

Title	The multiplicative genus associated with the formal group law $(x+y-2axy)/(1-(a^2+b^2)xy)$
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Citation	Osaka Journal of Mathematics. 1994, 31(2), p. 473-486
Version Type	VoR
URL	<a href="https://doi.org/10.18910/7447">https://doi.org/10.18910/7447</a>
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**THE MULTIPLICATIVE GENUS ASSOCIATED  
 WITH THE FORMAL GROUP LAW  
 $(x+y-2axy) / (1-(a^2+b^2)xy)$**

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(Received November 21, 1992)

(Revised January 25, 1993)

**1. Introduction**

The complex cobordism group  $MU^*(CP^\infty)$  is isomorphic to the ring of formal power series  $MU^*[[x]]$ , where  $x=e_{MU}(\eta)$  is the Euler class of the tautological line bundle over the infinite complex projective space  $CP^\infty$ . Since  $MU^*(CP^\infty \times CP^\infty) \cong MU^*[[x_1, x_2]]$ ,  $x_1=e_{MU}(\eta \hat{\otimes} 1)$  and  $x_2=e_{MU}(1 \hat{\otimes} \eta)$ , we can write

$$e_{MU}(\eta \hat{\otimes} \eta) = \sum a_{ij}^U x_1^i x_2^j.$$

The formal power series induces a formal group law over  $MU^*$

$$F_{MU}(x, y) = \sum a_{ij}^U x^i y^j.$$

The complex cobordism ring  $MU^*$  with the formal group law  $F_{MU}$  is isomorphic to Lazard's ring with the universal formal group law [8]. Given any formal group law  $F(x, y)$  over a commutative ring  $R$ , there is a ring homomorphism  $\varphi: MU^* \rightarrow R$  which is called a multiplicative genus. In this paper we study the multiplicative genus  $\varphi_{a,b}: MU^* \rightarrow Q$  associated with the formal group law

$$F(x, y) = \frac{x + y - 2axy}{1 - (a^2 + b^2)xy}$$

which is related to the following formal power series, called the logarithm for  $F(x, y)$ ,

$$l(z) = \int_0^z \frac{1}{1 - 2ax + (a^2 + b^2)x^2} dx,$$

which satisfies  $l(F(x, y)) = l(x) + l(y)$ . The characteristic power series  $Q(z) = z/l^{-1}(z)$  (cf. [3]) for the multiplicative genus is given by

$$Q(z) = \frac{z(b + a \tan bz)}{\tan bz}.$$

The cobordism classes of Milnor manifolds

$$H_{ij} = \{([z_0, z_1, \dots, z_i], [w_0, w_2, \dots, w_j]) \mid z_0 w_0 + z_1 w_1 + \dots + z_i w_i = 0\} \\ \subset CP^i \times CP^j,$$

where  $i \leq j$ , and the complex projective spaces  $CP^n$  generate  $MU^*$ . Let  $H(x, y) = \sum [H_{ij}] x^i y^j$ , and

$$\log_{MU}(z) = \frac{[CP^n]}{n+1} z^{n+1},$$

which is the logarithm for  $F_{MU}(x, y) = \sum a_{i,j}^U x^i y^j$ . Then relations on  $a_{ij}^U$ ,  $[H_{ij}]$  and  $[CP^n]$  are given by the following [2]:

$$H(x, y) = \frac{d \log_{MU}(x)}{dx} \frac{d \log_{MU}(y)}{dy} F_{MU}(x, y).$$

We use the relations to calculate the multiplicative genus  $\varphi_{a,b}: MU^* \rightarrow Q$  associated with the above formal group law for Milnor manifolds. The main theorem of this paper is the following.

**Theorem 1.1.** *Let  $\varphi_{a,b}: MU^* \rightarrow Q$  be the multiplicative genus associated with the formal group law  $F(x, y) = (x + y - 2axy)/(1 - (a^2 + b^2)xy)$ . The values of  $\varphi_{a,b}$  for the Milnor manifolds  $H_{s,k}$ ,  $s \leq k$ , are as follows:*

$$\varphi_{a,b}([H_{s,k}]) = \left( \frac{\alpha^{s+1} - \beta^{s+1}}{\alpha - \beta} \right) \left( \frac{\alpha^k - \beta^k}{\alpha - \beta} \right),$$

where  $\alpha = a + b\sqrt{-1}$  and  $\beta = a - b\sqrt{-1}$ .

