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NON-ORIENTABLE SURFACES IN 4-SPACE

Dedicated to Professor Junzo Tao on his sixtieth birthday

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A. Kawauchi, T. Shibuya and S. Suzuki proved that any closed connected oriented surface piecewise-linearly and locally-flatly embedded in Euclidean 4-space $\boldsymbol{R}^4$ can be deformed into a surface with a special configuration called a *normal form* [5]. In this paper we difine normal forms for closed connected non-orientable surfaces in $\boldsymbol{R}^4$ and prove that any closed connected non-orientable surface pieceaise-linearly and locally-flatly embedded in $\boldsymbol{R}^4$ is deformed into a normal form (Theorem 1.3).

It is known that the Euler number of a closed connected non-orientable surface in $\boldsymbol{R}^4$ can only take on the following values: 2χ-4, 2χ, $2\chi+4$, $\cdots$, $4-2\chi$, where χ is the Euler characteristic of the surface. This was conjectured by H. Whitney in 1940 [10] and proved by W.S. Massey in 1969 [8] using the Atiyah-Singer index theorem. We give, as an application of Theorem 1.3, a geometrical proof to it.

We prepare some definitions and state the main theorem (Theorem 1.3) in Section 1 and prove it in Section 2. In Section 3 we study the relationship between the Euler number and the normal form. Section 4 concerns unknotted non-orientable surfaces in $\boldsymbol{R}^4$. The above mentioned proof of the Whitney and Massey theorem are given in Section 5.

Throughout this paper, we work in the piecewise linear category. For the notation, we refer to K-S-S [5].

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1 Definitons and Main Theorem

In this section all links and bands are unoriented. Let l be a link in $\boldsymbol{R}^3$ and $B_1, \cdots, B_m$ be mutually disjoint bands spanning l. For each $i(=1, \cdots, m)$, the intersection of l and B_i is the union of two disjoint arcs α_i, α_i' on l with $B_i \cap l = \partial B_i \cap l = \alpha_i \cup \alpha_i'$. Then $\mathrm{Cl}(l \cup \partial B_1 \cup \cdots \cup \partial B_m - (\alpha_1 \cup \alpha_1' \cup \cdots \cup \alpha_m \cup \alpha_m'))$ is a link. We call the new link *the link obtained from l by the* (*unoriented*) *hyperbolic* *transformations along the bands* $B_1, \cdots, B_m$, and denote it by $\boldsymbol{h}(\ell; B_1, \cdots, B_m)$. Bands $B_1, \cdots, B_m$ spanning a knot k are said to be *cohernt to* k if they can be assigned orientations which are coherent to an orientation of k. Otherwise, $B_1, \cdots, B_m$ are said to be *noncoherent to* k. See Fig. 1. Further, noncoherent bands $B_1, \cdots, B_m$ spanning a knot k are said to be *in regular position to* k if there exist mutually disjoint m simple arcs $I_1, \cdots, I_m$ on k such that for each $i=1, \cdots, m$, the attaching arcs α_i, α'_i of the bands B_i are contained in I_l. See Fig. 2.

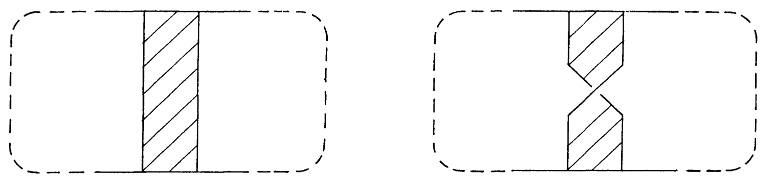

Fig. 1

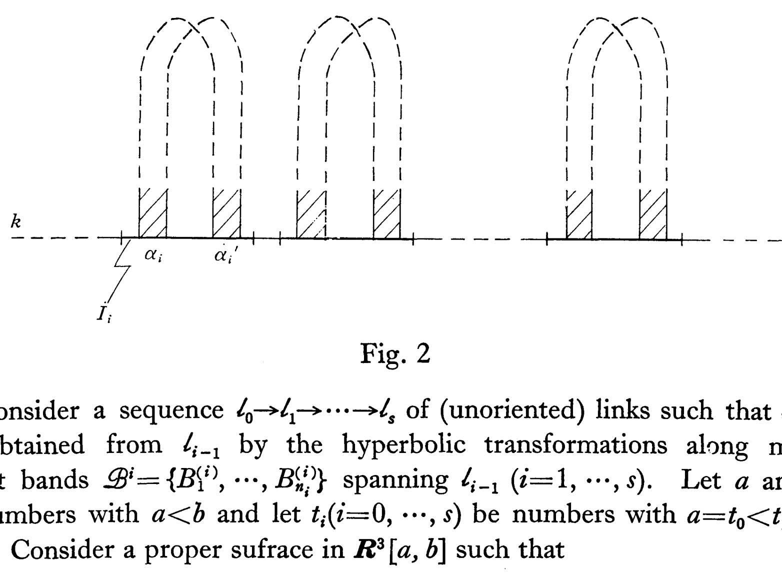

Fig. 2

Consider a sequence $\ell_0 \to \ell_1 \to \cdots \to \ell_s$ of (unoriented) links such that ℓ_i is the link obtained from ℓ_{i-1} by the hyperbolic transformations along mutually disjoint bands $\mathcal{B}^i = \{B_1^{(i)}, \cdots, B_{n_i}^{(i)}\}$ spanning ℓ_{i-1} $(i=1, \cdots, s)$. Let a and b be real numbers with $a<b$ and let $t_i (i=0, \cdots, s)$ be numbers with $a=t_0<t_1<\cdots<t_s=b$. Consider a proper sufrace in $\boldsymbol{R}^3[a, b]$ such that

$$F \cap \boldsymbol{R}^3[t] = \begin{cases} \ell_{i-1}[t] & \text{for} \quad t \in [t_{i-1}, (t_{i-1}+t_i)/2) \\ (\ell_{s-1} \cup \mathcal{B}^i)\,[t] & \text{for} \quad t = (t_{i-1}+t_i)/2 \\ \ell_i[t] & \text{for} \quad t \in ((t_{i-1}+t_i)/2, t_i]\,(i=1, 2, \cdots, s)\,. \end{cases}$$

We call F *the realizing surface of the sequence* $\ell_0 \to \ell_1 \to \cdots \to \ell_s$ and denote it by $F_a^b(\ell_0, \ell_1, \cdots, \ell_s; \mathcal{B}^1, \mathcal{B}^2, \cdots \mathcal{B}^s)$ or $F_a^b(\ell_0 \to \ell_1 \to \cdots \to \ell_s)$. (For a subset A of $\boldsymbol{R}^3$, we define $A[t_1] = \{(x, t) \in \boldsymbol{R}^3 \times \boldsymbol{R} \mid x \in A, t=t_1\}$ and $A[t_1, t_2] = \{(x, t) \in \boldsymbol{R}^3 \times \boldsymbol{R} \mid x \in A, t_1 \le t \le t_2\}$.) If both ℓ_0 and ℓ_s are trivial, there exist mutually disjoint disks $\mathcal{D}_-$ (resp. $\mathcal{D}_+$) in $\boldsymbol{R}^3$ with $\partial\mathcal{D}_- = \ell_0$ (resp. $\partial\mathcal{D}_+ = \ell_s$). Then the closed surface

$$\bar{F}_a^b = F_a^b(\ell_0, \ell_1, \cdots, \ell_s; \mathscr{B}^1, \mathscr{B}^2, \cdots, \mathscr{B}^s) \cup \mathscr{D}_-[a] \cup \mathscr{D}_+[b]$$

is called *the closed realizing surface of the sequence* $\ell_0 \to \ell_1 \to \cdots \to \ell_s$.

REMARK 1.1. The closed realizing surface $\bar{F}_a^b$ does not depend on the choices of disks $\mathscr{D}_-$ and $\mathscr{D}_+$ in the following sense. Suppose $\mathscr{D}'_-$ and $\mathscr{D}'_+$ are other disks with $\partial\mathscr{D}'_- = \ell_0$ and $\partial\mathscr{D}'_+ = \ell_s$, then the closed surface $\bar{F}'^b_a = F_a^b \cup \mathscr{D}'_-[a] \cup \mathscr{D}'_+[b]$ is ambient isotopic in $\boldsymbol{R}^3(-\infty, +\infty) = \boldsymbol{R}^4$ to $\bar{F}_a^b = F_a^b \cup \mathscr{D}_-[a] \cup \mathscr{D}_+[b]$ and the ambient isotopy may keep $\boldsymbol{R}^3[a+\varepsilon, b-\varepsilon]$ fixed for sufficiently small positive

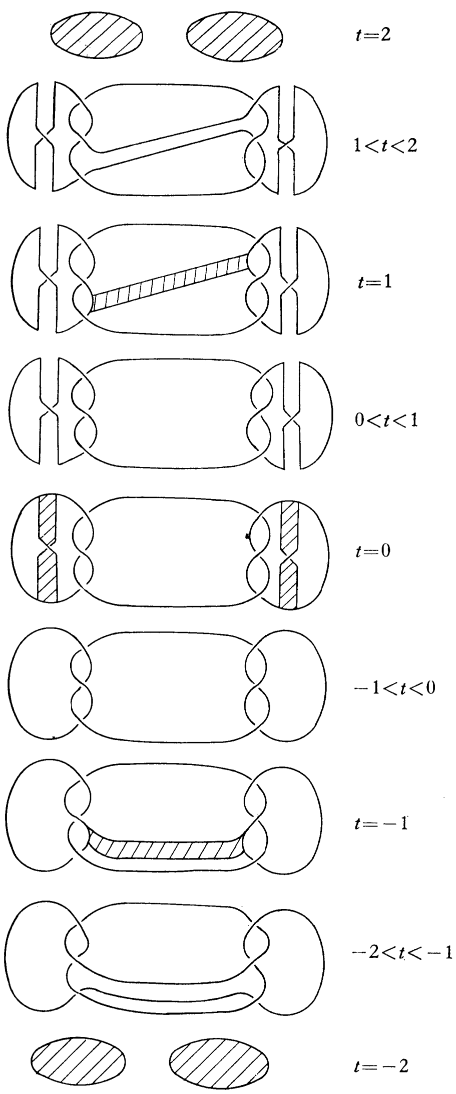

Fig. 3

number ε. (See for example K-S-S [5], Lemma 1.5.)

