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## ON A WAVE EQUATION CORRESPONDING TO GEODESICS

Dedicated to Professor Hideki Ozeki on his 60th birthday

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(Received September 5, 1994)

### 0. Introduction

For a closed curve  $\gamma(x)$  in a riemannian manifold  $M$  we define its energy  $E(\gamma)$  by  $\|\partial_x \gamma\|^2$ . The first variation  $(d/dt)_{t=0} E(\gamma(t, *))$  is given by  $-2\langle \partial_t \gamma, \nabla_x^2 \gamma \rangle$ . Therefore, its Euler-Lagrange equation is the equation of geodesics. We consider a corresponding hyperbolic equation of  $\gamma = \gamma(t, x)$ :

$$(H) \quad \nabla_t^2 \gamma + \mu \partial_t \gamma = \nabla_x^2 \gamma,$$

where the coefficient  $\mu$  represents the resistance and is usually a positive constant. This equation is locally expressed as

$$\partial_t^2 \gamma^i + \Gamma_{jk}^i(\gamma) \partial_t \gamma^j \partial_t \gamma^k + \mu \partial_t \gamma^i = \partial_x^2 \gamma^i + \Gamma_{jk}^i(\gamma) \partial_x \gamma^j \partial_x \gamma^k,$$

which is a semi-linear wave equation.

Eells and Sampson [1] introduced a corresponding heat equation

$$(P) \quad \partial_t \gamma = \nabla_x^2 \gamma.$$

We know that if the manifold  $M$  is compact and real analytic, then the solution of (P) exists for all time and converges to a geodesic [3].

Physically, equation (H) represents the equation of motion of a rubber band in viscous liquid. Therefore, it is likely that results similar to (P) hold. In fact we will prove the following result.

**Theorem.** *Let  $M$  be a complete riemannian manifold and  $\mu$  a constant. Then Cauchy problem (H) for closed curves has a unique solution  $\gamma(t, x)$  on  $\mathbf{R} \times S^1$ . If  $M$  is compact and  $\mu > 0$ , then the solution almost converges to geodesics; that is,  $\partial_t \gamma \rightarrow 0$  and  $\nabla_x^2 \gamma \rightarrow 0$  when  $t \rightarrow \infty$ .*

However the convergence of  $\gamma$  itself is still open, even on a manifold with negative sectional curvature.

REMARK. Gu [2, Theorem] proved that equation (H) without resistance (i.e.,  $\mu=0$ ) has an all time solution. He essentially used the equality  $(\nabla_t - \nabla_x)(\partial_t - \partial_x)\gamma = 0$ , which fails when  $\mu \neq 0$ . We will overcome this difficulty by systematic use of covariant derivation.

## 1. Preliminaries

Throughout this paper, we use the following notation. Let  $M$  be a riemannian manifold. We consider closed curves  $\gamma = \gamma(x)$  in  $M$  and families  $\gamma = \gamma(t, x)$  of closed curves. The partial derivation is denoted by  $\partial$  and the riemannian covariant derivation is denoted by  $\nabla$ . The pointwise norm  $|\ast|$ , the  $L_2$  norm  $\|\ast\|$  and the  $L_2$  inner product  $\langle \ast, \ast \rangle$  are defined by  $|\ast|^2 = g(\ast, \ast)$ ,  $\langle \ast, \ast \rangle = \int_{S_1} g(\ast, \ast) dx$  and  $\|\ast\|^2 = \langle \ast, \ast \rangle$ .

Let  $\gamma$  be a map:  $\mathbf{R}_t \times \mathbf{R}_x \rightarrow M$ . A  $(p+q)$ -th covariant derivation  $\nabla_\ast \nabla_\ast \cdots \nabla_\ast$  with  $p$   $\nabla_t$ 's and  $q$   $\nabla_x$ 's is denoted by  $P_{p,q}$ , regardless of the order of derivations. It is also denoted by  $P_n$  ( $n=p+q$ ), when we do not specify the numbers  $p$  and  $q$  separately.

