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**THE MULTIPLICATIVE GENUS ASSOCIATED
 WITH THE FORMAL GROUP LAW
 $(x+y-2axy) / (1-(a^2+b^2)xy)$**

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1. Introduction

The complex cobordism group $MU^*(CP^\infty)$ is isomorphic to the ring of formal power series $MU^*[[x]]$, where $x=e_{MU}(\eta)$ is the Euler class of the tautological line bundle over the infinite complex projective space CP^∞ . Since $MU^*(CP^\infty \times CP^\infty) \cong MU^*[[x_1, x_2]]$, $x_1=e_{MU}(\eta \hat{\otimes} 1)$ and $x_2=e_{MU}(1 \hat{\otimes} \eta)$, we can write

$$e_{MU}(\eta \hat{\otimes} \eta) = \sum a_{ij}^U x_1^i x_2^j.$$

The formal power series induces a formal group law over MU^*

$$F_{MU}(x, y) = \sum a_{ij}^U x^i y^j.$$

The complex cobordism ring MU^* with the formal group law F_{MU} is isomorphic to Lazard's ring with the universal formal group law [8]. Given any formal group law $F(x, y)$ over a commutative ring R , there is a ring homomorphism $\varphi: MU^* \rightarrow R$ which is called a multiplicative genus. In this paper we study the multiplicative genus $\varphi_{a,b}: MU^* \rightarrow Q$ associated with the formal group law

$$F(x, y) = \frac{x + y - 2axy}{1 - (a^2 + b^2)xy}$$

which is related to the following formal power series, called the logarithm for $F(x, y)$,

$$l(z) = \int_0^z \frac{1}{1 - 2ax + (a^2 + b^2)x^2} dx,$$

which satisfies $l(F(x, y)) = l(x) + l(y)$. The characteristic power series $Q(z) = z/l^{-1}(z)$ (cf. [3]) for the multiplicative genus is given by

$$Q(z) = \frac{z(b + a \tan bz)}{\tan bz}.$$

The cobordism classes of Milnor manifolds

$$H_{ij} = \{([z_0, z_1, \dots, z_i], [w_0, w_2, \dots, w_j]) \mid z_0 w_0 + z_1 w_1 + \dots + z_i w_i = 0\} \\ \subset CP^i \times CP^j,$$

where $i \leq j$, and the complex projective spaces CP^n generate MU^* . Let $H(x, y) = \sum [H_{ij}] x^i y^j$, and

$$\log_{MU}(z) = \frac{[CP^n]}{n+1} z^{n+1},$$

which is the logarithm for $F_{MU}(x, y) = \sum a_{i,j}^U x^i y^j$. Then relations on a_{ij}^U , $[H_{ij}]$ and $[CP^n]$ are given by the following [2]:

$$H(x, y) = \frac{d \log_{MU}(x)}{dx} \frac{d \log_{MU}(y)}{dy} F_{MU}(x, y).$$

We use the relations to calculate the multiplicative genus $\varphi_{a,b}: MU^* \rightarrow Q$ associated with the above formal group law for Milnor manifolds. The main theorem of this paper is the following.

Theorem 1.1. *Let $\varphi_{a,b}: MU^* \rightarrow Q$ be the multiplicative genus associated with the formal group law $F(x, y) = (x + y - 2axy)/(1 - (a^2 + b^2)xy)$. The values of $\varphi_{a,b}$ for the Milnor manifolds $H_{s,k}$, $s \leq k$, are as follows:*

$$\varphi_{a,b}([H_{s,k}]) = \left(\frac{\alpha^{s+1} - \beta^{s+1}}{\alpha - \beta} \right) \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} \right),$$

where $\alpha = a + b\sqrt{-1}$ and $\beta = a - b\sqrt{-1}$.

The paper is organized as follows. In Section 2 we study multiplicative idempotent natural transformations over the cobordism cohomology $MU^*(-) \otimes Q$ which induce multiplicative genera. In Section 3 we investigate the multiplicative genus $\varphi_{a,b}: MU^* \rightarrow Q$ and we give a proof of Theorem 1.1. In Section 4 we discuss multiplicative genus related to the logarithm given by the integral of $1/\{\text{a polynomial}\}$.

