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On Wendt's Theorem of Knots

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1. By the *unknotting number*[1] $s(k)$ of a knot[2] k we will mean the minimal number of cuts (See Fig. 1) in order to change the given knot k to an unknotted one. Let $M_g(k)$ be the g-fold branched cyclic covering

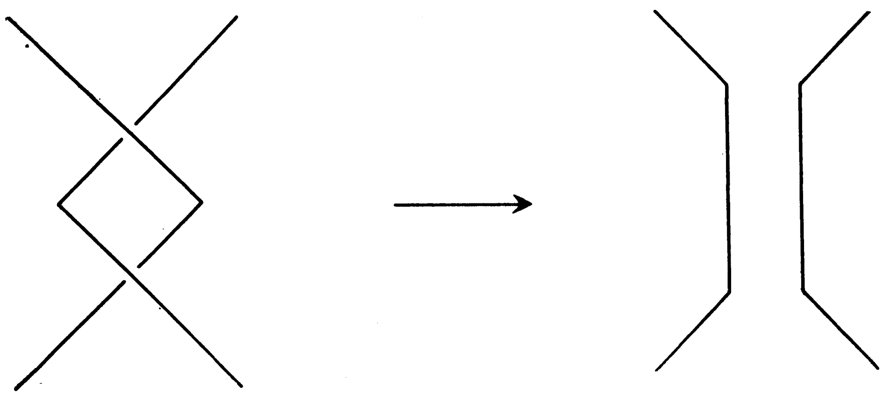

Fig. 1.

space of k and $e_g(k)$ the minimal number of essential generators of the 1-dimensional homology group $H_1(M_g(k))$ of $M_g(k)$. Then in 1937 H. Wendt [3] proved that

$$e_g \leqq s(g-1) .$$

Now we introduce the *weak unknotting number* $\bar{s}(k)$ defined very similarly to the above one as follows: $\bar{s}(k)$ is the minimal number of cuts as shown in Fig. 2, which change the given knot k to an unknotted one. The purpose of this note is to prove the following inequality

$$(*) \qquad e_g \leqq \bar{s}(g-1) .$$

Since $\bar{s} \leqq s$ by the definition, Wendt's inequality follows immediately from ours.

1) Überschneidungszahl in H. Wendt [3], see K. Reidemeister [2], p. 17.

2) A knot is a polygonal simple closed curve in the 3-sphere S^3.

