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ON A WAVE EQUATION CORRESPONDING TO GEODESICS

Dedicated to Professor Hideki Ozeki on his 60th birthday

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0. Introduction

For a closed curve $\gamma(x)$ in a riemannian manifold M we define its energy $E(\gamma)$ by $\|\partial_x\gamma\|^2$. The first variation $(d/dt)_{t=0}E(\gamma(t, *))$ is given by $-2\langle\partial_t\gamma, \nabla_x^2\gamma\rangle$. Therefore, its Euler-Lagrange equation is the equation of geodesics. We consider a corresponding hyperbolic equation of $\gamma=\gamma(t, x)$:

(H) $$\nabla_t^2\gamma+\mu\partial_t\gamma=\nabla_x^2\gamma,$$

where the coefficient μ represents the resistance and is usually a positive constant. This equation is locally expressed as

$$\partial_t^2\gamma^i+\Gamma_{j\,k}^{i}(\gamma)\partial_t\gamma^j\partial_t\gamma^k+\mu\partial_t\gamma^i=\partial_x^2\gamma^i+\Gamma_{j\,k}^{i}(\gamma)\partial_x\gamma^j\partial_x\gamma^k,$$

which is a semi-linear wave equation.

Eells and Sampson [1] introduced a corresponding heat equation

(P) $$\partial_t\gamma=\nabla_x^2\gamma.$$

We know that if the manifold M is compact and real analytic, then the solution of (P) exists for all time and converges to a geodesic [3].

Physically, equation (H) represents the equation of motion of a rubber band in viscous liquid. Therefore, it is likely that results similar to (P) hold. In fact we will prove the following result.

Theorem. *Let M be a complete riemannian manifold and μ a constant. Then Cauchy problem* (H) *for closed curves has a unique solution $\gamma(t, x)$ on $\boldsymbol{R}\times S^1$. If M is compact and $\mu>0$, then the solution almost converges to geodesics*; that is, *$\partial_t\gamma\to 0$ and $\nabla_x^2\gamma\to 0$ when $t\to\infty$.*

However the convergence of γ itself is still open, even on a manifold with negative sectional curvature.

REMARK. Gu [2, Theorem] proved that equation (H) without resistance (i.e., $\mu=0$) has an all time solution. He essentially used the equality $(\nabla_t-\nabla_x)(\partial_t-\partial_x)\gamma=0$, which fails when $\mu\neq 0$. We will overcome this difficulty by systematic use of covariant derivation.

1. Preliminaries

Throughout this paper, we use the following notation. Let M be a riemannian manifold. We consider closed curves $\gamma=\gamma(x)$ in M and families $\gamma=\gamma(t, x)$ of closed curves. The partial derivation is denoted by ∂ and the riemannian covariant derivation is denoted by ∇. The pointwise norm $|*|$, the L_2 norm $\|*\|$ and the L_2 inner product $\langle *, *\rangle$ are defined by $|*|^2=g(*, *)$, $\langle *, *\rangle=\int_{S^1} g(*, *)dx$ and $\|*\|^2=\langle *, *\rangle$.

Let γ be a map: $\boldsymbol{R}_t\times\boldsymbol{R}_x\to M$. A $(p+q)$-th covariant derivation $\nabla_*\nabla_*\cdots\nabla_*$ with p ∇_t's and q ∇_x's is denoted by $P_{p,q}$, regardless of the order of derivations. It is also denoted by P_n $(n=p+q)$, when we do not specify the numbers p and q separately.

Lemma 1.1. *If we denote by Q_{p+q-2} the difference $P_{p,q}\gamma-\nabla_t^p\nabla_x^q\gamma$ for $p+q\geq 2$, then Q_n has the following properties:*

a) *Q_n can be expressed as a linear combination*

$$\sum a_i\cdot(\nabla^k R)(P_{j_1}\gamma, \cdots, P_{j_k}\gamma)(P_{j_{k+1}}\gamma, P_{j_{k+2}}\gamma)P_{j_{k+3}}\gamma.$$

b) *In the above expression of Q_n, $\sum_{m=1}^{k+3} j_m=n+2$ for each term.*

c) *Q_n is a polynomial with respect to $P_i\gamma$'s $(i\leq n)$. Moreover, each term of Q_n can contain at most one $P_n\gamma$.*

Proof. Property c) is a consequence of properties a) and b). Therefore we have to check a) and b). They trivially hold for $p+q=2$. In fact, $Q_0=0$. Suppose that they hold for $p+q\leq n+2$. For induction, assuming $p+q=n+2$, we have to check the forms $P_{p+1,q}\gamma=\nabla_t P_{p,q}\gamma$ and $P_{p,q+1}\gamma=\nabla_x P_{p,q}\gamma$. For the first form, we have

$$\nabla_t P_{p,q}\gamma=\nabla_t(\nabla_t^p\nabla_x^q\gamma+Q_n)=\nabla_t^{p+1}\nabla_x^q\gamma+\nabla_t Q_n,$$

and the term $Q_{n+1}=\nabla_t Q_n$ has the desired properties.