**Lemma 1.1.** *If we denote by  $Q_{p+q-2}$  the difference  $P_{p,q}\gamma - \nabla_t^p \nabla_x^q \gamma$  for  $p+q \geq 2$ , then  $Q_n$  has the following properties:*

- a)  $Q_n$  can be expressed as a linear combination

$$\sum a_i \cdot (\nabla^k R)(P_{j_1}\gamma, \dots, P_{j_k}\gamma)(P_{j_{k+1}}\gamma, P_{j_{k+2}}\gamma)P_{j_{k+3}}\gamma.$$

- b) In the above expression of  $Q_n$ ,  $\sum_{m=1}^{k+3} j_m = n+2$  for each term.

- c)  $Q_n$  is a polynomial with respect to  $P_i\gamma$ 's ( $i \leq n$ ). Moreover, each term of  $Q_n$  can contain at most one  $P_n\gamma$ .

Proof. Property c) is a consequence of properties a) and b). Therefore we have to check a) and b). They trivially hold for  $p+q=2$ . In fact,  $Q_0=0$ . Suppose that they hold for  $p+q \leq n+2$ . For induction, assuming  $p+q=n+2$ , we have to check the forms  $P_{p+1,q}\gamma = \nabla_t P_{p,q}\gamma$  and  $P_{p,q+1}\gamma = \nabla_x P_{p,q}\gamma$ . For the first form, we have

$$\nabla_t P_{p,q}\gamma = \nabla_t (\nabla_t^p \nabla_x^q \gamma + Q_n) = \nabla_t^{p+1} \nabla_x^q \gamma + \nabla_t Q_n,$$

and the term  $Q_{n+1} = \nabla_t Q_n$  has the desired properties.

If the second form  $\nabla_x P_{p,q}\gamma$  only contains  $\nabla_x$ , the claim holds. If it contains  $\nabla_t$ , we have

$$\begin{aligned} \nabla_x P_{p,q}\gamma &= \nabla_x (\nabla_t^p \nabla_x^q \gamma + Q_n) = \nabla_x \nabla_t \nabla_t^{p-1} \nabla_x^q \gamma + \nabla_x Q_n \\ &= \nabla_t \nabla_x \nabla_t^{p-1} \nabla_x^q \gamma + R(\partial_x \gamma, \partial_t \gamma) \nabla_t^{p-1} \nabla_x^q \gamma + \nabla_x Q_n \\ &= \nabla_t (\nabla_t^{p-1} \nabla_x^{q+1} \gamma + Q_n) + \{R(\partial_x \gamma, \partial_t \gamma) \nabla_t^{p-1} \nabla_x^q \gamma + \nabla_x Q_n\} \end{aligned}$$

$$= \nabla_t^p \nabla_x^{q+1} \gamma + \{R(\partial_x \gamma, \partial_t \gamma) \nabla_t^{p-1} \nabla_x^q \gamma + \nabla_x Q_n + \nabla_t Q_n\}.$$

Q.E.D.

**Lemma 1.2.** *Let  $\gamma$  be a solution of (H) and  $\varphi$  a  $P_n \gamma$ . Then we have*

$$\nabla_t^2 \varphi + \mu \nabla_t \varphi - \nabla_x^2 \varphi = Q_n + Q_{n-1},$$

where  $Q_n$  has properties a)–c) in Lemma 1.1.

Proof. let  $\varphi$  be a  $P_{p,q} \gamma$  ( $p+q=n$ ). Then,

$$\begin{aligned} \nabla_t^2 \varphi &= P_{p+2,q} \gamma = P_{p,q} \nabla_t^2 \gamma + Q_n = P_{p,q} (\mu \partial_t \gamma + \nabla_x^2 \gamma) + Q_n \\ &= -\mu P_{p+1,q} \gamma + P_{p,q+2} \gamma + Q_n \\ &= -(\mu \nabla_t \varphi + Q_{n-1}) + (\nabla_x^2 \varphi + Q_n) + Q_n. \end{aligned}$$

Q.E.D.

## 2. All time existence

We start from a standard short time existence result in [4].