2. The decomposition of $MU^* \otimes Q$ and the multiplicative genus

Let $p(t_1, \dots, t_n)$ be a symmetric polynomial, and let

$$p(t_1, \dots, t_n) = P(\sigma_1, \dots, \sigma_n)$$

where σ_i is the i -th elementary symmetric polynomial. For a complex vector bundle ξ over X with $\dim_{\mathbb{C}} \xi = n$ we have

$$P(c_1(\xi), \dots, c_n(\xi)) \in MU^*(X)$$

where $c_i(\xi)$ is the i -th complex cobordism Chern class, and

$$S_P(\xi) = \Phi_{\xi}(P(c_1(\xi), \dots, c_n(\xi))) \in \widetilde{MU}^*(T(\xi))$$

where $T(\xi)$ is the Thom complex of ξ and Φ_{ξ} is the Thom isomorphism. Let

$$\alpha = \{h\} \in MU^k(X) = \lim_{n \rightarrow \infty} [S^{2n-k} X^+, MU(n)]_0,$$

$$h: S^{2n-k} X^+ \rightarrow MU(n) = T(\gamma^n),$$

where $\gamma^n \rightarrow BU(n)$ is the universal complex vector bundle over $BU(n)$. The complex cobordism cohomology operation $S_P: MU^*(X) \rightarrow MU^*(X)$ is given by

$$S_P(\alpha) = \sigma^{k-2n} h^*(S_P(\gamma^n))$$

where σ^{k-2n} denotes the $(k-2n)$ -fold suspension isomorphism. For any set $\omega = (i_1, \dots, i_q)$ of positive integers, denote $S_{\omega}(t)$ the smallest symmetric function of variable t_j , $1 \leq j \leq n$, which contains the monomial $t_1^{i_1} \dots t_q^{i_q}$ and write

$$S_{\omega}(t) = P_{\omega}(\sigma_1, \dots, \sigma_n).$$

Then we have the Landweber-Novikov operation $S_{\omega}(\alpha) = S_{P_{\omega}}(\alpha)$ (cf. [6] and [4]).

Given a formal power series

$$\begin{aligned} \hat{f}(x) &= 1 + v_1 x + v_2 x^2 + \dots \\ f(x) &= x \hat{f}(x) \end{aligned}$$

where $v_i \in MU^{-2i}$, we have a symmetric polynomial

$$\hat{f}(t_1) \cdots \hat{f}(t_n) = P_f(\sigma_1, \dots, \sigma_n).$$

The multiplicative natural operation $S_f: MU^*(X) \rightarrow MU^*(X)$ given by

$$S_f(\alpha) = S_{P_f}(\alpha)$$

satisfies

- (1) $S_f(g^*(\alpha)) = g^*(S_f(\alpha))$, for any map $g: Y \rightarrow X$
- (2) $S_f(\alpha\beta) = S_f(\alpha)S_f(\beta)$
- (3) $S_f(c_1(\gamma^1)) = f(c_1(\gamma^1))$.

If $v_i \in MU^* \otimes Q$ then we similarly obtain the multiplicative natural operation $S_f: MU^*(-) \otimes Q \rightarrow MU^*(-) \otimes Q$. We now define

$$\text{mog}_{MU}^f(x) = \sum \frac{S_f([CP^n])}{n+1} x^{n+1} \in MU^* \hat{\otimes} Q[[x]] \text{ (cf. [10]).}$$

Since the logarithm $\log_{MU}(x)$ for the formal group law $F_{MU}(x,y)$ uniquely exists,

$$S_f(\log_{MU}(x)) = \log_{MU}(x)$$

and for $x = c_1(\gamma^1)$,

$$\text{mog}_{MU}^f(f(x)) = \log_{MU}(x).$$

$MU^* \otimes Q$ is the polynomial ring over Q generated by $\{[CP^1], [CP^2], \dots, [CP^n], \dots\}$. Given a formal power series

$$g(x) = x + \frac{b_1}{2}x^2 + \dots + \frac{b_n}{n+1}x^{n+1} + \dots$$

in $MU^* \otimes Q[[x]]$, $b_i \in MU^{-2i} \otimes Q$, we can consider the ring homomorphism

$$\psi_g: MU^* \otimes Q \rightarrow MU^* \otimes Q, \psi_g([CP^n]) = b_n.$$

The formal power series $g(x)$ is said to be *projective* if ψ_g is the projection, namely $\psi_g \psi_g = \psi_g$.