If the second form $\nabla_x P_{p,q}\gamma$ only contains ∇_x, the claim holds. If it contains ∇_t, we have

$$\begin{aligned}\nabla_x P_{p,q}\gamma&=\nabla_x(\nabla_t^p\nabla_x^q\gamma+Q_n)=\nabla_x\nabla_t\nabla_t^{p-1}\nabla_x^q\gamma+\nabla_x Q_n\\&=\nabla_t\nabla_x\nabla_t^{p-1}\nabla_x^q\gamma+R(\partial_x\gamma, \partial_t\gamma)\nabla_t^{p-1}\nabla_x^q\gamma+\nabla_x Q_n\\&=\nabla_t(\nabla_t^{p-1}\nabla_x^{q+1}\gamma+Q_n)+\{R(\partial_x\gamma, \partial_t\gamma)\nabla_t^{p-1}\nabla_x^q\gamma+\nabla_x Q_n\}\end{aligned}$$

$$=\nabla_t^p\nabla_x^{q+1}\gamma+\{R(\partial_x\gamma,\ \partial_t\gamma)\nabla_t^{p-1}\nabla_x{}^q\gamma+\nabla_xQ_n+\nabla_tQ_n\}.$$

Q.E.D.

Lemma 1.2. *Let γ be a solution of* (H) *and φ a $P_n\gamma$. Then we have*

$$\nabla_t^2\varphi+\mu\nabla_t\varphi-\nabla_x^2\varphi=Q_n+Q_{n-1},$$

where Q_n has properties a)—c) *in* Lemma 1.1.

Proof. let φ be a $P_{p,q}\gamma$ $(p+q=n)$. Then,

$$\begin{aligned}\nabla_t^2\varphi&=P_{p+2,q}\gamma=P_{p,q}\nabla_t^2\gamma+Q_n=P_{p,q}(\mu\partial_t\gamma+\nabla_x^2\gamma)+Q_n\\&=-\mu P_{p+1,q}\gamma+P_{p,q+2}\gamma+Q_n\\&=-(\mu\nabla_t\varphi+Q_{n-1})+(\nabla_x^2\varphi+Q_n)+Q_n.\end{aligned}$$

Q.E.D.

2. All time existence

We start from a standard short time existence result in [4].

Theorem 2.1 [4, Theorem 7.5]. *For any closed C^3 curve $\gamma_0(x)$ and any C^2 vector field $\gamma_1(x)$ along γ_0, there is a positive constant T such that equation* (H) *with initial data $\gamma(0, x)=\gamma_0(x)$ and $\partial_t\gamma(0, x)=\gamma_1(x)$ has a unique solution $\gamma(t, x)$ on $0\leq t\leq T$.*

Let T be the largest number such that a solution $\gamma(t, x)$ with C^∞ initial data $\gamma_0(x)$ exists on $0\leq t<T$. If we can prove that the solution $\gamma(t, x)$ is uniformly bounded on $[0, T)\times S^1$ in C^n-norm for each n, then we can extend the solution beyond the time T. This implies that the maximal existence time is infinite. To consider negative time interval $(-T, 0]$, we change the time variable t to $-\tau$, and get the same equation with resistance $-\mu$. Therefore, the proof of all time existence is reduced to the following

Proposition 2.2. *Let $\gamma_0(x)$ be a C^∞ closed curve on M and $\gamma_1(x)$ a C^∞ vector field along γ_0. Let $\gamma(t, x)$ be a solution of* (H) *with initial data $\gamma(0, x)=\gamma_0(x)$ and $\partial_t\gamma(0, x)=\gamma_1(x)$ on $0\leq t<T$, where T is a finite positive number. Then, any $|P_n\gamma|$ is uniformly bounded on $[0,T)\times S^1$.*