The paper is organized as follows. In Section 2 we study multiplicative idempotent natural transformations over the cobordism cohomology  $MU^*(-) \otimes Q$  which induce multiplicative genera. In Section 3 we investigate the multiplicative genus  $\varphi_{a,b}: MU^* \rightarrow Q$  and we give a proof of Theorem 1.1. In Section 4 we discuss multiplicative genus related to the logarithm given by the integral of  $1/\{\text{a polynomial}\}$ .

**2. The decomposition of  $MU^* \otimes Q$  and the multiplicative genus**

Let  $p(t_1, \dots, t_n)$  be a symmetric polynomial, and let

$$p(t_1, \dots, t_n) = P(\sigma_1, \dots, \sigma_n)$$

where  $\sigma_i$  is the  $i$ -th elementary symmetric polynomial. For a complex vector bundle  $\xi$  over  $X$  with  $\dim_{\mathbb{C}} \xi = n$  we have

$$P(c_1(\xi), \dots, c_n(\xi)) \in MU^*(X)$$

where  $c_i(\xi)$  is the  $i$ -th complex cobordism Chern class, and

$$S_P(\xi) = \Phi_{\xi}(P(c_1(\xi), \dots, c_n(\xi))) \in \widetilde{MU}^*(T(\xi))$$

where  $T(\xi)$  is the Thom complex of  $\xi$  and  $\Phi_{\xi}$  is the Thom isomorphism. Let

$$\alpha = \{h\} \in MU^k(X) = \lim_{n \rightarrow \infty} [S^{2n-k} X^+, MU(n)]_0,$$

$$h: S^{2n-k} X^+ \rightarrow MU(n) = T(\gamma^n),$$

where  $\gamma^n \rightarrow BU(n)$  is the universal complex vector bundle over  $BU(n)$ . The complex cobordism cohomology operation  $S_P: MU^*(X) \rightarrow MU^*(X)$  is given by

$$S_P(\alpha) = \sigma^{k-2n} h^*(S_P(\gamma^n))$$

where  $\sigma^{k-2n}$  denotes the  $(k-2n)$ -fold suspension isomorphism. For any set  $\omega = (i_1, \dots, i_q)$  of positive integers, denote  $S_{\omega}(t)$  the smallest symmetric function of variable  $t_j$ ,  $1 \leq j \leq n$ , which contains the monomial  $t_1^{i_1} \dots t_q^{i_q}$  and write

$$S_{\omega}(t) = P_{\omega}(\sigma_1, \dots, \sigma_n).$$

Then we have the Landweber-Novikov operation  $S_{\omega}(\alpha) = S_{P_{\omega}}(\alpha)$  (cf. [6] and [4]).

Given a formal power series

$$\begin{aligned} \hat{f}(x) &= 1 + v_1 x + v_2 x^2 + \dots \\ f(x) &= x \hat{f}(x) \end{aligned}$$

where  $v_i \in MU^{-2i}$ , we have a symmetric polynomial

$$\hat{f}(t_1) \cdots \hat{f}(t_n) = P_f(\sigma_1, \dots, \sigma_n).$$

The multiplicative natural operation  $S_f: MU^*(X) \rightarrow MU^*(X)$  given by

$$S_f(\alpha) = S_{P_f}(\alpha)$$

satisfies

- (1)  $S_f(g^*(\alpha)) = g^*(S_f(\alpha))$ , for any map  $g: Y \rightarrow X$
- (2)  $S_f(\alpha\beta) = S_f(\alpha)S_f(\beta)$
- (3)  $S_f(c_1(\gamma^1)) = f(c_1(\gamma^1))$ .