DEFINITION 1.2. A closed, connected, locally-flat and non-orientable surface F in $\boldsymbol{R}^4$ is said to be *in a normal form* if F is the closed realizing surface of a sequence $\mathcal{O}_- \to k_- \to k_+ \to \mathcal{O}_+$ with the following properties.

(1) $\mathcal{O}_-$ and $\mathcal{O}_+$ are trivial links.

(2) $\mathcal{O}_- \to k_-$ is *complete fusion*. Namely k_- is the knot obtained from $\mathcal{O}_-$ by the hyperbolic transformations along $\mathcal{B}_- = \{B_i^-\}$ with $|\mathcal{B}_-| = |\mathcal{O}_-| - 1$, where $|\ \ |$ means the number of components.

(3) k_+ is the knot obtained from k_- by the hyperbolic tranfsormations along noncoherent bands, $\mathcal{B}_0 = \{B_i^0\}$, spanning k_- in regular position to k_-.

(4) $k_+ \to \mathcal{O}_+$ is *complete fission*. Namely $\mathcal{O}_+$ is the link obtained from the knot k_+ by the hyperbolic transformations along $\mathcal{B}_+ = \{B_i^+\}$, with $|\mathcal{B}_+| = |\mathcal{O}_+| - 1$.

We call $k_- \cup \mathcal{B}_0 = k_+ \cup \mathcal{B}_0$ *the middle cross-section of* F and k_- (resp. k_+) *the lower* (resp. *upper*) *cross-sectional knot*. Figure 3 is an example of a surface in the normal form. The following is the main theorem of this paper, whose proof is in Section 2.

Theorem 1.3. *Any closed, connected, locally-flat and non-orientable surface* F *in* $\boldsymbol{R}^3(-\infty, +\infty)$ *can be deformed into a surface in the normal form by an ambient isotopy of* $\boldsymbol{R}^3(-\infty, +\infty)$. *Further, the number of the middle cross-sectional bands is equal to the non-orientable genus of the surface* F $(=\operatorname{rank}_{\boldsymbol{Z}_2} H_1(F; \boldsymbol{Z}_2))$.

2. Proof of the Main Theorem

In this section we prove Theorem 1.3. It is a non-orientable version of K-S-S [5, Lemma 2.1]. We use some lemmas in [5], in which it is assumed that links, bands and surfaces have suitable orientations. But the following lemmas are still valid without changing proofs.

Lemma 2.1 (cf. K-S-S [5, Lemma 2.8]). *Let* F *be a closed connected locally-flat surface in* $\boldsymbol{R}^3(-\infty, +\infty)$. F *is ambient isotopic in* $\boldsymbol{R}^3(-\infty, +\infty)$ *to the closed realizing surface of a sequence of hyperbolic transformations* $\mathcal{O} = l_0 \to l_1 \to l_2 \to \cdots \to l_m = \mathcal{O}'$ *such that* $l_{i+1} = \boldsymbol{h}(l_i; B_{i+1})$ *for a band* B_{i+1}, $i = 0, 1, \cdots, m-1$, *and* $\mathcal{O}$, $\mathcal{O}'$ *are trivial links.*

Lemma 2.2. (cf. K-S-S [5, Lemma 1.10]). *Let* l *be a link in* $\boldsymbol{R}^3$. *And let* $B_1, \cdots, B_m, B_1', \cdots, B_{m'}'$ *be mutually disjoint bands spanning* l. *Consider two sequences*

$$l \to l' = \boldsymbol{h}(l; \mathcal{B}) \to l'' = \boldsymbol{h}(l'; \mathcal{B}')$$

and

$$\ell \to \ell'' = \boldsymbol{h}(\ell; \mathcal{B} \cup \mathcal{B}'),$$

where $\mathcal{B} = \{B_1, \cdots, B_m\}$ *and* $\mathcal{B}' = \{B'_1, \cdots, B'_{m'}\}$. *Then the realizing surfaces* $F^b_a(\ell, \ell', \ell''; \mathcal{B}, \mathcal{B}')$ *and* $F^b_a(\ell, \ell''; \mathcal{B} \cup \mathcal{B}')$ *are ambient isotopic in* $\boldsymbol{R}^3(-\infty, +\infty)$ *keeping* $\boldsymbol{R}^3(-\infty, a] \cup \boldsymbol{R}^3[b, +\infty)$ *fixed.*

An ambient isotopy $\{h_s\}_{s\in I}$ of $\boldsymbol{R}^3(-\infty, +\infty)$ is said to be *level-preserving* if $h_s(\boldsymbol{R}^3[t]) = \boldsymbol{R}^3[t]$ holds for each $s \in I$ and t with $-\infty < t < +\infty$, and $[a, b]$-*vertical-line-preserving* if for each $s \in I$ and $x \in \boldsymbol{R}^3$ there exists a unique point $x_s \in \boldsymbol{R}^3$ such that $h_s(x[t]) = x_s[t]$ holds for all $t \in [a, b]$.

Lemma 2.3 (cf. K-S-S [5, Lemma 1.12]). *Suppose* ℓ_1 *and* ℓ'_1 *are links obtained from a link* ℓ_0 *by the hyperbolic transformations along bands* $\{B_i\}$ *and* $\{B'_j\}$ *spanning* ℓ_0, *respectively. If* $\ell_0 \cup \cup_i B_i$ *is ambient isotopic in* $\boldsymbol{R}^3$ *to* $\ell_0 \cup \cup_j B'_j$ *keeping* ℓ_0 *fixed setwise, then the realizing surfaces* $F^b_a(\ell_0, \ell_1; \{B_i\})$ *and* $F^b_a(\ell_0, \ell'_1; \{B'_j\})$ *are ambient isotopic in* $\boldsymbol{R}^3(-\infty, +\infty)$. *Moreover we may assume the ambient isotopy is level-preserving and* $[\xi_1, \xi_2]$-*vertical-line preserving for arbitrarily given* ξ_1, ξ_2 *with* $a < \xi_1 \leq (a+b)/2 < b \leq \xi_2$ *and keeps* $\boldsymbol{R}^3(-\infty, a]$ *fixed.*

Lemma 2.4 (cf. K-S-S [5, Lemma 1.14]). *Suppose that* $\ell = \ell_0 \to \ell_1 \to \cdots \to \ell_n = \ell'$ *is a sequence of hyperbolic transformations from a link* ℓ *to a link* ℓ' *such that* $\ell_{i+1} = \boldsymbol{h}(\ell_i; B_{i+1})$ *for a band* B_{i+1}, $i = 0, 1, \cdots, n-1$. *Then there exist mutually disjoint bands* $B'_1, \cdots, B'_n$ *spanning* ℓ *such that the realizing surface* $F^b_a(\ell, \ell''; \{B'_1, \cdots, B'_n\})$ *with* $\ell'' = \boldsymbol{h}(\ell; B'_1, \cdots, B'_n)$ *is ambient isotopic in* $\boldsymbol{R}^3(-\infty, +\infty)$ *to the realizing surface* $F^b_a(\ell, \ell_1, \cdots, \ell_n; B_1, B_2, \cdots, B_n)$. *Moreover we may assume that the ambient isotopy is level-preserving on* $\boldsymbol{R}^3[b, +\infty)$ *and* $[b, \rho]$-*vertical-line-preserving for arbitrarily given* ρ *with* $b < \rho$ *and keeps* $\boldsymbol{R}^3(-\infty, a]$ *fixed.*

We need one more lemma to prove Theorem 1.3, which is a non-orientable version of [5, Lemma 1.16].

Lemma 2.5. *Let* $B_1, \cdots, B_n$ *be mutually disjoint bands spanning a knot* k. *And let* k' *be the knot obtained from* k *by the hyperbolic transformations along* $B_1, \cdots, B_n$. *If there exists at least one noncoherent band to* k *in* $B_1, \cdots, B_n$, *then there exist new mutually disjoint* n *noncoherent bands* $\bar{B}_1, \cdots, \bar{B}_n$ *spanning* k *in regular position such that the realizing surfaces* $F^b_a(k, k''; \{\bar{B}_1, \cdots, \bar{B}_n\})$ *and* $F^b_a(k, k'; \{B_1, \cdots, B_n\})$ *are ambient isotopic by an ambient isotopy of* $\boldsymbol{R}^3(-\infty, +\infty)$ *keeping* $\boldsymbol{R}^3(-\infty, a]$ *fixed, where* k'' *is the knot obtained from* k *by the hyperbolic transformations along* $\bar{B}_1, \cdots, \bar{B}_n$. *This ambient isotopy may be level-preserving on* $\boldsymbol{R}^3[b, +\infty)$ *and* $[b, \rho]$-*vertical-line-preserving for an arbitrary* $\rho > b$.

