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# On Wendt's Theorem of Knots 

By Shin'ichi Kinoshita

1. By the unknotting number ${ }^{1)} s(k)$ of a $\operatorname{knot}^{2)} k$ we will mean the minimal number of cuts (See Fig. 1) in order to change the given knot $k$ to an unknotted one. Let $M_{g}(k)$ be the $g$-fold branched cyclic covering


Fig. 1.
space of $k$ and $e_{g}(k)$ the minimal number of essential generators of the 1-dimensional homology group $H_{1}\left(M_{g}(k)\right)$ of $M_{g}(k)$. Then in 1937 H . Wendt [3] proved that

$$
e_{g} \leq s(g-1)
$$

Now we introduce the weak unknotting number $\bar{s}(k)$ defined very similarly to the above one as follows: $\bar{s}(k)$ is the minimal number of cuts as shown in Fig. 2, which change the given knot $k$ to an unknotted one. The purpose of this note is to prove the following inequality

$$
\begin{equation*}
e_{g} \leqq \bar{s}(g-1) \tag{*}
\end{equation*}
$$

Since $\bar{s} \leqq s$ by the definition, Wendt's inequality follows immediately from ours.

[^0]

Fig. 2.
2. Now we prove our inequality (*). Let $k$ be a knot. Suppose that $k$ is deformed into $k^{\prime}$ by one cut, where we assume that $k$ and $k^{\prime}$ remain the same except for the neighbourhood of the cut. Then we are only to prove that

$$
\left.e_{g}(k) \leqq e_{g}\left(k^{\prime}\right)+(g-1)^{3}\right)
$$

Suppose that $k$ is oriented. Then there are two different cases ${ }^{1)}$ as shown in Fig. 3 and Fig. 4 respectively.

Firstly we prove for the case 1 . Suppose that the crossing number of $k^{\prime}$ is equal to $m+2$. Then the crossing number of $k$ is $m+2 n$, where $2 n$ is the crossing number of two sub-polygons of $k$ as shown in Fig. 3. The regions on the plane decomposed by the projection of $k$ will be denoted by letters as in Fig. 3. Then the Alexander presentation ${ }^{5}$ of the group $F\left(S^{3}-k\right)^{6)}$ is as follows:
3) See H. Wendt [3].
4) Other cases can be proved by the similar method to one of these two cases.
5) See J. W. Alexander [1]. See also R. H. Fox, Ann. of Math. 59 (1954) p. 196.
6) $F\left(S^{3}-k\right)$ is the fundamental group of $S^{3}-k$.


Fig. 3.
Fig. 4.

Generators: $\left\{\begin{array}{l}x, s_{0}, s_{1}, \cdots, s_{m}, s_{a}, s_{b}, \\ s_{c_{1}}, s_{c_{2}}, \cdots, s_{c_{2 n-1}} .\end{array}\right.$

This presentation can be transformed to the following one.
Generators: $\quad x, s_{0}, s_{1}, \cdots, s_{m}, s_{a}, s_{b}$.
Relations: $\left\{\begin{array}{l}s_{0}=s_{1}=1, \\ x s_{b}^{-n} x^{-1} s_{s}^{n} s_{a}^{-1}=1, \\ x s_{t} s_{u}^{-1} x^{-1} s_{v} s_{w}^{-1}=1, \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . .\end{array}\right\} m$

Similarly a presentation of $F\left(S^{3}-k^{\prime}\right)$ is as follows:

$$
\begin{array}{ll}
\text { Generators: } & x, s_{0}, s_{1}, \cdots, s_{m}, s_{a}, s_{b} \\
\text { Relations: } & \left\{\begin{array}{l}
s_{0}=s_{1}=s_{a}=1, \\
x s_{t} s_{u}^{-1} x^{-1} s_{v} s_{w}^{-1}=1, \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . .
\end{array}\right\} m
\end{array}
$$

Putting

$$
s_{i, j}=x^{j} s_{i} x^{-j}, \quad\left\{\begin{array}{l}
i=0,1, \cdots, m, a, b \\
j=0, \pm 1, \pm 2, \cdots
\end{array}\right.
$$

we know that $H_{1}\left(M_{g}(k)\right)$ is given by the following square matrix:

$$
\begin{aligned}
& s_{a, 0}, \cdots, s_{a, g-1}, \quad s_{b, 0}, \cdots, s_{b, g-1},
\end{aligned}
$$