**Theorem 2.1** [4, Theorem 7.5]. *For any closed  $C^3$  curve  $\gamma_0(x)$  and any  $C^2$  vector field  $\gamma_1(x)$  along  $\gamma_0$ , there is a positive constant  $T$  such that equation (H) with initial data  $\gamma(0, x) = \gamma_0(x)$  and  $\partial_t \gamma(0, x) = \gamma_1(x)$  has a unique solution  $\gamma(t, x)$  on  $0 \leq t \leq T$ .*

Let  $T$  be the largest number such that a solution  $\gamma(t, x)$  with  $C^\infty$  initial data  $\gamma_0(x)$  exists on  $0 \leq t < T$ . If we can prove that the solution  $\gamma(t, x)$  is uniformly bounded on  $[0, T) \times S^1$  in  $C^n$ -norm for each  $n$ , then we can extend the solution beyond the time  $T$ . This implies that the maximal existence time is infinite. To consider negative time interval  $(-T, 0]$ , we change the time variable  $t$  to  $-\tau$ , and get the same equation with resistance  $-\mu$ . Therefore, the proof of all time existence is reduced to the following

**Proposition 2.2.** *Let  $\gamma_0(x)$  be a  $C^\infty$  closed curve on  $M$  and  $\gamma_1(x)$  a  $C^\infty$  vector field along  $\gamma_0$ . Let  $\gamma(t, x)$  be a solution of (H) with initial data  $\gamma(0, x) = \gamma_0(x)$  and  $\partial_t \gamma(0, x) = \gamma_1(x)$  on  $0 \leq t < T$ , where  $T$  is a finite positive number. Then, any  $|P_n \gamma|$  is uniformly bounded on  $[0, T) \times S^1$ .*

Proof. To prove this, we change the coordinate system  $\{t, x\}$  to  $\{\xi = t + x, \eta = t - x\}$ . Then we have  $\partial_t = \partial_\xi + \partial_\eta$  and  $\partial_x = \partial_\xi - \partial_\eta$ . Therefore, for  $\varphi = P_n \gamma$ ,

$$\begin{aligned} \nabla_t^2 \varphi &= \nabla_{\partial_t + \partial_x} (\nabla_\xi \varphi + \nabla_\eta \varphi) = \nabla_\xi^2 \varphi + \nabla_\xi \nabla_\eta \varphi + \nabla_\eta \nabla_\xi \varphi + \nabla_\eta^2 \varphi \\ &= \nabla_\xi^2 \varphi + 2 \nabla_\xi \nabla_\eta \varphi + \nabla_\eta^2 \varphi + Q_n, \\ \nabla_x^2 \varphi &= \nabla_\xi^2 \varphi - 2 \nabla_\xi \nabla_\eta \varphi + \nabla_\eta^2 \varphi + Q_n. \end{aligned}$$

Hence, by Lemma 1.2,

$$4\nabla_\varepsilon\nabla_\eta\varphi=\nabla_t^2\varphi-\nabla_x^2\varphi+Q_n=-\mu(\nabla_\varepsilon\varphi+\nabla_\eta\varphi)+Q'_n,$$

where  $Q'_n$  denotes a form  $Q_n+Q_{n-1}$ . Note that  $Q'_0=0$ . From this equation, we have

$$\begin{aligned} 2\partial_\varepsilon|\nabla_\eta\varphi|^2 &= 4(\nabla_\eta\varphi, \nabla_\varepsilon\nabla_\eta\varphi) \\ &= -\mu(\nabla_\eta\varphi, \nabla_\varepsilon\varphi+\nabla_\eta\varphi)+(\nabla_\eta\varphi, Q'_n) \\ &\leq |\mu||\nabla_\eta\varphi|^2+|\mu||\nabla_\varepsilon\varphi||\nabla_\eta\varphi|+|\nabla_\eta\varphi||Q'_n|. \end{aligned}$$