If  $v_i \in MU^* \otimes Q$  then we similarly obtain the multiplicative natural operation  $S_f: MU^*(-) \otimes Q \rightarrow MU^*(-) \otimes Q$ . We now define

$$\text{mog}_{MU}^f(x) = \sum \frac{S_f([CP^n])}{n+1} x^{n+1} \in MU^* \hat{\otimes} Q[[x]] \text{ (cf. [10]).}$$

Since the logarithm  $\log_{MU}(x)$  for the formal group law  $F_{MU}(x,y)$  uniquely exists,

$$S_f(\log_{MU}(x)) = \log_{MU}(x)$$

and for  $x = c_1(\gamma^1)$ ,

$$\text{mog}_{MU}^f(f(x)) = \log_{MU}(x).$$

$MU^* \otimes Q$  is the polynomial ring over  $Q$  generated by  $\{[CP^1], [CP^2], \dots, [CP^n], \dots\}$ . Given a formal power series

$$g(x) = x + \frac{b_1}{2}x^2 + \dots + \frac{b_n}{n+1}x^{n+1} + \dots$$

in  $MU^* \otimes Q[[x]]$ ,  $b_i \in MU^{-2i} \otimes Q$ , we can consider the ring homomorphism

$$\psi_g: MU^* \otimes Q \rightarrow MU^* \otimes Q, \psi_g([CP^n]) = b_n.$$

The formal power series  $g(x)$  is said to be *projective* if  $\psi_g$  is the projection, namely  $\psi_g \psi_g = \psi_g$ .

**Proposition 2.1.** *Suppose that  $g(x) \in MU^* \otimes Q[[x]]$  is projective. Let*

$$f(x) = g^{-1}(\log_{MU}(x)) = x + v_1x^2 + v_2x^3 + \dots + v_nx^{n+1} + \dots.$$

*Then the multiplicative operation  $S_f: MU^*(-) \otimes Q \rightarrow MU^*(-) \otimes Q$  satisfies the following properties.*

- (1)  $S_f(S_f(c_1(\gamma^1))) = S_f(c_1(\gamma^1))$
- (2)  $S_f(v_n) = 0$ , for any  $n \geq 1$ .

**Proof.** Since  $g(f(x)) = \log_{MU}(x) = \text{mog}_{MU}^f(f(x))$  for  $x = c_1(\gamma^1)$ , we have

$S_f([CP^n])=b_n$ . Since the operation  $S_f$  on  $MU^* \otimes Q$  coincides with  $\psi_g$  and  $g(x)$  is projective, we get  $S_f(S_f([CP^n]))=S_f([CP^n])$ . Apply  $S_f$  to  $\text{mog}_{MU}^f(S_f(x))=\text{log}_{MU}(x)$ ,  $x=c_1(\gamma^1)$ , to obtain

$$\text{mog}_{MU}^f(S_f(S_f(x)))=\text{mog}_{MU}^f(S_f(x)).$$

Hence (1) follows. Apply  $S_f$  to  $f(x)=x+v_1x^2+v_2x^3+\dots+v_nx^{n+1}+\dots$ ,  $x=c_1(\gamma^1)$ , to get

$$f(x)=S_f(S_f(x))=f(x)+S_f(v_1)(f(x))^2+S_f(v_2)(f(x))^3+\dots$$

and  $S_f(v_i)=0$ ,  $i > 0$ .  $\square$

Suppose that  $g(x)(\in MU^* \otimes Q[[x]])$  is projective. Put

$$f(x)=g^{-1}(\text{log}_{MU}(x))$$

and

$$f(x)=x+v_1x^2+v_2x^3+\dots+v_nx^{n+1}+\dots, v_i \in MU^{-2i} \otimes Q.$$

Then we have a natural transformation

$$\begin{aligned} \varepsilon_f: MU^*(X) \otimes Q &\rightarrow MU^*(X) \otimes Q, \\ \varepsilon_f(\alpha) &= \sum v_1^{r_1} \cdots v_k^{r_k} S_{\underbrace{(1, \dots, 1)}_{r_1}, \dots, \underbrace{(k, \dots, k)}_{r_k}}(\alpha). \end{aligned}$$

**Theorem 2.2.** *The natural transformation  $\varepsilon_f$  satisfies the following:*

- (1)  $\varepsilon_f$  is multiplicative.
- (2)  $\varepsilon_f \varepsilon_f = \varepsilon_f$ .
- (3)  $\varepsilon_f(MU^*(-) \otimes Q)$  is a generalized cohomology.