Proposition 2.1. *Suppose that $g(x) \in MU^* \otimes Q[[x]]$ is projective. Let*

$$f(x) = g^{-1}(\log_{MU}(x)) = x + v_1x^2 + v_2x^3 + \dots + v_nx^{n+1} + \dots.$$

Then the multiplicative operation $S_f: MU^(-) \otimes Q \rightarrow MU^*(-) \otimes Q$ satisfies the following properties.*

- (1) $S_f(S_f(c_1(\gamma^1))) = S_f(c_1(\gamma^1))$
- (2) $S_f(v_n) = 0$, for any $n \geq 1$.

Proof. Since $g(f(x)) = \log_{MU}(x) = \text{mog}_{MU}^f(f(x))$ for $x = c_1(\gamma^1)$, we have

$S_f([CP^n])=b_n$. Since the operation S_f on $MU^* \otimes Q$ coincides with ψ_g and $g(x)$ is projective, we get $S_f(S_f([CP^n]))=S_f([CP^n])$. Apply S_f to $\text{mog}_{MU}^f(S_f(x))=\text{log}_{MU}(x)$, $x=c_1(\gamma^1)$, to obtain

$$\text{mog}_{MU}^f(S_f(S_f(x)))=\text{mog}_{MU}^f(S_f(x)).$$

Hence (1) follows. Apply S_f to $f(x)=x+v_1x^2+v_2x^3+\dots+v_nx^{n+1}+\dots$, $x=c_1(\gamma^1)$, to get

$$f(x)=S_f(S_f(x))=f(x)+S_f(v_1)(f(x))^2+S_f(v_2)(f(x))^3+\dots$$

and $S_f(v_i)=0$, $i > 0$. \square

Suppose that $g(x)(\in MU^* \otimes Q[[x]])$ is projective. Put

$$f(x)=g^{-1}(\text{log}_{MU}(x))$$

and

$$f(x)=x+v_1x^2+v_2x^3+\dots+v_nx^{n+1}+\dots, v_i \in MU^{-2i} \otimes Q.$$

Then we have a natural transformation

$$\begin{aligned} \varepsilon_f: MU^*(X) \otimes Q &\rightarrow MU^*(X) \otimes Q, \\ \varepsilon_f(\alpha) &= \sum v_1^{r_1} \cdots v_k^{r_k} S_{\underbrace{(1, \dots, 1)}_{r_1}, \dots, \underbrace{(k, \dots, k)}_{r_k}}^{(\alpha)}. \end{aligned}$$

Theorem 2.2. *The natural transformation ε_f satisfies the following:*

- (1) ε_f is multiplicative.
- (2) $\varepsilon_f \varepsilon_f = \varepsilon_f$.
- (3) $\varepsilon_f(MU^*(-) \otimes Q)$ is a generalized cohomology.

Proof. Let

$$S^{r_1, \dots, r_k}(\alpha) = S_{\underbrace{(1, \dots, 1)}_{r_1}, \dots, \underbrace{(k, \dots, k)}_{r_k}}^{(\alpha)}$$

We then have

$$S^R(\alpha\beta) = \sum_{R'+R''=R} S^{R'}(\alpha)S^{R''}(\beta),$$

which completes the proof of (1). We can see that $\varepsilon_f(x)=f(x)$, for

$x=c_1(\gamma^1)$, and so $\varepsilon_f=S_f$. Therefore (2) is an immediate result of Proposition 3.1(2) and $\varepsilon_f(MU^*(-)\otimes Q)$ is a direct summand of $MU^*(-)\otimes Q$. (3) follows from (2). \square

EXAMPLE. (1) (Brown-Peterson) The formal power series

$$g(x) = \sum_{n=0}^{\infty} \frac{[CP^{p^n-1}]}{p^n} x^{p^n}$$

is projective. Since the coefficients of $f(x)=g^{-1}(\log_{MU}(x))$ exist in $MU^*\otimes Z_{(p)}$ (cf. [1] and [10]), we have a natural idempotent operation

$$\varepsilon_f: MU^*(-)\otimes Z_{(p)} \rightarrow MU^*(-)\otimes Z_{(p)}$$

and the Brown-Peterson cohomology $BP^*(-)=\varepsilon_f(MU^*(-)\otimes Z_{(p)})$.

