OUKA
The University of Osaka
Institutional Knowledge Archive

Title	Mass distributions on the ideal boundaries of abstract Riemann surfaces. II
Author(s)	Kuramochi, Zenjiro
Citation	Osaka Mathematical Journal. 1956, 8(2), p. 145-186
Version Type	VoR
URL	https://doi.org/10.18910/7471
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/

The University of Osaka

Osaka Mathematical Journal
Vol. 8, No. 2, December, 1956.

Mass Distributions on the Ideal Boundaries of Abstract Riemann Surfaces, II[1)]

By Zenjiro KURAMOCHI

The present article is concerned with the equilibrium potential on Riemann surfaces with positive boundary.

1. Let R^* be a Riemann surface with positive boundary and let $\{R_n\}$ $(n=0, 1, 2, \cdots)$ be its exhaustion with compact relative boundaries $\{\partial R_n\}$. Put $R=R^*-R_0$. Let $N_n(z, p)$ be a positive function in R_n-R_0 harmonic in R_n-R_0 except one point $p\in R$ such that $N_n(z, p)=0$ on ∂R_0, $\frac{\partial N_n(z, p)}{\partial n}=0$ on ∂R_n and $N_n(z, p)+\log|z-p|$ is harmonic in a neighbourhood of p. Then the $*$-Dirichlet integral of $N_n(z, p)$ taken over R_n-R_0 is $D^*(N_n(z, p))=U_n(p)$, where $U_n(p)=\lim_{z\to p}(N_n(z, p)+\log|z-p|)$ and the $*$-Dirichlet integral is taken with respect to $N_n(z, p)+\log|z-p|$ in the neighbourhood of p. For $N_n(z, p)$ and $N_{n+i}(z, p)$, we have

$$D^*_{R_n-R_0}(N_n(z, p), N_{n+i}(z, p))=D^*_{R_{n+i}-R_0}(N_{n+i}(z, p))=2\pi U_{n+i}(p)^{2)},$$

$$\begin{aligned}D_{R_n-R_0}(N_n(z, p)-N_{n+i}(z, p)) &= D^*_{R_n-R_0}(N_n(z, p))-2D^*_{R_n-R_0}(N_n(z, p), N_{n+i}(z, p))\\ &\quad +D^*_{R_n-R_0}(N_{n+i}(z, p))<D^*_{R_n-R_0}(N_n(z, p))-D^*_{R_{n+i}-R_0}(N_{n+i}(z, p))\\ &= 2\pi(U_n(p)-U_{n+i}(p)).\end{aligned}$$

Hence $\{U_n(p)\}$ is decreasing with respect to n. Since $\int_{\partial R_0}\frac{\partial N_n(z, p)}{\partial n}ds=2\pi$ for every n, $\lim_{n=\infty}U_n(p)>-\infty$, whence $\{U_n(p)\}$ converges. Therefore $D_{R_{n+i}-R_0}(N_{n+i}(z, p)-N_n(z, p))$ tends to zero if n and i tend to ∞, which implies that $\{N_n(z, p)\}$ converges in mean. Further $N_n(z, p)=0$ on ∂R_0 yields that $\{N_n(z, p)\}$ converges uniformly to a function $N(z, p)$, which clearly has the minimal $*$-Dirichlet integral over R, in every compact part of R. Clearly by the compactness of ∂R_0, we have $\int_{\partial R_0}\frac{\partial N(z, p)}{\partial n}ds=$

1) Resumé of this article appeared in Proc. Japan Acad. **32**, 1956.

2) Let $v_r(p)$ be a circular neighbourhood of p with respect to the local parameter: $v_r(p)=E[z\in R: |z-p|<r]$. Then $D^*(N_n(z, p), N_{n+i}(z, p))=\int_{\partial v(p)}(N_{n+i}(z, p)+\log|z-p|)\frac{\partial N_n(z, p)}{\partial n}ds$. By letting $r\to 0$, we have $D^*(N_{n+i}(z, p), N_n(z, p))=2\pi U_{n+i}(p)$. Clearly $*$-Dirichlet integral reduces to Dirichlet integral when the functions have no pole.

$\int_{\partial R_0} \lim_{n=\infty} \frac{\partial N_n(z, p)}{\partial n} ds = 2\pi$. We call $N(z, p)$ the $*$-Green's function of R with pole at p.

As in case of a Riemann surface with null-boundary, we define for R^* the ideal boundary point, by making use of $\{N(z, p_i)\}$, that is, if $\{p_i\}$ is a sequence of points in R having no point of accumulation in $R+\partial R_0$ for which the corresponding functions $N(z, p_i)$ $(i=1, 2, 3, \cdots)$ converge uniformly in every compact set of R, we say that $\{p_i\}$ is a fundamental sequence determining an *ideal boundary point*. The set of all the ideal boundary points will be denoted by B and the set $R+B$, by $\bar{R}$. The domain of definition of $N(z, p)$ may now be extended by writing $N(z, p)=\lim_{i=\infty} N(z, p_i)$ $(z \in R$ and $p \in B)$, where $\{p_i\}$ is any fundamental sequence determining p. For p in B, the flux of $N(z, p)$ along ∂R_0 is also 2π. The distance between two points p_1 and p_2 of $\bar{R}$ is defined by

$$\delta(p_1, p_2) = \sup_{z \in R_1 - R_0} \left| \frac{N(z, p_1)}{1+N(z, p_1)} - \frac{N(z, p_2)}{1+N(z, p_2)} \right| .$$

The topology induced by this metric is homeomorphic to the original topology in R and we see easily that $R-R_1+\partial R_1+B$ and B are closed and compact. Evidently, if $\{p_i\}$ tends to p in δ-sense (with respect to δ-metric), then $N(z, p_i)$ tends to $N(z, p)$, that is $N(z, p)$ is continuous with respect to this metric and derivatives of $N(z, p_i)$ converges to those of $N(z, p)$ at every point z of R.

First, we shall prove the following

Lemma 1. *Let G be a compact or non-compact closed set containing a relatively closed set F and suppose that there exists at least one harmonic function $U(z)$ such that $U(z)=\varphi$ on $\partial R_0+\partial F$ and whose Dirichlet integral taken over $R-F$ is finite. Let $U_F(z)$ be the harmonic function in $R-F$ having the minimal Dirichlet integral over $R-F$ with boundary value φ on $\partial R_0+\partial F$ among all functioh $\{U_\alpha(z)\}$ having the same boundary value φ on $\partial R_0+\partial F$. Let $U_G(z)$ be a harmonic function in $R-G$ with the boundary value $U_F(z)$ on $\partial G+\partial R_0$ such that $U_G(z)$ has the minimal Dirichlet integral taken over $R-G$ among all functions with the boundary value $U_F(z)$ on $\partial G+\partial R_0$. Then*

$$U_G(z) = U_F(z) .$$

Proof. Let $U_n'(z)$ be a harmonic function in R_n-R_0-G such that $U_n'(z)=U_F(z)$ on $\partial G+\partial R_0$ and $\frac{\partial U_n'(z)}{\partial n}=0$ on ∂R_n-G. Then we see as

in case of $N(z, p)$ that $\{U_n'(z)\}$ converges to a function $U'(z)$ in mean and that $U'(z)$ has the minimal Dirichlet integral (we shortly it denote by M.D.I) among all functions with boundary value $U_F(z)$ on $\partial R_0+\partial G$. Assume $D_{R-G}(U'(z)) \leqq D_{R-G}(U_F(z))-d$ $(d>0)$. Then $D_{R_n-R_0-G}(U_n'(z))< D_{R-G}(U_F(z))-d$ $(n=1,2,3,\cdots)$. Now let $U_n''(z)$ be a harmonic function in R_n-R_0-F such that $U_n''(z)=U_F(z)$ on $\partial R_n\cap(G-F)+\partial R_0$ and $U_n''(z)=U'(z)$ on ∂R_n-G. Then by Dirichlet principle, $D_{R_n-R_0-F}(U_n''(z)) \leqq D_{R_n-R_0-G}(U_n'(z)) + D_{(R_n-R_0)\cap(G-F)}(U_F(z)) \leqq D_{R_n-R_0-F}(U(z))-d$.

Choose a subsequence $\{U_{n'}''(z)\}$ of $\{U_n''(z)\}$ which converges uniformly in every compact set of $R-F$ to a function $U^*(z)$. Then we have also $D_{R-F}(U^*(z)) \leqq \lim_{n'=\infty} D_{R_{n'}-R_0}(U_{n'}''(z)) \leqq D_{R-F}(U_F(z))-d$. This contradicts the minimality of $D_{R-F}(U_F(z))$. Hence $D_{R-G}(U'(z))=D_{R-G}(U_F(z))$ and $U'(z)$ is clearly the harmonic continuation of $U_F(z)$ by Dirichlet principle. On the other hand, it is clear that such $U'(z)$ is determined uniquely[3] by the boundary value on $\partial R_0+\partial G$. Hence $U_F(z)=U'(z)=U_G(z)$. Next, we consider the Dirichlet integral of $N(z, p)$.

Lemma 2. *Put* $N^M(z, p)=min\,[M, N(z, p)]p\in\bar{R}$. *Then the Dirichlet integral of* $N^M(z, p)$ *over* R *satisfies*

$$D_R(N^M(z, p)) \leqq 2\pi M : M \geqq 0 .$$

Proof. We shall prove the lemma in three cases as follows:

Case 1. $p\in R$ *and the set* $V_M(p)=E[z\in R: N(z, p)\geqq M]$ *is compact.*

Case 2. $p\in R$ *and* $V_M(p)$ *is non-compact.*

Case 3. $p\in B$.

Case 1. $p\in R$ *and* $V_M(p)$ *is compact.* Let $N_n(z, p)$ be a function in R_n-R_0 such that $N_n(z, p)$ is harmonic in R_n-R_0 except p, $N_n(z, p)+\log|z-p|$ is harmonic in a neighbourhood of p, $N_n(z, p)=0$ on ∂R_0 and $\dfrac{\partial N_n(z, p)}{\partial n}=0$ on ∂R_n. Let $N_n'(z, p)$ be a harmonic function in $R_n-R_0-V_M(p)$ such that $N_n'(z, p)=M$ on $\partial V_M(p)$, $N_n'(z, p)=0$ on ∂R_0 and $\dfrac{\partial N_n'(z, p)}{\partial n}=0$ on ∂R_n. Then the Dirichlet integral is $D_{R_n-R_0-V_M(p)}(N_n'(z, p)) = \int_{\partial R_0} M\dfrac{\partial N_n'(z, p)}{\partial n}ds$. Clearly, $\{D_{R_n-R_0-V_M(p)}(N_n'(z, p)\}$ is increasing with

3) Let $U_i(z)$ $(i=1, 2)$ be a harmonic function in $R-G$ such that $U_1(z)=U_2(z)$ on $\partial G+\partial R_0$ and $U_i(z)$ has the finitely minimal Dirichlet integrals over $R-G$. Then by the minimality of $D(U_i(z))$, we have $D(U_i(z), V(z))=0$, where $V(z)$ is a harmonic function in $R-G$ such that $V(z)=0$ on $\partial R_0+\partial G$ and $D(V(z))<\infty$. We can consider $U_1(z)-U_2(z)$ as $V(z)$. Hence

$$D(U_1(z)-U_2(z),\ U_1(z))=D(U_1(z)-U_2(z),\ U_2(z))=0$$

whence $D(U_1(z)-U_2(z))=0$, i.e. $U_1(z)=U_2(z)$.

respect to n and $N_n'(z, p)$ converges in mean and also converges uniformly in every compact set of $R-V_M(p)$ to a function $N'(z, p)$ and $D_{R-V_M(p)}(N'(z, p))=2\pi M$ and further $N'(z, p)$ has M.D.I over $R-V_M(p)$ among all functions having the value M on $\partial V_M(p)$ and zero on ∂R_0. Let R' be a compact component of R bounded by ∂R_0 and a compact analytic curve γ which separates $V_M(p)$ from ∂R_0. Denote by $\omega^*(z)$ a harmonic function in R' such that $\omega^*(z)=0$ on ∂R_0 and $\omega^*(z)=1$ on γ and let $\omega_n(z)$ be a harmonic function in $R_n-R_0-V_M(p)$ such that $\omega_n(z)=1$ on $\partial V_M(p)$, $\omega_n(z)=0$ on ∂R_0 and $\frac{\partial\omega_n(z)}{\partial n}=0$ on ∂R_0. Then clearly, $D_{R_n-R_0-V_M(p)}(\omega_n(z))\leqq D_{R'}(\omega^*(z))$. On the other hand, by the maximum principle

$$|N_n(z, p)-N_n'(z, p)|<\delta_n\omega_n(z),$$

where $\delta_n=\max[|N_n(z, p)-M|]$ on $\partial V_M(p)$.
Let $n\to\infty$. Then $N_n(z, p)$ tends to $M(=N'(z, p))$ on $\partial V_M(p)$ and consequently $\delta_n\to 0$ as $n\to\infty$. Since $\delta_n\omega_n(z)\to 0$ as $n\to\infty$, we have $N(z, p)=N'(z, p)$ and $D_R(N^M(z, p))=D_{R-V_M(p)}(N(z, p))=\lim_{n=\infty} M\int_{\partial R_0}\frac{\partial N_n'(z, p)}{\partial n}ds=2\pi M$.

Case 2. *$p\in R$ and $V_M(p)$ is non-compact.* Take M' large so that $V_{M'}(p)$ is compact. Then since $N(z, p)(p\in R)$ has the M.D.I over $R-V_{M'}(p)$, $N(z, p)$ also has M.D.I over $R-V_M(p)$ by lemma 1. Therefore $N(z, p)=\lim_{n=\infty} N_n'(z, p)$ in $R-V_M(p)$, where $N_n'(z, p)$ is harmonic in $R-R_0-V_M(p)$, $N_n'(z, p)=0$ on ∂R_0, $N_n'(z, p)=M$ on $\partial V_M(p)$ and $\frac{\partial N_n'(z, p)}{\partial n}=0$ $\partial R_n-V_M(p)$. Hence

$$D_R(N^M(z, p))=D_{R-V_M(p)}(\lim_{n=\infty} N_n'(z, p))=\lim_{n=\infty} M\int_{\partial R_0}\frac{\partial N_n'(z, p)}{\partial n}ds=2\pi M.$$

Case 3. $p\in B$. Let $\{p_i\}$ be a fundamental sequence determining p. Then for any given positive number ε, we can find a narrow strip S[4] such that the interior of S contains $\partial V_M(p)\cap(R_n-R_0)$ and that $D_{R_n-R_0-V_M(p)-S}(N(z, p))\geqq D_{R_n-R_0-V_M(p)}(N(z, p))-\varepsilon$ and further $(V_M(p_i)\cap(R_n-R_0))\subset(S+V_M(p))$ for any $i\geqq i_0(S)$, where $V_M(p_i)=E[z\in R: N(z, p_i)\geqq M]$ and $i_0(S)$ is a suitable number depending on S and ε, because $N(z, p_i)$ converges uniformly in every compact part of R to $N(z, p)$. On the other hand, since the derivatives of $N(z, p_i)$ converge to those of $N(z, p)$ uniformly in R_n-R_0, we have

4) S may consist of a finite number of components.

$$D_{R_n-R_0-V_M(p)-S}(N(z,\ p)) \leqq \varlimsup_{i=\infty} D_{R-V_M(p_i)}(N(z,\ p_i)) \leqq 2\pi M .$$

Hence, by letting $\varepsilon \to 0$ and then $n \to \infty$,

$$D_R(N^M(z,\ p)) = D_{R-V_M(p)}(N(z,\ p)) \leqq 2\pi M .$$

In the present part, we consider only positive continuous function $U(z)$ such that $U(z)=0$ on ∂R_0 and $D_R(U^M(z))<\infty$ for every M, where $U^M(z) = \min[M,\ U(z)]$. In what follows, in order to introduce the harmonicity or superharmonicity in $\bar{R}$, we make some preparations:

2. Capacity and the Equilibrium Potential of Relatively closed Sets in R.

Let F be a compact or non-compact relatively closed set in R having no common point with R_1. Denote by $\omega_n(z)$ a harmonic function in R_n-R_0-F such that $\omega_n(z)=0$ on ∂R_0, $\omega_n(z)=1$ on F except possibly a subset of capacity zero of F and $\dfrac{\partial \omega_n(z)}{\partial n}=0$ on $\partial R_n - F$. Then the Dirichlet integral of $\omega_n(z)$ and $\omega_{n+i}(z)$ taken over R_n-R_0-F is $D_{R_n-R_0-F}(\omega_n(z)-\omega_{n+i}(z),\ \omega_n(z))=0$, whence

$$D_{R_n-R_0-F}(\omega_{n+i}(z)) = D_{R_n-R_0-F}(\omega_n(z)) + D_{R_n-R_0-F}(\omega_{n+i}(z)-\omega_n(z)) ,$$
$$D_{R_n-R_0-F}(\omega_n(z)) < D_{R_{n+i}-R_0-F}(\omega_{n+i}(z)) < D_{R_1-R_0}(\omega^*(z)) ,$$

where $\omega_*(z)$ is a harmonic function in R_1-R_0 such that $\omega^*(z)=0$ on ∂R_0 and $\omega^*(z)=1$ on ∂R_1. Hence $\{D_{R_n-R_0-F}(\omega_n(z))\}$ is convergent, which implies that

$$D_{R_n-R_0}(\omega_{n+i}(z)-\omega_n(z)) = D_{R_n-R_0}(\omega_{n+i}(z)) - D_{R_n-R_0}(\omega_n(z)) ,$$

tends to zero as n and i tend to ∞.

Hence $\omega_n(z)$ converges to a harmonic function $\omega_F(z)$ in mean. Since $\omega_n(z)=0$ on ∂R_0, $\omega_n(z)$ converges to $\omega_F(z)$ uniformly in every compact set of $R-F$. Evidently, $\omega_F(z)$ has M.D.I over $R-F$ among all functions having the value 1 on F except possibly a subset of capacity zero of F. We call such $\omega_F(z)$ the *equilibrium potential of* F and $D(\omega_F(z)) = \int_{\partial R_0} \dfrac{\partial \omega_F(z)}{\partial n} ds$ the capacity of F. Then we have the following

Theorem 1.

1) *If* $F_n \uparrow F$, *then* $\omega_{F_n}(z) \uparrow \omega_F(z)$ *and* $\mathrm{Cap}\,(F_n) \uparrow \mathrm{Cap}\,(F)$.

2) *Let* G_ε *be the domain such that* $G_\varepsilon = E[z \in R:\ \omega_F(z) \geqq 1-\varepsilon]$ *and let* $\omega_{G_\varepsilon}(z)$ *be the equilibrium potential of* G_ε. *Then*

$$\omega_F(z) = (1-\varepsilon)\omega_{G_\varepsilon}(z) .$$

3) *Let ∂G_ε be the niveau curve of $\omega_F(z)$ with height $1-\varepsilon$. Then there exists a set H in the interval $(0,1)$ such that* $\operatorname{mes} H=1$ *and that* $1-\varepsilon\in H$ *implies*

$$\operatorname{Cap}(F)=\int_{\partial G_\varepsilon}\frac{\partial\omega_F(z)}{\partial n}ds=\int_{\partial R_0}\frac{\partial\omega_F(z)}{\partial n}ds\,.$$

Proof. Let $\omega_F(z)$ and $\omega_{F_n}(z)$ be the equilibrium potentials of F and F_n respectively. Then $\omega_F(z)\geqq\omega_{F_n}(z)$ and $D(\omega_F(z))\geqq D(\omega_{F_n}(z))$. On the other hand, clearly $\omega_{F_n}(z)$ is increasing with respect to n and $\lim_{n=\infty}\omega_{F_n}(z)$ attains 1 on F except possibly a subst of F of capacity zero. Since $\omega_F(z)$ has the M.D.I, we have $D(\omega_F(z))=\lim_{n=\infty}D(\omega_{F_n}(z))$ and $\omega_F(z)=\lim_{n=\infty}\omega_{F_n}(z)$, because such a function is determined uniquely by its boundary value on F.