**Proof.** Let

$$S^{r_1, \dots, r_k}(\alpha) = S_{\underbrace{(1, \dots, 1)}_{r_1}, \dots, \underbrace{(k, \dots, k)}_{r_k}}(\alpha)$$

We then have

$$S^R(\alpha\beta) = \sum_{R'+R''=R} S^{R'}(\alpha)S^{R''}(\beta),$$

which completes the proof of (1). We can see that  $\varepsilon_f(x)=f(x)$ , for

$x=c_1(\gamma^1)$ , and so  $\varepsilon_f=S_f$ . Therefore (2) is an immediate result of Proposition 3.1(2) and  $\varepsilon_f(MU^*(-)\otimes Q)$  is a direct summand of  $MU^*(-)\otimes Q$ . (3) follows from (2).  $\square$

EXAMPLE. (1) (Brown-Peterson) The formal power series

$$g(x) = \sum_{n=0}^{\infty} \frac{[CP^{p^n-1}]}{p^n} x^{p^n}$$

is projective. Since the coefficients of  $f(x)=g^{-1}(\log_{MU}(x))$  exist in  $MU^*\otimes Z_{(p)}$  (cf. [1] and [10]), we have a natural idempotent operation

$$\varepsilon_f: MU^*(-)\otimes Z_{(p)} \rightarrow MU^*(-)\otimes Z_{(p)}$$

and the Brown-Peterson cohomology  $BP^*(-)=\varepsilon_f(MU^*(-)\otimes Z_{(p)})$ .

(2) (Ochanine) The formal power series

$$g(x) = \int_0^x \frac{1}{\sqrt{1-2[CP^2]x^2+(3[CP^2]^2-2[CP^4])x^4}} dx$$

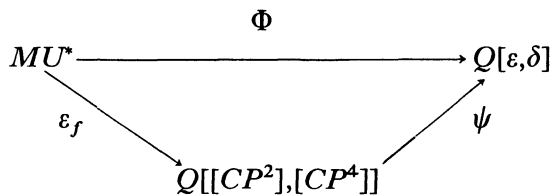
is projective. The multiplicative idempotent operation  $\varepsilon_f$  for  $f(x)=g^{-1}(\log_{MU}(x))$  gives rise to a generalized cohomology  $h^*(-)=\varepsilon_f(MU^*(-)\otimes Q)$  with  $h^*(a \text{ point})=Q[[CP^2], [CP^4]]$  and the multiplicative genus

$$\varphi = \varepsilon_f: MU^* \rightarrow Q[[CP^2],[CP^4]].$$

The Ochanine genus  $\Phi: MU^* \rightarrow Q[\varepsilon,\delta]$  (cf.[7] and [5]) is the multiplicative genus associated with the formal group law  $F(x,y)=l^{-1}(l(x)+l(y))$  with

$$l(x) = \int_0^x \frac{1}{\sqrt{1-2\delta x^2 + \varepsilon x^4}} dx,$$

and  $\Phi([CP^2])=\delta$  and  $\Phi([H_{3,2}])=\varepsilon$  (cf. Proposition 3 of [7]). Thus the Ochanine multiplicative genus is represented as the following composite.



Here  $\psi$  is the ring homomorphism defined by

$$\psi([CP^2]) = \delta \text{ and } \psi([CP^4]) = (3\delta^2 - \varepsilon)/2.$$

(3) The formal power series

$$g(x) = \int_0^x \frac{1}{1 - [CP^1]x + [CP^1]^2x^2} dx$$

is projective. The multiplicative idempotent operation  $\varepsilon_f$  for  $f(x) = g^{-1}(\log_{MU}(x))$  induces the multiplicative genus  $\varepsilon_f: MU^* \rightarrow Q[[CP^1]]$ . The values for complex projective spaces are as follows.