(2) (Ochanine) The formal power series

$$g(x) = \int_0^x \frac{1}{\sqrt{1-2[CP^2]x^2+(3[CP^2]^2-2[CP^4])x^4}} dx$$

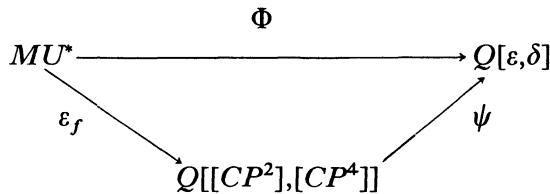
is projective. The multiplicative idempotent operation ε_f for $f(x)=g^{-1}(\log_{MU}(x))$ gives rise to a generalized cohomology $h^*(-)=\varepsilon_f(MU^*(-)\otimes Q)$ with $h^*(a \text{ point})=Q[[CP^2], [CP^4]]$ and the multiplicative genus

$$\varphi = \varepsilon_f: MU^* \rightarrow Q[[CP^2],[CP^4]].$$

The Ochanine genus $\Phi: MU^* \rightarrow Q[\varepsilon,\delta]$ (cf.[7] and [5]) is the multiplicative genus associated with the formal group law $F(x,y)=l^{-1}(l(x)+l(y))$ with

$$l(x) = \int_0^x \frac{1}{\sqrt{1-2\delta x^2 + \varepsilon x^4}} dx,$$

and $\Phi([CP^2])=\delta$ and $\Phi([H_{3,2}])=\varepsilon$ (cf. Proposition 3 of [7]). Thus the Ochanine multiplicative genus is represented as the following composite.



Here ψ is the ring homomorphism defined by

$$\psi([CP^2]) = \delta \text{ and } \psi([CP^4]) = (3\delta^2 - \varepsilon)/2.$$

(3) The formal power series

$$g(x) = \int_0^x \frac{1}{1 - [CP^1]x + [CP^1]^2x^2} dx$$

is projective. The multiplicative idempotent operation ε_f for $f(x) = g^{-1}(\log_{MU}(x))$ induces the multiplicative genus $\varepsilon_f: MU^* \rightarrow Q[[CP^1]]$. The values for complex projective spaces are as follows.

$$\begin{aligned} \varepsilon_f([CP^{3n}]) &= (-1)^n [CP^1]^{3n}, \quad \varepsilon_f([CP^{3n+1}]) = (-1)^n [CP^1]^{3n+1}, \\ \varepsilon_f([CP^{3n+2}]) &= 0. \end{aligned}$$

3. The genus associated with $(x+y-2axy)/(1-(a^2+b^2)xy)$

Let $\varphi_{a,b}: MU^* \rightarrow Q$ be the multiplicative genus associated with the formal group law $F(x,y) = (x+y-2axy)/(1-(a^2+b^2)xy)$ for rational numbers a and b . The logarithm of $\varphi_{a,b}$ is

$$l(x) = \int_0^x \frac{1}{1 - 2ax + (a^2 + b^2)x^2} dx.$$

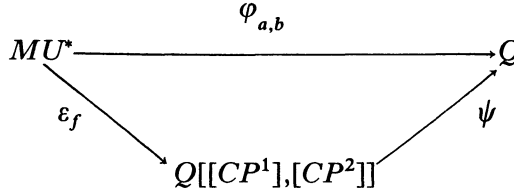
Consider the formal power series

$$g(x) = \int_0^x \frac{1}{1 - [CP]x + ([CP^1]^2 - [CP^2])x^2} dx$$

which is projective. By Theorem 2.2 it induces a multiplicative idempotent operation ε_f for $f(x) = g^{-1}(\log_{MU}(x))$ which gives a generalized cohomology $h^*(-) = \varepsilon_f(MU^*(-) \otimes Q)$ with $h^*(a \text{ point}) = Q[[CP^1], [CP^2]]$ and the multiplicative genus

$$\varphi = \varepsilon_f: MU^* \rightarrow Q[[CP^1], [CP^2]].$$

The multiplicative genus $\varphi_{a,b}$ is represented as the following composite.



Here ψ is the ring homomorphism defined by $\psi([CP^1])=2a$ and $\psi([CP^2])=3a^2-b^2$.

