



Title	On Wendt's theorem of knots
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Citation	Osaka Mathematical Journal. 1957, 9(1), p. 61-66
Version Type	VoR
URL	https://doi.org/10.18910/7456
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On Wendt's Theorem of Knots

By Shin'ichi KINOSHITA

1. By the *unknotting number*¹⁾ $s(k)$ of a knot²⁾ k we will mean the minimal number of cuts (See Fig. 1) in order to change the given knot k to an unknotted one. Let $M_g(k)$ be the g -fold branched cyclic covering

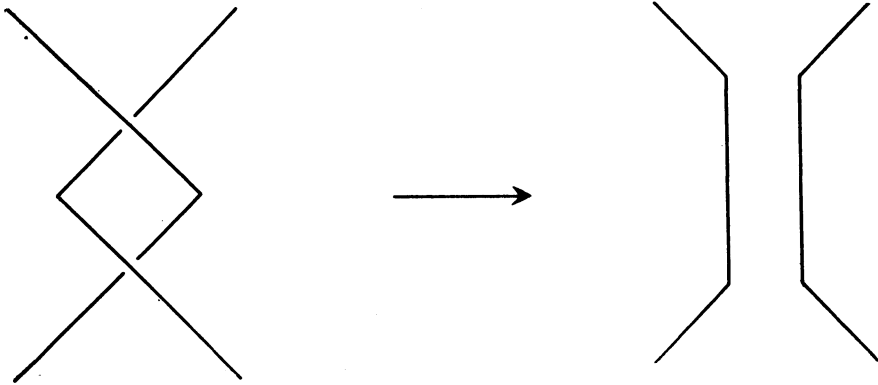


Fig. 1.

space of k and $e_g(k)$ the minimal number of essential generators of the 1-dimensional homology group $H_1(M_g(k))$ of $M_g(k)$. Then in 1937 H. Wendt [3] proved that

$$e_g \leq s(g-1).$$

Now we introduce the *weak unknotting number* $\bar{s}(k)$ defined very similarly to the above one as follows: $\bar{s}(k)$ is the minimal number of cuts as shown in Fig. 2, which change the given knot k to an unknotted one. The purpose of this note is to prove the following inequality

$$(*) \quad e_g \leq \bar{s}(g-1).$$

Since $\bar{s} \leq s$ by the definition, Wendt's inequality follows immediately from ours.

1) Überschneidungszahl in H. Wendt [3], see K. Reidemeister [2], p. 17.
 2) A knot is a polygonal simple closed curve in the 3-sphere S^3 .