$$\begin{aligned} \varepsilon_f([CP^{3n}]) &= (-1)^n [CP^1]^{3n}, \quad \varepsilon_f([CP^{3n+1}]) = (-1)^n [CP^1]^{3n+1}, \\ \varepsilon_f([CP^{3n+2}]) &= 0. \end{aligned}$$

**3. The genus associated with  $(x+y-2axy)/(1-(a^2+b^2)xy)$**

Let  $\varphi_{a,b}: MU^* \rightarrow Q$  be the multiplicative genus associated with the formal group law  $F(x,y) = (x+y-2axy)/(1-(a^2+b^2)xy)$  for rational numbers  $a$  and  $b$ . The logarithm of  $\varphi_{a,b}$  is

$$l(x) = \int_0^x \frac{1}{1 - 2ax + (a^2 + b^2)x^2} dx.$$

Consider the formal power series

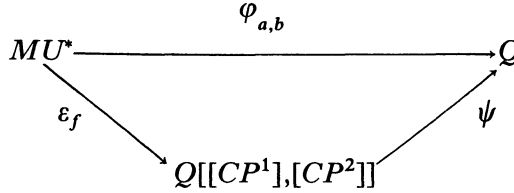
$$g(x) = \int_0^x \frac{1}{1 - [CP]x + ([CP^1]^2 - [CP^2])x^2} dx$$

which is projective. By Theorem 2.2 it induces a multiplicative idempotent operation  $\varepsilon_f$  for  $f(x) = g^{-1}(\log_{MU}(x))$  which gives a generalized cohomology  $h^*(-) = \varepsilon_f(MU^*(-) \otimes Q)$  with  $h^*(a \text{ point}) = Q[[CP^1], [CP^2]]$  and the multiplicative genus

$$\varphi = \varepsilon_f: MU^* \rightarrow Q[[CP^1], [CP^2]].$$

The multiplicative genus  $\varphi_{a,b}$  is represented as the following composite.





Here  $\psi$  is the ring homomorphism defined by  $\psi([CP^1])=2a$  and  $\psi([CP^2])=3a^2-b^2$ .

The characteristic power series for the multiplicative genus  $\varphi_{a,b}$  is given by  $Q(z)=z/l^{-1}(z)$ . Since

$$l(x) = \frac{1}{b} \left( \arctan \left( \frac{a^2+b^2}{b} \left( x - \frac{a}{a^2+b^2} \right) \right) + \arctan \frac{a}{b} \right),$$

it follows that

$$Q(z) = \frac{z(b + a \tan bz)}{\tan bz}.$$

For rational numbers  $a$  and  $b$ , put

$$h(x,y) = \frac{x+y-2axy}{1-(a^2+b^2)xy} \cdot \frac{1}{1-2ax+(a^2+b^2)x^2} \cdot \frac{1}{1-2ay+(a^2+b^2)y^2}.$$

Then

$$(1-(a^2+b^2)xy)(1-2ax+(a^2+b^2)x^2)(1-2ay+(a^2+b^2)y^2)h(x,y) = x+y-2axy$$

and for  $k \geq 3$

$$\begin{aligned} & \frac{\partial^k h}{\partial y^k}(x,0) - k(2a+(a^2+b^2)x) \frac{\partial^{k-1} h}{\partial y^{k-1}}(x,0) \\ & + k(k-1)(a^2+b^2)(1+2ax) \frac{\partial^{k-2} h}{\partial y^{k-2}}(x,0) \\ & - k(k-1)(k-2)(a^2+b^2)^2 x \frac{\partial^{k-3} h}{\partial y^{k-3}}(x,0) = 0. \end{aligned}$$

**Proposition 3.1.** Let  $\alpha = a + b\sqrt{-1}$  and  $\beta = a - b\sqrt{-1}$ . Then

$$\frac{\partial^k h}{\partial y^k}(x,0) = \frac{k!}{(1-\alpha x)(1-\beta x)} \left( \frac{\alpha^k - \beta^k}{\alpha - \beta} + \alpha^k \beta^k x^{k+1} \right).$$