The characteristic power series for the multiplicative genus $\varphi_{a,b}$ is given by $Q(z)=z/l^{-1}(z)$. Since

$$l(x) = \frac{1}{b} \left(\arctan \left(\frac{a^2+b^2}{b} \left(x - \frac{a}{a^2+b^2} \right) \right) + \arctan \frac{a}{b} \right),$$

it follows that

$$Q(z) = \frac{z(b + a \tan bz)}{\tan bz}.$$

For rational numbers a and b , put

$$h(x,y) = \frac{x+y-2axy}{1-(a^2+b^2)xy} \cdot \frac{1}{1-2ax+(a^2+b^2)x^2} \cdot \frac{1}{1-2ay+(a^2+b^2)y^2}.$$

Then

$$(1-(a^2+b^2)xy)(1-2ax+(a^2+b^2)x^2)(1-2ay+(a^2+b^2)y^2)h(x,y) = x+y-2axy$$

and for $k \geq 3$

$$\begin{aligned} & \frac{\partial^k h}{\partial y^k}(x,0) - k(2a+(a^2+b^2)x) \frac{\partial^{k-1} h}{\partial y^{k-1}}(x,0) \\ & + k(k-1)(a^2+b^2)(1+2ax) \frac{\partial^{k-2} h}{\partial y^{k-2}}(x,0) \\ & - k(k-1)(k-2)(a^2+b^2)^2 x \frac{\partial^{k-3} h}{\partial y^{k-3}}(x,0) = 0. \end{aligned}$$

Proposition 3.1. Let $\alpha = a + b\sqrt{-1}$ and $\beta = a - b\sqrt{-1}$. Then

$$\frac{\partial^k h}{\partial y^k}(x,0) = \frac{k!}{(1-\alpha x)(1-\beta x)} \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} + \alpha^k \beta^k x^{k+1} \right).$$

Proof. Put

$$P_k = \frac{\partial^k h}{\partial y^k}(x, 0)$$

$$Q_k = P_k - 2akP_{k-1} + k(k-1)(a^2 + b^2)P_{k-2}.$$

We then have

$$Q_k - k(a^2 + b^2)xQ_{k-1} = 0$$

and

$$Q_k = \frac{k!}{2}(a^2 + b^2)^{k-2}x^{k-2}Q_2.$$

Thus it follows that

$$P_k - 2akP_{k-1} + k(k-1)(a^2 + b^2)P_{k-2} = \frac{k!}{2}(a^2 + b^2)^{k-2}x^{k-2}Q_2.$$

Let $\alpha = a + b\sqrt{-1}$, $\beta = a - b\sqrt{-1}$ and $R_k = P_k - k\alpha P_{k-1}$. Then we get

$$R_k - k\beta R_{k-1} = (\alpha\beta x)^{k-2} \frac{k!}{2} Q_2$$

and

$$R_k - \beta^{k-2} \frac{k!}{2} R_2 = \frac{k!}{2} \beta^{k-2} (P_2 - 2(\alpha + \beta)P_1 + 2\alpha\beta P_0) \sum_{i=1}^{k-2} (\alpha x)^i.$$

Note that

$$P_0 = \frac{x}{(1 - \alpha x)(1 - \beta x)}$$

$$P_1 = \frac{1 + \alpha\beta x^2}{(1 - \alpha x)(1 - \beta x)}$$

$$P_2 = \frac{2(\alpha + \beta) + 2\alpha^2\beta^2 x^3}{(1 - \alpha x)(1 - \beta x)}.$$

Then we get

$$R^k = k! \beta^{k-1} \frac{1 - (\alpha x)^k (1 - \beta x)}{(1 - \alpha x)(1 - \beta x)}$$

and

$$P_k = \frac{k!}{(1-\alpha x)(1-\beta x)} \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} + \alpha^k \beta^k x^{k+1} \right). \quad \square$$

We utilize Proposition 3.1 to obtain Theorem 1.1.

Proof of Theorem 1.1.

The $h(x,y)$ is described as

$$h(x,y) = h(x,0) + \frac{\partial h}{\partial y}(x,0)y + \frac{1}{2!} \frac{\partial^2 h}{\partial y^2}(x,0)y^2 + \dots + \frac{1}{k!} \frac{\partial^k h}{\partial y^k}(x,0)y^k + \dots.$$

From Proposition 3.1 it follows that

$$\frac{1}{k!} \frac{\partial^k h}{\partial y^k}(x,0) = \frac{\alpha^k - \beta^k}{\alpha - \beta} \sum_{s=0}^{\infty} \left(\sum_{i+j=s} \alpha^i \beta^j \right) x^s + \alpha^k \beta^k \sum_{s=0}^{\infty} \left(\sum_{i+j=s} \alpha^i \beta^j \right) x^{s+k+1}$$

and the coefficient of $x^s y^k$ is

$$\frac{\alpha^k - \beta^k}{\alpha - \beta} \sum_{i+j=s} \alpha^i \beta^j = \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} \right) \left(\frac{\alpha^{s+1} - \beta^{s+1}}{\alpha - \beta} \right). \quad \square$$

REMARK. When $b=0$ in this theorem,

$$\varphi_{a,0}([H_{s,k}]) = (s+1)k a^{s+k-1}.$$

Especially $\varphi_{1,0}$ corresponds to the Euler characteristic and $\varphi_{0,b}$ is the Ochanine multiplicative genus of case $\delta = -b^2$ and $\varepsilon = b^4$.