Proof of 2). If we replace $U_F(z)$ in lemma 1 by $\omega_F(z)$ in this Theorem, then we have at once 2).

Proof of 3). Let $\omega_n'(z)$ be a harmonic function in $R_n-R_0-G_\varepsilon$ such that $\omega_n'(z)=0$ on ∂R_0, $\omega_n'(z)=1-\varepsilon$ on ∂G_ε and $\frac{\partial\omega_n'(z)}{\partial n}=0$ on $\partial R_n-G_\varepsilon$. Then, since $\lim_{n=\infty}\omega_n'(z)$ has M.D.I over $R-G_\varepsilon$, we have $\lim_{n=\infty}\omega_n'(z)=\omega_F(z)$ by 2). On the other hand, since $\int_{\partial G_\varepsilon\cap(R_n-R_0)}\frac{\partial\omega_n'(z)}{\partial n}ds=\int_{\partial R_0}\frac{\partial\omega_n'(z)}{\partial n}ds$, $\frac{\partial\omega_n'(z)}{\partial n}\geqq 0$ on ∂G_ε and $\lim_{n=\infty}\int_{\partial R_0}\frac{\partial\omega_n'(z)}{\partial n}ds=\int_{\partial R_0}\lim_{n=\infty}\frac{\partial\omega_n'(z)}{\partial n}ds$, we have by Fatou's lemma

$$L_\varepsilon=\int_{\partial G_\varepsilon}\frac{\partial\omega_F(z)}{\partial n}ds\leqq\lim_{n=\infty}\int_{\partial G_\varepsilon}\frac{\partial\omega_n'(z)}{\partial n}ds=\int_{\partial R_0}\frac{\partial\omega_F(z)}{\partial n}ds=L=D(\omega_F(z))\,.$$

Now we can take $p+iq=\omega_F(z)+i\bar{\omega}_F(z)$ as the local parameter at every point of $R-F$, where $\bar{\omega}_F(z)$ is the conjugate function of $\omega_F(z)$. Then $\frac{\partial\omega_F(z)}{\partial q}=0$ and $\frac{\partial\omega_F(z)}{\partial p}=1$ at every point of the niveau of $\omega_F(z)$ and the Dirichlet integral is

$$L=D(\omega_F(z))=\iint_{R-F}\left\{\left(\frac{\partial\omega_F(z)}{\partial p}\right)^2+\left(\frac{\partial\omega_F(z)}{\partial q}\right)^2\right\}dpdq=\int_0^1 L_\varepsilon dp\,.$$

If there were a set E of positive measure in $(0,1)$ such that $1-\varepsilon\in E$ implies $L_\varepsilon<L$, we have $D(\omega_F(z))<L$. This is absurd. Hence we have 3).

Regular Domains. Let F be a compact or non-compact relatively closed domain in R and let $\omega_F(z)$ be its equilibrium potential of F. If

$\int_{\partial F} \frac{\partial \omega_F(z)}{\partial n} ds = \int_{\partial R_0} \frac{\partial \omega_F(z)}{\partial n} ds$, F is called a *regular domain.* We see at once by 3) of Theorem 1 that there exists a sequence of regular domains $G_\varepsilon = E[z \in R : \omega_F(z) \geqq 1-\varepsilon]$ which we call the *regular domains generated by the equilibrium potential,* containing any closed set F of positive capacity and that any compact closed domain with analytic relative boundaries is always regular.

3. Definition of $U_D(z)$ for compact or non-compact Domain D.

Suppose a continuous function $U(z)$ in R such that $U(z)=0$ on ∂R_0, $D(U^M(z)) < \infty$ and a domain D. Let $U_D^M(z)$ be a harmonic function in $R-D$ such that $U_D^M(z) = U^M(z)$ on $\partial R_0 + \partial D$ and $U_D^M(z)$ has M.D.I over $R-D$. Then evidently, $U_D^M(z)$ is determined uniquely. We define $U_D(z)$ by $\lim_{M=\infty} U_D^M(z)$.

Theorem 3. *Let D be a regular domain and let $N^D(z, p)$ be a function in $R-D$ such that $N^D(z, p)$ is harmonic in $R-D$ except p where $N(z, p)+\log|z-p|$ is harmonic, $N^D(z, p)=0$ on $\partial R_0+\partial D$ and $N^D(z, p)$ has the minimal $*$-Dirichlet integral (it is taken with respect to $N(z, p)+\log|z-p|$ in a neighbourhood of p). Then we have the following*

$$U_D(p) = \frac{1}{2\pi} \int_{\partial D} U(z) \frac{\partial N^D(z, p)}{\partial n} ds . \tag{1}$$

Proof. Let $\omega_n(z)$ be a harmonic functionin R_n-R_0-D such that $\omega_n(z)=0$ on ∂R_0, $\omega_n(z)=1$ on ∂D and $\frac{\partial \omega_n(z)}{\partial n}=0$ on $\partial R_n - D$ and let $N_n^D(z, p)$ be a harmonic function in R_n-R_0-D with one positive logarithmic singularity at p such that $N_n^D(z, p)=0$ on $\partial R_0+\partial D \cap (R_n-R_0)$ and $\frac{\partial N_n^D(z, p)}{\partial n}=0$ on $\partial R_n - D$. Then by the maximum principle there exist constants M' and n such that $N_n^D(z, p) < M'$ for $n \geqq n_0$ outside of a neighbourhood of p. Hence there exists a constant M'' such that $N_n^D(z, p) \leqq M''(1-\omega_n(z))$ in R_n-R_0 outside of a neighbourhood of p for every $n \geqq n_0$, whence $0 \leqq \frac{\partial N_n^D(z, p)}{\partial n} < -M'' \frac{\partial \omega_n(z)}{\partial n}$ on $\partial D \cap (R_n-R_0)$. Now since D is regular, we have $\int_{\partial R_0} \frac{\partial \omega_D(z)}{\partial n} ds = \int_{\partial D} \frac{\partial \omega_D(z)}{\partial n} ds = \int_{\partial D} \lim_{n=\infty} \frac{\partial \omega_n(z)}{\partial n} ds$, where $\omega_0(z) = \lim_{n=\infty} \omega_n(z)$ is the equilibrium potential of D.

Assume that there exists a positive constant δ such that for infinitely many numbers m and $n (n > m)$ such that $\int_{\partial D \cap (R_n - R_m)} \frac{\partial \omega_n(z)}{\partial n} ds > \delta$. Then

$$\int_{\partial D\cap(R_m-R_0)} \frac{\partial \omega_n(z)}{\partial n} ds < \int_{\partial D\cap(R_n-R_0)} \frac{\partial \omega_n(z)}{\partial n} ds - \delta .$$

Let n tend to ∞. Then by Fatou's lemma

$$\int_{\partial D\cap(R_m-R_0)} \frac{\partial \omega_D(z)}{\partial n} ds \leqq \lim_{n=\infty} \int_{\partial D\cap(R_n-R_0)} \frac{\partial \omega_n(z)}{\partial n} ds - \delta \leqq \lim_{n=\infty} \int_{\partial R_0} \frac{\partial \omega_n(z)}{\partial n} ds - \delta .$$

Let m tend to ∞. Then $\int_{\partial D} \frac{\partial \omega_D(z)}{\partial n} ds \leqq \int_{\partial R_0} \frac{\partial \omega_D(z)}{dn} ds - \delta$. This contradicts the regularity of D. Hence, for any given positive number ε, there exist numbers m and $n_0(\varepsilon, m)$ such that $0 \leqq \int_{\partial D\cap(R_n-R_m)} \frac{\partial \omega_n(z)}{\partial n} ds < \varepsilon$, for $n \geqq n_0$. It follows that $\int_{\partial D\cap(R_n-R_0)} \frac{\partial N_n^D(z, p)}{\partial n} ds < M''\varepsilon$, for $n \geqq n_0$. (2)

Let $U_n^M(z)$ be a harmonic function in $R_n - R_0 - D$ such that $U_n^M(z) = U^M(z)$ on $\partial R_0 + \partial D$ and $\frac{\partial U_n^M(z)}{\partial n} = 0$ on $\partial R_n - D$. Then by Green's formula

$$U_n^M(p) = \frac{1}{2\pi} \int_{\partial D\cap(R_n-R_0)} U^M(z) \frac{\partial N_n^D(z, p)}{\partial n} ds .$$

Let n tend to ∞. Then since $U_n^M(z)$ tends to $U_D^M(z)$ and by (2), we have

$$U_D^M(p) = \frac{1}{2\pi} \int_{\partial D} U^M(z) \frac{\partial N^D(z, p)}{\partial n} ds .$$

Hence by letting $M \to \infty$, we have $U_D(p) = \frac{1}{2\pi} \int_{\partial D} U(z) \frac{\partial N^D(z, p)}{\partial n} ds$.

5. Harmonicity and Superharmonicity in $\bar{R}$. If $U(z)$ is superharmonic in R and further, for any compact domain D, if $U(z) = U_D(z)$ or $U(z) > U_D(z)$, we say that $U(z)$ is *harmonic or superharmonic in* $\bar{R}$ respectively.

Theorem 3. *If $U(z)$ and $V(z)$ are positive, $U(z) = V(z) = 0$ on ∂R_0 and harmonic in R and superharmonic in $\bar{R}$, then for a domain D*

1) $U_D(z) \leqq U(z)$.
2) $U(z) \geqq V(z)$ *implies* $U_D(z) \geqq V_D(z)$.
3) $U_D(z) + V_D(z) = {}_D(U+V)(z)$.
4) $(CU_D(z)) = {}_D(CU)(z)$ *for* $C \geqq 0$.
5) $U_{D_1+D_2}(z) \leqq U_{D_1}(z) + U_{D_2}(z)$ *for two domains* D_1 *and* D_2.
6) *If* $D_1 \supset D_2$, *then* ${}_{D_1}(U_{D_2}(z)) = U_{D_2}(z)$ *and* $U_{D_1}(z) \geqq U_{D_2}(z)$.

The first five assertions are clear by definition. We shall prove 6). We see easily that $U^M(z)$ is superharmonic in $\bar{R}$ by the superharmonicity

of $U(z)$ in $\bar{R}$. Assume $D_1 \supset D_2$. Then by lemma 1 $U_{D_1}^M(z) = {}_{D_2}(U_{D_1}^M(z))$. Hence by letting $M \to \infty$ $U_{D_1}(z) = {}_{D_2}(U_{D_1}(z)) \leqq {}_{D_2}(U(z)) = U_{D_2}(z)$.

Another Definition of $U_D(z)$. If $U(z)$ is superharmonic in $\bar{R}$, $U_D(z)$ is given as follows: Put $D_n = D \cap (R_n - R_0)$. Then

$$U_D(z) = \lim_{n=\infty} U_{D_n}(z) .$$

Proof. $U_{D_n}(z)$ is increasing with respect to n by 6) of the above Theorem. Hence $\{U_{D_n}(z)\}$ converge. Since $D(U_D^M(z)) \leqq D(U^M(z)) < \infty$, for any given positive number ε there exists a number n_0 such that $D_{D \cap (R-R_0)}(U_D^M(z)) < \varepsilon$ for $n \geqq n_0(M)$. On the other hand, since $U_{D_n}^M(z)$ has M.D.I over $R - D_n$ with boundary value $U^M(z) = U_{D_n}^M(z)$ on ∂D_n,

$$D_{R-D_n}(U_{D_n}^M(z)) \leqq D_{R-D_n}(U_D^M(z)) \leqq D_{R-D}(U^M(z)) + \varepsilon \quad \text{for} \quad n \geqq n_0(M) .$$

Let $n \to \infty$ and then $\varepsilon \to 0$. Then

$$D_{R-D}(U_D^M(z)) \geqq \lim_{n=\infty} (D_{R-D_n}(U_{D_n}^M(z)) \geqq D_{R-D}(\lim_{n=\infty} U_{D_n}^M(z)) .$$

Hence $\lim_{n=\infty} U_{D_n}^M(z)$ has M.D.I over $R - D$ with boundary value $U^M(z)$ on ∂D, whence $\lim_{n=\infty} U_{D_n}^M(z) = U_D^M(z)$ and $\lim_{n=\infty} U_{D_n}(z) \geqq U_D^M(z)$. Let $M \to \infty$. Then

$$\lim_{n=\infty} U_{D_n}(z) \geqq U_D(z) .$$

Next, put $M_n = \sup_{z \in R_n - R_0} U(z)$. Then clearly $U_{D_n}(z) = U_{D_n}^{M_n}(z) \leqq U_D^{M_n}(z)$. Let $n \to \infty$. Then $\lim_{n=\infty} U_{D_n}(z) \leqq U_D(z)$. Thus we have $\lim_{n=\infty} U_{D_n}(z) = U_D(z)$.

6. Equilibrium Potential of a closed subset A of B. Let A be a δ-closed set of B. Put $A_m = E\left[z \in \bar{R} : \delta(z, A) \leqq \frac{1}{m}\right]$. Then $R \cap A_m$ is a relatively closed set of R and $\bigcap_{m>0} A_m = A$. Let $\omega_{A_m,n}(z)$ be a harmonic function in $R_n - R_0 - A_m$ such that $\omega_{A_m,n}(z) = 0$ on ∂R_0, $\omega_{A_m,n}(z) = 1$ on ∂A_m and $\frac{\partial \omega_{A_m,n}(z)}{\partial n} = 0$ on $\partial R_n - A_m$. Then

$$D_{R_n - R_0 - A_m}(\omega_{A_m,n}(z), \omega_{A_{m+i},n}(z)) = \int_{\partial A_m \cap (R_n - R_0)} \frac{\partial \omega_{A_{m+i},n}(z)}{\partial n} ds$$

$$= \int_{\partial A_{m+i} \cap (R_n - R_0)} \frac{\partial \omega_{A_{m+i},n}(z)}{\partial n} ds = D_{R_n - R_0 - A_{m+i}}(\omega_{A_{m+i},n}(z)) .$$

Since $D(\omega_{A_m,n}(z))$ and $D(\omega_{A_{m+i}\,n}(z))$ converge as $n \to \infty$, we have $D_{R-R_0-A_m}(\omega_{A_{m+i},n}(z), \omega_{A_m}(z)) = D_{R-R_0-A_{m+i}}(\omega_{A_{m+i}}(z))$. Hence $D_{R-R_0-A_m}(\omega_{A_m}(z) - \omega_{A_{m+i}}(z)) = D_{R-R_0-A_m}(\omega_{A_m}(z)) - 2D_{R-R_0-A_m}(\omega_{A_m}(z), \omega_{A_{m+i}}(z)) + D_{R-R_0-A_m}$

$(\omega_{A_{m+i}}(z)) < D_{R-R_0-A_m}(\omega_{A_m}(z)) - D_{R-R_0-A_m}(\omega_{A_{m+i}}(z))$ and $D_{R-R_0-A_m}(\omega_{A_m}(z))$ is decreasing with respect to m. Therefore $\omega_{A_m}(z)$ converges to a function $\omega_A(z)$ in mean as $m \to \infty$. We call $\omega_A(z) = \lim_{m=\infty} \omega_{A_m}(z)$ the *equilibrium* potential of A. Suppose $\omega_A(z) > 0$. Let $V(z)$ be a harmonic function in $R-G$ such that $V(z) = 0$ on $\partial R_0 + \partial G$ and $D(V(z)) < \infty$, where G is a relatively closed set containing A. Then by lemma 1 $\omega_{A_m}(z)$ $(A_m \subset G)$ has M.D.I over $R-G$ among all functions having the boundary value $\omega_{A_m}(z)$ on ∂G. Hence

$$D(\omega_{A_n}(z) \pm \varepsilon V(z)) \geqq D(\omega_{A_m}(z)) ,$$

for every small positive number ε. Since $\omega_{A_m}(z)$ converges to $\omega_A(z)$ in mean,

$$D(\omega_{A_m}(z) - \omega_A(z),\ V(z)) \leqq \sqrt{D(\omega_{A_m}(z) - \omega_A(z)) D(V(z))} ,$$

which implies $D(V(z),\ \omega_A(z)) = 0$. Since $V(z)$ is arbitrary, $\omega_A(z)$ has also M.D.I over $R-G$ among all functions having the boundary value $\omega_A(z)$ on ∂G. Therefore ${}_A\omega_A(z) = \omega_A(z)$. Hence if we take $G_\varepsilon = [z \in R:\ \omega_A(z) \geqq 1-\varepsilon]$, $\dfrac{\omega_A(z)}{1-\varepsilon}$ is *the equilibrium potential* of G_ε.

7. Integral Representation of Superharmonic Functions in $\bar{R}$.

Definition of $U_A(z)$ for a δ-closed subset A of B. $A_m = E\left[z \in R:\ \delta(z, A) \leqq \dfrac{1}{m}\right]$. Then A_m is relatively closed set and clearly $U_{A_m}(z)$ is decreasing as $m \to \infty$. We define $U_A(z)$ by $\lim_{m=\infty} U_{A_m}(z)$.

Theorem 4.

1) *$N(z, p)$ $(p \in \bar{R})$ is superharmonic in R and superharmonic in $\bar{R}$, more generally $\int N(z, p) d\mu(p)$ is superharmonic in $\bar{R}$ for $\mu > 0$.*

2) *$\omega_D(z)$ and $\omega_A(z)$ are superharmonic in $\bar{R}$.*

Proof of 1). First, suppose $p \in R$. Since clearly $N(z, p)$ is superharmonic in R, it is sufficient to prove that $N(z, p) \geqq N_D(z, p)$ for every *compact* domain D. Since $N(z, p)$ has the minimal $*$-Dirichlet integral over R, we have by Green's formula and by Theorem 2

$$N(z, p) = \text{or} > \frac{1}{2\pi} \int_{\partial D} N(\zeta, p) \frac{\partial N^D(\zeta, z)}{\partial n} ds = N_D(z, p) ,$$

according as $p \in D$ or $p \notin D$.

Next, consider $p \in B$. Let $\{p_i\}$ be a fundamental sequence determining p. Then $N(z, p_i)$ tends to $N(z, p)$ on ∂D, hence

$$N(z,\ p)=\lim_{i=\infty}N(z,\ p_i)\geqq\frac{1}{2\pi}\int_{\partial D}\lim_{i=\infty}N(\zeta,\ p_i)\frac{\partial N^D(\zeta,\ z)}{\partial n}ds=N_D(z,\ p)\ .$$

Thus $N(z,\ p)(p\in\bar{R})$ is superharmonic in $\bar{R}$.