Proof of Lemma 2.5. Let B_1 be a noncoherent band to k in $B_1, \cdots, B_n$. Take an arc I_1 on k such that the attaching arcs α_1, α'_1 of B_1 are contained in I_1. Consider the realizing surface $F^b_a(k, k^{(1)}, k'; B_1, \{B_2, \cdots, B_n\})$, where $k^{(1)}$ is the

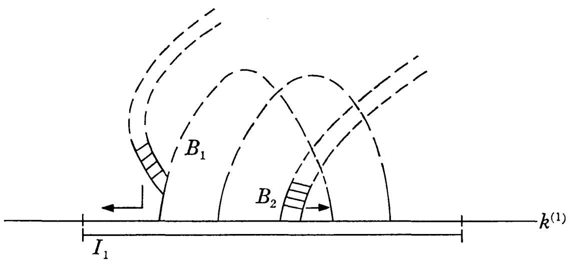

Fig. 4

knot obtained from k by the hyperbolic transformation along B_1. By Lemma 2.2, it is ambient isotopic to $F_a^b(k, k'; \{B_1, \cdots, B_n\})$ by an ambient isotopy of $\boldsymbol{R}^3(-\infty, +\infty)$ keeping $\boldsymbol{R}^3(-\infty, a] \cup \boldsymbol{R}^3[b, +\infty)$ fixed. Slide the attaching arcs $\{\alpha_i, \alpha_i'; i=2, \cdots, n\}$ of $B_2, \cdots, B_n$ along $k^{(1)}$ and deform $B_2, \cdots, B_n$ into $B_2^{(1)}, \cdots, B_n^{(1)}$ such that they are disjoint from B_1 and I_1. (In the process, they may intersect $B_1 - k^{(1)}$.) See Fig. 4. By Lemma 2.3, the realizing surface $F_a^b(k, k^{(1)}, k^{(1)\prime}; B_1, \{B_2^{(1)}, \cdots, B_n^{(1)}\})$ is ambient isotopic to $F_a^b(k, k^{(1)}, k'; B_1, \{B_2, \cdots, B_n\})$ by an ambient isotopy as in Lemma 2.3, where $k^{(1)\prime} = \boldsymbol{h}(k^{(1)}; B_2^{(1)}, \cdots, B_n^{(1)})$. Again by Lemma 2.2, the realizing surface $F_a^b(k, k^{(1)}, k^{(1)\prime}; B_1, \{B_2^{(1)}, \cdots, B_n^{(1)}\})$ is ambient isotopic in $\boldsymbol{R}^3(-\infty, +\infty)$ to $F_a^b(k, k^{(1)\prime}; \{B_1, B_2^{(1)}, \cdots, B_n^{(1)}\})$. We can get the desired bands by repeating analogous process. This completes the proof of 2.5.

Proof of Theorem 1.3. Let F be a closed, connected, locally-flat and non-orientable surface in $\boldsymbol{R}^3(-\infty, +\infty)$. By Lemma 2.1, F is ambient isotopic in $\boldsymbol{R}^3(-\infty, +\infty)$ to the closed realizing surface of a sequence $\mathcal{O}_- = l_0 \to l_1 \to \cdots \to l_m = \mathcal{O}_+$. Applying Lemma 2.4, we get mutually disjoint m bands $B_1, \cdots, B_m$ spanning $\mathcal{O}_-$ such that the closed realizing surface $\bar{F}_a^b(\mathcal{O}_- \to l_1 \to \cdots \to l_m = \mathcal{O}_+)$ is ambient isotopic in $\boldsymbol{R}^3(-\infty, +\infty)$ to the realizing surface $\bar{F}_a^{b(1)} = \bar{F}_a^b(\mathcal{O}_-, \mathcal{O}_+'; \{B_1, \cdots, B_m\})$, where $\mathcal{O}_+' = \boldsymbol{h}(\mathcal{O}_-; B_1, \cdots, B_m)$.

Let $\mathcal{O}_-$(resp. $\mathcal{O}_+'$) have μ (resp. μ') components. Since F is connected, there exist at least $\mu-1$ bands, say $B_1^-, \cdots, B_{\mu-1}^-$, in $\{B_1, \cdots, B_m\}$ such that $\mathcal{O}_- \cup \bigcup_{i=1}^{\mu-1} B_i^-$ is connected. And there exist at least $\mu'-1$ bands, say $B_1^+, \cdots, B_{\mu'-1}^+$, in $\{B_1, \cdots, B_m\} - \{B_1^-, \cdots, B_{\mu-1}^-\}$ such that $\mathcal{O}_+ \cup \bigcup_{i=1}^{\mu'-1} B_i^+$ is connected. Put $\mathcal{B}_- = \{B_1^-, \cdots, B_{\mu-1}^-\}$, $\mathcal{B}_+ = \{B_1^+, \cdots, B_{\mu'-1}^+\}$ and $\mathcal{B}_0 = \{B_1, \cdots, B_m\} - \mathcal{B}_- - \mathcal{B}_+$. By Lemma 2.2, $\bar{F}_a^{b(1)}$ is ambient isotopic in $\boldsymbol{R}^3(-\infty, +\infty)$ to $\bar{F}_a^{b(2)} = \bar{F}_a^b(\mathcal{O}_-, k_-, k_+, \mathcal{O}_+'; \mathcal{B}_-, \mathcal{B}_0, \mathcal{B}_+)$ with $k_- = \boldsymbol{h}(\mathcal{O}_-; \mathcal{B}_-)$ and $k_+ = \boldsymbol{h}(\mathcal{O}_+'; \mathcal{B}_+)$.

Since F is non-orientable, there exists at least one noncoherent band to k_- in $\mathcal{B}_0$. [If $\mathcal{B}_0$ has no noncoherent bands to k_-, then F is orientable. cf. K-S-S [5].] By Lemma 2.5, there exists a family of noncoherent bands spanning k_- in regular position, say $\mathcal{B}_0'$, so that $\bar{F}_a^{b(2)}$ is ambient isotopic in $\boldsymbol{R}^3(-\infty, +\infty)$ to $\bar{F}_a^{b(3)} = \bar{F}_a^b(\mathcal{O}_-, k_-, k_+', \mathcal{O}_+''; \mathcal{B}_-, \mathcal{B}_0', \mathcal{B}_+')$, where $k_+' = \boldsymbol{h}(k_-; \mathcal{B}_0')$ and $\mathcal{B}_+'$ is a family of $\mu'-1$ bands spanning k_+' and $\mathcal{O}_+'' = \boldsymbol{h}(k_+'; \mathcal{B}_+')$. Thus we get the required surface $\bar{F}_a^{b(3)}$ which is in the normal form and ambient isotopic to F in

$\boldsymbol{R}^3(-\infty, +\infty)$.

Let χ be the Euler characteristic of F. The equality

$$\chi = \mu - m + \mu'$$

is easily verified. So the number of middle cross-sectional bands $\mathscr{B}_0'$ is

$$\begin{aligned} |\mathscr{B}_0'| &= m - (\mu - 1 + \mu' - 1) \\ &= 2 - \chi\,. \end{aligned}$$

This is the non-orientable genus of F. This completes the proof.

3 Euler Number

Let F be a closed, connected, locally-flat and non-orientable surface in $\boldsymbol{R}^4$. The regular neighborhood N of F may be regarded as a normal disk bundle over F. Let $\tilde{p}\colon \tilde{F} \to F$ be the orientable double covering of F. Consider the induced bundle $\tilde{N}$ over $\tilde{F}$ and assign $\tilde{N}$ by the map $\tilde{p}\colon \tilde{N} \to N$ the orientation associated with that of N in $\boldsymbol{R}^4$.

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\tilde{p}} & N \\ \downarrow & & \downarrow \\ \tilde{F} & \xrightarrow[p]{} & F \end{array}$$

Definition 3.1. The Euler number of the surface F is defined by

$$e(F) = \mathrm{Int}_{\tilde{N}}(\tilde{F}, \tilde{F})/2\,,$$

where $\mathrm{Int}_{\tilde{N}}(\tilde{F}, \tilde{F})$ denotes the self-intersection number of $\tilde{F}$ in $\tilde{N}$.

The Euler number defined above is equal to the Euler number of the normal bundle of F in $\boldsymbol{R}^4$ using local coefficients that is an invariant of embedded surfaces in $\boldsymbol{R}^4$ (cf. [10]). In this section we study the relationship between the Euler number of F and its middle cross-section.