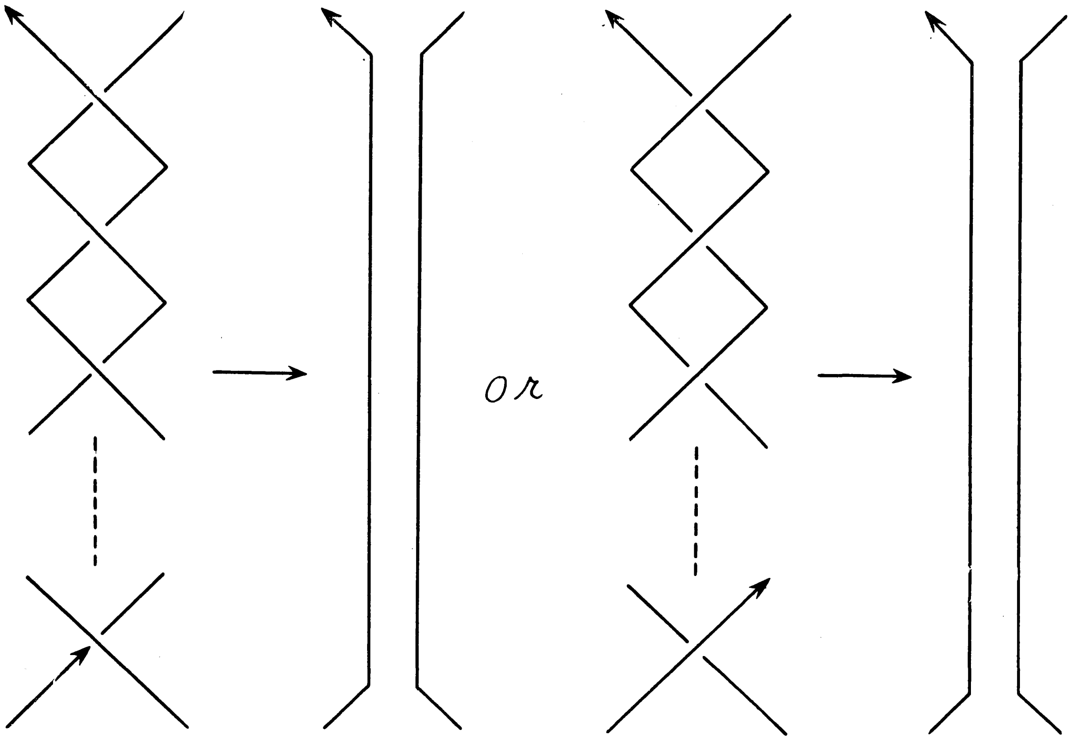

Fig. 2.

2. Now we prove our inequality (*). Let k be a knot. Suppose that k is deformed into k' by one cut, where we assume that k and k' remain the same except for the neighbourhood of the cut. Then we are only to prove that

$$e_g(k) \leqq e_g(k') + (g-1)^{3)} .$$

Suppose that k is oriented. Then there are two different cases[4] as shown in Fig. 3 and Fig. 4 respectively.

Firstly we prove for the case 1. Suppose that the crossing number of k' is equal to $m+2$. Then the crossing number of k is $m+2n$, where $2n$ is the crossing number of two sub-polygons of k as shown in Fig. 3. The regions on the plane decomposed by the projection of k will be denoted by letters as in Fig. 3. Then the Alexander presentation[5] of the group $F(S^3-k)$[6] is as follows:

3) See H. Wendt [3].

4) Other cases can be proved by the similar method to one of these two cases.

5) See J. W. Alexander [1]. See also R. H. Fox, Ann. of Math. 59 (1954) p. 196.

6) $F(S^3-k)$ is the fundamental group of S^3-k.

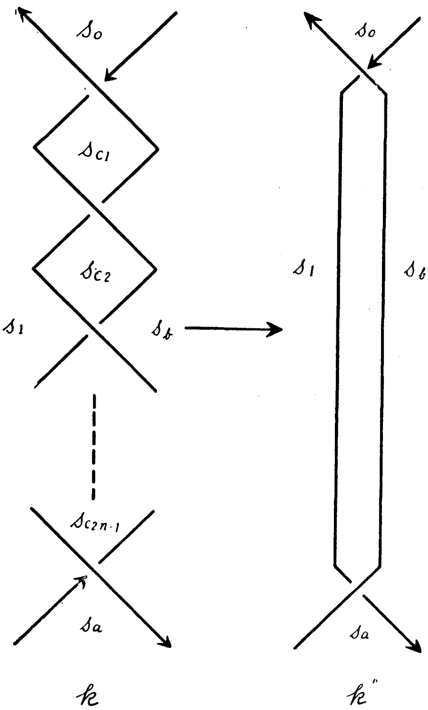

Fig. 3.

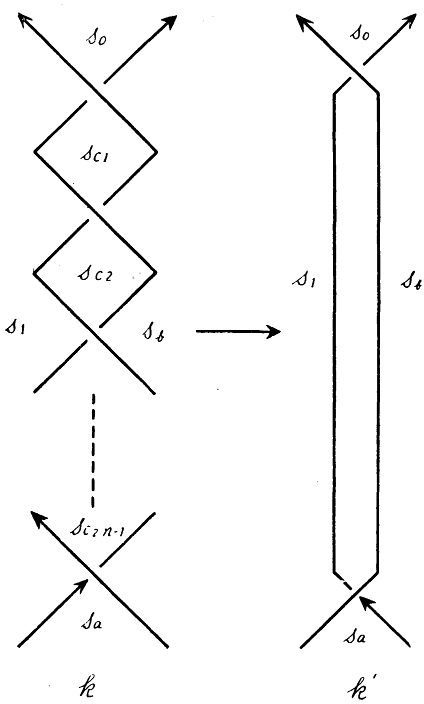

Fig. 4.