The approximation to $V(z)=\int N(z,\ p)\,d\mu(p)$ by a sequence of functions $V_n(z)$ $(n=1,\ 2,\ \cdots)$ of the form $V_n(z)=\sum_{i=1}^{n}c_iN(z,\ p_i)$ can be done in every compact part of R. $V_n(z)=\frac{1}{2\pi}\int_{\partial D}V_n(\zeta)\frac{\partial N^D(\zeta,\ z)}{\partial n}ds$, which implies by letting $n\to\infty$ $V(z)=\frac{1}{2\pi}\int_{\partial D}V(\zeta)\frac{\partial N^D(\zeta,\ z)}{\partial n}ds=V_D(z)$. Therefore $V(z)$ is superharmonic in $\bar{R}$.

Proof of 2). Let G be a compact domain and let $\omega_D^n(z)$ be a harmonic function in $R-R_n-D$ such that $\omega_D^n(z)=0$ on ∂R_0, $\omega_D^n(z)=1$ on $\partial D\cap(R_n-R_0)$ and $\frac{\partial\omega_D^n(z)}{\partial n}=0$ on ∂R_n-D. Then

$$\omega_D^n(z)\geqq\frac{1}{2\pi}\int_{\partial G\cap(R_n-R_0)}\omega_D^n(\zeta)\frac{\partial N_n^G(\zeta,\ z)}{\partial n}ds\ ,$$

where $N_n^G(\zeta,\ z)$ is the $*$-Green's function of R_n-R_0-G with pole at z. Let $n\to\infty$. Then

$$\omega_D(z)\geqq\frac{1}{2\pi}\int_{\partial G}\omega_D(\zeta)\frac{\partial N^G(\zeta,\ z)}{\partial n}ds={}_G\omega_D(z)\ .$$

Hence $\omega_D(z)$ is superharmonic in $\bar{R}$.

Put $G=A_m$. Then $\omega_{A_m}(z)\geqq\frac{1}{2\pi}\int_{\partial G}\omega_{A_m}(\zeta)\frac{\partial N^G(\zeta,\ z)}{\partial n}ds={}_G\omega_{A_m}(z)\ .$

Let $m\to\infty$. Then $\omega_A(z)\geqq\frac{1}{2\pi}\int_{\partial G}\omega_A(\zeta)\frac{\partial N^G(\zeta,\ z)}{\partial n}ds={}_G\omega_A(z)\ .$

Thus $\omega_A(z)$ is also superharmonic in $\bar{R}$.

Theorem 5. *If $U(z)$ is positive harmonicin R and superharmonic in $\bar{R}$, then for a δ-closed subset A of B, we have*

1) *There exists a mass distribution μ on A such that*

$$U_A(z)=\frac{1}{2\pi}\int_A N(z,\ p)\,d\mu(p)\ ,$$

for all point z in R. The total mass $\mu(A)$ is given by $\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial U_A(z)}{\partial n}ds$.

2) ${}_A\omega_A(z)=\omega_A(z)=\frac{1}{2\pi}\int_A N(z,\ p)\,d\mu(p)$ for $\omega_A(z)>0$.

2′) *If p is an ideal boundary point such that $\omega_p(z)>0$, then*

$$\omega_p(z)=KN(z,\ p)\ ,\qquad K>0\ .$$

3) $$U(z)=\frac{1}{2\pi}\int_B N(z,\ p)\,d\mu(p)\ .$$

Proof. Put $A_m=E\left[z\in R:\ \delta(z,\ A)\leqq\frac{1}{m}\right]$ and $A_{m,n}=A_m\cap(R_n-R_0)$. Then by 5. $U_A(z)=\lim_{m=\infty}\lim_{n=\infty}U_{A_{m,n}}(z)$. Now $U(z)\geqq U_{A_m}(z)\geqq U_{A_{m,n}}(z)$ for $z\notin A_{m,n}$, $U(z)=U_{A_{m,n}}(z)$ for $z\in A_{m,n}(z)$ is continuous on $A_{m,n}(z)$, whence $U_{A_{m,n}}(z)$ is superharmonic at every point of $A_{m,n}$. Hence it can be proved by the method of F. Riesz-Frostmann that the functional

$$J(\mu)=\frac{1}{2}\frac{1}{4\pi^2}\int\!\!\int_{A_{m,n}}N(z,\ p)\,d\mu(p)\,d\mu(z)-\frac{1}{2\pi}\int_{A_{m,n}}U_{A_{m,n}}(z)\,d\mu(z)\ ,$$

is minimized by a unique mass distribution on $\mu(A_{m,n})$ on $A_{m,n}$ among all non negative mass distributions. The function $V(z)$ given by $\frac{1}{2\pi}\int_{A_{m,n}}N(z,\ p)\,d\mu(p)$ is equal to $U(z)$ on $A_{m,n}$ except possibly a subset of capacity zero of $A_{m,n}$ and has the M.D.I, because $V(z)$ is a linear form of $N(z,\ p)$ $(p\in R)$. Therefore $U_{A_{m,n}}(z)=V(z)$, where the total mass is given by $\frac{1}{2\pi}\int\frac{\partial U_{A_{m,n}}(z)}{\partial n}ds$ for every n and m. Since $N(z,\ p)$ is a δ-continuous function of p for fixed z and the total mass is less than $\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds$, $\mu(A_{m,n})$ has an weak limit $\mu(A_m)$ on A_m as $n\to\infty$. Hence $U_{A_m}(z)=\frac{1}{2\pi}\int_{A_m}N(z,\ p)d\mu(p)$ and by letting $m\to\infty$, $U_A(z)=\frac{1}{2\pi}\int_A N(z,p)d\mu(p)$. 2) and 2') are clear by the property of $\omega_A(z)$ and 3) is also clear, if we consider B as A.

8. Classifications of the Ideal Boundary Points.

Regular or Singular ideal Boundary Point. Take an ideal boundary point p as a closed subst A of B. Then we call p a *regular or singular* ideal boundary point according as $\omega_p(z)=0$ or $\omega_p(z)>0$.

In what follows, we shall consider another classification. We shall prove the following

Theorem 6. *Let $U(z)$ be a harmonic in R and superharmonic function in $\bar{R}$ and let A be a closed subset of capacity zero of $\bar{R}$. Then*

$$_AU_A(z)=U_A(z)\ .$$

Proof. Let G be a compact domain in R. Then

$$U(z)=V_G(z)+U'(z)\qquad\text{for}\quad z\in R-G\ ,\qquad\qquad(a)$$

where $V_G(z)$ is a harmonic function in $R-G$ such that $V_G(z)=U(z)$ on

$\partial G+\partial R_0$ and $V_G(z)$ has M.D.I over $R-G$ and $U'(z)$ is a harmonic function in $R-G$ such that $U'(z)=0$ on $\partial G+\partial R_0$ and $U'(z)$ is superharmonic in $\overline{R-G}$. In fact, let D be a domain in R. Then since $D+G \supset G$, by Lemma 1, $V_G(z)=V_{D+G}(z)$, where $V_{D+G}(z)$ is a harmonic function in $R-G-D$ such that $V_{D+G}(z)=V_G(z)$ on $\partial D+\partial G+\partial R_0$ and $V_{D+G}(z)$ has M.D.I over $R-G-D$. Now, since $U(z)$ is superharmonic in $\bar{R}$ and $V_G(z)=V_{G+D}(z)$,

$$U(z) = U'(z)+V_G(z) \geqq \frac{1}{2\pi}\int_{(\partial G-D)+(\partial D-G)} U(\zeta)\frac{\partial N^{D+G}(\eta, z)}{\partial n}ds$$

$$= \frac{1}{2\pi}\int_{(\partial G-D)+(\partial D-G)} (V_G(z)+U'(z))\frac{\partial N^{D+G}(\zeta, z)}{\partial n}ds = V_{G+D}(z)+U_D'(z)\,.$$

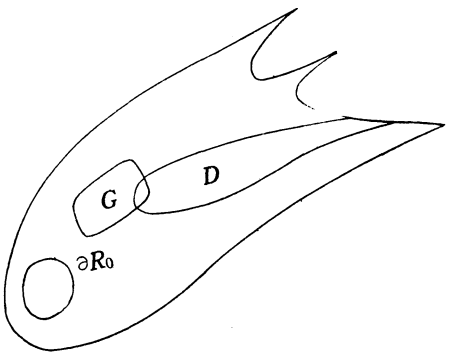
Hence $$U'(z) \geqq U_D'(z)\,, \qquad \text{(b)}$$

where $U_D'(z)$ is a harmonic function in $R-G-D$ such that $U_D'(z)=0=U'(z)$ on $\partial G+\partial R_0-D$, $U_D'(z)=U'(z)$ on $\partial D-G$ and $U'(z)$ has M.D.I over $R-G-D$. This means that $U'(z)$ is superharmonic in $\overline{R-G}$.

Consider $A_{m,n}=A_m\cap(R_n-R_0)$ as D in (a). Then by (a)

$$U_{A_{m,n}}(z) = V_{A_{m,n}}(z)+U'_{A_{m,n}}(z)+(V_G-V_{A_{m,n}})(z) \quad \text{for } z\in R-A_{m,n}-G, \qquad \text{(c)}$$

where $V_{A_{m,n}}(z)$ is a harmonic function in $R-G$ such that $V_{A_{m,n}}(z)=U_{A_{m,n}}(z)$ on $\partial R_0+\partial G$ and $V_{A_{m,n}}(z)$ has M.D.I over $R-G$ and $U'_{A_{m,n}}(z)$ is a harmonic function in $R-G-A_{m,n}$ such that $U'_{A_{m,n}}(z)=0$ on $\partial R_0+\partial G-A_{m,n}$, $U'_{A_{m,n}}(z)=U'(z)$ on $\partial A_{m,n}-G$ and $U'_{A_{m,n}}(z)$ has M.D.I over $R-G-A_{m,n}$. Hence by (b) $U'_{A_{m,n}}(z)\leqq U'(z)$.



And $(V_G-V_{A_{m,n}})(z)$ is a harmonic function in $R-G-A_{m,n}$ such that $(V_G-V_{A_{m,n}})(z)=0$ on $\partial R_0+\partial G-D$, $(V_G-V_{A_{m,n}})(z)=V_G(z)-V_{A_{m,n}}(z)$ ($V_G(z)=U(z)$ and ∂G) on $\partial A_{m,n}$ and $(V_G-V_{A_{m,n}})(z)$ has M.D.I over $R-G-A_{m,n}$. Clearly since $U(z)\geqq U_{A_{m,n}}(z)$, $0\leqq(V_G-V_{A_{m,n}}(z)\leqq M\omega'_{A_{m,n}}(z)$, where $M=\max_{z\in\partial G} V_G(z)$ and $\omega'_{A_{m,n}}(z)$ is the equilibrium potential of $A_{m,n}$ with respect to $R-G$.

Let $n\to\infty$. Then $U'_{A_{m,n}}(z)\uparrow U'_{A_m}(z)$, since $U'(z)$ is superharmonic in $R-G$. $U_{A_{m,n}}(z)\uparrow U_{A_m}(z)$ implies $V_{A_{m,n}}(z)\uparrow V_{A_m}(z)$. $(V_G-V_{A_{m,n}})(z)\to(V_G-V_{A_m})(z)$. Here $V_{A_{m,n}}(z)$ converges to $V_{A_m}(z)$ in mean, because $D_{R-G}(V_{A_{m,n}}(z))=\int_{\partial G} V_{A_{m,n}}(z)\frac{\partial V_{A_{m,n}}(z)}{\partial n}ds$ and ∂G is compact. Hence

$V_{A_m}(z)$ has also M.D.I over $R-G$ with boundary value $U_{A_m}(z)$ on ∂G and 0 on ∂R_0. Therefore

$$U_{A_m}(z) = V_{A_m}(z) + U'_{A_m}(z) + (V_G - V_{A_m})(z) . \qquad (d)$$

Let $m \to \infty$. Then $V_{A_m}(z) \downarrow U_A(z)$, $V_{A_m}(z) \downarrow V_A(z)$, $U'_{A_m}(z) \downarrow U_A(z)$ and $0 = \lim_{m=\infty} (V_G - V_{A_m})(z) \leqq M\omega_A'(z) = 0$. Hence

$$U_A(z) = V_A(z) + U_A'(z) . \qquad (e)$$

By (d) and (e), we have

$$U_{A_m}(z) - U_A(z) = V_{A_m}(z) - V_A(z) + (U'_{A_m}(z) - U_A'(z)) + (V_G - V_{A_m})(z) ,$$

where $V_{A_m}(z) = {}_G U_{A_m}(z)$ and $V_A(z) = {}_G U_A(z)$ by definition and the last two terms on the right hand side are non negative. Hence

$$U_{A_m}(z) - U_A(z) \geqq {}_G(U_{A_m}(z) - U_A(z)) .$$

Suppose $G = A_{m',n'}$ $(n' < m)$. Then by letting $n' \to \infty$, we have

$$U_{A_m}(z) - U_A(z) \geqq {}_{A_{m'}}(U_{A_m}(z) - U_A(z)) . \qquad (f)$$

Proof of the theorem. Since $U_A(z)$ is representable in the form (e) for any compact domain G, $U_A(z)$ is clearly superharmonic in $\bar{R}$, that is $U_A(z) \geqq {}_G U_A(z) = V_A(z)$ for domain G. Hence ${}_{A_{m'}} U_A(z) \leqq U_A(z)$ for every m' and ${}_A U_A(z) \leqq U_A(z)$.

Let z be a point of R. Then, since $U_{A_m}(z) \downarrow U_A(z)$ as $m \to \infty$, for any given positive number ε, there exists a number m_0 depending on z such that

$$\varepsilon > U_{A_{m+i}}(z) - U_A(z) > 0 \qquad \text{for} \quad m+i \geqq m_0 .$$

Then by (f)

$$0 < {}_{A_{m'}}(U_{A_{m+i}}(z)) - U_A(z)) < U_{A_{m+i}}(z) - U_A(z) < \varepsilon .$$

On the other hand, by 6) of Theorem 3 ${}_{A_{m'}}(U_{A_{m+i}}(z)) = U_{A_{m+i}}(z)$ for $m+i \geqq m'$. Hence

$${}_{A_{m'}}(U_{A_{m+i}}(z) + U_A(z) - U_{A_{m+i}}(z)) \geqq U_{A_{m+i}}(z) - \varepsilon .$$

Thus by letting $\varepsilon \to 0$, ${}_{A_{m'}}(U_A(z)) \geqq U_A(z)$. Therefore ${}_A U_A(z) = U_A(z)$.

Putting $A = q$, we define the function $\Psi(q)$ of q in B as $\frac{1}{2\pi} \int_{\partial R_0} \frac{\partial N_q(z, q)}{\partial n} ds$. Then we have

Theorem 7.

1) $\Phi(q)$ *has only two possible values* 1 *and* 0.

2) *Denote by* B_0 *and* B_1 *the sets of points of* B *for which* $\Psi(q)=0$ *and* $\Psi(q)=1$ *respectively. Then* $B=B_0+B_1$ *and* B_0 *is void or an* F_σ.

We shall prove 1) in two cases as follows:

Case 1. q *is regular ideal boundary point,* i.e. $\omega_q(z)=0$.

Case 2. q *is a singular ideal boundary point,* i.e. $\omega_q(z)>0$.

Case 1. $\omega_q(z)=0$. We have $N_q(z, q)=\Psi(q)N(z, q)$ by 2) of Theorem 5 and ${}_qN_q(z, q)=\Psi^2(q)N(z, q)=\Psi(q)N(z, q)=N_q(z, q)$ by Theorem 6. Hence we have $\Psi(q)=0$ or 1.

Case 2. $\omega_q(z)>0$. In this case we have $N(z, q)=K\omega_q(z)=N_q(z, q)$ $=K_q\omega_q(z)=K\Psi(q)N_q(z, q)$ by 2′) of Theorem 5. Hence $\omega_q(z)>0$ implies $\Psi(q)=1$.

Proof of 2). The set Γ_m is defined as the set (possible void) of all points q of B such that $\Psi(A_m(q))=\dfrac{1}{2\pi}\int_{\partial R_0}\dfrac{\partial N_{A_m{}^{(q)}}(z, q)}{\partial n}ds\leqq\dfrac{1}{2}$ (this means $\Psi(q)=0$), where $A_m(q)=E\left[z\in\bar{R}:\delta(z, q)\leqq\dfrac{1}{m}\right]$. Then clearly $B_0=\bigcup_{m>1}\Gamma_m$. We shall show that $\Psi(A_m(q))$ is a lower semicontinuous function of q.

By definition $N_{A_m{}^{(q)}}(z, q)=\lim_{n=\infty}N_{A_{m,n}{}^{(q)}}(z, q)$, where $A_{m,n}(q)=A_m(q)\cap(R_n-R_0)$. Hence, for any given positive number ε, there exists a number n such that $\Psi(A_{m,n}(q))=\dfrac{1}{2\pi}\int_{\partial R_0}\dfrac{\partial N_{A_{m,n}{}^{(q)}}(z, q)}{\partial n}ds\geqq\dfrac{1}{2\pi}\int_{\partial R_0}\dfrac{\partial N_{A_m{}^{(q)}}(z, q)}{\partial n}ds$ $-\varepsilon=\Psi(A_m(q))-\varepsilon$. Suppose $q_i\to q$. Then $A_{m,n}(q_i)\to A_{m,n}(q)$. Hence by the compactness of $A_{m,n}(q)$

$$\lim_{i=\infty}N_{A_{m,}{}^{(q_i)}}(z, q_i)\geqq\lim_{i=\infty}\int_{\partial A_{m,n}{}^{(q_i)}}N(\zeta, q_i)\frac{\partial N^{A_{m,n}{}^{(q_i)}}(\zeta, z)}{\partial n}ds$$

$$=\int_{\partial A_{m,n}{}^{(q)}}N(\zeta, q)\frac{\partial N^{A_{m,n}{}^{(q)}}(\zeta, z)}{\partial n}ds=N_{A_{m,n}{}^{(q)}}(z, q)\,.$$

Consequently $\varliminf_{i=\infty}\Psi(A_m(q_i))\geqq\Psi(A_m(q))-\varepsilon$, whence by letting $\varepsilon\to0$

$$\varliminf_{i=\infty}\Psi(A_m(q_i))\geqq\Psi(A_m(q))\,.$$

Therefore $\Psi(A_m(q))$ is lower semicontinuous with respect to q, whence Γ_m is closed and B_0 is an F_σ.

9. Canonical Distributions. We shall consider properties of B_0 and B_1.

Theorem 8.

1) *Cap* $(B_0)=0$.

2) *If* $U(z)$ *is given by* $\dfrac{1}{2\pi}\int_{B_0}N(z, p)\,d\mu(p)$, $U_{B_0}(z)=0$

3) *Every function $U(z)$ which is harmonic in R and superharmonic in $\bar{R}$ is representable by a mass distribution on B_1 such that*

$$U(z)=\frac{1}{2\pi}\int_{B_1} N(z, p)\,d\mu(p)\,.$$

Proof of 1). The set Γ_m, being closed and compact, may be covered by a finite number of its closed subsets whose diameters are less than $\frac{1}{m}$. It is sufficient, by 5) of Theorem 4, to prove 1) for any closed subset A whose diameter is less than $\frac{1}{m}$, of Γ_m. Assume $\mathrm{Cap}(A)>0$. Then $0<{}_A\omega_A(z)=\omega_A(z)=\frac{1}{2\pi}\int_A N(z, p)\,d\mu(p)$ by 1) of Theorem 5. On the other hand, since ${}_A\omega_A(z)=\lim_{m=\infty}\lim_{n=\infty}{}_{A_{m,n}}\omega_A(z)$, for any given positive number ε, there exist numbers m and n such that

$$\mathrm{Cap}\,(A)=\int_{\partial R_0}\frac{\partial\omega_A(z)}{\partial n}ds\leqq\int_{\partial R_0}\frac{\partial({}_{A_{m,n}}\omega_A(z))}{\partial n}ds+\varepsilon\,,$$

where $A_m=E\left[z\in\bar{R}:\delta(z, A)\leqq\frac{1}{m}\right]$ and $A_{m,n}=A_m\cap(R_n-R_0)$.