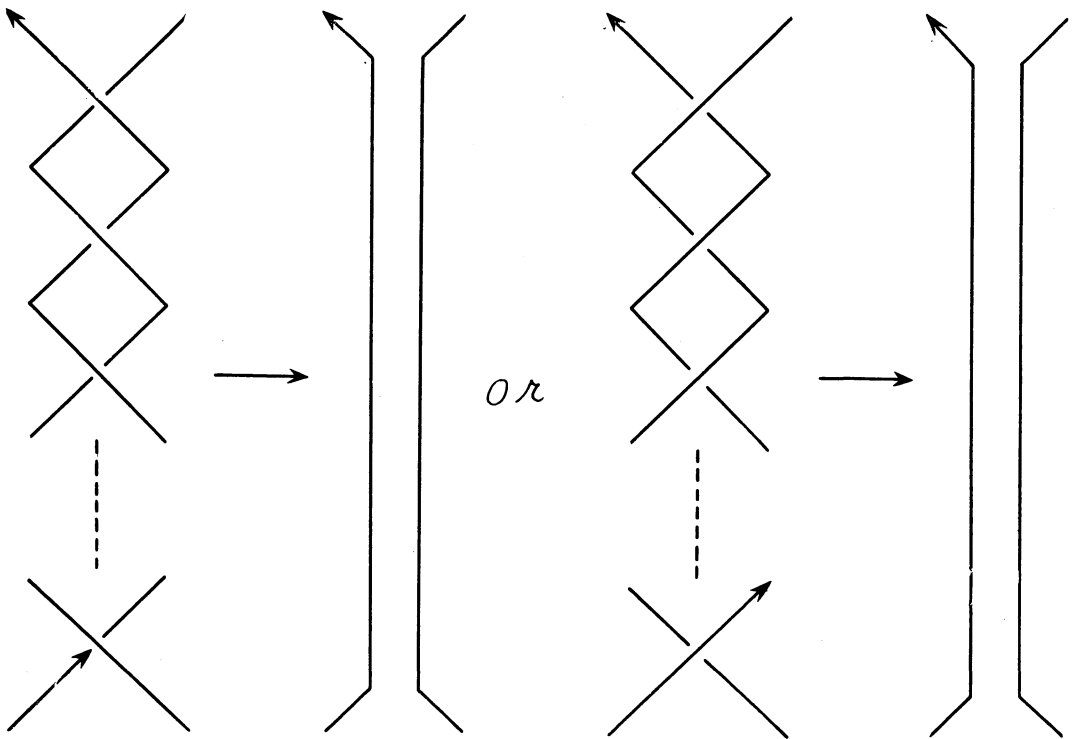


Fig. 2.

2. Now we prove our inequality (*). Let k be a knot. Suppose that k is deformed into k' by one cut, where we assume that k and k' remain the same except for the neighbourhood of the cut. Then we are only to prove that

$$e_g(k) \leq e_g(k') + (g-1)^3.$$

Suppose that k is oriented. Then there are two different cases⁴⁾ as shown in Fig. 3 and Fig. 4 respectively.

Firstly we prove for the case 1. Suppose that the crossing number of k' is equal to $m+2$. Then the crossing number of k is $m+2n$, where $2n$ is the crossing number of two sub-polygons of k as shown in Fig. 3. The regions on the plane decomposed by the projection of k will be denoted by letters as in Fig. 3. Then the Alexander presentation⁵⁾ of the group $F(S^3-k)$ ⁶⁾ is as follows:

3) See H. Wendt [3].

4) Other cases can be proved by the similar method to one of these two cases.

5) See J. W. Alexander [1]. See also R. H. Fox, Ann. of Math. 59 (1954) p. 196.

6) $F(S^3-k)$ is the fundamental group of S^3-k .

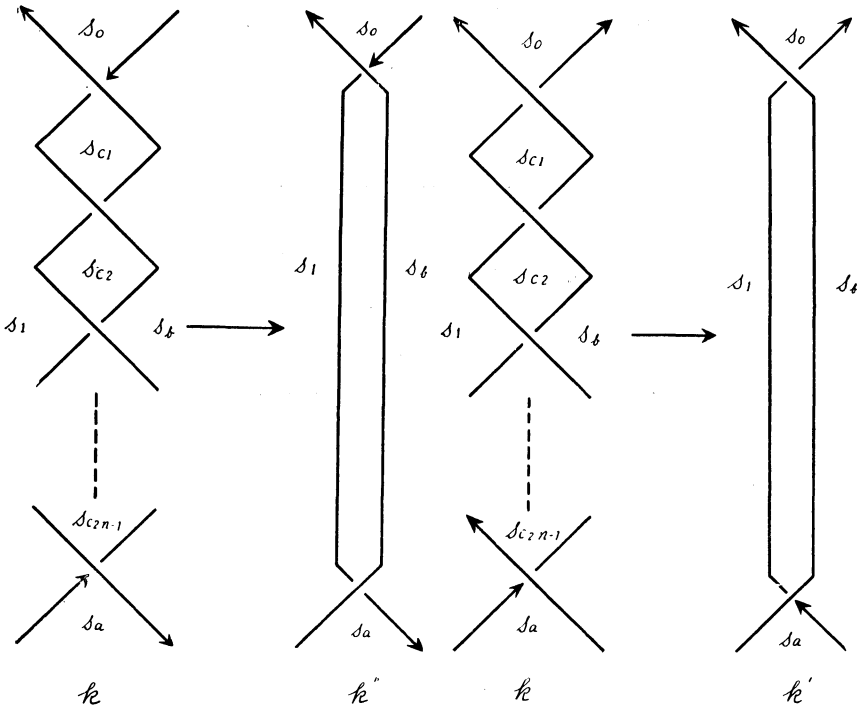


Fig. 3.

Fig. 4.