$$\text{Generators:}\quad \begin{cases} x,\ s_0,\ s_1, \cdots, s_m,\ s_a,\ s_b, \\ s_{c_1},\ s_{c_2}, \cdots, s_{c_{2n-1}}. \end{cases}$$

$$\text{Relations:}\quad \begin{cases} s_0 = s_1 = 1, \\ \left.\begin{array}{l} x s_{c_1} s_b^{-1} x^{-1} s_0 s_1^{-1} = 1, \\ x s_{c_1} s_1^{-1} x^{-1} s_{c_2} s_b^{-1} = 1, \\ \cdots\cdots\cdots\cdots\cdots\cdots, \\ x s_{c_{2n-1}} s_1^{-1} x^{-1} s_a s_b^{-1} = 1, \end{array}\right\} 2n \\ \left.\begin{array}{l} x s_t s_u^{-1} x^{-1} s_v s_w^{-1} = 1, \\ \cdots\cdots\cdots\cdots\cdots\cdots. \end{array}\right\} m \end{cases}$$

This presentation can be transformed to the following one.

$$\text{Generators:}\quad x,\ s_0,\ s_1, \cdots, s_m,\ s_a,\ s_b.$$

$$\text{Relations:}\quad \begin{cases} s_0 = s_1 = 1, \\ x s_b^{-n} x^{-1} s_b^n s_a^{-1} = 1, \\ \left.\begin{array}{l} x s_t s_u^{-1} x^{-1} s_v s_w^{-1} = 1, \\ \cdots\cdots\cdots\cdots\cdots\cdots. \end{array}\right\} m \end{cases}$$

Similarly a presentation of $F(S^3-k')$ is as follows:

$$\text{Generators:}\quad x,\ s_0,\ s_1,\ \cdots,\ s_m,\ s_a,\ s_b\,.$$

$$\text{Relations:}\quad \begin{cases} s_0=s_1=s_a=1\,, \\ \left.\begin{array}{l} x s_t s_u^{-1} x^{-1} s_v s_w^{-1}=1\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots\,. \end{array}\right\} m \end{cases}$$

Putting

$$s_{i,j}=x^j s_i x^{-j}\,,\qquad \begin{cases} i=0, 1, \cdots, m, a, b \\ j=0,\pm 1,\pm 2, \cdots \end{cases}$$

we know that $H_1(M_g(k))$ is given by the following square matrix:

$$\begin{array}{c} \quad s_{a,0},\ \cdots,\ s_{a,g-1},\ \ s_{b,0},\ \cdots,\ s_{b,g-1},\ \ \cdots\cdots\cdots \\ \left(\begin{array}{ccc|ccc|c} -1 & & 0 & n & -n & 0 & \\ & \ddots & & & \ddots & & \\ & & \ddots & & & \ddots & 0 \\ 0 & & -1 & -n & 0 & n & \\ \hline & * & & & & d_{ij} & \end{array}\right). \end{array}$$

Adding each $(1+i)$-th row to the first row $(i=1, 2, \cdots, g-1)$, we have

$$\begin{array}{c} \quad s_{a,0},\ \cdots,\ s_{a,g-1},\ \ s_{b,0},\ \cdots,\ s_{b,g-1},\ \ \cdots\cdots\cdots \\ \left(\begin{array}{ccc|cccc|c} -1 & -1\cdots & -1 & 0 & 0 & 0\cdots & 0 & \\ 0 & -1 & 0 & 0 & n & -n & 0 & \\ \vdots & \ddots & & & & \ddots & & 0 \\ \vdots & & \ddots & & & & \ddots & \\ 0 & 0 & -1 & -n & 0 & 0 & n & \\ \hline & * & & & & d_{ij} & & \end{array}\right), \end{array}$$

which can be deforemed to

$$\left(\begin{array}{ccc|cccc|c} -1 & 0\cdots\cdots & 0 & 0 & 0 & 0\cdots & 0 & \\ 0 & -1 & 0 & 0 & n & -n & 0 & \\ \vdots & \ddots & & & & \ddots & & 0 \\ \vdots & & \ddots & & & & \ddots & \\ 0 & 0 & -1 & -n & 0 & 0 & n & \\ \hline & *' & & & & d_{ij} & & \end{array}\right).$$

On the other hand $H_1(M_g(k'))$ is given by (d_{ij}). From this it is easy to see that

$$e_g(k) \leqq e_g(k') + (g-1)\,,$$

and the proof for the case 1 is complete.