Fix a time  $t$  and take a maximal point  $(t, x)$  of  $|\nabla_\eta\varphi|^2$ . Then, at that point,

$$\begin{aligned} \partial_t|\nabla_\eta\varphi|^2 &= (\partial_t+\partial_x)|\nabla_\eta\varphi|^2=2\partial_\varepsilon|\nabla_\eta\varphi|^2 \\ &\leq |\mu||\nabla_\eta\varphi|^2+|\mu||\nabla_\varepsilon\varphi||\nabla_\eta\varphi|+|\nabla_\eta\varphi||Q'_n|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{d}{dt}\max|\nabla_\eta\varphi|^2 \\ \leq |\mu|\max|\nabla_\eta\varphi|^2+|\mu|\max|\nabla_\varepsilon\varphi|\max|\nabla_\eta\varphi|+\max|\nabla_\eta\varphi|\max|Q'_n|. \end{aligned}$$

Adding a symmetric formula for  $|\nabla_\varepsilon\varphi|$  to this, we get

$$\begin{aligned} \frac{d}{dt}\{\max|\nabla_\varepsilon\varphi|^2+\max|\nabla_\eta\varphi|^2\} \\ \leq |\mu|\{\max|\nabla_\varepsilon\varphi|+\max|\nabla_\eta\varphi|\}^2+\max|Q'_n|\{\max|\nabla_\varepsilon\varphi|+\max|\nabla_\eta\varphi|\} \\ \leq 2(|\mu|+1)\{\max|\nabla_\varepsilon\varphi|^2+\max|\nabla_\eta\varphi|^2\}+\max|Q'_n|^2 \end{aligned}$$

Here, the derivation  $(d/dt)u(t)$  means

$$\limsup_{h\rightarrow+\infty}\frac{u(t)-u(t-h)}{h}.$$

Now we can prove our claim by induction. Noting that  $Q'_0=0$  in the above inequality, we see that the  $C^1$  norm of  $\gamma$  is bounded by the initial data. In particular, the norm of any covariant derivatives of curvature tensor of  $M$  is bounded on the image of  $\gamma$ . Thus the claim holds for  $n=1$ . Suppose that the claim holds up to  $n$ . Then,  $\varphi$  and  $Q'_n$  in the above inequality are uniformly bounded on  $[0, T)\times S^1$ . Therefore, the above inequality implies that  $\max|\nabla_\varepsilon\varphi|^2+\max|\nabla_\eta\varphi|^2$  is bounded on  $[0, T)$ , hence  $\nabla\varphi=P_{n+1}\gamma$  is uniformly bounded on  $[0, T)\times S^1$ .

Q.E.D.

### 3. Convergence

Now, we suppose that  $\mu>0$  and show the convergence. Let  $\gamma$  be the solution of (H) and put  $\varphi=P_n\gamma$ . We use an energy inequality for wave equations. Using Lemma 1.2, we have

$$\begin{aligned}
 & \frac{d}{dt}(\|\nabla_t\varphi\|^2 + \|\nabla_x\varphi\|^2) + 2\mu\|\nabla_t\varphi\|^2 \\
 &= 2\langle\nabla_t\varphi, \nabla_t^2\varphi\rangle + 2\langle\nabla_x\varphi, \nabla_t\nabla_x\varphi\rangle + 2\mu\|\nabla_t\varphi\|^2 \\
 &= 2\langle\nabla_t\varphi, \nabla_x^2\varphi - \mu\nabla_t\varphi + Q'_n\rangle + 2\langle\nabla_x\varphi, \nabla_t\nabla_x\varphi\rangle + 2\mu\|\nabla_t\varphi\|^2 \\
 &= -2\langle\nabla_x\nabla_t\varphi, \nabla_x\varphi\rangle + 2\langle\nabla_x\varphi, \nabla_t\nabla_x\varphi\rangle + 2\langle Q'_n, \nabla_t\varphi\rangle \\
 &= 2\langle Q_n, \nabla_x\varphi\rangle + 2\langle Q'_n, \nabla_t\varphi\rangle \\
 &\leq C\|Q'_n\|(\|\nabla_x\varphi\| + \|\nabla_t\varphi\|),
 \end{aligned}$$

where  $Q'_n$  denotes a form  $Q_n + Q_{n-1}$ . Note that  $Q'_0 = 0$ .

