ψ^k is given by*

$$\psi^k(r_2(x)) = k^u r_2(x) + ((\varepsilon k^{j/2} - k^u)/8) r_2(f_1(8f_3(x))) + k^{j/2} w,$$

where $u = (n + j + 1)/2$, $w = -(\varepsilon/3)h_4(\varepsilon + 5, \varepsilon + 1)W$ and

$$W = \begin{cases} r_2(f_1(S_3 + S_5)) & (n \equiv 1 \pmod{8}) \\ r_2(f_1(2S_7 - S_3)) & (n \equiv 5 \pmod{8}). \end{cases}$$

Suppose $n \equiv 3 \pmod{4}$. Then using (5.11) and (5.12) (1), we see that the short exact sequence

$$0 \rightarrow \widetilde{KO}(S^j(L_8^m/L_8^{n+1})) \rightarrow \widetilde{KO}(S^j(L_8^m/L_8^n)) \rightarrow \widetilde{KO}(S^{n+j+1}) \rightarrow 0$$

of ψ -maps splits. This implies that

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \widetilde{KO}(S^j(L_8^m/L_8^{n+1})) \oplus \widetilde{KO}(S^{n+j+1})$$

and

$$\widetilde{J}(S^j(L_8^m/L_8^n)) \cong \widetilde{J}(S^j(L_8^m/L_8^{n+1})) \oplus \widetilde{J}(S^{n+j+1}).$$

Thus, results of the case $j \equiv 2 \pmod{4}$, $n \equiv 3 \pmod{4}$ and $m \geq n+3$ follow from those of the case $j \equiv 2 \pmod{4}$ and $n \equiv 0 \pmod{4}$.

Now, we turn to the case $n \equiv 1 \pmod{4}$. If $m \geq n+5$, then we have

$$\text{Im } r_2 \cong \mathbf{Z} \oplus \mathbf{Z}/2^{h_3(m+j, n+j)} \oplus (\oplus_{i=0}^1 \mathbf{Z}/2^{h_4(m+j-4i, n+j-4i+5)}).$$

If $n+5 > m \geq n+3$, then we have $\text{Im } r_2 \cong \mathbf{Z}$. Under the assumption of (5.12) (2), we have

$$\begin{aligned} & ((\varepsilon k^{j/2} - k^u)/8) r_2(f_1(8f_3(x))) + k^{j/2} w \\ & \equiv ((n+1)/2) N(u/2^{v_2(u)}) ((k^u - 1)/2^{v_2(u)+2}) W \pmod{g_1(U_{m, n+1}^j)}, \end{aligned}$$

where $u = (n+j+1)/2$. Thus we have $J(S^j(L_8^m/L_8^n)) \cong \widetilde{KO}(S^j(L_8^m/L_8^n))/U_1$, where U_1 is the subgroup of $\widetilde{KO}(S^j(L_8^m/L_8^n))$ generated by

$$g_1(U_{m, n+1}^j) \cup \{m((n+j+1)/2) r_2(x) + MW\}$$

with $M \equiv 1 \pmod{2}$. By (5.8), we have

$$(5.13) \quad U_1 = \begin{cases} \langle \{4v_1, 2v_2, 4v_3, M_0 v_0 + M(v_1 + v_2)\} \rangle & (n \equiv 1 \pmod{8}) \\ \langle \{8v_2, 2v_2 + v_1, 4v_3, M_0 v_0 + 2M(v_2 + v_3)\} \rangle & (n \equiv 5 \pmod{8}), \end{cases}$$

where $M_0 = m((n+j+1)/2)$, $v_2(M) = 0$, $v_i = g_1(r_1(S_{2i+1}))$ ($1 \leq i \leq 3$) and $v_0 = r_2(x)$.