Proof. To prove this, we change the coordinate system $\{t, x\}$ to $\{\xi=t+x, \eta=t-x\}$. Then we have $\partial_t=\partial_\xi+\partial_\eta$ and $\partial_x=\partial_\xi-\partial_\eta$. Therefore, for $\varphi=P_n\gamma$,

$$\begin{aligned}\nabla_t^2\varphi&=\nabla_{\partial_\xi+\partial_\eta}(\nabla_\xi\varphi+\nabla_\eta\varphi)=\nabla_\xi^2\varphi+\nabla_\xi\nabla_\eta\varphi+\nabla_\eta\nabla_\xi\varphi+\nabla_\eta^2\varphi\\&=\nabla_\xi^2\varphi+2\nabla_\xi\nabla_\eta\varphi+\nabla_\eta^2\varphi+Q_n,\\\nabla_x^2\varphi&=\nabla_\xi^2\varphi-2\nabla_\xi\nabla_\eta\varphi+\nabla_\eta^2\varphi+Q_n.\end{aligned}$$

Hence, by Lemma 1.2,

$$4\nabla_\xi\nabla_\eta\varphi=\nabla_t^2\varphi-\nabla_x^2\varphi+Q_n=-\mu(\nabla_\xi\varphi+\nabla_\eta\varphi)+Q'_n,$$

where Q'_n denotes a form Q_n+Q_{n-1}. Note that $Q'_0=0$. From this equation, we have

$$\begin{aligned}2\partial_\xi|\nabla_\eta\varphi|^2&=4(\nabla_\eta\varphi,\ \nabla_\xi\nabla_\eta\varphi)\\&=-\mu(\nabla_\eta\varphi,\ \nabla_\xi\varphi+\nabla_\eta\varphi)+(\nabla_\eta\varphi,\ Q'_n)\\&\leq|\mu||\nabla_\eta\varphi|^2+|\mu||\nabla_\xi\varphi||\nabla_\eta\varphi|+|\nabla_\eta\varphi||Q'_n|.\end{aligned}$$

Fix a time t and take a maximal point $(t,\ x)$ of $|\nabla_\eta\varphi|^2$. Then, at that point,

$$\begin{aligned}\partial_t|\nabla_\eta\varphi|^2&=(\partial_t+\partial_x)|\nabla_\eta\varphi|^2=2\partial_\xi|\nabla_\eta\varphi|^2\\&\leq|\mu||\nabla_\eta\varphi|^2+|\mu||\nabla_\xi\varphi||\nabla_\eta\varphi|+|\nabla_\eta\varphi||Q'_n|.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\frac{d}{dt}\max|\nabla_\eta\varphi|^2\\&\quad\leq|\mu|\max|\nabla_\eta\varphi|^2+|\mu|\max|\nabla_\xi\varphi|\max|\nabla_\eta\varphi|+\max|\nabla_\eta\varphi|\max|Q'_n|.\end{aligned}$$

Adding a symmetric formula for $|\nabla_\xi\varphi|$ to this, we get

$$\begin{aligned}&\frac{d}{dt}\{\max|\nabla_\xi\varphi|^2+\max|\nabla_\eta\varphi|^2\}\\&\quad\leq|\mu|\{\max|\nabla_\xi\varphi|+\max|\nabla_\eta\varphi|\}^2+\max|Q'_n|\{\max|\nabla_\xi\varphi|+\max|\nabla_\eta\varphi|\}\\&\quad\leq2(|\mu|+1)\{\max|\nabla_\xi\varphi|^2+\max|\nabla_\eta\varphi|^2\}+\max|Q'_n|^2\end{aligned}$$

Here, the derivation $(d/dt)u(t)$ means

$$\limsup_{h\to+\infty}\frac{u(t)-u(t-h)}{h}.$$

Now we can prove our claim by induction. Noting that $Q'_0=0$ in the above inequality, we see that the C^1 norm of γ is bounded by the initial data. In particular, the norm of any covariant derivatives of curvature tensor of M is bounded on the image of γ. Thus the claim holds for $n=1$. Suppose that the claim holds up to n. Then, φ and Q'_n in the above inequality are uniformly bounded on $[0,\ T)\times S^1$. Therefore, the above inequality implies that $\max|\nabla_\xi\varphi|^2+\max|\nabla_\eta\varphi|^2$ is bounded on $[0,\ T)$, hence $\nabla\varphi=P_{n+1}\gamma$ is uniformly bounded on $[0,\ T)\times S^1$.

Q.E.D.