Generators: $\{ x, s_0, s_1, \dots, s_m, s_a, s_b, s_{c_1}, s_{c_2}, \dots, s_{c_{2n-1}} \}$

Relations: $\left\{ \begin{array}{l} s_0 = s_1 = 1, \\ x s_{c_1} s_b^{-1} x^{-1} s_0 s_1^{-1} = 1, \\ x s_{c_1} s_1^{-1} x^{-1} s_{c_2} s_b^{-1} = 1, \\ \dots, \\ x s_{c_{2n-1}} s_1^{-1} x^{-1} s_a s_b^{-1} = 1, \\ x s_t s_u^{-1} x^{-1} s_v s_w^{-1} = 1, \\ \dots \end{array} \right\} \begin{array}{l} 2n \\ m \end{array}$

This presentation can be transformed to the following one.

Generators: $x, s_0, s_1, \dots, s_m, s_a, s_b.$

Relations: $\left\{ \begin{array}{l} s_0 = s_1 = 1, \\ x s_b^{-n} x^{-1} s_b^n s_a^{-1} = 1, \\ x s_t s_u^{-1} x^{-1} s_v s_w^{-1} = 1, \\ \dots \end{array} \right\} m$

Similarly a presentation of $F(S^g - k')$ is as follows:

Generators: $x, s_0, s_1, \dots, s_m, s_a, s_b.$

$$\text{Relations: } \left\{ \begin{array}{l} s_0 = s_1 = s_a = 1, \\ x s_u s_u^{-1} x^{-1} s_v s_v^{-1} = 1, \\ \dots \dots \dots \end{array} \right\} m$$

Putting

$$s_{i,j} = x^j s_i x^{-j}, \quad \begin{cases} i = 0, 1, \dots, m, a, b \\ j = 0, \pm 1, \pm 2, \dots \end{cases}$$

we know that $H_1(M_g(k))$ is given by the following square matrix:

$$\left(\begin{array}{ccc|ccc} s_{a,0}, \dots, s_{a,g-1}, & s_{b,0}, \dots, s_{b,g-1}, & \dots & & & \\ \hline -1 & & 0 & n & -n & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & & -1 & -n & 0 & n \\ \hline & * & & & d_{ij} & \\ & & & & & 0 \end{array} \right).$$

Adding each $(1+i)$ -th row to the first row ($i=1, 2, \dots, g-1$), we have

$$\left(\begin{array}{ccc|ccc} s_{a,0}, \dots, s_{a,g-1}, & s_{b,0}, \dots, s_{b,g-1}, & \dots & & & \\ \hline -1 & -1 \dots -1 & 0 & 0 & 0 & 0 \dots 0 \\ 0 & -1 & 0 & 0 & n & -n & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & -n & 0 & 0 & n \\ \hline & * & & & d_{ij} & \\ & & & & & 0 \end{array} \right),$$

which can be deformed to

$$\left(\begin{array}{ccc|ccc} -1 & 0 \dots 0 & 0 & 0 & 0 & 0 \dots 0 \\ 0 & -1 & 0 & 0 & n & -n & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & -n & 0 & 0 & n \\ \hline & *' & & & d_{ij} & \\ & & & & & 0 \end{array} \right).$$

Adding each $(1+i)$ -th row to the first row ($i=1, 2, \dots, g-1$), we have

$$\left(\begin{array}{cccc|cccc} s_{a,0}, & \dots, & s_{a,g-1}, & s_{b,0}, & \dots, & s_{b,g-1}, & \dots & \dots \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & & 0 & b_{g-1} & b_0 & & b_{g-2} \\ \vdots & & \ddots & & \vdots & & \ddots & \\ 0 & 0 & & 1 & b_1 & b_2 & & b_0 \end{array} \middle| \begin{array}{c} 0 \\ \\ \\ \\ \end{array} \right),$$

* d_{ij}

which can be deformed to

$$\left(\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & & 0 & b_{g-1} & b_0 & & b_{g-2} \\ \vdots & & \ddots & & \vdots & & \ddots & \\ 0 & 0 & & 1 & b_1 & b_2 & & b_0 \end{array} \middle| \begin{array}{c} 0 \\ \\ \\ \\ \end{array} \right).$$

*' d_{ij}

On the other hand $H_1(M_g(k'))$ is given by (d_{ij}) . Thus

$$e_g(k) \leq e_g(k') + (g-1),$$

which completes the proof of our inequality (*).

(Received March 25, 1957)

References

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