This implies that we have

$$J''(\text{Im } r_2) \cong F(v) / \langle \{B_i \mid 0 \leq i \leq 3\} \rangle,$$

where $F(v)$ is the free abelian group generated by $\{v_i \mid 0 \leq i \leq 3\}$,

$$\begin{aligned} B_0 &= \begin{cases} 4M_0 v_0 & (m \geq n+9 + 12h_4(n, n-4)) \\ 2M_0 v_0 & (n+9 + 12h_4(n, n-4) > m \geq n+5 + 4h_4(n, n-4)) \\ M_0 v_0 & (n+5 + 4h_4(n, n-4) > m), \end{cases} \\ B_1 &= \begin{cases} v_1 + 2v_2 & (n \equiv 5 \pmod{8}) \\ 2M_0 v_0 + 2v_1 & (n \equiv 1 \pmod{8} \text{ and } m \geq n+13) \\ v_1 - 2v_3 & (n \equiv 1 \pmod{8} \text{ and } n+13 > m), \end{cases} \\ B_2 &= \begin{cases} 2M_0 v_0 + 4v_2 & (n \equiv 5 \pmod{8} \text{ and } m \geq n+17) \\ MM_0 v_0 + 2v_2 & (n \equiv 5 \pmod{8} \text{ and } n+17 > m \geq n+5) \\ M_0 v_0 + Mv_1 + v_2 & (n \equiv 1 \pmod{8} \text{ and } m \geq n+9) \\ v_2 + 2h_4(n, n-4) v_3 & (n+5 + 4h_4(n, n-4) > m) \end{cases} \end{aligned}$$

and

$$B_3 = \begin{cases} M_0 v_0 + 2Mv_0 + 2v_3 & (n \equiv 5 \pmod{8} \text{ and } m \geq n+9) \\ 4v_3 & (n \equiv 1 \pmod{8} \text{ and } m \geq n+5) \\ v_3 + 2h_4(n+4, n) v_2 & (n+5 + 4h_4(n+4, n) > m). \end{cases}$$

By the proof for the case $n \equiv 2 \pmod{4}$, we obtain

(5.14) Suppose $j-1 \equiv n \equiv 1 \pmod{4}$ and $m \geq n+3$.

(1) If $m+j \equiv 0 \pmod{8}$, then we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \begin{cases} (\text{Im } r_2) \oplus \mathbf{Z}/2 & (m \geq n+17) \\ \mathbf{Z} \oplus \mathbf{Z}/16 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2 & (m = n+13) \\ \mathbf{Z} \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/4 & (m = n+9) \\ \mathbf{Z} \oplus \mathbf{Z}/8 & (m = n+5) \end{cases}$$

and

$$\check{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} J''(\text{Im } r_2) \oplus \mathbf{Z}/2 & (m \geq n+17) \\ \mathbf{Z}/2M_0 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/4 & (h_4(n+4, n) = m-n-13 = 0) \\ \mathbf{Z}/4M_0 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 & (h_4(n, n-4) = m-n-13 = 0) \\ \mathbf{Z}/2M_0 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2 & (h_4(n+4, n) = m-n-9 = 0) \\ \mathbf{Z}/4M_0 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2 & (h_4(n, n-4) = m-n-9 = 0) \\ \mathbf{Z}/M_0 \oplus \mathbf{Z}/8 & (h_4(n+4, n) = m-n-5 = 0) \\ \mathbf{Z}/2M_0 \oplus \mathbf{Z}/4 & (h_4(n, n-4) = m-n-5 = 0), \end{cases}$$

where $M_0 = m((n+j+1)/2)$.

(2) If $m+j \equiv 1 \pmod{8}$, then we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \widetilde{KO}(S^j(L_8^{m-1}/L_8^n)) \oplus \mathbf{Z}/2$$

and

$$\check{J}(S^j(L_8^m/L_8^n)) \cong \check{J}(S^j(L_8^{m-1}/L_8^n)) \oplus \mathbf{Z}/2.$$

(3) If $m+j \equiv 2 \pmod{8}$, then we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong (\text{Im } r_2) \oplus \mathbf{Z}/2$$

and

$$\check{J}(S^j(L_8^m/L_8^n)) \cong J''(\text{Im } r_2) \oplus \mathbf{Z}/2.$$

Noting the fact that we have $S^j(L_8^{n+1}/L_8^n) \approx S^{n+j+1}$ and

$$S^j(L_8^{n+2}/L_8^n) \simeq \begin{cases} S^{n+j+2} \vee S^{n+j+1} & (n \equiv 1 \pmod{2}) \\ S^{n+j}L_8^2 & (n \equiv 0 \pmod{2}), \end{cases}$$

we obtain the result for the case $n \equiv 1 \pmod{2}$.

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