Let (k_0, k_0^*) be the pair of a knot k_0 and its longitude k_0^*, where k_0^* is a longitude of the boundary of the tubular neighborhood V_0 of k_0 in $\boldsymbol{R}^3$. (It is not required to be preferred.) Consider a noncoherent band B spanning k_0 with attaching arcs $\{\alpha, \alpha'\}$. Let B' be the band obtained by spreading B along k_0 slightly and let V be the thin tubular neighborhood of k_0 such that the intersection of B' and V consists of two rectangles. V may be different from V_0. The intersection of B' and ∂V is the union of two disjoint arcs α^* and α'^*. Modify k_0^* slightly so that it coincides with α^* and α'^* near α and α' by an ambient isotopy of $\boldsymbol{R}^3$ keeping k_0 fixed. See Fig. 5. Then the band $B^* = B' - V$ spans the new k_0^*, obtained from k_0^* by that modification, with the attaching arcs $\{\alpha^*, \alpha'^*\}$. Let k and k^* be the knots obtained from k_0 and k_0^* by the hyperbolic

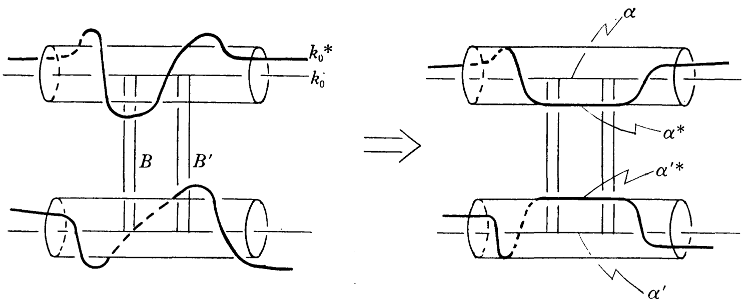

Fig. 5

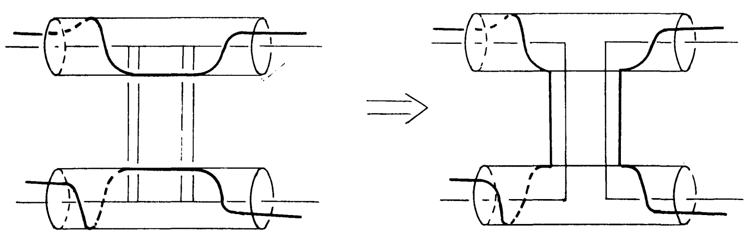

Fig. 6

transformations along the bands B and B^* respectively. See Fig. 6. k^* is uniquely determined up to ambient isotopy of $\boldsymbol{R}^3$ keeping k fixed.

DEFINITION 3.2. Under the above notations, the pair (k, k^*) is called *the pair obtained from* (k_0, k_0^*) *by the hyperbolic transformation along the noncoherent band* B.

Let F be a closed, connected, locally-flat and non-orientable surface in $\boldsymbol{R}^4$. By Theorem 1.3, we can deform it into the closed realizing surface of a sequence $\mathcal{O}_- \to k_- \to k_+ \to \mathcal{O}_+$ satisfying the conditions in Definition 1.2. We also denote by F the deformed surface in the normal form. Consider the preferred longitude k_-^* of the knot k_-, i.e. the linking number of (k_-, k_-^*) is zero, and the pair (k_+, k_+^*) obtained from (k_-, k_-^*) by the hyperbolic transformations along the middle cross-sectional bands $\mathcal{B}_0$ which are noncoherent to k_-. Then the following proposition holds.

Proposition 3.3. *The linking number of the pair* (k_+, k_+^*) *is equal to* $e(F)$.

Proof. Let $\{h_t\}_{t\in I}$ be the ambient isotopy of $\boldsymbol{R}^3$ keeping k_- fixed that carries k_-^* onto $h_1(k_-^*)$ from which k_+^* is obtained (see Fig. 5.). Let ε be a sufficiently small positive number. Assume that the levels of $\mathcal{B}_-$, $\mathcal{B}_0$ and $\mathcal{B}_+$ are -1, 0 and 1 respectively. We consider a proper surface $F^*{}_{-1+\varepsilon}^{1-\varepsilon}$ in $\boldsymbol{R}^3[-1+\varepsilon, 1-\varepsilon]$ such that

$$F^{*}{}^{1-\varepsilon}_{-1+\varepsilon} \cap \boldsymbol{R}^3[t] = \begin{cases} k^*_-[t] & \text{for } t=-1+\varepsilon \\ h_s(k^*_-)[t], \text{ where } & s=(t+1-\varepsilon)/(1-2\varepsilon) \\ & \text{for } -1+\varepsilon<t<-\varepsilon \\ (h_1(k^*_-)\cup \mathcal{B}^*_0)[t] & \text{for } t=-\varepsilon \\ k^*_+[t] & \text{for } -\varepsilon<t\leq 1-\varepsilon . \end{cases}$$

Evidently the proper surface $F^{*}{}^{1-\varepsilon}_{-1+\varepsilon}$ is a nowhere zero section of $N(F)\cap \boldsymbol{R}^3[-1+\varepsilon, 1-\varepsilon]$ which is regarded as a normal disk bundle over $F\cap \boldsymbol{R}^3[-1+\varepsilon, 1-\varepsilon]$, where $N(F)$ is the regular neighborhood of F in $\boldsymbol{R}^4$. Since $F\cap \boldsymbol{R}^3(-\infty, -1+\varepsilon]$ is homeomorphic to a disk and the linking number of k_- and k^*_- is zero, we can extend $F^{*}{}^{1-\varepsilon}_{-1+\varepsilon}$ to a proper surface $F^{*}{}^{1-\varepsilon}_{-\infty}$ in $\boldsymbol{R}^3(-\infty, 1-\varepsilon]$ as a nowhere zero section of $N(F)\cap \boldsymbol{R}^3(-\infty, 1-\varepsilon]$. Let $N^{1-\varepsilon}_{-\infty}$ (resp. $N^{+\infty}_{1-\varepsilon}$) be $N(F)\cap \boldsymbol{R}^3(-\infty, 1-\varepsilon]$ (resp. $N(F)\cap \boldsymbol{R}^3[1-\varepsilon, +\infty)$). And let $\tilde{N}^{1-\varepsilon}_{-\infty}$ (resp. $\tilde{N}^{+\infty}_{1-\varepsilon}$) be the preimage of $N^{1-\varepsilon}_{-\infty}$ (resp. $N^{+\infty}_{1-\varepsilon}$) under the map $\tilde{p}\colon \tilde{N}\to N$ as in Definition 3.1. Since $F\cap \boldsymbol{R}^3[1-\varepsilon, +\infty)$ is homeomorophic to a disk, we can regard $\tilde{N}^{+\infty}_{1-\varepsilon}$ as $D_1\times D^2\cup D_2\times D^2$ and identify them. The preimage of $F\cap \boldsymbol{R}^3[1-\varepsilon, +\infty)$ is $D_1\times\{0\}\cup D_2\times\{0\}$ and the preimage of $k_+[1-\varepsilon]=\partial(F\cap \boldsymbol{R}^3[1-\varepsilon, +\infty))$ is $(\partial D_1)\times\{0\}\cup(\partial D_2)\times\{0\}$. And the preimage of $k^*_+[1-\varepsilon]$ is two disjoint loops lying in $(\partial D_1)\times D^2\cup(\partial D_2)\times D^2$, say k^{*1}_+ and k^{*2}_+. For any section $D^*_i (i=1, 2)$ of $D_i\times D^2$ with $D^*_i|_{\partial D_i}=k^{*i}_+$, the intersection number of $D_i\times\{0\}$ and D^*_i is equal to the linking number of $\partial D_i\times\{0\}$ and k^{*i}_+. Hence it is equal to $\operatorname{link}(k_+, k^*_+)$. So we have

$$\begin{aligned} \operatorname{Int}_{\tilde{N}}(\tilde{F}, \tilde{F}) &= \operatorname{Int}_{\tilde{N}^{1-\varepsilon}_{-\infty}}(\tilde{F}^{1-\varepsilon}_{-\infty}, \tilde{F}^{1-\varepsilon}_{-\infty})+\operatorname{Int}_{\tilde{N}^{+\infty}_{1-\varepsilon}}(\tilde{F}^{+\infty}_{1-\varepsilon}, \tilde{F}^{+\infty}_{1-\varepsilon}) \\ &= \operatorname{Int}_{\tilde{N}^{1-\varepsilon}_{-\infty}}(F^{1-\varepsilon}_{-\infty}, F^{*1-\varepsilon}_{-\infty})+\operatorname{Int}_{D_1\times D^2}(D_1\times\{0\}, D^*_1) \\ &\quad +\operatorname{Int}_{D_2\times D^2}(D_2\times\{0\}, D^*_2) \\ &= \operatorname{link}(\partial D_1\times\{0\}, k^{*1}_+)+\operatorname{link}(\partial D_2\times\{0\}, k^{*2}_+) \\ &= 2\operatorname{link}(k_+, k^*_+) . \end{aligned}$$

This completes the proof.

3.4 Example. Here are two projective planes P_+ and P_- in $\boldsymbol{R}^4$, called *standard projective planes*. Then $e(P_+)=2$ and $e(P_-)=-2$. Thus they are not ambient isotopic in $\boldsymbol{R}^4$. The Euler number of a surface depends upon the orientation of $\boldsymbol{R}^4$(or S^4). We fix the orientation and promise that the standard projective plane in $\boldsymbol{R}^4$ (or S^4) with the Euler number 2 is *positive* and the other is *negative*.