Now $\omega_A(z)$ can be approximated on $A_{m,n}$ by a sequence of functions $V_l(z)=\sum_{i=1}^{l}c_iN(z, q_i)\,(q_i\in A)$ $(l=1, 2, \cdots)$. Then by Fatou's lemma

$$\begin{aligned}\mathrm{Cap}\,(A)&=\int_{\partial R_0}\frac{\partial\omega_A(z)}{\partial n}ds\leqq\varlimsup_{l=\infty}\int_{\partial R_0}\frac{\partial V_l(z)}{\partial n}ds\leqq\frac{1}{2}\int_{\partial R_0}\frac{\partial\omega_A(z)}{\partial n}ds+\varepsilon\\&=\frac{1}{2}\mathrm{Cap}\,(A)+\varepsilon\,,\end{aligned}$$

because $A_m\subset v_m(q_i)=E\left[z\in\bar{R}:\delta(z, q_i)\leqq\frac{1}{m}\right]$ for every $q_i\in A$ implies $\int_{\partial R_0}\frac{\partial N_{v_m(q_i)}(z, q_i)}{\partial n}ds\leqq\frac{1}{2}\int_{\partial R_0}\frac{\partial N(z, q_i)}{\partial n}ds$. This is absurd. Hence $\mathrm{Cap}\,(A)=0$, $\mathrm{Cap}\,(\Gamma_m)=0$ and $\mathrm{Cap}\,(B_0)=0$.

Proof of 2). As above, we have for $A\subset\Gamma_m$, $U_A(z)\leqq U_{A_m}(z)$ and $\int_{\partial R_0}\frac{\partial U_A(z)}{\partial n}ds\leqq\int_{\partial R_0}\frac{\partial U_{A_m}(z)}{\partial n}ds\leqq\frac{1}{2}\int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds$, whence *mass of* $U_A(z)\leqq\frac{1}{2}$ *mass of* $U(z)$ and *mass of* ${}_AU_A(z)\leqq\frac{1}{2}$ *mass of* $U_A(z)$. On the other hand, since $\mathrm{Cap}\,(A)=0$, we have by Theorem 6 ${}_AU_A(z)=U_A(z)$. Hence $U_A(z)=0$, $U_{\Gamma_m}(z)=0$ and $U_{B_0}(z)=0$.

Proof of 3. Suppose $U(z)=\frac{1}{2\pi}\int_{B_0}N(z, p)\,d\mu(p)$. Put $\Gamma_{m,n}=E\left[z\in B:\delta(z, \Gamma_m)\leqq\frac{1}{n}\right]$. Let z be a point R. Since $U_{\Gamma_m}(z)=0$, for any given

positive number ε, there exists a number $n(m)$ such that $U_{\Gamma_{m,n}}(z) \leqq \frac{\varepsilon}{2^m}$ for $n \geqq n(m)$. For each m select $\Gamma_m'(=\Gamma_{m,n})$ in this fashion. Put $C_m = \sum_{i=1}^{m} \Gamma_i'$. Then C_m are closed and form a increasing sequence as $m \to \infty$. Denote by $\tilde{A}_m$ the closure of the complement of C_m in B. Then the distance between $\tilde{A}_m$ and Γ_m is at least $\frac{1}{n(m)}$. Thus $\{\tilde{A}_m\}$, which forms a descending sequence, has an intersection $\tilde{A}$ which is closed and, having no point in common with any Γ_m, is a subset of B_1.

Now $U_{C_m}(z) \leqq \sum_{i=1}^{m} U_{\Gamma_i'}(z) \leqq \sum_{i=1}^{m} 2^{-i}\varepsilon \leqq \varepsilon$. Observing $\tilde{A}_m + C_m = B$, we obtain

$$U(z) = U_B(z) = U_{\tilde{A}_m + C_m}(z) \leqq U_{\tilde{A}_m}(z) + U_{C_m}(z) \leqq U_{\tilde{A}_m}(z) + \varepsilon .$$

Let $m \to \infty$ and then $\varepsilon \to 0$. Then $\bigcap_{m>1} \tilde{A}_m \subset B_1$ and $U(z) = U_{B_1}(z) = \frac{1}{2\pi} \int_{B_1} N(z, p)\, d\mu(p)$. Thus $U(z)$ is representable by a mass distribution on B_1 without any change of $U(z)$.

Proof of 3). Suppose that $U(z)$ is harmonic in R and superharmonic in $\bar{R}$. Then $U(z) = \frac{1}{2\pi} \int_B N(z, p)\, d\mu(p) = \frac{1}{2\pi} \int_{B_1} N(z, p)\, d\mu_1(p) + \frac{1}{2\pi} \int_{B_0} N(z, p)\, d\mu_0(p)$ by 3) of Theorem 5. As above $\int_{B_0} N(z, p)\, d\mu_0(p) = \int_{B_1} N(z, p)\, d\mu_1'(p)$. Then $U(z) = \frac{1}{2\pi} \int_{B_1} N(z, p)\, d(\mu_1 + \mu_1')\,(p)$. Thus we have 3). We call such distribution on B_1 *canonical.*

10. Minimal Functions. Let $U(z)$ be a function which is harmonic in R and superharmonic in $\bar{R}$. If $U(z) \geqq V(z) \geqq 0$ implies $V(z) = KU(z)$ $(0 \leqq K \leqq 1)$ for every function $V(z)$ such that both $U(z) - V(z)$ and $V(z)$ are harmonic in R and superharmonic in $\bar{R}$, $U(z)$ is called a *minimal function.*

Theorem 9.

1) *Let $U(z)$ be a minimal function such that $U_A(z) > 0$ and $U(z) - U_A(z)$ are superharmonic function in $\bar{R}$. Then* $U(z) = \left(\frac{1}{2\pi} \int_{\partial \bar{R}_0} \frac{\partial U(z)}{\partial n} ds\right) N(z, p)$ *$(p \in A)$.*

2) *Every minimal function is a multiple of some $N(z, p)$ $(p \in B_1)$.*

3) *$N(z, p)$ is minimal or not according as $\Psi(p) = 1$ or $= 0$.*

Proof of 1). $U_A(z) = \frac{1}{2\pi} \int_A N(z, p)\, d\mu(p) > 0$ implies $\mu(A) > 0$ and $A \cap B_1 \neq 0$. Hence A has a closed subset A_1 for which $\mu(A_1) > 0$. A_1, being compact, can be covered by a finite number of its closed subsets,

all of them having diameters less than some selected positive number. At least one such subset has a positive μ mass. We select a particular such and call it A_2. By proceeding in this way inductively, it is possible to construct a descending sequence $A_1, A_2, \cdots$, of closed sets of A whose diameters approach zero and each of which has a positive μ mass. Let p be the unique point common to all A_n and B_1. Now since $\mu(A_n)>0$, the integral $\frac{1}{2\pi}\int_{A_n} N(z, p)\,d\mu(A_n)$ extended over A_n instead of A represents a superharmonic function $U_n(z)$ such that mass of $U(z)\geqq$ mass of $U_A(z)\geqq$ mass of $U_n(z)$, because $U(z)-U_A(z)$ is superharmonic in $\bar{R}$, i.e. $U(z)-U_A(z)$ is represented by a positive mass distribution. Hence the minimality of $U(z)$ implies $U_n(z)=C_nU(z)$ $(0<C_n\leqq 1)$. If we write $\mu_n'(e)=\mu\cdot\frac{1}{C_n}$, $\{\mu_n'(e)\}$ has as an weak limit a point mass of amount $\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds$ located at p. Thus we have $U(z)=\left(\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds\right)N(z, p)$ $(p\in A)$.

Proof of 2). Take B as A. Then we have at once 2).

Proof of 3). Suppose $p\in B_1$ and a function $U(z)$ such that both $U(z)$ and $0<N(z, p)-U(z)=V(z)$ are harmonic in R and superharmonic in $\bar{R}$. Then

$$N_p(z, p)=U_p(z)+V_p(z)=U(z)+V(z)=N(z, p)\,,$$

$$U_p(z)\leqq U(z),\ V_p(z)\leqq V(z),\ \text{whence}\ U_p(z)=U(z)\ \text{and}\ V_p(z)=V(z)\,.$$

Hence by 1) of Theorem 5 $U(z)=U_p(z)=K_1V(z, p)$ and $V(z)=V_p(z)=K_2N(z, p)$. Thus $N(z, p)$ is minimal.

Next, suppose that $p\in B_0$ and $N(z, p)$ is minimal. Then $N(z, p)$ is representable by 3) of Theorem 8 by a mass distribution on B_1, that is $N(z, p)=\int_{B_1} N(z, p)\,d\mu(p)$. If μ is a point mass at $q\in B_1$, $N(z, p)=N(z, q)$. This implies $p=q\in B_1$. This is absurd. Hence μ is not a point mass. As 1) of this Theorem we can select a decesending sequence of closed subsets $\{A_n\}$ of B_1 such that $\mu(A_n)>0$ and diameters of $\{A_n\}$ tend to zero as $n\to\infty$. Then the restriction of μ mass on A_n represents a superharmonic function $V_n(z)$ such that $N(z, p)-V_n(z)$ is superharmonic in $\bar{R}$. Hence as 1) we have $N(z, p)=N(z, p^*)$, i.e. $p^*=p$, where $p^*=\bigcap_{n>0}A_n\subset B_1$. This contradicts $p\in B_0$. Hence $N(z, p)$ is non-minial.

By preceeding paragraphs we have the shema as follows:

Ideal boundary point $\begin{cases}\nearrow \text{Regular I.B.P} \longrightarrow B_0 \text{ (non-minim point)} \\ \searrow \text{Singular I.B.P} \longrightarrow B_1 \text{ (minimal point)}\end{cases}$

We see easily that if $R \notin O_{AD}$,[5] R has no singular ideal boundary point and if R is a Riemann surface of finite connectivity, R has no point of B_0.

In what follow, we shall prove useful properties of points of $R+B_1$.

Theorem 10.

1) *Let* $V_m(p)=E[z\in R:\ N(z,p)\geqq m]$ *and* $v_n(p)=E\left[z\in \bar{R}:\ \delta(z,p)\leqq\frac{1}{n}\right]$ *and suppose* $p\in R+B_1$. *Then*

$$V_{V_m(p)}(z,p)=N(z,p) \text{ for very } m \text{ less than } M^*=\sup_{z\in R}N(z,p).$$

Hence $N(z,p)=m\omega_{V_m(p)}(z)$.

2) *For every* $V_m(p)$ $p\in R+B_1$ *there exists a number* n *such that*

$$V_m(p)\supset(R\cap v_n(p))\,.$$

Proof. Since $N(z,p)$ $p\in R$ has the minimal $*$–Dirichlet integral over R, 1) is clear for $p\in R$ and since $N(z,p)$ has its pole at p, 2) is also evident for $p\in R$. Hence we have only to prove for $N(z,p)$ $p\in B_1$.

Proof of 1). First we remark that $p\in B_1$ and $\omega_p(z)=0$ imply $\sup_{z\in R}N(z,p)=M^*=\infty$. In fact, suppose $N(z,p)\leqq M<\infty$ and $\omega_p(z)=0$. Then $N_p(z,p)\leqq M\omega_p(z)=0$, whence $p\in B_0$.

Therefore we shall prove 1) in two cases as follows:

Case 1. $p\in B_1$, $\omega_p(z)=0$ *and* $\sup_{z\in R}N(z,p)=\infty$.

Case 2. $p\in B_1$ *and* $\omega_p(z)>0$.

Case 1. $p\in B_1$, $\omega_p(z)=0$ *and* $\sup_{z\in R}N(z,p)=\infty$. Put $\lim_{n=\infty}N_{v_n(p)-V_m(p)}(z,p)=N'(z,p)$. Then, since $v_n(p)\supset v_n(p)-V_m(p)$, $N'(z,p)$ has no mass excep p. Hence $N'(z,p)=KN(z,p)$ $(0\leqq K<1)$. But $\sup_{z\in R}N(z,p)=\infty$ and $\sup_{z\in R}N'(z,p)\leqq m$ implies $N'(z,p)=0$. On the other hand, $N(z,p)=N_p(z,p)\leqq\lim_{n=\infty}N_{v_n(p)\cap V_m(p)}(z,p)+N'(z,p)\leqq N(z,p)$. Therefore

$$N(z,p)\geqq N_{V_m(p)}(z,p)\geqq\lim_{n=\infty}N_{v_n(p)\cap V_m(p)}(z,p)\geqq N(z,p)\,,$$

whence $N(z,p)=N_{V_m(p)}(z,p)$.

Case 2. $p\in B_1$ *and* $\omega_p(z)>0$. In this case $N(z,p)=K\omega_p(z)$. Hence our assertion is evident.

---

5) O_{AD} is the class of Riemann surfaces on which no non constant Dirichlet Bounded analytic function exists.

Proof of 2). Since $N_{V_m(p)}(z, p)=N(z, p)$ has M.D.I over $R-V_m(p)$, $\frac{N(z, p)}{m}$ can be considerd as the equilibrium potential of $V_m(p)$. Hence we can suppose by 1) of Theorem 1 that $V_m(p)$ is regular, that is, $\int_{\partial V_m(p)} \frac{\partial N(z, p)}{\partial n} ds=2\pi$.

Let q be a point R not contained in $V_m(p)$. Let $N_n(z, p)$ be a harmonic function in $R_n-R_0-V_m(p)$ such that $N_n(z, p)=0$ on ∂R_0, $N_n(z, p)=m$ on $\partial V_m(p)$ and $\frac{\partial N_n(z, p)}{\partial n}=0$ on $\partial R_n-V_m(p)$. Let $N_n(z, q)$ be a function in R_n-R_0 such that $N_n(z, q)=0$ on ∂R_0, $\frac{\partial N_n(z, q)}{\partial n}=0$ on ∂R_n and $N_n(z, q)$ is harmonic in R_n-R_0 except q where $N_n(z, q)$ has a logarithmic singularity. Then clearly $\lim_{n=\infty} N_n(z, p)=N(z, p)$, because $\frac{N(z, p)}{m}=\omega_{V_m(p)}(z)$. $N_n(z, q)$ converges to a function $N(z, q)$.

By Green's formula

$$\int_{\partial V_m(p)\cap(R_n-R_0)} N_n(z, q)\frac{\partial N_n(z, p)}{\partial n} ds=2\pi N_n(q, p) .$$

Since $V_m(p)$ is regular and $N_n(z, q)$ is uniformly bounded on $\partial V_m(p)$, we have by letting $n\to\infty$

$$\frac{1}{2\pi}\lim_{n=\infty}\int_{\partial V_m(p)\cap(R_n-R_0)} N_n(z, p)\frac{\partial N_n(z, p)}{\partial n} ds=\frac{1}{2\pi}\int_{\partial V_m(p)} N(z, q)\frac{\partial N(z, p)}{\partial n} ds$$
$$=N(q, p) . \qquad (5)$$

Assume that 2) is fales. Then there exists a sequence of point $\{q_i\}$ such that $q_i\notin V_m(p)$ and $\lim_{i=\infty}\delta(q_i, p)=0$. If $M^*=\infty$ (resp. $M^*<\infty$), let $m'=2m$ $\left(\text{resp. } m'=m^*: M^*-\frac{\delta}{2}>m^*>m+\frac{\delta}{2}, \text{ where } \delta=\frac{M^*-m}{2}\right)$ and suppose that $V_{m'}(p)$ is regular. Then $V_m(p)\supset V_{m'}(p)\not\ni q_i$. Since $\int_{\partial V_{m'}(p)} \frac{\partial N(z, p)}{\partial n} ds=2\pi$, there exists a number n_0 such that

$$\int_{\partial V_{m'}(p)\cap(R_n-R_0)} \frac{\partial N(z, p)}{\partial n} ds>\frac{3\pi}{2}\left(\text{resp. } 2\pi-\varepsilon_0, \text{ where } 0<\varepsilon_0<\frac{\pi\delta}{2\left(m+\frac{\delta}{4}\right)}\right)$$
$$\text{for } n\geqq n_0 ,$$

Now by 5)

$$\int_{\partial V_{m'}(p)\cap(R_{n_0}-R)} N(z, q_i)\frac{\partial N(z, p)}{\partial n} ds<\int_{\partial V_{m'}(p)} N(z, q_i)\frac{\partial N(z, p)}{\partial n} ds=N(q_i, p)<m .$$

Hence there exists at least one point z_i on $\partial V_{m'}(p)\cap(R_{n_0}-R_0)$ such that $N(z, q_i)<\frac{4m}{3}\left(\text{resp.}<m\left(\frac{2\pi}{2\pi-\varepsilon_0}\right)\leqq m+\frac{\delta}{4}\right)$. Let i tend ∞. Then we

have $N(z_0, p) < \frac{4m}{3}\left(\text{resp.} < m + \frac{\delta}{4}\right)$, where z_0 is one of the limiting points of $\{z_i\}$. This contradicts $N(z_0, p) = m'$. Hence we have 2).

11. The $*$-Green's Function $N(z, q)$ in $\bar{R}$.

We give definition of $N(p, q)$ in three cases as follows:

Case 1. $N(p, q)$ *when* p *or* $q \in R$.

Case 2. $N(p, q)$ *for* $p \in (R + B_1)$ *and* $q \in \bar{R}$.

Case 3. $N(p, q)$ *for* $p \in B_0$ *and* $q \in \bar{R}$.

Definition of $N(p, q)$ in case 1: p or q *is contained* in R. If two points p and q are contained in R, we have by definition $N_n(p, q) = N_n(q, p)$, where $N_n(z, p)$ and $N_n(z, q)$ are $*$-Green's functions of $R_n - R_0$ with poles at p and q respectively. Hence, by letting $n \to \infty$, $N(p, q) = N(q, p)$. Next, suppose $p \in B$ and $q \in R$. Let $\{p_i\}$ be one of fundamental sequences determining p. Then, since $N(p_i, q) = N(q, p_i)$ and since $N(z, p_i)$ converges to $N(z, p)$ uniformly in every compact set of R, $N(p_i, q)$ has a limit denoted by $N(p, q)$ as $p_i \to p$. More generally, suppose that a sequence $\{p_i\}$ of $\bar{R}$ tends to p with respect to δ-metric and that q belongs to R. Then we have $N(q, p) = \lim_{i=\infty} N(q, p_i) = \lim_{i=\infty} N(p_i, q)$. Hence $N(z, q)$ $(q \in R)$ has a limit when z tends to $p \in \bar{R}$. In this case we define the value of $N(z, q)$ at p by this limit denoted by $N(p, q)$. Thus we have the following

Lemma 1. *If at least one of two points* p *and* q *is contained in* R, *then*

$$N(p, q) = N(q, p) .$$

$N(z, q)$ is defined in $\bar{R}$ for $q \in R$ but $N(z, q)$ has been defined only in R for $q \in B$. In the sequel, we shall define $N(z, q)$ in $\bar{R}$ for $q \in B$. At first, consider case 2. For this purpose, we shall prove the following

Lemma. 2. *Let* $V_m(p)$ *be the set* $E[z \in R: N(z, p) \geqq m]$ *for* $p \in B_1$. *Then* $V_m(p)$ *may consist of at most enumerably infinite number of domains* D_l $(l = 1, 2, \cdots)$. *Then*

1) *The Dirichlet integral of* $N(z, p)$ *taken over* $R - V_m(p)$ *is* $2\pi m$ *for every* $m < M^* = \sup_{z \in R} N(z, p)$.