3. Convergence

Now, we suppose that $\mu>0$ and show the convergence. Let γ be the solution of (H) and put $\varphi=P_n\gamma$. We use an energy inequality for wave equations. Using Lemma 1.2, we have

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla_t\varphi\|^2+\|\nabla_x\varphi\|^2)+2\mu\|\nabla_t\varphi\|^2 \\
&\quad=2\langle\nabla_t\varphi,\ \nabla_t^2\varphi\rangle+2\langle\nabla_x\varphi,\ \nabla_t\nabla_x\varphi\rangle+2\mu\|\nabla_t\varphi\|^2 \\
&\quad=2\langle\nabla_t\varphi,\ \nabla_x^2\varphi-\mu\nabla_t\varphi+Q'_n\rangle+2\langle\nabla_x\varphi,\ \nabla_t\nabla_x\varphi\rangle+2\mu\|\nabla_t\varphi\|^2 \\
&\quad=-2\langle\nabla_x\nabla_t\varphi,\ \nabla_x\varphi\rangle+2\langle\nabla_x\varphi,\ \nabla_t\nabla_x\varphi\rangle+2\langle Q'_n,\ \nabla_t\varphi\rangle \\
&\quad=2\langle Q_n,\ \nabla_x\varphi\rangle+2\langle Q'_n,\ \nabla_t\varphi\rangle \\
&\quad\leq C\|Q'_n\|(\|\nabla_x\varphi\|+\|\nabla_t\varphi\|),
\end{aligned}$$

where Q'_n denotes a form Q_n+Q_{n-1}. Note that $Q'_0=0$.

We show the following proposition by induction.

Proposition 3.1.

1) *Any* $\|P_n\gamma\|$ *is bounded on* $[0, \infty)$.

2) *Any* $\int_0^\infty\|P_n\gamma\|^2dt$ *is finite except* $P_n\gamma=\partial_x\gamma$ *(and* $P_0\gamma$*).*

Proof. Nothing that $Q'_0=0$ in the above inequality, we have

$$\frac{d}{dt}(\|\partial_t\gamma\|^2+\|\partial_x\gamma\|^2)=-2\mu\|\partial_t\gamma\|^2\leq 0.$$

Intergrating both hand sides by t, we see that the claim holds for $n=1$. Suppose that the claim holds up to n. In the above inequality, if the factor $\nabla_x\varphi=\nabla_xP_n\gamma$ contains ∇_t, then

$$\nabla_x\varphi=\nabla_tP_n\gamma+Q_{n-1}.$$

And if not,

$$\nabla_x\varphi=\nabla_x^{n+1}\gamma=\nabla_x^{n-1}(\nabla_t^2\gamma+\mu\partial_t\gamma)=\nabla_tP_n\gamma+Q_{n-1}+\mu P_n\gamma.$$

In both cases, we have

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla_t\varphi\|^2+\|\nabla_x\varphi\|^2)+2\mu\|\nabla_t\varphi\|^2 \\
&\qquad\leq C\|Q'_n\|(2\|\nabla_tP_n\gamma\|+\|Q_{n-1}\|+\|P_n\gamma\|).
\end{aligned}$$

Summing up this inequality for all $\varphi=P_n\gamma$, we have

$$\begin{aligned}
&\frac{d}{dt}\sum_{P_n}(\|\nabla_tP_n\gamma\|^2+\|\nabla_xP_n\gamma\|^2)+2\mu\sum_{P_n}\|\nabla_tP_n\gamma\|^2 \\
&\quad\leq C\{\sum\|Q'_n\|^2+\sum\|Q_{n-1}\|^2+\sum\|P_n\gamma\|^2\}+\varepsilon\sum\|\nabla_tP_n\gamma\|^2,
\end{aligned}$$

where ε can be taken arbitrarily small. Here, $\sum_{P_n}$ means summation $\sum_\varphi$ with respect to all φ of form $P_n\gamma$. Take $\varepsilon=\mu$. Then we see that

$$\frac{d}{dt}\sum_{P_n}(\|\nabla_tP_n\gamma\|^2+\|\nabla_xP_n\gamma\|^2)+\mu\sum_{P_n}\|\nabla_tP_n\gamma\|^2$$

is dominated by a bounded L_1 function, because at least one $P_i\gamma$ is not $\partial_x\gamma$ in each term of Q_n'. Thus, integrating by t, we see that any $\|P_{n+1}\gamma\|^2$ is uniformly bounded and that any $\|\nabla_t P_n \gamma\|$ is L_2. Then, also any $\|\nabla_x P_n \gamma\|$ is L_2, because it can be expressed by a $\|\nabla_t P_n \gamma\|$ and Q_n''s. Therefore, the claim holds for $n+1$. Q.E.D.

End of the proof of Theorem. Finally, we remark that the derivative $(d/dt)\|P_n \gamma\|^2$ is expressed by $P_{n+1}\gamma$, and hence is uniformly bounded on $[0, \infty)$ by Lemma 3.1. Therefore, any $\|P_n \gamma\|^2$ (except $\partial_x \gamma$) converges to 0 when $t \to \infty$.

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