Adding each ( $1+i$ )-th row to the first row ( $i=1,2, \cdots, g-1$ ), we have

$$
\begin{aligned}
& s_{a, 0}, \cdots, s_{a, g-1}, s_{b, 0}, \cdots, s_{b, g-1},
\end{aligned}
$$

which can be deforemed to

$$
\left(\left.\begin{array}{rrrr|rrrr}
-1 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\
0 & \cdots & 0 \\
\vdots & 1 & & 0 & 0 & n & -n & 0 \\
\vdots & & \ddots & & & \ddots & \\
0 & 0 & -1 & -n & 0 & 0 & n
\end{array} \right\rvert\, \quad 0\right.
$$

On the other hand $H_{1}\left(M_{g}\left(k^{\prime}\right)\right)$ is given by $\left(d_{i j}\right)$. From this it is easy to see that

$$
e_{g}(k) \leqq e_{g}\left(k^{\prime}\right)+(g-1),
$$

and the proof for the case 1 is complete.
3. Now we prove for the case 2 . The same notation as Nr. 2 will be used. Then the Alexander presentation of $F\left(S^{3}-k\right)$ is as follows:

$$
\begin{aligned}
& \text { Generators: }\left\{\begin{array}{l}
x, s_{0}, s_{1}, \cdots, s_{m}, s_{a}, s_{b}, \\
s_{c_{1}}, s_{c_{2}}, \cdots, s_{2_{n-1}} .
\end{array}\right. \\
& \text { Relations: }\left\{\begin{array}{l}
s_{0}=s_{1}=1, \\
x s_{0} s_{1}^{-1} x^{-1} s_{c_{1}} s_{b}^{-1}=1, \\
x s_{c_{1}} s_{1}^{-1} x^{-1} s_{c_{2}} s_{b}^{-1}=1, \\
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots, \ldots \ldots \ldots, \\
x s_{c_{2 n-1}} s_{1}^{-1} x^{-1} s_{a} s_{b}^{-1}=1, \\
x s_{t} s_{u}^{-1} x^{-1} s_{v} s_{w}^{-1}=1, \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .
\end{array}\right\} m n
\end{aligned}
$$

This presentation can be transformed to the following one:

$$
\begin{aligned}
& \text { Generators: } \quad x, s_{0}, s_{1}, \cdots, s_{m}, s_{a}, s_{b} \text {. } \\
& \text { Relations: }\left\{\begin{array}{l}
s_{0}=s_{1}=1, \\
s_{a}^{-1}\left(x s_{b}^{-1} x\right)^{2 n}\left(x^{-2} s_{b}\right)^{2 n}=1, \\
x s_{t} s_{b}^{-1} x^{-1} s_{v} s_{w}^{-1}=1, \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . .
\end{array}\right\} m
\end{aligned}
$$

Put

$$
-\left(1-x+x^{2}-\cdots-x^{2 n-1}\right)=b_{0}+b_{1} x+\cdots+b_{g-1} x^{g-1} \quad\left(\bmod x^{g}-1\right)
$$

Then $\sum_{i=0}^{g-1} b_{i}=0$. It can easily be seen that $H_{1}\left(M_{g}(k)\right)$ is given by the following square matrix:

Adding each $(1+i)$-th row to the first row $(i=1,2, \cdots, g-1)$, we have

$$
\begin{aligned}
& s_{a, 0}, \cdots, s_{a, g-1}, s_{b, 0}, \cdots, s_{b, g-1}, \\
& \left(\left.\begin{array}{ccccccccc}
1 & 1 & \cdots \cdots \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 1 & & & 0 & b_{g-1} & b_{0} & & b_{g-2} \\
\vdots & & \ddots & & & \vdots & & \ddots & \\
\vdots & & \ddots & \ddots & & & & & \\
0 & 0 & & & 1 & b_{1} & b_{2} & & \\
b_{0}
\end{array} \right\rvert\, \quad \begin{array}{l}
0 \\
\hline
\end{array}\right.
\end{aligned}
$$

which can be deformed to

On the other hand $H_{1}\left(M_{g}\left(k^{\prime}\right)\right)$ is given by $\left(d_{i j}\right)$. Thus

$$
e_{g}(k) \leqq e_{g}\left(k^{\prime}\right)+(g-1)
$$

which completes the proof of our inequality (*).
(Received March 25, 1957)

## References

[1] J. W. Alexander: Topological invariants of knots and links, Trans. Amer. Math Soc. 30, 275-306 (1928).
[2] K. Reidemeister: Knotentheorie, Berlin (1932).
[3] H. Wendt: Die gordische Auflösung von Knoten, Math. Z. 42, 680-696 (1937).


[^0]:    1) Überschneidungszahl in H. Wendt [3], see K. Reidemeister [2], p. 17.
    2) A knot is a polygonal simple closed curve in the 3 -sphere $S^{3}$.