2) *Let* D_l *be a component of* $V_m(p)$. *Then* D_l *contains a subset* D_l' *of* $V_{m'}(p)$ *for* $m' : m < m' < M^*$.

3) *For D_l of regular domain $V_m(p)$, the Dirichlet integral of $N(z, p)$ taken over $D_l - D_l'$ is $2\pi(m'-m)\int_{\partial D_l}\frac{\partial N(z,p)}{\partial n}ds$ and*

$$\lim_{n=\infty}\int_{\partial D_l\cap(R_n-R_0)}\frac{\partial U_n(z)}{\partial n}ds=\lim_{n=\infty}\int_{\partial D_l'\cap(R_n-R_0)}\frac{\partial U_n(z)}{\partial n}ds=\int_{\partial D_l}\lim_{n=\infty}\frac{\partial U_n(z)}{\partial n}ds$$
$$=\int_{\partial D_l'}\lim_{n=\infty}\frac{\partial U_n(z)}{\partial n}ds=\int_{\partial D_l}\frac{\partial N(z,p)}{\partial n}ds,$$

where $U_n(z)$ is a harmonic function in $(D_l-D_l')\cap(R_n-R_0)$ such that $U_n(z)=m$ on ∂D_l, $U_n(z)=m'$ on $\partial D_l'$ and $\frac{\partial U_n(z)}{\partial n}=0$ on $\partial R_n\cap(D_l-D_l')$.

Proof of 1). $p\in B_1$ implies by 1) of Theorem 10, that $N_{V_m(p)}(z, p)=N(z, p)$. Hence $\frac{N(z,p)}{m}$ is the equilibrium potential of $V_m(p)$. Therefore, $N(z, p)=\lim_{n=\infty}U_n'(z)$, where $U_n'(z)$ is a harmonic function in $R_n-R_0-V_m(p)$ such that $U_n'(z)=0$ on ∂R_0, $U_n'(z)=m$ on $\partial V_m(p)$ and $\frac{\partial U_n'(z)}{\partial n}=0$ on $\partial R_0\cap(R-V_m(p))$. The Dirichlet integral of $U_n'(z)$ over $R_n-R_0-V_m(p)$ is $m\int_{\partial V_m(p)\cap(R_n-R_0)}\frac{\partial U_n(z)}{\partial n}ds$. Since $D(U_n'(z))$ is increasing with respect to n and $U_n'(z)$ tends to $N(z, p)$ as $n\to\infty$,

$$\lim_{n=\infty}D_{R_n-R_0-V_m(p)}(U_n'(z))=D_{R-V_m(p)}(N(z,p))=\lim_{n=\infty}m\int_{\partial R_0}\frac{\partial U_n'(z)}{\partial n}ds$$
$$=m\int_{\partial R_0}\frac{\partial N(z,p)}{\partial n}ds=2\pi m.$$

Proof of 2). Assume that D_l has no point of $V_{m'}(p)$ $(m'>m)$. Put $N'(z, p)\equiv m$ in D_l and $N'(z, p)=N(z, p)$ for $z\in(R-D_l)$. Then $D(N'(z, p))<D(N(z, p))$. This contradicts that $\frac{N(z,p)}{m}$ is the equilibrium potential of $V_m(p)$. Hence we have 2).

Proof of 3). Since $\frac{N(z,p)}{m}$ can be considered as the equilibrium potential of $V_m(p)$, $N(z, p)$ has M.D.I over $V_m(p)-V_m'(p)$ among all functions having the boundary values m on $\partial V_m(p)$ and m' on $\partial V_{m'}(p)$ respectively, whence $N(z, p)$ has also M.D.I over D_l-D_l' among all functions with values m on ∂D_l and m' on $\partial D_l'$. Hence $U_n(z)\to N(z, p)$ as $n\to\infty$. Since $D(U_n(z))$ is increasing with respect to n and by Fatou's lemma, we have

$$D_{D_l-D_l'}(N(z,p))=\lim_{n=\infty}D_{(D_l-D_l')\cap(R_n-R_0)}(U_n(z))=(m'-m)\lim_{n=\infty}\int_{\partial D_l\cap(R_n-R_0)}\frac{\partial U_n(z)}{\partial n}ds$$

$$= (m'-m)\lim_{n=\infty}\int_{\partial D_l'\cap(R_n-R_0)}\frac{\partial U_n(z)}{\partial n}ds \geqq (m'-m)\int_{\partial D_l}\lim_{n=\infty}\frac{\partial U_n(z)}{\partial n}ds$$

$$= (m'-m)\int_{\partial D_l}\frac{\partial N(z,p)}{\partial n}ds. \tag{6}$$

$$D_{D_l-D_l'}(N(z,p)) \geqq (m'-m)\int_{\partial D_l}\lim_{n=\infty}\frac{\partial N(z,p)}{\partial n}ds = (m'-m)\int_{\partial D_l'}\frac{\partial N(z,p)}{\partial n}ds.$$

On the other hand, by 1) and by the regularity of $V_m(p)$ and $V_{m'}(p)$

$$\sum_l D_{D_l-D_l'}(N(z,p)) = D_{V_m(p)-V_{m'}(p)}(N(z,p)) = 2\pi(m'-m)$$

$$= (m'-m)\int_{\partial R_0}\frac{\partial N(z,p)}{\partial n}ds = (m'-m)\int_{\partial V_m(p)}\frac{\partial N(z,p)}{\partial n}ds$$

$$= (m'-m)\int_{\partial V_{m'}(p)}\frac{\partial N(z,p)}{\partial n}ds = (m-m')\sum_l\int_{\partial D_l}\frac{\partial N(z,p)}{\partial n}ds$$

$$= (m'-m)\sum_l\int_{\partial D_l'}\frac{\partial N(z,p)}{\partial n}ds. \tag{7}$$

If $D_{D_l-D_l'}(N(z,p)) > (m'-m)\int_{\partial D_l}\frac{\partial N(z,p)}{\partial n}ds$ or $(m'-m)\int_{\partial D_l'}\frac{\partial N(z,p)}{\partial n}ds$ for at least one D_l or D_l' respectively, (6) will be a contradiction. Hence

$$D_{D_l-D_l'}(N(z,p)) = (m'-m)\int_{\partial D_l}\frac{\partial N(z,p)}{\partial n}ds = (m'-m)\int_{\partial D_l'}\frac{\partial N(z,p)}{\partial n}ds$$

for every D_l and D_l'. Therefore

$$D_{D_l-D_l'}(N(z,p)) = \lim_{n=\infty} D_{D_l-D_l'}(U_n(z)) = (m'-m)\lim_{n=\infty}\int_{\partial D_l\cap(R_n-R_0)}\frac{\partial U_n(z)}{\partial n}ds$$

$$= (m'-m)\lim_{n=\infty}\int_{\partial D_l'\cap(R_n-R_0)}\frac{\partial U_n(z)}{\partial n}ds = (m'-m)\int_{\partial D_l}\frac{\partial N(z,p)}{\partial n}ds$$

$$= (m'-m)\int_{\partial D_l}\lim_{n=\infty}\frac{\partial U_n(z)}{\partial n}ds = (m'-m)\int_{\partial D_l'}\lim_{n=\infty}\frac{\partial U_n(z)}{\partial n}ds.$$

Thus we have 3).

Lemma. 3. *Suppose $p\in B_1$ and $q\in\bar{R}$. Let $V_m(p)$ and $V_{m'}(p)$ be regular domains with m and m' such that $\sup_{z\in R} N(z,p) > m' > m$, i.e. $V_m(p) \supset V_{m'}(p)$. Then*

$$2\pi N^{V_{m'}(p)}(p,q) = \int_{\partial V_{m'}(p)} N(z,q)\frac{\partial N(z,p)}{\partial n}ds \geqq \int_{\partial V_m(p)} N(z,q)\frac{\partial N(z,p)}{\partial n}ds$$

$$= 2\pi N^{V_m(p)}(p,q).$$

Proof. Let D be one of D_l which is a component of $V_m(p)$ and D' be the set of $V_{m'}(p)$ contained in D. Let $N_n^D(\zeta,z)$ be the $*$–Green's func-

tion of $D\cap(R_n-R)$, that is, $N_n^D(\zeta, z)=0$ on $\partial D\cap(R_n-R_0)$, $\frac{\partial N_n^D(\zeta, z)}{\partial n}ds=0$ on ∂R_n-D and $N_n^D(\zeta, z)$ is harmonic in $D\cap(R_n-R_0)$ except a logarithmic singularity at z. Then for given n_0 there exist constants L and n_1 such that $N_n^D(\zeta, z)\leqq L$ in $D\cap(R_{n_0}-R_0)$ for $n\geqq n_1$.

Let $U_n(\zeta)$ be the function defined in 3) of lemma e, i.e. $U_n(z)=m$ on ∂D, $U_n(\zeta)=m'$ on $\partial D'$ and $\frac{\partial U_n(\zeta)}{\partial n}=0$ on $\partial R_n\cap(D-D')$. Then, since $U_n(\zeta)-m=m'-m$ on $\partial D'$ and $\frac{\partial N_n^D(\zeta, z)}{\partial n}ds=\frac{\partial U_n(\zeta)}{\partial n}=0$ on $\partial R_n\cap(D-D')$, there exist suitable constants δ, and n_1' by the maximum principle such that

$$N_n^D(\zeta, z)<\frac{L}{\delta}(U_n(\zeta)-m) \quad \text{in} \quad D\subset(R_{n_0}-R_0) \quad \text{for } n\geqq n_1'.$$

Hence

$$0\leqq\frac{\partial N_n^D(\zeta, z)}{\partial n}<\frac{L}{\delta}\frac{\partial U_n(\zeta)}{\partial n} \quad \text{on} \quad \partial D\cap(R_{n_0}-R_0) \quad \text{for } n\geqq n_1'.$$

Therefore by 3) of lemma 2

$$\lim_{n=\infty}\int_{\partial D\cap(R_n-R_0)}\frac{\partial N_n^D(\zeta, z)}{\partial n}ds=\int_{\partial D}\lim_{n=\infty}\frac{\partial N_n^D(\zeta, z)}{\partial n}ds. \tag{8}$$

Suppose $q\in R$ and let $N_{D,n}(z, q)$ be a harmonic function in $D\cap(R_n-R_0)$ such that $N_{D,n}(z, q)=N(z, q)$ $\partial D\cap(R_n-R_0)+\partial R_0$ and $\frac{\partial N_{D,n}(z, q)}{\partial n}=0$ on $\partial R_n\cap D$. Then $N_{D,n}(z, q)$ converges (converges in mean) to a function $N_D(z, q)$ which is called *the solution of the $*$-Dirichlet problem with boundary value $N(z, q)$ on ∂D.*

Since $N(z, q)$ is uniformly bounded on $\partial D\cap(R-R_{n''})$, where n'' is a suitable number, it can be proved in the same manner as Theorem 2, by (8) that

$$\lim_{n=\infty}N_{D,n}(z, q)=\lim_{n=\infty}\frac{1}{2\pi}\int_{\partial D\cap(R_n-R_0)}N(\zeta, q)\frac{\partial N_n^D(\zeta, z)}{\partial n}ds$$

$$=\frac{1}{2\pi}\int_{\partial D}N(\zeta, q)\frac{\partial N^D(\zeta, z)}{\partial n}ds=N_D(z, q).$$

where $N^D(\zeta, z)=\lim_{n=\infty}N_n^D(\zeta, z)$.

Now, since $N(z, q)$ has M.D.I or minimal $*$-Dirichlet integral over D according as $q\notin D$ or $q\in D$, $N(z, q)=\lim_{n=\infty}N_n'(z, q)$, where $N_n'(z, q)$ is a harmonic function in $D\cap(R_n-R_0)$ or harmonic except a logarithmic singularity at q such that $N_n'(z, q)=N(z, q)$ on $\partial D\cap(R_n-R_0)$ and $\frac{\partial N_n'(z, q)}{\partial n}$

$=0$ on $\partial R_n \cap D$. Hence $N_D(z, q) = \lim_{n=\infty} N_{D,n}(z, q) = \lim_{n=\infty} N_n'(z, q) = N(z, q)$ or $< N(z, q) = \lim_{n=\infty} N_n'(z, q)$ according as $q \notin D$ or not. Thus

$$N(z, q) \geqq N_D(z, q) = \frac{1}{2\pi} \int_{\partial D} N(\zeta, q) \frac{\partial N^D(\zeta, z)}{\partial n} ds . \tag{9}$$

Let $\{q_i\}$ be a fundamental sequence determining a point $q \in B$. Then, since $N(\zeta, q_i)$ tends to $N(\zeta, q)$ as $i \to \infty$, by Fatou's lemma and by (9)

$$N(z, q) \geqq N_D(z, q) = \frac{1}{2\pi} \int_{\partial D} N(\zeta, q) \frac{\partial N^D(\zeta, z)}{\partial n} ds , \tag{9'}$$

where $N_D(z, q)$ is the solution of $*$–Dirichlet problem in D with boundary value $N(z, q)$ on ∂D.

$N^M_{D,n}(z, q)$ be a harmonic function in $D \cap (R_n - R_0)$ such that $N^M_{D,n}(z, q) = N^M(z, q)$ on $\partial R_0 + \partial D \cap (R_n - R_0)$ and $\frac{\partial N^M_{D,n}(z, q)}{\partial n} = 0$ $\partial R_n \cap D$. Then $N^M_{D,n}(z, q)$ converges to a function $N^M_D(z, q)$ as $n \to \infty$. Clearly, as in case of $N_D(z, q)$, $N^M_D(z, q)$ is given by

$$N^M_D(z, q) = \frac{1}{2\pi} \int_{\partial D} N^M(\zeta, z) \frac{\partial N^D(\zeta, z)}{\partial n} ds ,$$

i. e. $N^M_D(z, q)$ is the solution of $*$–Dirichlet problem in D with boundary value $N^M(z, q)$, whence $\lim_{M=\infty} N^M_D(z, q) = N_D(z, q)$.

The Dirichlet integral $\sum_l D_{D_l}(N^M_{D_l,n}(z, q)) \leqq \sum_l D_{D_l}(N^M(z, q)) \leqq 2\pi M$. Hence by letting $n \to \infty$ $\sum_l D_{D_l}(N^M_D(z, q)) \leqq 2\pi M$. For simplicity, we denote by $N^M_{V_m(p)}(z, q)$ the function being equal to $N^M_{D_l}(z, q)$ (solution of $*$–Dirichlet problem in D_l) in every domain D_l with boundary value $N^M(z, q)$.

Next, as in 3) of Lemma 2, it is proved that $N(z, p) = \lim_{n=\infty} U_n(z)$ in $V_m(p) - V_{m'}(p)$, $\frac{\partial U_n(z)}{\partial n} \to \frac{\partial N(z, p)}{\partial n}$ on $\partial V_m(p) + \partial V_{m'}(p)$

$$\int_{\partial V_m(p)} \frac{\partial N(z, p)}{\partial n} ds = \lim_{n=\infty} \int_{\partial V_m(p)} \frac{\partial U_n(z)}{\partial n} ds \quad \text{and}$$

$$\int_{\partial V_{m'}(p)} \frac{\partial N(z, p)}{\partial n} ds = \lim_{n=\infty} \int_{\partial V_{m'}(p)} \frac{\partial U_n(z)}{\partial n} ds , \tag{10}$$

where $U_n(z)$ is a harmonic function in $(V_m(p) - V_{m'}(p)) \cap (R_n - R_0)$ such that $U_n(z) = m$ on $\partial V_m(p)$, $U_n(z) = m'$ on $\partial V_{m'}(p)$ and $\frac{\partial U_n(z)}{\partial n} = 0$ on $\partial R_n \cap (V_m(p) - V_{m'}(p))$.

Let $N^M_{V_m(p),n}(z,q)=N^M_{D_l,n}(z,q)$ in every domain $D_l\cap(R_n-R_0)$. Then we have by Green's formula

$$\int_{\partial V_m(p)\cap(R_n-R_0)} N^M_{V_m(p),n}(z,q)\frac{\partial U_n(z)}{\partial n}ds=\int_{\partial V_{m'}(p)\cap(R_n-R_0)} N^M_{V_m(p),n}(z,q)\frac{\partial U_n(z)}{\partial n}ds,$$

because $U_n(z)=m$ and m' on $\partial V_m(p)$ and $\partial V_{m'}(p)$ respectively and $\int_{\partial V_m(p)\cap(R_n-R_0)}\frac{\partial N_{V_m(p),n}(z,q)}{\partial n}ds=\int_{\partial R_n\cap V_m(p)}\frac{\partial N^M_{V_m(p),n}(z,q)}{\partial n}ds=0$ and $\int_{\partial V_{m'}(p)\cap(R_n-R_0)}\frac{\partial N^M_{V_m(p),n}(z,q)}{\partial n}ds=\int_{\partial R_n V_{m'}(p)}\frac{\partial N^M_{V_m(p),n}(z,q)}{\partial n}ds=0$. Let $n\to\infty$. Then by (10)

$$\int_{\partial V_m(p)} N^M_{V_m(p)}(z,q)\frac{\partial N(z,p)}{\partial n}ds=\int_{\partial V_{m'}(p)} N^M_{V_m(p)}(z,q)\frac{\partial N(z,p)}{\partial n}ds. \tag{11}$$

Therefore by letting $M\to M^*$, by (9′) and (11) we have

$$2\pi N^{V_{m'}(p)}(p,q)=\int_{\partial V_{m'}(p)} N(z,p)\frac{\partial N(z,p)}{\partial n}ds$$

$$\geqq\int_{\partial V_m(p)} N(z,p)\frac{\partial N(z,p)}{\partial n}ds=2\pi N^{V_m(p)}(p,q).$$

Definition of $N(p,q)$ in Case 2: for $p\in R+B_1$ and $q\in\bar{R}$. Since $N^{V_m(p)}(p,q)$ is increasing with respect to m, $N^{V_m(p)}(p,q)$ has a limit denoted by $N(p,q)$ as $m\uparrow M^*=\sup_{z\in\bar{R}}N(z,p)$. We define the value of $N(z,q)$ at $p\in B_1$ by this limit. It is easily proved that, in case 1) this definition of $N(p,q)$ coincides with what has been given previously. In fact, it is evident that $N(p,q)=\frac{1}{2\pi}\int_{\partial V_m(p)}N(z,p)\frac{\partial N(z,p)}{\partial n}ds$ for $p\in R$ and $V_m(p)\ni q$ and that, by (5) $N^{V_m(p)}(p,q)=\frac{1}{2\pi}\int_{\partial V_m(p)}N(z,q)\frac{\partial N(z,p)}{\partial n}ds$ $=N(q,p)=\lim_{i=\infty}N(q,p_i)=\lim_{i=\infty}N(p_i,q)=N(p,q)$ for $p\in B$ and $q\in R$, where $\{p_i\}$ is a fundamental sequence determining p.