Let (S^4_i, F_i) be a pair of an oriented 4-sphere S^4_i and a closed, connected, locally-flat non-orientable surface F_i in S^4_i $(i=1, 2)$. Consider the connected sum $(S^4_1\# S^4_2, F_1\# F_2)$ of the pairs (S^4_1, F_1) and (S^4_2, F_2) with respect to the orientations of S^4_1 and S^4_2. The surface $F_1\# F_2$ in the oriented 4-sphere $S^4=S^4_1\# S^4_2$ is called the *knot sum of* F_1 *and* F_2. For non-orientable surfaces in oriented

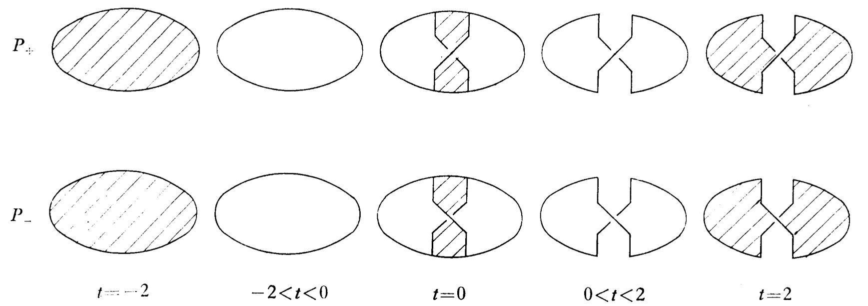

Fig. 7

Euclidean 4-spaces $\boldsymbol{R}^4_i$, define the knot sum in a similar way. By the definition of the Euler number, we see the following proposition:

Proposition 3.5. *Let F_1 and F_2 be closed, connected, locally-flat surfaces in $\boldsymbol{R}^4$ (or S^4). Then $e(F_1 \# F_2)=e(F_1)+e(F_2)$.*

Theorem 3.6 [The Whitney and Massey theorem]. *Let F be a closed, connected, locally-flat and non-orientable surface with the Euler characteristic χ in $\boldsymbol{R}^4$. Then the possible values of the Euler number of F, $e(F)$, are $2\chi-4$, 2χ, $\cdots$, and $4-2\chi$.*

This theorem was conjectured by H. Whitney in his paper [10] and proved by W.S. Massey [8]. We give another proof in 5.1. A knot k is a *slice knot* if $k[0]\subset\boldsymbol{R}^3[0]$ bounds a locally-flat proper disk D in $\boldsymbol{R}^3[0,+\infty)$. The following proposition is a corollary to Theorem 3.6.

Proposition 3.7. *Let k be the knot obtained from a slice knot k_0 by the hyperbolic transformations along n noncoherent bands $\mathcal{B}$ spanning k_0. And let (k, k^*) be the pair obtained from (k_0, k_0^*) by the hyperbolic transformations along $\mathcal{B}$, where k_0^* is the preferred longitude of k_0. Then if the absolute value of the linking number of (k, k^*) is greater than $2n$, the knot k cannot be slice.*

Proof. Assume k is a slice knot. We consider the realizing surface F^1_{-1} in $\boldsymbol{R}^3[-1,1]$ of $k_0\to k$. Since k_0 and k are slice knots, there exist two locallly-flat proper disks D_0 in $\boldsymbol{R}^3(-\infty,-1]$ and D in $\boldsymbol{R}^3[1,+\infty)$ with $\partial D_0=k_0[-1]$ and $\partial D=k[1]$. The closed surface $D_0\cup F^1_{-1}\cup D$ is a non-orientable surface with the non-orientable genus n, so by Theorem 3.6 the absolute value of the Euler number is less than or equal to $4-2\chi$ $(=2n)$. On the other hand by the same argument of the proof of Proposition 3.3 we can show $e(D_0\cup F^1_{-1}\cup D)=\text{link}(k, k^*)$ and $|e(D_0\cup F^1_{-1}\cup D)|\leqq 2n$. This is a contradiction. This completes the proof.

It is known that any closed, connected, locally-flat and orientable surface in $\boldsymbol{R}^4$ bounds a 3-manifold in $\boldsymbol{R}^4$. For non-orientable surfaces, the following holds:

Theorem 3.8. *A closed, connected, locally-flat and non-orientable surface F in $\boldsymbol{R}^4$ bounds a 3-manifold in $\boldsymbol{R}^4$, if and only if $e(F)=0$.*

This result is claimed by K. Asano [1] (cf. Hosokawa-Kawauchi [3]) and shown by C. McA. Gordon and R.A. Litherland in [2]. We shall give here a proof near to Asano's approach in [1]. We use the following lemma which is a non-orientable version of Kawauchi's in [4].

Lemma 3.9. *Let G and $\tilde{G}$ be a compact orientable surface and a compact non-orientable surface in $\boldsymbol{R}^3$ with the same boundary a knot k respectively. Suppose that they intersect transversally and does not intersect in the neighborhood of k except for k. Then for any proper arc γ on G which intersects $\tilde{G}$ transversally, $|\mathring{\gamma}\cap\tilde{G}|$ is even.*

Proof of Lemma 3.9. We join γ and an arc on k cut by $\partial\gamma$ in order to get a loop c on G with $c\cap\mathring{G}=\mathring{\gamma}$. Let c' be a loop obtained from c by pushing off in the normal direction of G. We can assume that $|c'\cap\tilde{G}|=|\mathring{\gamma}\cap\tilde{G}|$. Since c' is disjoint from G, $|c'\cap\tilde{G}|=|c'\cap(G\cup\tilde{G})|$. On the other hand we can regard c' and $G\cup\tilde{G}$ as $\boldsymbol{Z}_2$-cycles in $\boldsymbol{R}^3$. Then $[G\cup\tilde{G}]=0$ in $H_2(\boldsymbol{R}^3;\boldsymbol{Z}_2)$, so their $\boldsymbol{Z}_2$-intersection number $\mathrm{Int}([G\cup\tilde{G}],c')=0$. Hence $|\mathring{\gamma}\cap G|$ is even.

Proof of Theorem 3.8. If F bounds a 3-manifold W, we can get a nowhere zero cross-section of $N(F)$ by making use of W. So $e(F)=0$. We shall show the sufficiency. Let F be a closed, connected, locally-flat and non-orientable surface in $\boldsymbol{R}^4$ with the Euler number zero. By Theorem 1.3, we can assume F is in a normal form and the critical bands $\mathcal{D}_+$, $\mathcal{D}_-$, $\mathcal{B}_+$, $\mathcal{B}_-$ and $\mathcal{B}_0$ are in the levels $t=2$, -2, 1, -1 and 0 respectively (Definition 1.2). Let k_+ be the upper cross-sectional knot of F. The bands $\mathcal{B}_+=\{B_1,\cdots,B_m\}$ span the boundary $\mathcal{O}_+$ of $\mathcal{D}_+$ which is a trivial link, and k_+ is obtained from $\mathcal{O}_+$ by the hyperbolic transformations along $\mathcal{B}_+$. We may assume that $\mathcal{D}_+\cup\mathcal{B}_+$ is a normal singular surface in $\boldsymbol{R}^3$ whose singularity consists of mutually disjoint simple ribbon singularities. Assign orientations to $\mathcal{D}_+$ such that all of them are coherent to an orientation of k_+. For eact i, $i=1,\cdots,m$, let α_i and α_i' be the attaching arcs of B_i to $\mathcal{O}_+$. Then we give a sign to each simple ribbon singularity of $\mathcal{D}_+\cup\mathcal{B}_+$ as in Fig. 8. We can assume that the bands $\mathcal{B}_+=\{B_1,\cdots,B_m\}$ satisfy the condition that, for each i $(=1,\cdots,m)$, the sum of the signs of ribbon singularities of $\mathcal{D}_+\cup\mathcal{B}_+$ along B_i is zero. (Modify them as in Fig. 9, if necessary.) Then there exist mutually disjoint annuli $A_1,\cdots,A_\lambda$, surrounding $\mathcal{B}_+$, attached to $\mathring{\mathcal{D}}_+$ with attaching sets $\partial\Delta_1\cup\partial\Delta_1',\cdots,\partial\Delta_\lambda\cup\partial\Delta_\lambda'$ such that the surface $G=\mathrm{Cl}\,(\mathcal{D}_+\cup\mathcal{B}_+\cup A_1\cup\cdots\cup A_\lambda-(\Delta_1\cup\Delta_1'\cup\cdots\cup\Delta_\lambda\cup\Delta_\lambda'))$ is a compact orientable surface embedded

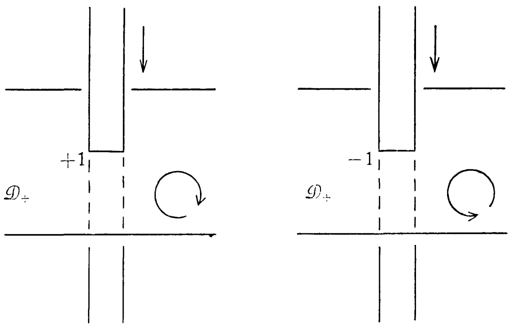

Fig. 8

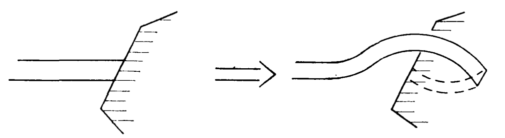

Fig. 9

in $\boldsymbol{R}^3$ with boundary k_+, where $\Delta_1, \Delta_1', \cdots, \Delta_\lambda, \Delta_\lambda'$ are neighborhoods of the ribbon singularities of $\mathscr{D}_+ \cup \mathscr{B}_+$ in $\mathring{\mathscr{D}}_+$. See Fig. 10. (Here Cl means the closure in $\boldsymbol{R}^3$.)