Proof. Put

$$P_k = \frac{\partial^k h}{\partial y^k}(x, 0)$$

$$Q_k = P_k - 2akP_{k-1} + k(k-1)(a^2 + b^2)P_{k-2}.$$

We then have

$$Q_k - k(a^2 + b^2)xQ_{k-1} = 0$$

and

$$Q_k = \frac{k!}{2}(a^2 + b^2)^{k-2}x^{k-2}Q_2.$$

Thus it follows that

$$P_k - 2akP_{k-1} + k(k-1)(a^2 + b^2)P_{k-2} = \frac{k!}{2}(a^2 + b^2)^{k-2}x^{k-2}Q_2.$$

Let  $\alpha = a + b\sqrt{-1}$ ,  $\beta = a - b\sqrt{-1}$  and  $R_k = P_k - k\alpha P_{k-1}$ . Then we get

$$R_k - k\beta R_{k-1} = (\alpha\beta x)^{k-2} \frac{k!}{2} Q_2$$

and

$$R_k - \beta^{k-2} \frac{k!}{2} R_2 = \frac{k!}{2} \beta^{k-2} (P_2 - 2(\alpha + \beta)P_1 + 2\alpha\beta P_0) \sum_{i=1}^{k-2} (\alpha x)^i.$$

Note that

$$P_0 = \frac{x}{(1 - \alpha x)(1 - \beta x)}$$

$$P_1 = \frac{1 + \alpha\beta x^2}{(1 - \alpha x)(1 - \beta x)}$$

$$P_2 = \frac{2(\alpha + \beta) + 2\alpha^2\beta^2 x^3}{(1 - \alpha x)(1 - \beta x)}.$$

Then we get

$$R^k = k! \beta^{k-1} \frac{1 - (\alpha x)^k (1 - \beta x)}{(1 - \alpha x)(1 - \beta x)}$$

and

$$P_k = \frac{k!}{(1-\alpha x)(1-\beta x)} \left( \frac{\alpha^k - \beta^k}{\alpha - \beta} + \alpha^k \beta^k x^{k+1} \right). \quad \square$$

We utilize Proposition 3.1 to obtain Theorem 1.1.

**Proof of Theorem 1.1.**

The  $h(x,y)$  is described as

$$h(x,y) = h(x,0) + \frac{\partial h}{\partial y}(x,0)y + \frac{1}{2!} \frac{\partial^2 h}{\partial y^2}(x,0)y^2 + \dots + \frac{1}{k!} \frac{\partial^k h}{\partial y^k}(x,0)y^k + \dots.$$

From Proposition 3.1 it follows that

$$\frac{1}{k!} \frac{\partial^k h}{\partial y^k}(x,0) = \frac{\alpha^k - \beta^k}{\alpha - \beta} \sum_{s=0}^{\infty} \left( \sum_{i+j=s} \alpha^i \beta^j \right) x^s + \alpha^k \beta^k \sum_{s=0}^{\infty} \left( \sum_{i+j=s} \alpha^i \beta^j \right) x^{s+k+1}$$

and the coefficient of  $x^s y^k$  is

$$\frac{\alpha^k - \beta^k}{\alpha - \beta} \sum_{i+j=s} \alpha^i \beta^j = \left( \frac{\alpha^k - \beta^k}{\alpha - \beta} \right) \left( \frac{\alpha^{s+1} - \beta^{s+1}}{\alpha - \beta} \right). \quad \square$$

REMARK. When  $b=0$  in this theorem,

$$\varphi_{a,0}([H_{s,k}]) = (s+1)k a^{s+k-1}.$$

Especially  $\varphi_{1,0}$  corresponds to the Euler characteristic and  $\varphi_{0,b}$  is the Ochanine multiplicative genus of case  $\delta = -b^2$  and  $\varepsilon = b^4$ .

Let  $\xi$  be the canonical line bundle over  $CP^2$  and let  $P(m,n)$  denote the projective space bundle associated with  $\xi^m \oplus \xi^n$ .