Remark. *Let $V_m(p)$ be a regular domain and let $\{V_{m_i}(p)\}$ be a sequence of regular domain with $m_i\uparrow m$. Then $N^{V_m(p)}(p,q)=\lim_{i=\infty}N^{V_{m_i}(p)}(p,q)$.*

In fact, there exists a number n, for any given positive number ε, such that

$$\int_{\partial V_m(p)\cap(R_n-R_0)} N(z,q)\frac{\partial N(z,p)}{\partial n}ds\geqq\int_{\partial V_m(p)} N(z,q)\frac{\partial N(z,p)}{\partial n}ds-\varepsilon.$$

On the other hand, suppose $z_i\in\partial V_{m_i}(p)$, $z_0\in\partial V_m(p)$ and $z_i\to z$. Then

$\frac{\partial N(z_i, p)}{\partial n} ds \to \frac{\partial N(z_0, p)}{\partial n} ds$ and $N(z_i, q) \to N(z_0, q)$, hence

$$\lim_{i=\infty} \int_{\partial V_{m_i}(p)} N(z, q) \frac{\partial N(z, p)}{\partial n} ds \geqq \lim_{i=\infty} \int_{\partial V_{m_i}(p) \cap (R_n - R_0)} N(z, q) \frac{\partial N(z, p)}{\partial n} ds$$

$$= \int_{\partial V_m(p) \cap (R_n - R_0)} N(z, q) \frac{\partial N(z, p)}{\partial n} ds \geqq \int_{\partial V_m} N(z, p) \frac{\partial N(z, p)}{\partial n} ds - \varepsilon .$$

By letting $\varepsilon \to 0$, $\lim_{i=\infty} N^{V_{m_i}(p)}(p, q) \geqq N^{V_m(p)}(p, q)$. Next, $m_i < m$ implies $N^{V_{m_i}(p)}(p, q) \leqq N^{V_m(p)}(p, q)$ and $\overline{\lim_{i=\infty}} N^{V_{m_i}(p)}(p, q) \leqq N^{V_m(p)}(p, q)$. Thus we have $N^{V_m(p)}(p, q) = \lim_{i=\infty} N^{V_{m_i}(p)}(p, q)$.

We define $N^{V_m(p)}(p, q)$ for any domain $V_m(p)$ by $\lim_{i=\infty} N^{V_{m_i}(p)}(z, p)$ as above. This definition coincides with what has been defined previousely for regular domain $V_n(p)$. Hence $N^{V_m(p)}(p, q)$ is defined for every $m < \sup_{z \in R} N(z, p)$.

Definition of Superharmonicity at a point $p \in R + B_1$. Suppose a function $U(z)$ in $\bar{R}$. If $U(p) \geqq \frac{1}{2\pi} \int_{\partial V_m(p)} U(z) \frac{\partial N(z, p)}{\partial n} ds$ holds for regular $V_m(p)$ of $N(z, p)$, we say that $U(z)$ is *superharmonic in the weak sense at a point* p. Thus we shall have the following

Theorem 11.

1). $N(p, p) = \sup_{z \in R} N(z, p)$ *for* $p \in R + B_1$.

2). $N(z, q) (q \in \bar{R})$ *is* δ*-lower semicontinuous in* $R + B_1$.

3). $N(z, q)$ *is superharmonic in the weak sense at every point of* $R + B_1$.

4). $N(p, q) = N(q, p)$ *for two points* p *and* q *belonging to* $R + B_1$.

Proof. 1) and 3) are clear by definition.

Proof of 2). Let $\{p_i\}$ be a sequence of points of $R + B_1$ tending to p. Since by the above remark $N^{V_m(p)}(p, q) = \lim_{m \to m'} N^{V_{m'}(p)}(p, q) (m' \uparrow m)$, there exists a number m', for any given positive number ε, such that $V_{m'}(p)$ is regular and $N^{V_m(p)}(p, q) \leqq N^{V_{m'}(p)}(p, q) + \varepsilon$. Hence there exists a number n_0 such that

$$N^{V_m(p)}(p, q) \leqq \frac{1}{2\pi} \int_{\partial V_{m'}(p) \cap (R_n - R_0)} N(z, q) \frac{\partial N(z, p)}{\partial n} ds + 2\varepsilon \qquad \text{for} \quad n \geqq n_0 .$$

Let $V_{m''}(p_i)$ be a sequence of regular domains such that $p_i \to p$ and $m'' \uparrow m$. Replace $G_{V_m(p)}(p, q)$ by $N^{V_m(p)}(p, q)$ in 3) of Theorem 1 of Part I.

Then $N(\alpha_i, q)$ on $\partial V_{m''}(p_i)$ tends to $N(\alpha, q)$ on $\partial V_m(p)$ and $\frac{\partial N(\alpha_i, p)}{\partial n} ds$ tends to $\frac{\partial N(\alpha, p)}{\partial n} ds$, whence $\varlimsup_{i=\infty} N^{V_{m''}(p)}(p_i, q) \geqq \varlimsup_{i=\infty} N^{V_{m''}(p_i)}(p_i, q) \geqq N^{V_m(p)}(p, q) - 2\varepsilon$ and $\varlimsup_{i=\infty} N(p_i, q) \geqq N(p, q)$. Hence we have 2).

Proof of 4). Replace $G_{V_m(q)}(p, q)$ and $G_{V_n(p)}(q, p)$ by $N^{V_m(p)}(p, q)$ and $N^{V_n(q)}(q, p)$ respectively and consider that $\{V_m(p)\}$ clusters at B as $m \uparrow M^* = \sup_{z \in R} N(z, p)$. Then we at once 4), where $V_m(p)$ and $V_n(q)$ are regular. Now we define $N(z, q)$ not only in $R + B_1$ but also in B_0.

Definition of $N(z, q)$ in Case 3: for $p \in B_0$ and $q \in \bar{R}$. At first, if $p \in B_0$, $N(z, p)$ is represented by $\int_{B_1} N(z, p_\alpha)\, d\mu(p_\alpha) (p_\alpha \in B_1)$ by Theorem 8 for $z \in R$, where $\mu(p_\alpha)$ is an weak limit and its uniqueness cannot be proved by the present author.

Let $p_{\alpha_i} (\in \bar{R})$ $(i = 1, 2, \cdots)$ tend to p_α with respect to δ-metric. Then, since $N(z, p_{\alpha_i}) \to N(z, p_\alpha)$ on $\partial V_m(q)$ for $q \in R + B_1$. Hence, by Fatou's lemma

$$N^{V_m(q)}(q, p_\alpha) = \frac{1}{2\pi} \int_{\partial V_m(q)} N(z, p_\alpha) \frac{\partial N(z, q)}{\partial n} ds$$

$$\leqq \varliminf_{i=\infty} \int_{\partial V_m(q)} N(z, p_{\alpha_i}) \frac{\partial N(z, q)}{\partial n} ds = \varliminf_{i=\infty} N^{V_m(q)}(q, p_{\alpha_i}) .$$

Hence $N^{V_m(q)}(q, p_\alpha)$ is lower semicontinuous with respect to p_α for fixed $q \in R + B_1$. Since $N^{V_m(q)}(q, p) \uparrow N(q, p)$ at every point p, $\lim_{m \to M^*} \int N^{V_m(q)}(q, p_\alpha)\, d\mu(p_\alpha) = \int N(q, p_\alpha)\, d\mu(p_\alpha)$ $(M^* = \sup_{z \in R} (z, q))$, whence

$$N(q, p) = \lim_{m \to M^*} N^{V_m(q)}(q, p) = \lim_{m \to M^*} \frac{1}{2\pi} \int_{\partial V_m(q)} \left(\int_{B_1} N(z, p) \frac{\partial N(z, q)}{\partial n} d\mu(p_\alpha) \right) ds$$

$$= \frac{1}{2\pi} \int_{B_1} \left(\lim_{m \to M^*} \int_{\partial V_m(q)} N(z, p_\alpha) \frac{\partial N(z, q)}{\partial n} ds \right) d\mu(p_\alpha) = \int_{B_1} N(q, p_\alpha)\, d\mu(p_\alpha) . \quad (13)$$

Hence the representation

$$N(z, p) = \int_{B_1} N(z, p_\alpha)\, d\mu(p_\alpha) \quad (14)$$

is valid not only in R but also in B_1.

The value of $N(q, p)$ ($q \in R + B_1$ and $p \in B_0$) does not depend on a particular choice of distribution $\mu(p_\alpha)$, because the left hand side of (13) is given by $\lim_{m \to M^*} N^{V_m(q)}(q, p)$, that is $N(q, p)$ depends only on the value of $N(z, p)$ in R. Now (14) means that the potential of a unit mass on $p \in B_0$ has the same behaviour in $R + B_1$ as the potential of mass distribution $\int_{B_1} d\mu(p_\alpha)$. From this point of view, we may consider that a

point $p \in B_0$ is spanned by points $p_\alpha \in B_1$ with weight $\mu(p_\alpha)$. Hence it is natural to define the value of $N(z, q)(q \in \bar{R})$ at $z = p \in B_0$ by

$$\int_{B_1} N(p_\alpha, q)\, d\mu(p_\alpha)\,. \tag{15}$$

we shall prove the following

Theorem 12.

1). *$N(p, q) = N(q, p)$ for $p \in \bar{R}$ and $q \in R + B_1$. Hence $N(q, p)$ and $N(p, q)$ does not depend on a particular choice of distribution $\mu(p_\alpha)$.*

2). *$N(q, z)(q \in R + B_1)$ is δ-lower semicontinuous in $\bar{R}$.*

1'). *$N(p, q) = N(q, p)$ for p and q belonging to $\bar{R}$.*

2') *$N(z, q)(q \in \bar{R})$ is δ-lower semicontinuous in $\bar{R}$.*

Proof of 1). For $p \in R + B_1$ our assertion is evident by 4) of Theorem 11. We show for $p \in B_0$. In this case, since $N(p_\alpha, q) = N(q, p_\alpha)$ by 4) of Theorem 11, we have by (14) and (15)

$$N(q, p) = \int_{B_1} N(q, p_0)\, d\mu(p_\alpha) = \int_{B_1} N(p_\alpha, q)\, d\mu(p_\alpha) = N(p, q)\,.$$

Since $N(q, p)$ does not depend on a particular distribution, $N(p, q)$ also does not depend on it.

Proof of 2). If $p \in R + B_1$, is clear by Theorem 11. Let $\{p_i\}$ be a sequence of points tending to $p \in B_0$. They by 1) of this theorem $N(q, p_i) = N(p_i, q)$ and $N(p, q) = N(p, q)$. On the other hand, by Fatou's lemma $\varliminf_{i=\infty} N^{V_m(q)}(q, p_i) \geqq N^{V_m(q)}(q, p)$, which implies $\varliminf_{i=\infty} N(q, p_i) \geqq N(q, p)$. Hence

$$\varliminf_{i=\infty} N(p_i, q) = \varliminf_{i=\infty} N(q, p_i) \geqq N(q, p) = N(p, q)\,.$$

This completes the proof of 2).

Proof of 1'). If at least one of p and q belongs to $R + B_1$, our assertion is 1). Suppose that both p and q belong to B_0. In this case

$$N(z, p) = \int_{B_1} N(z, p)\, d\mu(p_\alpha) \quad \text{and} \quad N(z, q) = \int_{B_1} N(z, q_\beta)\, d\mu(q_\beta) \quad (p_\alpha \text{ and } q_\beta \in B_1).$$

Hence by (14) and by 1) of the this theorem

$$\begin{aligned} N(q, p) &= \int N(q_\beta, p)\, d\mu(q_\beta) \\ &= \int \left(\int (N(p_\alpha, q_\beta))\, d\mu(p_\alpha) \right) d\mu(q_\beta) = \int N(p, q_\beta)\, d\mu(q_\beta) = N(p, q)\,. \end{aligned}$$

It is proved as in 1) that $N(p, q)$ does not depend on particular distributions

$$\mu(p_\alpha) \qquad \text{and} \qquad \mu(q_\beta)\,.$$

Proof of 2'). Let $\{p_i\}$ be a sequence tending to p. Then for every point q_β, $\varliminf_{i=\infty} N(p_i, q_\beta) \geqq N(p, q_\beta)$, which yields at once by Fatou's lemma

$$\varliminf_{i=\infty} N(p_i, q) = \varliminf_{i=\infty} \int N(p_i, q_\beta)\, d\mu(q_\beta) \geqq \int N(p, q_\beta)\, d\mu(q_\beta) = N(p, q)\,.$$

Remark. Let $U(z)$ be a function given by $\int N(z, p)\, d\mu(p) (\mu > 0)$. Then $U(z)$ is lower semicontinuous in $\bar{R}$.

12. Mass Distributions on $\bar{R}$.

We have seen that $N(z, p)$ has the essential properties of the logarithmic potential: lower semicontinuity on $\bar{R}$, symmetry and superharmonicity in the weak sense on $R+B_1$. But there exists a fatal difference between our space and the euclidean space, that is, in our space there may exist points of B_0 where we cannot distribute any *true mass.* A distribution μ on B_0 may be called a *pseudo distribution* in the sense that $U_{B_0}(z) = 0$ and μ can be replaced, by Theorem 8, by a distribution on B_1, where $U(z) = \int_{B_0} N(z, p)\, d\mu(p)$. In other words, even when B_0 is not empty, B_0 behaves as an empty set for mass distributions.

Mass Distributions on $R+B_1$. Since $N(z, p)$ has the above properties, it is easy to construct the potential theory on $R+B_1$.

The energy integral $I(\mu)$ of a mass distribution μ on a closed subset F of $R+B_1$ is defined as

$$I(\mu) = \int_F \int N(q, p)\, d\mu(p)\, d\mu(q)\,.$$

The *$*$-Capacity* $*\mathrm{Cap}\,(F)$ and the *transfinite diameter* D_F of F are defined as follows: $\dfrac{*\mathrm{Cap}\,(F)}{2\pi}$ is defined as the least upper bound of total mass of μ on F whose potential is not greater than 1 on F.
$D_F = \varlimsup_{n=\infty} {}_nD_F$, where

$$\frac{1}{{}_nD_F} = \frac{1}{2\pi\, {}_nC_2} \left(\inf_{p_i, p_j \in F} \sum_{\substack{i,j \\ i<j \\ i=1}}^{n,n} N(p_i, p_j) \right).$$

We see easily the following

Lemma. $\mathrm{Cap}\,(F) > 0$ *implies* $*\mathrm{Cap}\,(F) > 0$ *for a closed susbet* F *of* $R+B_1$.

In fact, if $\mathrm{Cap}\,(F) > 0$, $\omega_F(z) = {}_F\omega_F(z) > 0$ and $\omega_F(z) = \int_F N(z, p)\, d\mu(p)$. Now the total mass of μ is given by $\dfrac{1}{2\pi} \int_{\partial R_0} \dfrac{\partial \omega_F(z)}{\partial n}\, ds$ and $\omega_F(z) \leqq 1$,

whence $*\mathrm{Cap}\,(F)>0$.

Then we have as in space the following

Theorem 13. *Let F be a closed subset of positive $*$Capacity of $R+B_1$. Then there exists a unit mass distribution μ on F whose energy integral is minimal and its potential $U(z)$ satisfies the following conditions*:

1). *$U(z)$ is a constant C on the kernel of the distribution, whence* $I(\mu)=D(U(z))=2\pi C$.

2). $U(z)=U_F(z)$.

3). *$U(z)=C$ on F except possibly a subset of $*$-Capacity zero of F.*

4). $U(z)=C\omega_F(z)$.

Proof. 1) and 3) can be proved as in space.

Proof of 2). Since $p\in R+B_1$, $N(z,p)=N_{v_m(p)}(z,p)$ for every point of $R+B_1$, where $v_m(p)=E[z\in\bar{R}:\ \delta(z,p)\leqq\frac{1}{m}]$. This implies $N_{F_m}(z,p)=N(z,p)$, where $F_m=E[z\in\bar{R}:\ \delta(z,F)\leqq\frac{1}{m}]$, because $F_n\supset v_m(p)$. Hence we have $U_F(z)=U(z)$.

Proof of 4). Put $U(z)=C\omega^*(z)$. Then by 2) ${}_F(\omega^*(z))=\omega^*(z)$ and not greater than 1 on F. Hence $\omega^*(z)=\lim_{m=\infty}\lim_{n=\infty}\omega^*_{m,n}(z)$, where $\omega^*_{m,n}(z)$ is a harmonic function in $R_n-R_0-F_m$ such that $\omega^*_{m,n}(z)=\omega^*(z)$ on $\partial F_m\cap(R_n-R_0)$, $\omega^*_{m,n}(z)=0$ on ∂R_0 and $\frac{\partial\omega^*_{m,n}(z)}{\partial n}=0$ on ∂R_n-F_m. On the other hand, $\omega_F(z)=\lim_{m=\infty}\lim_{n=\infty}\omega_{m,n}(z)$, where $\omega_{m,n}(z)$ is a harmonic function in $R_n-R_0-F_m$ such that $\omega_{m,n}(z)=1$ on $\partial F_m\cap(R_n-R_0)$, $\omega_{m,n}(z)=0$ on ∂R_0 and $\frac{\partial\omega_{m,n}(z)}{\partial n}=0$ on ∂R_n-F_n. Hence $\omega_{m,n}(z)\geqq\omega^*_{m,n}(z)$, whence by letting $n\to\infty$ and then $m\to\infty$, $\omega_F(z)\geqq\omega^*(z)$. Next, the set $A_\lambda=E[z\in\bar{R}:\ \omega^*(z)\leqq 1-\lambda]\cap F$ is clearly closed by the lower semicontinuity of $\omega^*(z)$. $*\mathrm{Cap}\,(A_\lambda)=0$ implies $\mathrm{Cap}\,(A_\lambda)=0$ by Lemma. Hence $0=\omega_{A_\lambda}(z)=\lim_{m=\infty}\lim_{n=\infty}\omega_{A_\lambda,m,n}(z)$, where $A_{\lambda,m}=E[z\in\bar{R}:\ \delta(z,A_\lambda)\leqq\frac{1}{m}]$ and $\omega_{A_\lambda,m,n}(z)$ is a harmonic function in $R_n-R_0-A_{\lambda,m}$ such that $\omega_{A_\lambda,m,n}(z)=1$ on $\partial A_{\lambda,m}$, $\omega_{A_\lambda,m,n}(z)=0$ on ∂R_0 and $\frac{\partial\omega_{A_\lambda,m,n}(z)}{\partial n}=0$ on $\partial R_n-A_{\lambda,m}$. Let $\{\lambda_i\}$ be a sequence such that $\lambda_i\downarrow 0$. Then

$$\omega^*_{m,n}(z)+\sum_{\lambda_i}\omega_{A_{\lambda_i},m,n}(z)\geqq\omega_{m,n}(z)\,.$$

Hence by letting $n\to\infty$ and then $m\to\infty$, $\omega^*(z)\geqq\omega_F(z)$. Then $\omega^*(z)=\omega_F(z)$.

Corollary. $\mathrm{Cap}(F)=*\mathrm{Cap}(F)$ *for a closed subset of* $R+B_1$.