We show the following proposition by induction.

**Proposition 3.1.**

- 1) Any  $\|P_n\gamma\|$  is bounded on  $[0, \infty)$ .
- 2) Any  $\int_0^\infty \|P_n\gamma\|^2 dt$  is finite except  $P_n\gamma = \partial_x\gamma$  (and  $P_0\gamma$ ).

Proof. Nothing that  $Q'_0 = 0$  in the above inequality, we have

$$\frac{d}{dt}(\|\partial_t\gamma\|^2 + \|\partial_x\gamma\|^2) = -2\mu\|\partial_t\gamma\|^2 \leq 0.$$

Integrating both hand sides by  $t$ , we see that the claim holds for  $n=1$ . Suppose that the claim holds up to  $n$ . In the above inequality, if the factor  $\nabla_x\varphi = \nabla_x P_n\gamma$  contains  $\nabla_t$ , then

$$\nabla_x\varphi = \nabla_t P_n\gamma + Q_{n-1}.$$

And if not,

$$\nabla_x\varphi = \nabla_x^{n+1}\gamma = \nabla_x^{n-1}(\nabla_t^2\gamma + \mu\partial_t\gamma) = \nabla_t P_n\gamma + Q_{n-1} + \mu P_n\gamma.$$

In both cases, we have

$$\begin{aligned}
 & \frac{d}{dt}(\|\nabla_t\varphi\|^2 + \|\nabla_x\varphi\|^2) + 2\mu\|\nabla_t\varphi\|^2 \\
 & \leq C\|Q'_n\|(2\|\nabla_t P_n\gamma\| + \|Q_{n-1}\| + \|P_n\gamma\|).
 \end{aligned}$$

Summing up this inequality for all  $\varphi = P_n\gamma$ , we have

$$\begin{aligned}
 & \frac{d}{dt} \sum_{P_n} (\|\nabla_t P_n\gamma\|^2 + \|\nabla_x P_n\gamma\|^2) + 2\mu \sum_{P_n} \|\nabla_t P_n\gamma\|^2 \\
 & \leq C\{\sum\|Q'_n\|^2 + \sum\|Q_{n-1}\|^2 + \sum\|P_n\gamma\|^2\} + \varepsilon \sum\|\nabla_t P_n\gamma\|^2,
 \end{aligned}$$

where  $\varepsilon$  can be taken arbitrarily small. Here,  $\sum_{P_n}$  means summation  $\sum_\varphi$  with respect to all  $\varphi$  of form  $P_n\gamma$ . Take  $\varepsilon = \mu$ . Then we see that

$$\frac{d}{dt} \sum_{P_n} (\|\nabla_t P_n\gamma\|^2 + \|\nabla_x P_n\gamma\|^2) + \mu \sum_{P_n} \|\nabla_t P_n\gamma\|^2$$

is dominated by a bounded  $L_1$  function, because at least one  $P_i\gamma$  is not  $\partial_x\gamma$  in each

term of  $Q'_n$ . Thus, integrating by  $t$ , we see that any  $\|P_{n+1}\gamma\|^2$  is uniformly bounded and that any  $\|\nabla_t P_n \gamma\|$  is  $L_2$ . Then, also any  $\|\nabla_x P_n \gamma\|$  is  $L_2$ , because it can be expressed by a  $\|\nabla_t P_n \gamma\|$  and  $Q'_n$ 's. Therefore, the claim holds for  $n+1$ . Q.E.D.

End of the proof of Theorem. Finally, we remark that the derivative  $(d/dt)\|P_n \gamma\|^2$  is expressed by  $P_{n+1}\gamma$ , and hence is uniformly bounded on  $[0, \infty)$  by Lemma 3.1. Therefore, any  $\|P_n \gamma\|^2$  (except  $\partial_x \gamma$ ) converges to 0 when  $t \rightarrow \infty$ .

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