**Proposition 3.2.**

$$[P(m,n)] = -\frac{(m-n)^2}{3} [H_{2,2}] + \frac{(m-n)^2 + 3}{3} [CP^2][CP^1]$$

Proof. Put

$$[P(m,n)] = x[H_{2,2}] + y[CP^2][CP^1] + z[CP^1]^3.$$

We can determine  $x$ ,  $y$  and  $z$  by the facts that

$$\begin{aligned}
 S_{(3)}(P(m,n)) &= 2(m-n)^2, \quad S_{(2,1)}(P(m,n)) = 6, \quad S_{(1,1,1)}(P(m,n)) = 6, \\
 S_{(3)}(H_{2,2}) &= -6, \quad S_{(2,1)}(H_{2,2}) = 6, \quad S_{(1,1,1)}(H_{2,2}) = 6, \\
 S_{(3)}(CP^2 \times CP^1) &= 0, \quad S_{(2,1)}(CP^2 \times CP^1) = 6, \quad S_{(1,1,1)}(CP^2 \times CP^1) = 6, \\
 S_{(3)}((CP^1)^3) &= 0, \quad S_{(2,1)}((CP^1)^3) = 0, \quad S_{(1,1,1)}((CP^1)^3) = 8. \quad \square
 \end{aligned}$$

To the formal group law  $F(x,y) = (x+y-2axy)/(1-(a^2+b^2)xy)$  we apply

$$a = \frac{\delta}{2} \quad \text{and} \quad b = \frac{\sqrt{3}\delta}{2}.$$

The logarithm is

$$l(x) = \int_0^x \frac{1}{1-\delta x + \delta^2 x^2} dx.$$

Let  $\varphi_\delta: MU^* \rightarrow Q$  denote the multiplicative genus associated with the formal group law  $F(x,y) = (x+y-\delta xy)/(1-\delta^2 xy)$ . Then we have the following.

**Theorem 3.3.** *The ideal in  $MU^* \otimes Q$  consisting of  $\alpha$  with  $\varphi_\delta(\alpha) = 0$  for any  $\delta$  is generated by cobordism classes of fibre bundles over  $CP^2$ .*

Proof. Let  $S_{(n)}(M)$  denote the characteristic number of  $M$  corresponding to the symmetric polynomial  $\Sigma t_i^n$ . The characteristic number of the Milnor manifold (cf.[9]) is

$$S_{(n+m-1)}(H_{n,m}) = -\binom{n+m}{n}, \quad 2 \leq n \leq m.$$

We take a generating set  $\{[CP^1], [CP^2], [H_{2,j}] | j \geq 2\}$  of the ring  $MU^* \otimes Q$ . We use Theorem 1.1 to get  $\varphi_\delta([CP^1]) = \delta$ ,  $\varphi_\delta([CP^2]) = 0$  and  $\varphi_\delta([H_{2,j}]) = 0$ ,  $j \geq 2$ . Therefore the kernel of  $\varphi_\delta$  is generated by  $\{[CP^2], [H_{2,j}], j \geq 2\}$ . If  $j > 2$ ,  $H_{2,j}$  is a fibre bundle over  $CP^2$  with the fibre  $CP^{j-1}$ . By using Proposition 3.3 we see that  $[H_{2,2}]$  belongs to the ideal generated by cobordism classes of fibre bundles over  $CP^2$ .  $\square$

**4. Genera cancelling  $[H_{2,j}]$ ,  $j > n$**

We discuss the multiplicative genus associated with the formal group law  $F(x,y) = l^{-1}(l(x)+l(y))$  with the logarithm  $l(x)$  given by the integral

of  $1/\{\text{a polynomial}\}$  which is a generalization of the logarithm for the multiplicative genus  $\varphi_\delta$  in Section 3. For rational numbers  $\delta_1, \delta_2, \dots, \delta_n$ , we consider a formal power series

$$l(z) = \int_0^z \frac{1}{1 + \delta_1 x + \delta_1^2 x^2 + \delta_2 x^3 + \dots + \delta_n x^{n+1}} dx$$

and denote the multiplicative genus associated with the formal group law  $F(x, y) = l^{-1}(l(x) + l(y))$  by

$$\varphi_{\delta_1, \dots, \delta_n}: MU^* \rightarrow Q.$$

We then have the following

**Proposition 4.1.**

(1)  $\varphi_{\delta_1, \dots, \delta_n}([CP^1]) = -\delta_1, \varphi_{\delta_1, \dots, \delta_n}([CP^2]) = 0, \varphi_{\delta_1, \dots, \delta_n}([H_{2,1}]) = \delta_1^2$