In fact, since $\omega_F(z)=\dfrac{U(z)}{C}$, $*\mathrm{Cap}(F)=2\pi\dfrac{1}{2\pi C}\int_{\partial R_0}\dfrac{\partial U(z)}{\partial n}ds=\dfrac{2\pi}{C}$ $=\dfrac{4\pi^2}{I(\mu)}=\int_{\partial R_0}\dfrac{\partial\omega_F(z)}{\partial n}ds$. Hence $*\mathrm{Cap}(F)=\mathrm{Cap}(F)$ and $\mathrm{Cap}(F)=1/I(\mu)$, where μ is the equilibrium distribution of total mass unity on F.

Theorem 14. (*Extension of Evans-Selberg's Theorem*). *Let F be a closed subset of $R+B_1$. Then* $\mathrm{Cap}(F)=0$, *if and only if there exists a unit mass distribution on F whose potential $U(z)$ satisfies the following conditions*:

1). $U(z)=0$ *on* ∂R_0.

2). $U(z)=\infty$ *at every point of* F.

3). $U(z)=U_F(z)$ *and* $\dfrac{U(z)}{m}$ *is the equilibrium potential of the set* $G_n=E[z\in R:\ U(z)\geqq m]$ *for every* m.

Proof. If such $U(z)$ exists, clearly $\mathrm{Cap}(F)=0$. Next $\mathrm{Cap}(F)$ $=*\mathrm{Cap}(F)=0$ implies by 1) of Theorem 12 $D_F=0$. Replace $G(p_i, p_j)$ by $N(p_i, p_j)$ in Part I. Then we have 1) and 2). Since every point mass of $V^m(z)=\dfrac{1}{2\pi m}(\sum_{i=1}^{m}N(z, p_i))$ is contained in F, $V_F^m(z)=V^m(z)$. This implies $U(z)=(\sum_{i=1}^{\infty}\dfrac{V^i(z)}{2^i})=U_F(z)$. Hence $\dfrac{U(z)}{m}$ is the equilibrium potential of $G_m=E[z\in R:\ U(z)\geqq m]$.

Remark 1. *Let p be a point in B_0. Then* $N(z, p)=\int_{B_1}N(z, p_\alpha)\,d\mu(p_\alpha)$ *and* $U(p)=\int U(p_\alpha)\,d\mu(p_\alpha)$. *Hence $U(z)$ may be infinite on a larger set F' containing F.*

Remark 2. *Theorem 14 holds for an F_σ of $R+B_1$ of capacity zero.*

Remark 3. *We cannot omit the condition that $F\in R+B_1$, (See an example).*

Mass Distribution on $\bar{R}$. Definition of *Cap (F) and D_F for closed subset F of $\bar{R}$. Let F be a closed set of $\bar{R}$. Then $F\cap(R+B_1)$ is a G_δ, since B_0 is an F_σ. We define *Capacity and the transfinite diameter of F as follows: Put $F_m=E[z\in\bar{R}:\ \delta(z, F)\leqq\dfrac{1}{m}]$ and put $*\mathrm{Cap}(F_m)$ $=\sup_\alpha *\mathrm{Cap}(F_\alpha)$ and $D_{F_m}=\sup_\alpha D_{F_\alpha}$, where F_α is a closed subset of $R+B_1$ contained in F_m. Since clearly $*\mathrm{Cap}(F_m)$ and D_{F_m} are decreasing with respect to m. Put $*\mathrm{Cap}(F)=\lim_{m=\infty}*\mathrm{Cap}(F_m)$ and $D_F=\lim_{m=\infty}D_{F_m}$. Then we have the following

Theorem 15. $^*\mathrm{Cap}(F)=\mathrm{Cap}(F)=4\pi^2 D_F$ *for a closed set* F *of* $\bar{R}$.

In fact, let $\omega_\alpha(z)$ be the equilibrium potential of F_α. Since $F_\alpha \subset F \cap (R+B_1)$, ${}_{F_m}\omega_\alpha(z)=\omega_\alpha(z)$ for every F_α. We assume $F_\alpha \uparrow$. Then $\omega_{F_\alpha}(z)$ converges to a function $\hat{\omega}(z)$. Then ${}_{F_m}(\hat{\omega}(z)) \geqq {}_{F_m}\omega_{F_\alpha}(z)=\omega_{F_\alpha}(z)$ for every α. On the other hand, clearly ${}_{F_m}(\hat{\omega}(z)) \leqq \hat{\omega}(z)$, because $\hat{\omega}(z)$ is superharmonic in $\bar{R}$. Therefore ${}_{F_m}(\hat{\omega}(z)) \leqq \omega(z)$. This implies that $\hat{\omega}(z)$ has has M.D.I. over $R-F$. Hence $\hat{\omega}(z)=\omega_{F_m}(z)$, since $\hat{\omega}(z)=1$ on $F_m \cap R$. Hence $\mathrm{Cap}(F_m)=^*\mathrm{Cap}(F_m)$, whence $4\pi^2 D_F=^*\mathrm{Cap}(F)=\mathrm{Cap}(F)$. Particularly $\mathrm{Cap}(B_0)=^*\mathrm{Cap}(B_0)=0$. Thus two capacities coincide each other. We call them capacity. Since $\omega_F(z)={}_F\omega_F(z)$ and $\omega_F(z)$ is lower semicontinuous, we can prove as 3) of Theorem 13 the following

Corollary. *If* $\omega_F(z)\neq 0$, $\omega_F(z)=1$ *except possibly a subset of capacity zero of* F.

Hence $\omega_F(z)$ has the characteristic property of the equilibrium potential in space. The capacity of Borel sets of $\bar{R}$ is defined as usual.

An Example

We shall construct a Riemann surface with singular ideal boundary points and points of B_0 and further we show that the condition of theorem 13 is necessary.

Let r_n be a circle: $|z|=r_n$ $(n=1, 2, \cdots)$, where $r_1<r_2<r_3, \cdots, r_1=1$ and $\lim_{n=\infty} r_n=2$. Denote by $\overset{\vee}{R}_n$ a ring domain: $r_n<|z|<r_{n+1}$ and let A_n, B_n, C_n ring domains such that $A_n: r_{n+1}>|z|>r_{n,\alpha}$, $B_n: r_{n,\alpha}>|z|>r_{n,\beta}$, $C_n: r_{n,\beta}>|z|>r_n$ with $r_n<r_{n,\beta}<r_{n,\alpha}<r_{n+1}$.

$\{A_n\}$. Let $\Gamma_{A,n}$ be a circle: $|z|=\sqrt{r_{n+1}, r_{n,\alpha}}$. Then there exists a constant Q_n depending only on the modulus of A_n, i.e. $\log \frac{r_{n+1}}{r_{n,\alpha}}$ such that $\max_{z\in\Gamma_{A,n}} U(z) \leqq Q_n \min_{z\in\Gamma_{A,n}} U(z)$ for any positive harmonic function $U(z)$ in A_n. Choose a sequence P_n such that $\lim_{n=\infty} \frac{P_n}{Q_n}=\infty$, (Fig. 1).

$\{B_n^*\}$. In B_n we make so many radial slits and connect them so that every harmonic function $|U(z)| \leqq P_n$ in B_n satisfies the condition that the oscillation of $U(z)$ on $\Gamma_{B,n}$ is less than $\frac{1}{n}$, where $\Gamma_{B,n}$ is a circle in B_n such that $\Gamma_{B,n}: |z|=\sqrt{r_{n,\alpha}, r_{n,\beta}}$. We make the above slits as follows.

Put $B_n=B$, $\alpha=\log r_{n,\alpha}$ and $B=\log r_{n,\beta}$. Let $J(\supset \Gamma_{B,n})$ be a ring domain such that

$$J: \beta+\frac{\alpha-\beta}{3}<\log|z|<\beta+\frac{2(\alpha-\beta)}{3}.$$

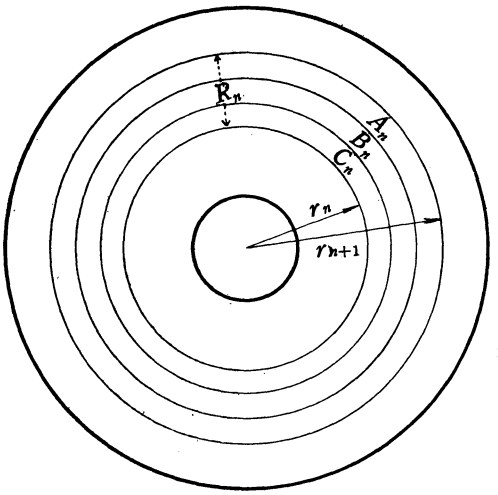


Fig. 1

Let $U(z)$ be a harmonic function in J such that $|U(z)| \leqq P_n$. Then $U(z) = \frac{1}{2\pi} \int_{\partial J} U(\zeta) \frac{\partial G(\zeta, z)}{\partial n} ds$, where $G(\zeta, z)$ is the Green's function of J with pole at z. Since $\frac{\partial G(\zeta, z)}{\partial n}$ is a continuous function of z in J for fixed ζ and since $U(z_1) - U(z_2) = \frac{1}{2\pi} \int_{\partial J} U(\zeta) \left(\frac{\partial G(\zeta, z_1)}{\partial n} - \frac{\partial G(\zeta, z_2)}{\partial n} \right) ds$, there exists a number m depending only on the modulus of J but on $U(z)$ such that $|\arg z_1 - \arg z_2| \leqq \frac{2\pi}{2^m}$ implies $|U(z_1) - U(z_2)| < \frac{1}{2n}$ for every pair of points z_1 and z_2 on the circle $\Gamma_{B,n}$.

Let H_i and H_i' $(i=1, 2, 3, \cdots, m)$ be ring domains as follows:

$$H_i: \alpha - (2i-1)s > \log|z| > \alpha - 2is,$$
$$H_i': \beta + (2i-1)s < \log|z| < \beta + 2is, \quad \text{where} \quad s = \frac{(\alpha - \beta)}{3 \cdot 2m}.$$

Let S_i^j and $S_i'^j$ $(j=1, 2, 3, \cdots, 2^{ml})$ slits in H_i and H_i' respectively as follows:

$$S_i^j: \alpha - (2i-1)s > \log|z| > -2is, \ \arg z = \frac{2\pi j}{2^{ml}}.$$
$$S_i'^j: \beta + (2i-1)s > \log|z| < \beta + 2is, \ \arg z = \frac{2\pi j}{2^{ml}}.$$

where l is a large integer so that $|U(z)| \leqq P_n$ and $U(z) = 0$ on $\sum_j S_i^j$ imply $|U(z)| < \frac{1}{2n \cdot m!}$ on a circle Γ_i for every harmonic function in $H_i - \sum_j S_i^j$.

Clearly $H_i-\sum_j S_i^j$ and $H_i'-\sum_j S_i'^j$ $(i=1, 2, \cdots, m)$ are conformally equivalent. Hence $|U(z)|\leqq P_n$ in H_i or H_i' and $U(z)=0$ on $\sum_i S_i^j$ or $\sum_j S_i'^j$ imply $|U(z)|<\frac{1}{2nm!}$ on Γ_i and Γ_i' respectively, where Γ_i and Γ_i' are circles as follows:

$$\Gamma_i : \log|z|=\alpha-(2i-1)s-\frac{s}{2},$$

$$\Gamma_i' : \log|z|=\beta+(2i-1)s+\frac{s}{2}.$$

In H_1 and H_1' identify the two edges of the slits S_1^j and $S_1'^j$ $(j=1, 2, 3, \cdots, 2^{m_1})$ lying symmetrically with respect to the real axis. Next, in H_2 and H_2' identify the two edges of S_2^j and $S_2'^j$ lying symmetrically with respect ta the imaginary axis. In H_3 and H_3', in every sector A_3^t: $\frac{(t-1)\pi}{2}<\operatorname{agr} z<\frac{t\pi}{2}$ identify two edges of slits S_3^j and $S_3'^j$ lying symmetrically with respect to the radius: $\arg z=\frac{(t-1)\pi}{2}+\frac{\pi}{4}$ $(t=1, 2, 3, 4)$. Generally speaking, let A_i^t be a sector as follows:

$$A_i^t : \frac{(t-1)\pi}{2^{i-2}}<\arg z<\frac{t\pi}{2^{i-2}}, \quad t=1, 2, 3, \cdots, 2^{i-1}.$$

In every A_i^t identify the two edgds of S_i^j and $S_i'^j$ lying symmetrically with respect to the radius: $\arg z=\frac{(t-1)\pi}{2^{i-2}}+\frac{\pi}{2^{i-1}}$. Then we have a Riemann surface $\{B_n^*\}$ with only two boundary components lying on $\log|z|=\alpha$ and $\log|z|=\beta$.

We shall show that $\{B_n^*\}$ has the property above stated. (Fig. 2).

Suppose a positive harmonic function $|U(z)|\leqq P_n$. Let $T_1(z)$ be a transformation such that $T_1(z)$ is the symmetric point of z with respect to the real axis. Then $U(z)-U(T_1(z))$ is harmonic in B_n^* and vanishes on $\sum_j (S_j'+S_1'^j)$, whence $|U(z)-U(T_1(z))|<\frac{1}{2n\cdot m!}$ on circles Γ_1 and Γ_1'. Hence by the maximum principle $|U(z)-U(T_1(z))|<\frac{1}{2n\cdot m!}$ in the ring domain bounded by Γ_1 and Γ_1'. Let $T_2(z)$ be a transformation such that $T_2(z)$ is the symmetric point of z with respect to the imaginary axis. Then as above $|U(z)-U(T_2(z))|<\frac{1}{2n\cdot m!}$ in the domain bounded by Γ_2 and Γ_2'. Next, consider $U(z)$ in a ring domain Γ_3: $\beta+5s<\log|z|<\alpha-5s$. Let T_3^1 be a transformation such that $T_3^1(z)$ is the symmetric point of z with respect to the radius: $\arg z=\frac{\pi}{4}$. Then $U(z)-U(T_3^1(z))$ is har-

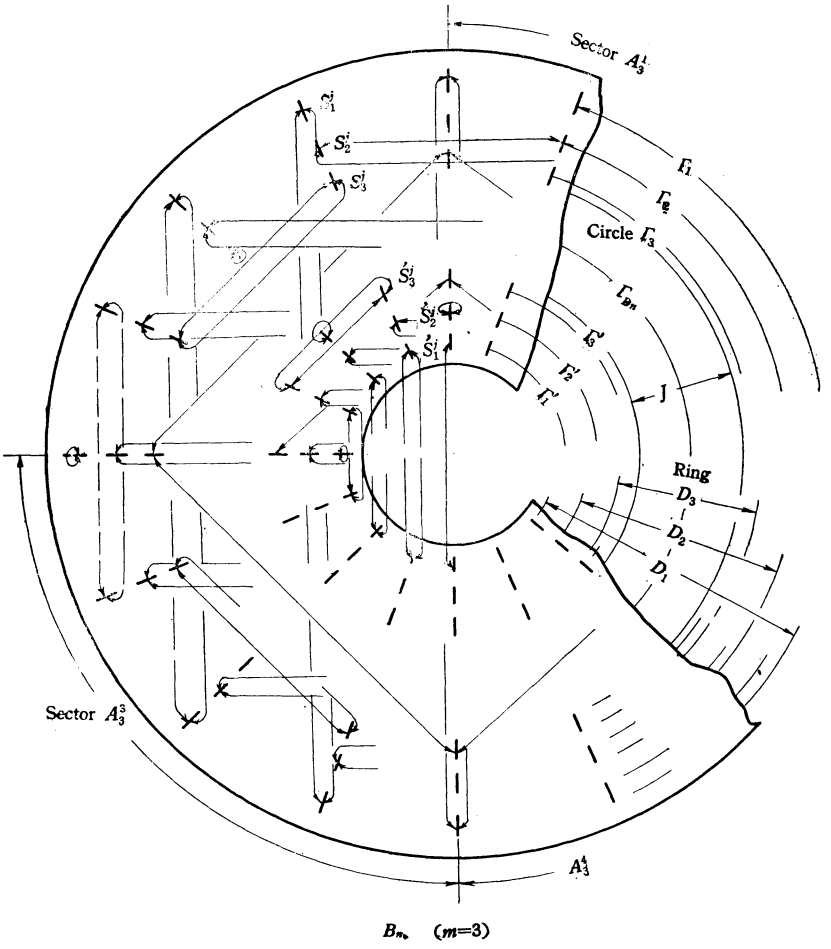


Fig. 2

monic in D_3 and $U(z)-U(T_3^1(z))=0$ on $\sum_j (S_3^j+S_3^j)\cap(A_3^1+A_3^3)$. Hence $|U(z)-U(T_3^1(z))|<\frac{1}{2n\cdot m!}$ for $z\in(A_3^1+A_3^3)\cap(\Gamma_3+\Gamma_3')$, similarly $|U(z)-U(T_3^2(z))|<\frac{1}{2n\cdot m!}$ for $z\in(A_3^2+A_3^4)\cap(\Gamma_3+\Gamma_3')$, where T_3^2 is a transformation with respect to $\arg z=\frac{3\pi}{4}$. Let z_1 and z_2 be two points in A_3^2 and A_3^4 such that $z_2=T_3^1(z_1)$. Then $z_2=T_3^2\cdot T_1\cdot T_2(z_1)$, where $T_3^2\cdot T_1\cdot T_2(z_1)$ and z_2 are contained in A_3^3. Hence by the property of T_1, T_2 and T_3^2 $|U((z_1)-U(z_2)|<\frac{3}{2n\cdot m!}$ on $\Gamma_3+\Gamma_3'$, whence by the maximum principle

$$|U(z)-U(T_3^1(z))|<\frac{3}{2n\cdot m!}<\frac{3!}{2n\cdot m!}$$

in the domain bounded by Γ_3 and Γ_3'. In the sequel, we say that T_3^1 has the *deviation* $<\frac{3!}{2n\cdot m!}$.

For every i, consider $U(z)$ in a ring domain D_i:

$$D_i:\ \beta+(2i-1)\,s<\log|z|<\alpha-(2i-1)\,s\,.$$

Let $T_i^t(z)$ $(t=1, 2, \cdots, 2)$ be a transformation such that $T_i^t(z)$ is the symmetric point of z with respect to the radius: $\arg z=\frac{2\pi(t-1)}{2^{i-1}}+\frac{\pi}{2^{i-1}}$. Then $U(z)-U(T_i^t(z))$ is harmonic in D_i. On the other hand, we have as above cases $|U(z)-U(T_i^t(z))|<\frac{1}{2n\cdot m!}$ on $A_i^t\cap(\Gamma_i+\Gamma_i')$ for every t. Now let z_1 and z_2 be two points not contained in A_i^t such that $T_i^t(z_1)=z_2$. Then there exists a system S_{z_1,z_2} of transformations satisfying the following conditions:

1°. S_{z_1,z_2} is composed of at most $i-1$ transformations contained in T_1, T_2, $\{T_3^t\}$, $\cdots$, $\{T_i^t\}$.