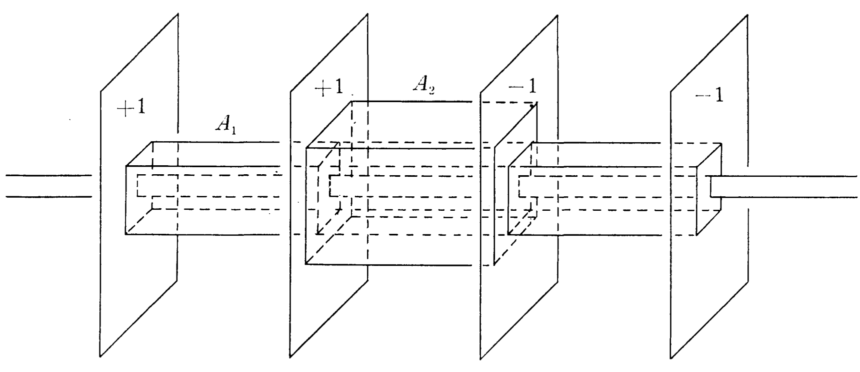

Fig. 10

We construct a 3-manifold $W_+ \subset \boldsymbol{R}^3[2/3, +\infty)$. Let $t_1, \cdots, t_\lambda$ be real numbers with $1 < t_1 < \cdots < t_\lambda < 2 = t_{\lambda+1}$. We define $W_+ \cap \boldsymbol{R}^3[2/3, t_1]$ as follows:

$$W_+ \cap \boldsymbol{R}^3[t] = \begin{cases} G[t] & \text{for } 2/3 \leq t \leq 1 \\ \mathrm{Cl}\,(G - \mathscr{B}_+)\,[t] & \text{for } 1 < t \leq t_1 . \end{cases}$$

Let $\Delta_1 \cup A_1 \cup \Delta_1'$ be an innermost 2-sphere in $\Delta_1 \cup A_1 \cup \Delta_1', \cdots, \Delta_\lambda \cup A_\lambda \cup \Delta_\lambda'$.

Then $\Delta_1 \cup A_1 \cup \Delta_1'$ bounds a unique 3-disk ∇_1 in $\boldsymbol{R}^3$ with $\nabla_1 \cap (A_2 \cup \cdots \cup A_\lambda) = \phi$. We define $W_+ \cap \boldsymbol{R}^3[t_1, t_2]$ as follows:

$$W_+ \cap \boldsymbol{R}^3[t] = \begin{cases} \mathrm{Cl}\,(G - \mathscr{B}_+)\,[t] & \text{for } t_1 \leq t < (t_2 - t_1)/2 \\ \mathrm{Cl}\,((G - \mathscr{B}_+) \cup \nabla_1)\,[t] & \text{for } t = (t_2 - t_1)/2 \\ \mathrm{Cl}\,((G - \mathscr{B}_+ - A_1) \cup \Delta_1 \cup \Delta_1')\,[t] & \\ & \text{for } (t_2 - t_1)/2 < t \leq t_2\,. \end{cases}$$

By the repetition of the procedure, we obtain $W_+ \cap \boldsymbol{R}^3[t] = \mathscr{D}_+[t]$ for $t = t_{\lambda+1} = 2$. And we define $W_+ \cap \boldsymbol{R}^3(2, +\infty)$ as empty. Then $\partial W_+ = (F \cap \boldsymbol{R}^3[2/3, +\infty)) \cup G[2/3]$. Similarly, we can obtain a 3-manifold W_- in $\boldsymbol{R}^3(-\infty, 1/3]$ with $\partial W_- = (F \cap \boldsymbol{R}^3(-\infty, 1/3]) \cup G'[1/3]$. G' is a non-orientable surface in $\boldsymbol{R}^3$ with $\partial G' = k_+$. Now G and G' have the same boundary k_+. Since $e(F) = 0$, there exists an ambient isotopy $\{f_s\}_{s \in I}$ of $\boldsymbol{R}^3$ keeping k_+ fixed such that $\tilde{G} = f_1(G')$ intersects G transversally and dose not intersect G in the tubular neighborhood of k_+ except for the boundary k_+. Then $G \cap \tilde{G}$ consists of k_+ and some simple loops. Let $\gamma_1, \cdots, \gamma_l$ be mutually disjoint simple proper arcs on the orientable surface G such that they cut G into a disk. We can assume that $\mathring{\gamma}_1, \cdots, \mathring{\gamma}_l$ intersect $\tilde{G}$ transversally. Then for each i, $i = 1, \cdots, l$, we see the number of the intersection points $\mathring{\gamma}_i \cap \tilde{G}$ is even by Lemma 3.9. So we can do a surgery for $\tilde{G}$ by some mutually disjoint 1-handles $H_1^1, \cdots, H_p^1$ along $\gamma_1, \cdots, \gamma_l$ spanning $\tilde{G}$ so that the resulting surface $\tilde{G}^{(0)} = \boldsymbol{h}^1(\tilde{G}; H_1^1, \cdots, H_p^1)$ (see Definition 4.1) is disjoint from $\gamma_1, \cdots, \gamma_l$. See Fig. 11. Then the intersection of $\tilde{G}^{(0)}$ and G consists of k_+ and $q(\geq 0)$ simple loops $C = \{c_1, \cdots, c_q\}$. C are disjoint from $\gamma_1, \cdots, \gamma_l$, so they bound disks on G. Assume that c_1 is an innermost loop on G. Let d_1 be the disk on G bounded by c_1 and let $\bar{d}_1$ be a 3-disk obtained by thickenning d_1 such

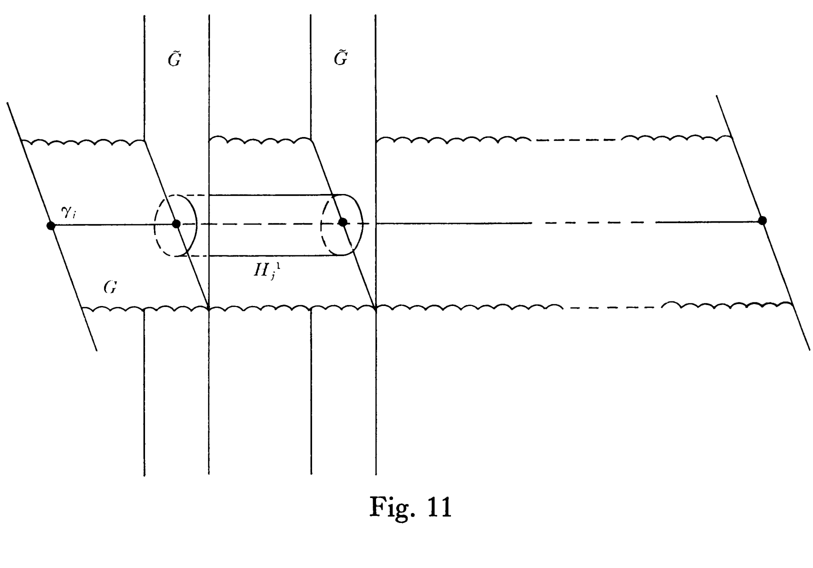

Fig. 11

that $\bar{d}_1 \cap \tilde{G}^{(0)} = (\partial \bar{d}_1) \cap \tilde{G}^{(0)}$ is an annulus and $\bar{d}_1 \cap G = d_1$. The surface $\tilde{G}^{(1)} = \boldsymbol{h}^2(\tilde{G}^{(0)}; \bar{d}_1)$ (see Definition 4.1) intersects G with k_+ and $C - \{c_1\}$. Inductively we assume that c_{i+1} is a loop in $C - \{c_1, \cdots, c_i\}$ which is innermost on G and let d_{i+1} be the disk on G bounded by c_{i+1}. And let $\bar{d}_{i+1}$ be a 3-disk in $\boldsymbol{R}^3$ such that $\bar{d}_{i+1} \cap \tilde{G}^{(i)} = (\partial \bar{d}_{i+1}) \cap \tilde{G}^{(i)}$ is an annulus and $\bar{d}_{i+1} \cap G = d_{i+1}$. Then $\tilde{G}^{(i+1)} = \boldsymbol{h}^2(\tilde{G}^{(i)}; d_{i+1})$ intersects G with k_+ and $C - \{c_1, \cdots, c_{i+1}\}$. Finally we obtain a surface $\tilde{G}^{(q)}$ which intersects G on k_+. The union of $\tilde{G}^{(q)}$ and G is a closed surface embedded in $\boldsymbol{R}^3$. But no closed non-orientable surface can be embedded in $\boldsymbol{R}^3$, so the above union is orientable. It bounds a 3-manifold in $\boldsymbol{R}^3$, say M. We construct a 3-manifold $W_0 \subset \boldsymbol{R}^3[1/3, 2/3]$ as follows:

$$W_0 \cap \boldsymbol{R}^3[t] = \begin{cases} G'[t] & \text{for } 1/3 \leq t < t_1' \\ f_s(G')[t], \text{ where } s = (t - t_1')/(t_2' - t_1') & \text{for } t_1' \leq t \leq t_2' \\ \tilde{G}[t] & \text{for } t_2' < t < t_3' \\ (\tilde{G} \cup H_1^1 \cup \cdots \cup H_p^1)[t] & \text{for } t = t_3' \\ \tilde{G}^{(0)}[t] & \text{for } t_3' < t < t_4' \\ (\tilde{G}^{(0)} \cup \bar{d}_1)[t] & \text{for } t = t_4' \\ \tilde{G}^{(1)}[t] & \text{for } t_4' < t < t_5' \\ (\tilde{G}^{(1)} \cup \bar{d}_2)[t] & \text{for } t = t_5' \\ \tilde{G}^{(2)}[t] & \text{for } t_5' < t < t_6' \\ \qquad \vdots & \\ (\tilde{G}^{(q-1)} \cup \bar{d}_q)[t] & \text{for } t = t_{q+3}' \\ \tilde{G}^{(q)}[t] & \text{for } t_{q+3}' < t < t_{q+4}' \\ M[t] & \text{for } t = t_{q+4}' \\ G[t] & \text{for } t_{q+4}' < t \leq 2/3 , \end{cases}$$

where $1/3 < t_1' < \cdots < t_{q+4}' < 2/3$. We get the required 3-manifold $W = W_+ \cup W_0 \cup W_-$ in $\boldsymbol{R}^4$, whose boundary is F. This completes the proof of Theorem 3.8.