Let ξ be the canonical line bundle over CP^2 and let $P(m,n)$ denote the projective space bundle associated with $\xi^m \oplus \xi^n$.

Proposition 3.2.

$$[P(m,n)] = -\frac{(m-n)^2}{3} [H_{2,2}] + \frac{(m-n)^2 + 3}{3} [CP^2][CP^1]$$

Proof. Put

$$[P(m,n)] = x[H_{2,2}] + y[CP^2][CP^1] + z[CP^1]^3.$$

We can determine x , y and z by the facts that

$$\begin{aligned}
 S_{(3)}(P(m,n)) &= 2(m-n)^2, \quad S_{(2,1)}(P(m,n)) = 6, \quad S_{(1,1,1)}(P(m,n)) = 6, \\
 S_{(3)}(H_{2,2}) &= -6, \quad S_{(2,1)}(H_{2,2}) = 6, \quad S_{(1,1,1)}(H_{2,2}) = 6, \\
 S_{(3)}(CP^2 \times CP^1) &= 0, \quad S_{(2,1)}(CP^2 \times CP^1) = 6, \quad S_{(1,1,1)}(CP^2 \times CP^1) = 6, \\
 S_{(3)}((CP^1)^3) &= 0, \quad S_{(2,1)}((CP^1)^3) = 0, \quad S_{(1,1,1)}((CP^1)^3) = 8. \quad \square
 \end{aligned}$$

To the formal group law $F(x,y) = (x+y-2axy)/(1-(a^2+b^2)xy)$ we apply

$$a = \frac{\delta}{2} \quad \text{and} \quad b = \frac{\sqrt{3}\delta}{2}.$$

The logarithm is

$$l(x) = \int_0^x \frac{1}{1-\delta x + \delta^2 x^2} dx.$$

Let $\varphi_\delta: MU^* \rightarrow Q$ denote the multiplicative genus associated with the formal group law $F(x,y) = (x+y-\delta xy)/(1-\delta^2 xy)$. Then we have the following.

Theorem 3.3. *The ideal in $MU^* \otimes Q$ consisting of α with $\varphi_\delta(\alpha) = 0$ for any δ is generated by cobordism classes of fibre bundles over CP^2 .*

Proof. Let $S_{(n)}(M)$ denote the characteristic number of M corresponding to the symmetric polynomial Σt_i^n . The characteristic number of the Milnor manifold (cf.[9]) is

$$S_{(n+m-1)}(H_{n,m}) = -\binom{n+m}{n}, \quad 2 \leq n \leq m.$$

We take a generating set $\{[CP^1], [CP^2], [H_{2,j}] | j \geq 2\}$ of the ring $MU^* \otimes Q$. We use Theorem 1.1 to get $\varphi_\delta([CP^1]) = \delta$, $\varphi_\delta([CP^2]) = 0$ and $\varphi_\delta([H_{2,j}]) = 0$, $j \geq 2$. Therefore the kernel of φ_δ is generated by $\{[CP^2], [H_{2,j}], j \geq 2\}$. If $j > 2$, $H_{2,j}$ is a fibre bundle over CP^2 with the fibre CP^{j-1} . By using Proposition 3.3 we see that $[H_{2,2}]$ belongs to the ideal generated by cobordism classes of fibre bundles over CP^2 . \square

4. Genera cancelling $[H_{2,j}]$, $j > n$

We discuss the multiplicative genus associated with the formal group law $F(x,y) = l^{-1}(l(x)+l(y))$ with the logarithm $l(x)$ given by the integral

of $1/\{\text{a polynomial}\}$ which is a generalization of the logarithm for the multiplicative genus φ_δ in Section 3. For rational numbers $\delta_1, \delta_2, \dots, \delta_n$, we consider a formal power series

$$l(z) = \int_0^z \frac{1}{1 + \delta_1 x + \delta_1^2 x^2 + \delta_2 x^3 + \dots + \delta_n x^{n+1}} dx$$

and denote the multiplicative genus associated with the formal group law $F(x, y) = l^{-1}(l(x) + l(y))$ by

$$\varphi_{\delta_1, \dots, \delta_n}: MU^* \rightarrow Q.$$

We then have the following

Proposition 4.1.