(2)  $\varphi_{\delta_1, \dots, \delta_n}([H_{2,j}]) = \begin{cases} \frac{j+1}{2} \delta_j, & 2 \leq j \leq n \\ 0, & j > n \end{cases}$

*Proof.* The logarithm  $l(x)$  for  $F(x, y)$  is described as

$$l(x) = \sum \frac{\varphi_{\delta_1, \dots, \delta_n}([CP^n])}{n+1} x^{n+1}.$$

We obtain (1) by the facts that

$$l(z) = z - \frac{\delta_1}{2} z^2 + \text{higher terms of degree} \geq 4$$

and  $[H_{2,1}] = [CP^1]^2$ . Consider the formal power series

$$h(x, y) = \sum \varphi_{\delta_1, \dots, \delta_n}([H_{i,j}]) x^i y^j.$$

The Buchstaber formula [2]

$$H(x, y) = (\sum [CP^n] x^n) (\sum [CP^n] y^n) F_{MU}(x, y)$$

implies

$$\begin{aligned} h(x, y) &= l'(x)l'(y)F(x, y) \\ &= \left( l'(0) + \frac{l''(0)}{1!}x + \dots + \frac{l^{(n+1)}(0)}{n!}x^n + \dots \right) \cdot l'(y). \end{aligned}$$

$$\left( F(0,y) + \frac{1}{1!} \frac{\partial F}{\partial x}(0,y)x + \frac{1}{2!} \frac{\partial^2 F}{\partial x^2}(0,y)x^2 + \dots + \frac{1}{n!} \frac{\partial^n F}{\partial x^n}(0,y)x^n + \dots \right).$$

Comparing the coefficients of  $x^2$ , we have

$$\begin{aligned} & h_{2,0} + h_{2,1}y + h_{2,2}y^2 + \dots + h_{2,n}y^n + \dots \\ &= l'(y) \cdot \left( \frac{l'(0)}{2} \frac{\partial^2 F}{\partial x^2}(0,y) + l''(0) \frac{\partial F}{\partial x}(0,y) + \frac{l^{(3)}(0)}{2} F(0,y) \right), \end{aligned}$$

where  $h_{i,j} = \varphi_{\delta_1, \dots, \delta_n}([H_{i,j}])$ . Since  $l(F(x,y)) = l(x) + l(y)$ , it follows that

$$l'(F(x,y)) \frac{\partial F}{\partial x} = l'(x)$$

and

$$l''(F(x,y)) \left( \frac{\partial F}{\partial x} \right)^2 + l'(F(x,y)) \frac{\partial^2 F}{\partial x^2} = l''(x).$$

From the facts that  $l'(0) = 1$ ,  $l''(0) = -\delta_1$  and  $l^{(3)}(0) = 0$ , it follows that

$$\begin{aligned} & h_{2,0} + h_{2,1}y + h_{2,2}y^2 + \dots + h_{2,n}y^n + \dots \\ &= -\delta_1 + \delta_1^2 y + \frac{3}{2} \delta_2 y^2 + \dots + \frac{n+1}{2} \delta_n y^n \end{aligned}$$

and we complete the proof of (2). □

We can take a generating set  $\{[CP^1], [CP^2], [H_{2,2}], \dots, [H_{2,n}], \dots\}$  for the polynomial ring  $MU^* \otimes Q$ . Therefore it follows from Proposition 4.1 that

**Theorem 4.2.**  *$[M] (\in MU^* \otimes Q)$  belongs to the ideal generated by  $\{[CP^2], [H_{2,j}] (j > n)\}$  in  $MU^* \otimes Q$  if and only if for any  $\delta_1, \dots, \delta_{n-1}$  and  $\delta_n (\in Q)$*

$$\varphi_{\delta_1, \dots, \delta_n}([M]) = 0.$$

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