4 Unknotted Non-orientable Surface

Let F be a closed, locally-flat surface (possibly disconnected) in $\boldsymbol{R}^4$. A 3-cell B in $\boldsymbol{R}^4$ is said to span F as a *1-handle,* if the intersection $B \cap F$ is a pair of disjoint 2-cells on ∂B. A 3-cell B in $\boldsymbol{R}^4$ is said to span F as a *2-handle,* if $B \cap F = (\partial B) \cap F$ and the intersection is homeomorphic to an annulus $S^1 \times [0, 1]$.

DEFINITION 4.1. Let $B_1, \cdots, B_m$ be mutually disjoint 3-cells in $\boldsymbol{R}^4$ which span F as 1-handles, then the surface $\boldsymbol{h}^1(F; B_1, \cdots, B_m) = \mathrm{Cl}(F \cup \partial B_1 \cup \cdots \cup \partial B_m - [F \cap (\partial B_1 \cup \cdots \cup \partial B_m)])$ is called *the surface obtained from F by the hyperboloidal transformations along* 1-*handles* $B_1, \cdots, B_m$. When $B_1, \cdots, B_m$ span F as 2-handles, then the resulting surface $\boldsymbol{h}^2(F; B_1, \cdots, B_m)$ is called *the surface obtained from F* *by the hyperboloidal transformations along 2-handles* $B_1, \cdots, B_m$.

Proposition 4.2. *Let* F *be a closed, locally-flat surface (possibly disconnected) in* $\boldsymbol{R}^4$. *And let* $B_1, \cdots, B_m$ *be mutually disjoint 3-cells spanning* F *as 1-handles (or 2-handles). Then the Euler number of the resulting surface* $F'=\boldsymbol{h}^1(F; B_1, \cdots, B_m)$ *(or* $F'=\boldsymbol{h}^2(F; B_1, \cdots, B_m)$*) is equal to that of* F.

Proof. (I) The case when $B_1, \cdots, B_m$ are 1-handles. Let F have $\mu \geq 1$ components, say $F_1, \cdots, F_\mu$. For each $i=1, \cdots, \mu$, take a 2-disk D_i on F_i. We can transform $B_1, \cdots, B_m$ so that they attach F in $D_1 \cup \cdots \cup D_\mu$ by sliding the attaching disks $F \cap (\partial B_1 \cup \cdots \cup \partial B_m)$ along F and by deforming the attaching disks into smaller subdisks, if necessary (see Fig. 12).

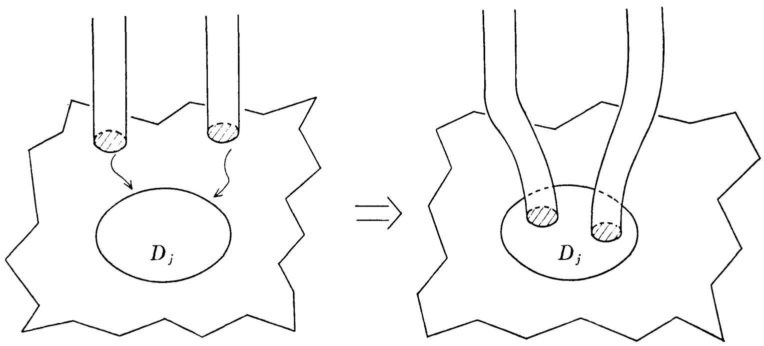

Fig. 12

Let $\tilde{p}: \tilde{N}(F) \to N(F)$ be the double covering space defined in 3.1, where $N(F)$ is the regular neighbourhood of F in $\boldsymbol{R}^4$ which is the normal disk bundle over F. For any section F^* of $N(F)$, by the definition, $\mathrm{Int}_{\tilde{N}(F)}(\tilde{p}^{-1}(F^*), \tilde{p}^{-1}(F))=2e(F)$. We may assume the intersecting points of $\tilde{p}^{-1}(F^*)$ and $\tilde{p}^{-1}(F)$ are over $\tilde{p}^{-1}(F-D_1 \cup \cdots \cup D_\mu)$. For F', construct a section F'^* of the normal disk bundle $N(F')$ over F', so that F'^* coincides with F^* over $F-\mathrm{int}(D_1 \cup \cdots \cup D_\mu) \subset F'$ and F'^* does not intersect F' over $F'-(F-D_1 \cup \cdots \cup D_\mu)$, by making

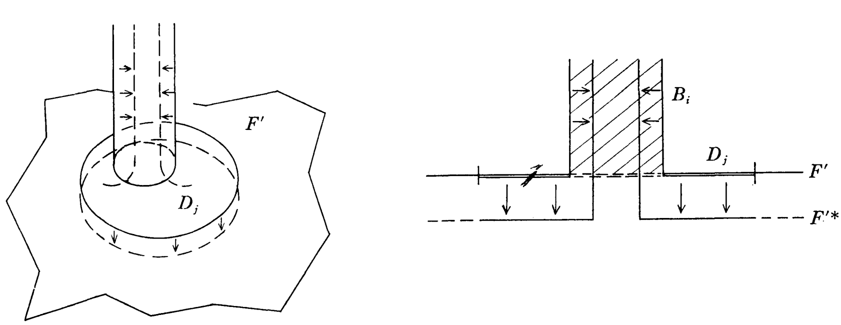

Fig. 13

use of 3-cells $B_1, \cdots, B_m$ (Fig. 13). This implies that $e(F')=e(F)$.

(II) The case when $B_1, \cdots, B_m$ are 2-handles. $B_1, \cdots, B_m$ are considered to span $F'=\boldsymbol{h}^2(F; B_1, \cdots, B_m)$ as 1-handles. Then F is obtained from F' by the hyperboloidal transformations along 1-handles $B_1, \cdots, B_m$. So $e(F)=e(F')$. This completes the proof.

DEFINITION 4.3. A 1-handle B in $\boldsymbol{R}^4$ which spans F is said to be a *trivial 1-handle* (resp. *non-orientable trivial 1-handle*), if there exists a 4-cell N in $\boldsymbol{R}^4$ containing B such that $N \cap F=(\partial N) \cap F$ is a 2-cell and the proper surface $\mathrm{Cl}((N \cap F) \cup \partial B-(F \cap \partial B))$ in N is orientable (resp. non-orientable).

DEFINITION 4.4. Let F be a closed, connected, locally-flat and non-orientable surface in $\boldsymbol{R}^4$. If the knot type of F is the knot sum of some copies of standard projective planes P_+ or P_- (see 3.4), we say that F is *unknotted*.

Theorem 4.5 (Hosokawa-Kawauchi [3]). *Let F be a closed, connected, locally-flat and non-orientable surface in $\boldsymbol{R}^4$. F bounds a non-orientable handle-body in $\boldsymbol{R}^4$, if and only if F is unknotted and $e(F)=0$.*

Lemma 4.6. *Let F be a closed, connected, locally-flat surface in $\boldsymbol{R}^4$ and let B be a non-orientable trivial 1-handle spanning F. Then the knot type of the resulting surface $\boldsymbol{h}^1(F; B)$ is the knot sum of F, P_+ and P_-, where P_+, P_- are the standard projective planes. Conversely if F' is the knot sum of F, P_+ and P_-, F' is the resulting surface $\boldsymbol{h}^1(F; B)$ by a non-orientable trivial 1-handle B spanning F.*

Theorem 4.7. *For any closed, connected, locally-flat and non-orientable surface F in $\boldsymbol{R}^4$, there exist mutually disjoint 1-handles $B_1, \cdots, B_m$ spanning F such that the resulting surface $\boldsymbol{h}^1(F; B_1, \cdots, B_m)$ is unknotted.*

Proof. We first note that $e(F) \equiv 0 \pmod 2$ (cf. Massey [8]).

(I) The case when $e(F)=0$. Then, by Theorem 3.8, F bounds a compact non-orientable 3-manifold M in $\boldsymbol{R}^4$. We can find mutually disjoint 3-cells $B_1, \cdots, B_m$ in M such that they span F as 1-handles and $K=\mathrm{Cl}(M-B_1 \cup \cdots \cup B_m)$ in $\boldsymbol{R}^4$ is a non-orientable handle-body with some genus (cf. Hosokawa-Kawauchi [3; 2.9]). Theorem 4.5 asserts that the boundary of K is an unknotted non-orientable surface in $\boldsymbol{R}^4$.

(II) The case when $e(F)=2s$ $(s=1, 2, \cdots)$. Let F' be the knot sum of F and s copies of negative standard projective plane. By the definition of the knot sum, we may assume F' coincides with F in $\boldsymbol{R}^4-D$, where D is a small 4-cell in $\boldsymbol{R}^4$ such that $(D, F \cap D)$ is homeomorphic to the standard disk pair. Since $e(F')=0$, there exist mutually disjoint 3-cells $B_1, \cdots, B_m$ spanning F' as 1-handles such that $\boldsymbol{h}^1(F'; B_1, \cdots, B_m)$ is unknotted. Let α_i be the core of 1-handle $B_i (i=1, \cdots, m)$. Since $\pi_1(D-F') \simeq \boldsymbol{Z}_2$, we can assume the arcs $\alpha_1, \cdots, \alpha_m$ are disjoint from D by sliding $\alpha_1, \cdots, \alpha_m$ along F' and by moving them. Hence $B_1, \cdots, B_m$ may be assumed to be disjoint from D. Further consider the knot sum F'' of F' and s copies of positive standard projective planes P_+ such that $F''-D=F'-D=F-D$ in $\boldsymbol{R}^4-D$. Since F'' is the knot sum of F, s copies of P_+ and s copies of P_-, applying Lemma 4.6, we get mutually disjoint 3-cells $B_1', \cdots, B_s'$ in D which span F with $F''=\boldsymbol{h}^1(F; B_1', \cdots, B_s')$. Then the surface $\boldsymbol{h}^1(F; B_1, \cdots, B_m, B_1', \cdots, B_s')$ is unknotted. The desired $B_1, \cdots, B_m, B_1', \cdots, B_s'$ are thus obtained.