3. Now we prove for the case 2. The same notation as Nr. 2 will be used. Then the Alexander presentation of $F(S^3-k)$ is as follows:

$$\text{Generators:}\quad \begin{cases} x,\ s_0,\ s_1,\ \cdots,\ s_m,\ s_a,\ s_b\,, \\ s_{c_1},\ s_{c_2},\ \cdots,\ s_{2n-1}\,. \end{cases}$$

$$\text{Relations:}\quad \begin{cases} s_0 = s_1 = 1\,, \\ \left.\begin{array}{l} x s_0 s_1^{-1} x^{-1} s_{c_1} s_b^{-1} = 1\,, \\ x s_{c_1} s_1^{-1} x^{-1} s_{c_2} s_b^{-1} = 1\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots\,, \\ x s_{c_{2n-1}} s_1^{-1} x^{-1} s_a s_b^{-1} = 1\,, \end{array}\right\} 2n \\ \left.\begin{array}{l} x s_t s_u^{-1} x^{-1} s_v s_w^{-1} = 1\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots\,. \end{array}\right\} m \end{cases}$$

This presentation can be transformed to the following one:

$$\text{Generators:}\quad x,\ s_0,\ s_1,\ \cdots,\ s_m,\ s_a,\ s_b\,.$$

$$\text{Relations:}\quad \begin{cases} s_0 = s_1 = 1\,, \\ s_a^{-1}(x s_b^{-1} x)^{2n}\,(x^{-2} s_b)^{2n} = 1\,, \\ \left.\begin{array}{l} x s_t s_u^{-1} x^{-1} s_v s_w^{-1} = 1\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots\,. \end{array}\right\} m \end{cases}$$

Put

$$-(1-x+x^2-\cdots-x^{2n-1}) = b_0 + b_1 x + \cdots + b_{g-1} x^{g-1} \qquad (\mathrm{mod}\ x^g - 1)\,.$$

Then $\sum_{i=0}^{g-1} b_i = 0$. It can easily be seen that $H_1(M_g(k))$ is given by the following square matrix:

$$\begin{array}{c} s_{a,0},\ \cdots,\ s_{a,g-1},\ s_{b,0},\ \cdots,\ s_{b,g-1},\ \cdots\cdots\cdots \\ \left(\begin{array}{cccc|cccc|c} 1 & 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_{g-1} & \\ 0 & 1 & & 0 & b_{g-1} & b_0 & & b_{g-2} & \\ \vdots & & \ddots & & \vdots & & \ddots & & 0 \\ 0 & 0 & & 1 & b_1 & b_2 & & b_0 & \\ \hline & & * & & & & d_{ij} & & \end{array}\right). \end{array}$$

Adding each $(1+i)$-th row to the first row $(i=1, 2, \cdots, g-1)$, we have

$$\begin{array}{c} s_{a,0},\ \cdots,\ s_{a,g-1},\ s_{b,0},\ \cdots,\ s_{b,g-1},\ \cdots\cdots \\ \left(\begin{array}{cccc|cccc|c} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \\ 0 & 1 & & 0 & b_{g-1} & b_0 & & b_{g-2} & \\ \vdots & & \ddots & & \vdots & & \ddots & & 0 \\ 0 & 0 & & 1 & b_1 & b_2 & & b_0 & \\ \hline & & * & & & & d_{ij} & & \end{array}\right), \end{array}$$

which can be deformed to

$$\left(\begin{array}{cccc|cccc|c} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \\ 0 & 1 & & 0 & b_{g-1} & b_0 & & b_{g-2} & \\ \vdots & & \ddots & & \vdots & & \ddots & & 0 \\ 0 & 0 & & 1 & b_1 & b_2 & & b_0 & \\ \hline & & *' & & & & d_{ij} & & \end{array}\right).$$

On the other hand $H_1(M_g(k'))$ is given by (d_{ij}). Thus

$$e_g(k) \leqq e_g(k') + (g-1),$$

which completes the proof of our inequality (*).

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References

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[2] K. Reidemeister: Knotentheorie, Berlin (1932).
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