(1) $\varphi_{\delta_1, \dots, \delta_n}([CP^1]) = -\delta_1, \varphi_{\delta_1, \dots, \delta_n}([CP^2]) = 0, \varphi_{\delta_1, \dots, \delta_n}([H_{2,1}]) = \delta_1^2$

(2) $\varphi_{\delta_1, \dots, \delta_n}([H_{2,j}]) = \begin{cases} \frac{j+1}{2} \delta_j, & 2 \leq j \leq n \\ 0, & j > n \end{cases}$

Proof. The logarithm $l(x)$ for $F(x, y)$ is described as

$$l(x) = \sum \frac{\varphi_{\delta_1, \dots, \delta_n}([CP^n])}{n+1} x^{n+1}.$$

We obtain (1) by the facts that

$$l(z) = z - \frac{\delta_1}{2} z^2 + \text{higher terms of degree} \geq 4$$

and $[H_{2,1}] = [CP^1]^2$. Consider the formal power series

$$h(x, y) = \sum \varphi_{\delta_1, \dots, \delta_n}([H_{i,j}]) x^i y^j.$$

The Buchstaber formula [2]

$$H(x, y) = (\sum [CP^n] x^n) (\sum [CP^n] y^n) F_{MU}(x, y)$$

implies

$$\begin{aligned} h(x, y) &= l'(x)l'(y)F(x, y) \\ &= \left(l'(0) + \frac{l''(0)}{1!}x + \dots + \frac{l^{(n+1)}(0)}{n!}x^n + \dots \right) \cdot l'(y). \end{aligned}$$

$$\left(F(0,y) + \frac{1}{1!} \frac{\partial F}{\partial x}(0,y)x + \frac{1}{2!} \frac{\partial^2 F}{\partial x^2}(0,y)x^2 + \dots + \frac{1}{n!} \frac{\partial^n F}{\partial x^n}(0,y)x^n + \dots \right).$$

Comparing the coefficients of x^2 , we have

$$\begin{aligned} & h_{2,0} + h_{2,1}y + h_{2,2}y^2 + \dots + h_{2,n}y^n + \dots \\ &= l'(y) \cdot \left(\frac{l'(0)}{2} \frac{\partial^2 F}{\partial x^2}(0,y) + l''(0) \frac{\partial F}{\partial x}(0,y) + \frac{l^{(3)}(0)}{2} F(0,y) \right), \end{aligned}$$

where $h_{i,j} = \varphi_{\delta_1, \dots, \delta_n}([H_{i,j}])$. Since $l(F(x,y)) = l(x) + l(y)$, it follows that

$$l'(F(x,y)) \frac{\partial F}{\partial x} = l'(x)$$

and

$$l''(F(x,y)) \left(\frac{\partial F}{\partial x} \right)^2 + l'(F(x,y)) \frac{\partial^2 F}{\partial x^2} = l''(x).$$

From the facts that $l'(0) = 1$, $l''(0) = -\delta_1$ and $l^{(3)}(0) = 0$, it follows that

$$\begin{aligned} & h_{2,0} + h_{2,1}y + h_{2,2}y^2 + \dots + h_{2,n}y^n + \dots \\ &= -\delta_1 + \delta_1^2 y + \frac{3}{2} \delta_2 y^2 + \dots + \frac{n+1}{2} \delta_n y^n \end{aligned}$$

and we complete the proof of (2). □

We can take a generating set $\{[CP^1], [CP^2], [H_{2,2}], \dots, [H_{2,n}], \dots\}$ for the polynomial ring $MU^* \otimes Q$. Therefore it follows from Proposition 4.1 that

Theorem 4.2. *$[M] (\in MU^* \otimes Q)$ belongs to the ideal generated by $\{[CP^2], [H_{2,j}] (j > n)\}$ in $MU^* \otimes Q$ if and only if for any $\delta_1, \dots, \delta_{n-1}$ and $\delta_n (\in Q)$*

$$\varphi_{\delta_1, \dots, \delta_n}([M]) = 0.$$

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