(III) The case when $e(F)=-2s\ (s=1, 2, \cdots)$. Change the roles of P_+ and P_- in the above argument. This completes the proof of Theorem 4.7.

5 A proof of the Whitney and Massey theorem

Let F be a closed, connected, locally-flat and non-orientable surface with non-orientable genus n in S^4. Since $H_1(S^4-F; \boldsymbol{Z})\simeq \boldsymbol{Z}_2$, we can consider the 2-fold branched covering $M(F)$ of (S^4, F). Let $\sigma(M(F))$ and $\beta_i(M(F))$ denote the signature and the i-th integral Betti number of $M(F)$ respectively.

5.1 Proof of Theorem 3.6. We show the following (, which is the same approach as Massey's in [8]):

$$\sigma(M(F)) = \varepsilon e(F)/2 \quad (\varepsilon = \pm 1)\,, \tag{1}$$

$$\beta_2(M(F)) = n\,. \tag{2}$$

From the definition of the signature, (1) and (2) imply that the possible values of $e(F)$ are $-2n$, $-2n+4$, $\cdots$, and $2n$. Hence they are $2\chi-4$, 2χ, $\cdots$, and $4-2\chi$.

Proof of (1). (I) The case when F is unknotted (4.4). Suppose F is the knot sum of λ copies of positive standard projective plane P_+ and μ copies of negative standard projective plane P_- ($\lambda\geq 0$, $\mu\geq 0$, $\lambda+\mu\neq 0$). Let $M(P_+)$ be the 2-fold branched cover of (S^4, P_+). Since $\pi_1(S^4-P_+)\simeq \boldsymbol{Z}_2$ (which is generated by the meridian of P_+), $H_1(M(P_+); \boldsymbol{Z})=0$. By the Poincare duality theorem, $\beta_3(M(P_+))=0$. And since $\beta_0(M(P_+))=\beta_4(M(P_+))=1$ and $\chi(M(P_+))=2\chi(S^4)-\chi(P_+)=3$, we have $\beta_2(M(P_+))=1$ by the Euler-Poincaré formula, where χ means the Euler characteristic. Hence the signature of $M(P_+)$ is 1 or -1. Put $\sigma(M(P_+))=\varepsilon\,(\varepsilon=\pm 1)$. $M(P_-)$ is the same manifold as $M(P_+)$, but the orientation is the opposite. Thus $\sigma(M(P_-))=-\varepsilon$. [In fact, $M(P_+)$ and $M(P_-)$ are the complex projective spaces $\boldsymbol{CP}^2$ and $\overline{\boldsymbol{CP}}^2$ respectively.] Since F is the knot sum of λ copies of P_+ and μ copies of P_-, the manifold $M(F)$ is the connected sum of λ copies of $M(P_+)$ and μ copies of $M(P_-)$. By the additivity of the signature, we have

$$\sigma(M(F)) = \lambda\sigma(M(P_+))+\mu\sigma(M(P_-))$$

$$= \lambda\varepsilon + \mu(-\varepsilon)$$
$$= \varepsilon(\lambda - \mu).$$

On the other hand, by the additivity of the Euler number (3.5),

$$\begin{aligned} e(F) &= \lambda e(P_+) + \mu e(P_-) \\ &= 2\lambda - 2\mu \\ &= 2(\lambda - \mu). \end{aligned}$$

Thus we have the equality (1) for unknotted surfaces.

(II) The case when F is knotted. Then, by Theorem 4.7, there exist mutually disjoint 1-handles $B_1, \cdots, B_m$ in S^4 spanning F such that the resulting surface $F' = \boldsymbol{h}^1(F; B_1, \cdots, B_m)$ is unknotted. We construct a cobordism (W, M) between (S^4, F) and (S^4, F') as follows:

$$\begin{aligned} &W \simeq S^4 \times [0, 1], \\ &(W, M) = (S^4, F) \times [0, 1/2) \cup (S^4, F \cup B_1 \cup \cdots \cup B_m) \times [1/2] \\ &\quad \cup (S^4, F') \times (1/2, 1]. \end{aligned}$$

M is a locally-flat, proper 3-manifold in W. Let α_i be the core of 1-handle B_i $(i=1, \cdots, m)$. The union $\alpha_1 \cup \cdots \cup \alpha_m$ is 1-dimensional, so $H_1(S^4 - (F \cup \alpha_1 \cup \cdots \cup \alpha_m); \boldsymbol{Z}) \simeq H_1(S^4 - F; \boldsymbol{Z})$. Since $W - M$ is homotopic to $S^4 - (F \cup B_1 \cup \cdots \cup B_m)$ and $H_1(S^4 - F; \boldsymbol{Z}) \simeq \boldsymbol{Z}_2$, $H_1(W - M; \boldsymbol{Z}) \simeq Z_2$. [$H_1(W - M; \boldsymbol{Z})$ is generated by the meridian of $(S^4, F) \times [0]$.] Consider the 2-fold branched covering of the pair (W, M). The boundary of this manifold is exactly the union $(-M(F)) \cup M(F')$, when we assign W the orientation induced from $S^4 \times [1]$. Hence $\sigma(M(F)) = \sigma(M(F'))$. On the other hand, Proposition 4.2 asserts that $e(F') = e(F)$. Since F' is unknotted, the equality (1) holds. Hence $\sigma(M(F)) = \varepsilon e(F)/2$ $(\varepsilon = \pm 1)$.

Proof of (2). By Theorem 1.3 we can assume that F is in a normal form in S^4, where we regard S^4 as the one-point compactification of $\boldsymbol{R}^4$. Let k_+ be the upper cross-sectional knot of F (Definition 1.2). We need the following lemma to prove (2):

Lemma 5.2. *Let F be a closed, connected, locally-flat and non-orientable surface in a normal form in $\boldsymbol{R}^4$. And let k_+ be the upper cross-sectional knot of F. Then the inclusion map i from $(\boldsymbol{R}^3 - k_+)$ $[1/2]$ to $\boldsymbol{R}^4 - F$ induces an epimorphism i_* from $\pi_1((\boldsymbol{R}^3 - k_+)[1/2])$ to $\pi_1(\boldsymbol{R}^4 - F)$.*

Proof of 5.2. Suppose F in $\boldsymbol{R}^4$ is the closed realizing surface of a sequence $\mathcal{O}_- \to k_- \to k_+ \to \mathcal{O}_+$ with $F \cap \boldsymbol{R}^3[1/2] = k_+[1/2]$. Let X_1, X_2 and X be the intersections $(\boldsymbol{R}^4 - F) \cap \boldsymbol{R}^3[1/2, +\infty)$, $(\boldsymbol{R}^4 - F) \cap \boldsymbol{R}^3(-\infty, 1/2]$ and $(\boldsymbol{R}^4 - F) \cap \boldsymbol{R}^3[1/2]$ respectively. Recall the configulation of F. The surface $F \cap \boldsymbol{R}^3[1/2, +\infty)$ in

$\boldsymbol{R}^3[1/2, +\infty)$ (resp. $F \cap \boldsymbol{R}^3(-\infty, 1/2]$ in $\boldsymbol{R}^3(-\infty, 1/2])$ has no minimal (resp. maximal) points. Thus the inclusion map i_1: $X \to X_1$ (resp. i_2: $X \to X_2$) induces an epimorphism i_{1*}: $\pi_1(X) \to \pi_1(X_1)$ (resp. $i_{2*}: \pi_1(X) \to \pi_1(X_2)$). Using Van Kampen's theorem, we have the result.

REMARK. When we assumed that S^4 is the one-point compactification of $\boldsymbol{R}^3(-\infty, +\infty)$, there exists an epimorphism from $\pi^1(S^3 - k_+)$ to $\pi_1(S^4 - F)$.

Now we continue the proof of (2). Let $\Sigma_2(k_+)$ be the 2-fold branched cover of S^3 branching over k_+. By Lemma 5.2, there exists an epimorphism from $\pi_1(S^3 - k_+)$ to $\pi_1(S^4 - F)$. It induces an epimorphism from $H_1(\Sigma_2(k^+); \boldsymbol{Z})$ to $H_1(M(F); \boldsymbol{Z})$. Since $H_1(\Sigma_2(k_+); \boldsymbol{Z})$ is a finite group of odd order (cf. Rolfsen [7]), we see $\beta_1(M(F)) = \mathrm{rank}_{\boldsymbol{Z}} H_1(M(F)); \boldsymbol{Z}) = 0$. By the Poincaré duality theorem, $\beta_3(M(F)) = 0$. Since $\beta_0(M(F)) = \beta_4(M(F)) = 1$ and $\chi(M(F)) = 2\chi(S^4) - \chi(F) = n + 2$, we obtain $\beta_2(M(F)) = n$. This completes the proof.

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