2°. S_{z_1,z_2} has the form $z_2=T_{n_1}^{s_1}T_{n_2}^{s_2},\cdots,T_{n_k}^{s_k}(T_i^{s_i})\,T_{n_{k+1}}^{s_{k+2}},\cdots,T_{n_L}^{s_L}$,

$$L\leqq i-1 \quad\text{and}\quad n_p\neq i \text{ for } p=1, 2, \cdots, k,\ k+2, \cdots, L)$$

3. $T_{n_{k+2}}^{s_{k+2}}T_{n_{k+3}}^{s_{k+3}},\cdots,T_{n_L}^{s_L}(z_1)$ is contained in $A_i^{s_i}$ with the same index s_i as that of $T_i^{s_i}$. Now suppose that the deviation of T_j^t is less than $\frac{j!}{2n\cdot m!}$ for every $j\leqq i-1$ (this is clear for $j=1, 2, 3$). But the deviation of of S_{z_1,z_2} is less than the sum of deviations of $\{T_j\}$ contained in S_{z_1,z_2}. Hence the deviation of T_i^t is less than $\frac{i!}{2n\cdot m!}$, that is $|U(z)-U(T_i^t(z)|<\frac{i!}{2n\cdot m!}$ on $\Gamma_i+\Gamma_i'$ for every t. This implies $|U(z)-U(T_i^t(z))|<\frac{i!}{2n\cdot m!}$ in the ring domain bounded by $\Gamma_i+\Gamma_i'$. Hence the deviation of T_i^t is less than $\frac{i!}{2n\cdot m!}$ in J for every i and t. On the other hand, $|U(z_1)-U(z_2)|<\frac{1}{2n}$ for z_1 and z_2 on $\Gamma_{B,n}$ with $|\arg z_1-\arg z_2|<\frac{2\pi}{2^m}$. Therefore the oscillation of $U(z)$ on $\Gamma_{B,n}$ is less than $\frac{1}{n}$.

Let R_n be a domain bounded by Γ $(|z|=1)$ and $\Gamma_{B,n}$. Then $\bigcap_{n\geqq 1}R_n$ is a Riemann surface with one compact boundary component Γ and one ideal boundary camponent.

Let $\{k_n\}$ be slits on the radius: $\arg z=0$ in C_n and let $w_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-k_n$ such that $w_{n,n+i}(z)=0$ on $\Gamma+\partial R_{n+i}$ $(=\Gamma_{B,n+i})$, $w_{n,n+i}(z)=1$ on k_n. Put $w_n(z)=\lim_{i=\infty}w_{n,n+i}(z)$. Let $w_{n,n+i}^*(z)$ be a

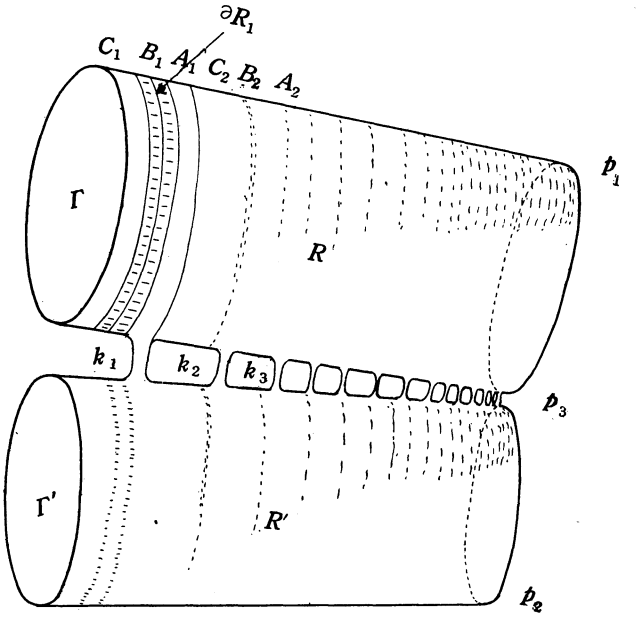
harmonic function in $R_{n+i}-k_n$ such that $w^*_{n,n+i}(z)=0$ on Γ, $w^*_{n,n+i}(z)=1$ on k_n and $\frac{\partial \omega^*_{n+i}(z)}{\partial n}=0$ on ∂R_{n+i}. Put $\lim_{i=\infty} w^*_{n,n+i}(z)=w^*_n(z)$. If we make every k_n sufficiently small, we have

$$\lim_{n=\infty}\left(\max_{z\in\Gamma_{B,n}}\sum_{n}^{\infty} w_n(z)\right)=0\,, \tag{1}$$

$$\overline{\lim_{n=\infty}}\left(\max_{z\in\Gamma_{B,n}}\sum_{n}^{\infty} w^*_n(z)\right)\leqq\frac{1}{4}\,. \tag{2}$$

Therefore we can suppose that $\{k_n\}$ have been chosen small so that the above conditions are satisfied.

Riemann surface $\tilde{R}$. Let R' be one more Riemann surface which is identical to R. From now, we denote by $V'(z), k', \cdots$ the function, figure, $\cdots$, on R' which corresponds to the function $V(z)$, figure k, $\cdots$ on R respectively. Identify k_n and k_n' for every n. Put $R+\tilde{R}'=\tilde{R}$. Then $\tilde{R}$ is a Riemann surface with two compact boundary compenent Γ and Γ' and has only one ideal boundary component. In what follows, we show that $\tilde{R}$ has the following properties, (Fig. 3).



Riemann surface $\tilde{R}$.

Fig. 3

1). *$\tilde{R}$ has no unbounded positive harmonic functions.*

Let R_n^A be the compact surface of R bounded by Γ and $\Gamma_{A,n}$. Clearly $\bigcup_n R_n^A = R$. Let $\hat{V}_n^A(z)$ be a harmonic function in $R_n^A + R_n^{A\prime}$ such that $\hat{V}_n^A(z)=0$ on $\Gamma+\Gamma'$ and $V_n^A(z)=1$ on $\Gamma_{A,n}+\Gamma'_{A,n}$. Then $\lim_{n=\infty} \hat{V}_n^A(z) = \hat{V}(z) = \dfrac{\log|z|}{\log 2}$ in the ring domain: $1<|z|<2$. Hence $V(z)$ tends to 1 as z converges to the ideal boundary of R. Let $V_{n,n+i}^A(z)$ be a harmonic function in $R_n^A + R_{n+i}^{A\prime} - \sum_{n+1}^{n+i} k_j$ such that $V_{n,n+i}^A(z)=0$ on $\Gamma+\Gamma'+\sum_{n+1}^{n+i} k_j + \Gamma_{A,n+i}$ and $V_{n,n+i}^A(z)=1$ on $\Gamma_{A,n}$. Consider $V_{n,n+i}^A(z)$ in $R-\sum_{j=1}^{\infty} k_j$. Then $V_{n,n+i}^A(z) \geqq \hat{V}_n^A(z) - \sum_{j=1}^{\infty} w_j(z)$. Hence by letting $i\to\infty$ and then $n\to\infty$ we have $\lim_{n=\infty}\lim_{i=\infty} \hat{V}_{n,n+i}^A(z) = V^A(z) \geqq \hat{V}(z) - \sum_{j=1}^{\infty} w_j(z)$ in $R-\sum_{j=1}^{\infty} k_j$. Therefore by (1) $V^A(z)>0$. (Fig. 4)

Consider a positive harmonic function $U(z)$ in $\tilde{R}$ vanishing on $\Gamma+\Gamma'$. Assume $\max_{z\in\Gamma_{A,n}} U(z) \geqq P_n$ for infinitely many numbers n. Then $\min U(z) \geqq \dfrac{P_n}{Q_n}$. Hence by the maximum principle $U(z) \geqq \dfrac{P_n}{Q_n}$ $(V_{n,n+i}^A(z))$ in $R-\sum k_i$. Thus we have by letting $i\to\infty$ and then $n\to\infty$, $U(z)=\infty$. This is absurd. Hence by the maximum principle $U(z) \leqq \max_{z\in B_n+B_{n'}} U(z) \leqq \max_{z\in\Gamma_{A,n}+\Gamma_{A,n}} U(z) \geqq P_n$ except for finitely many numbers. This implies by the property of B_n^* and $B_n^{*\prime}$ the oscillations of $U(z)$ on $\Gamma_{B,n}$ and $\Gamma'_{B,n}$ tend to zero as $n\to\infty$.

Let $\hat{V}_n(z)$ be a harmonic function in R_n+R_n' such that $\hat{V}_n(z)=0$ on $\Gamma+\Gamma'$ and $\hat{V}_n(z)=1$ on $\partial R_n+\partial R_n'$. Then $\lim_{n=\infty} \hat{V}_n(z) = \hat{V}(z) = \lim_{n=\infty} \hat{V}_n^A(z)$. Let $V_{n,n+i}(z)$ be a harmonic function in $R_n+R'_{n+i}-\sum_{n+1}^{n+i} k_j'$ such that $V_{n,n+i}(z)=0$ on $\Gamma+\Gamma'+\partial R'_{n+i}+\sum_{n+1}^{n+i} k_j$ and $V_{n,n+i}(z)=1$ on ∂R_n. Consider $V_{n,n+i}(z)$ in $R-\sum_1^{\infty} k_j$. Then as above, we have $V(z) = \lim_{n=\infty}\lim_{i=\infty} V_{n,n+i}(z) \geqq \hat{V}(z) - \sum_1^{\infty} w_j(z)$ in $R-\sum_1^{\infty} k_j$. Therefore by (1)

$$\lim_{n=\infty} (\min_{z\in\Gamma_{B,n}} V(z)) = 1 . \tag{3}$$

Next, consider $V(z)$ in $R'-\sum_1^{\infty} k_j$. Then also we have $V(z) = \lim_n \lim_i V_{n+i}(z) \leqq \sum_1^{\infty} w_j'(z)$ in $R'-\sum_1^{\infty} k_j'$. Hence by (1)

$$\overline{\lim_{n=\infty}} (\max_{z\in\Gamma_{B,n}} V(z)) = 0 . \tag{4}$$

We call such $V(z)$ the harmonic measure of the ideal boundary determined by a non-compact domain $G=R-\sum_1^\infty k_j$, (Fig. 5).

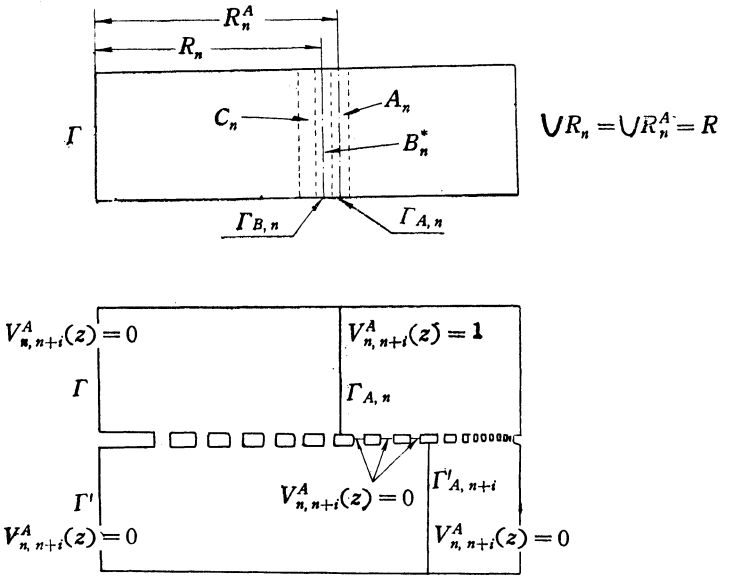


Fig. 4

If $\sup_{z\in\tilde{R}} U(z)=\infty$, $\max_{z\in\Gamma_{B,n}+\Gamma'_{B,n}} U(z)$ tends to ∞ as $n\to\infty$. This implies by property of B_n^* and $B_n^{*\prime}$ that at least one of $M_n=\min_{z\in\Gamma_{B,n}} U(z)$ and $M_n'=\min_{z\in\Gamma'_{B,n}} U(z)$ tends to ∞ as $n\to\infty$. Assume $M_n\uparrow\infty$. Then clearly

$$U(z)\geqq M_n(V_{n,n+i}(z))-\sum_1^\infty w_j(z)) \qquad \text{in } R-\sum_1^\infty k_j,$$

whence we have by letting $i\to\infty$ and then $n\to\infty$, $U(z)=\infty$. Therefore $U(z)$ is bounded $\leqq M$ in $\tilde{R}$.

2) *There exist only two linearly independint positive harmonic functions vanishing on* $\Gamma+\Gamma'$. Consider $U(z)$ in $R-\sum_1^\infty k_j$. Put $L=\overline{\lim_{n=\infty}}(\max_{z\in\Gamma_{B,n}} U(z))$ $=\overline{\lim_{n=\infty}}(\min_{z\in\Gamma_{B,n}} U(z))$. Then for any given positive number ε, there exist infinitely many numbers n such that

$$L+\varepsilon\geqq\max_{z\in\Gamma_{B,n}} U(z)\geqq\min_{z\in\Gamma_{B,n}} U(z)\geqq L-\varepsilon.$$

Since $U(z)>0$, $(L+\varepsilon)(V_{n,n+i}(z)+\sum_1^\infty w_j(z))\geqq U(z)\geqq(L-\varepsilon)(V_{n,n+i}(z)-\sum_1^\infty w_j(z))$ in R. Let $i\to\infty$ and then $n\to\infty$ and further let $\varepsilon\to 0$. Then

$$L(V(z)+\sum_1^\infty w_j(z)) \geqq U(z) \geqq L(V(z)-\sum_1^\infty w_j(z)) .$$

Hence by (1) and (3) we have $\lim_{n=\infty}(\max_{z\in\Gamma_{B,n}} U(z))=\lim_{n=\infty}(\min_{z\in\Gamma_{B,n}} U(z))=L$. Similarly we have $\lim_{n=\infty}(\max_{z\in\Gamma'_{B,n}} U(z))=\lim_{n\infty z}(\min_{\in\Gamma_{B,n}} U(z))=L'$.

Consider $U(z)$ in $\tilde{R}$. Then by (1), (3) and (4) we have as above, for any given positive number ε,

$$(L+\varepsilon)V(z)+(L'+\varepsilon)V'(z) \geqq U(z) \geqq (L-\varepsilon)V(z)+(L'-\varepsilon)V'(z) ,$$

where $V'(z)$ is the harmonic measure of the ideal boundary determined by G'. Hence $U(z)=LV(z)+L'V'(z)$. *Thus we have*

3) *There exists no function* $N(z, p)$ *such that* $\sup_{z\in R} N(z, p)=\infty$.

4) *There exists at least two singular ideal boundary points* $\in B_1$. Let $V^*_{n,n+i}(z)$ be a harmonic function in $R'_{n+i}+R_n-\sum_{n+1}^{n+i} k_j$ such that $V^*_{n,n+i}(z)=0$ on $\Gamma+\Gamma'$, $V^*_{n,n+i}(z)=1$ on ∂R_n and $\frac{\partial V^*_{n,n+i}(z)}{\partial n}=0$ $\sum_{n+1}^{n+i} k_j+\partial R'_{n+i}$. Put $V^*(z)=\lim_{i=\infty}\lim_{n=\infty} V^*_{n,n+i}(z)$. $V^*(z)$ is called the equilibrium potential of the ideal boundary determined by non-compact domain G and it is proved as $\omega_F(z)$ is superharmonic in R $(\tilde{R}+B)$. Clearly $V^*(z)\geqq V(z)$,, whence $\lim_{n=\infty}(\min_{z\in\Gamma_{B,n}} V(z))=1$. On the other hand, since $V^*(z)\leqq\sum_1^\infty w^*_n(z)$ in $R'-\sum_1^\infty k'_j$, we have by (4) $\lim_{n=\infty}(\max_{z\in\Gamma_{B,n}} V^*(z))\leqq\frac{1}{4}$. Hence $V^*(z)\neq V'^*(z)$, (Fig. 5). Now $V^*(z)$ and $V^{*\prime}(z)$ are superharmonic in $\tilde{\tilde{R}}$, that is $V^*(z)$

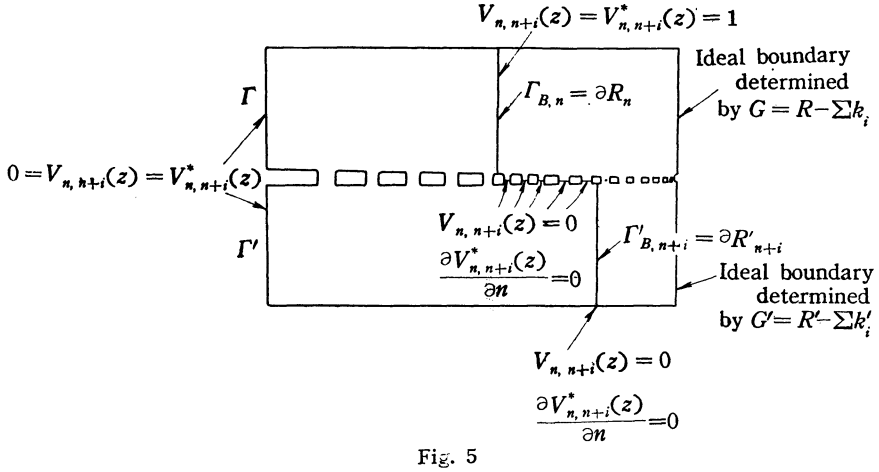


Fig. 5

$= \int_{B_1} N(z, p_\omega)\, d\mu(p_\omega)$ $V'^*(z) = \int_{B_1} N(z, p_\omega')\, d\mu'(p_\omega)$. Hence by the symmetric structure of $\tilde{R}$ there must exist at least two singular points p_1 and p_2 in B_1 such that $N(z, p_1) \neq N(z, p_2)$ and $N(z, p_1) = N(T(z), p_2)$, where $T(z)$ is the symmetric point of z with respect to $\sum_1^\infty k_j$. On the other hand, by 2), $N(z, p_1) = N(z, p_1)) = \lambda V(z) + \mu V'(z)$ and $N(z, p_2) = \mu V(z) + \lambda V'(z)$ ($\lambda \neq \mu$, $\mu \geqq 0$, $\lambda \geqq 0$).

5) *There exists at least one ideal boundary point belonging to* B_0. Let $\{p_1^i\}$ and $\{p_2^i\}$ be fundamental sequences determining p_1 and p_2 respectively. Then $\{p_1^i\}$ and $\{p_2^i\}$ are not contained in $\sum^\infty k_i$, because the symmetric structure of R implies $N(z, p_1) = N(z, p_2)$. Connect p_1^i and p_1^i with a curve C^i. Then there exists a point p_3^i on k_i. Choose a subsequence $\{p_3^i\}$ for which $N(z, p_3^i)$ converges to a function $N(z, p_3)$. Then $N(z, p_3) = \frac{1}{2}(N(z, p_1) + N(z, p_2))$, because $N(z, p_1) = N(T(z), p_2)$, i.e. $N(z, p_3) = K(V(z) + V'(z))$ and $\int_{\partial R_0} \frac{\partial N(z, p_i)}{\partial n} ds = 2\pi$ $(i = 1, 2, 3)$. Then $N(z, p_3)$ and $N(z, p_3) - \frac{1}{2} N(z, p_1) = \frac{1}{2} N(z, p_2)$ are superharmonic and $N(z, p_1)$ is not a multiple of $N(z, p_3)$. Hence $N(z, p_3)$ is not minimal, i.e. $p_3 \in B_0$.

$0 = \mathrm{Cap}(B_0) = \mathrm{Cap}(p_3)$ and p_3 is a closed set. But there exists no unbounded positive superharmonic function in $\tilde{R}$. *Therefore the condition of Theorem* 14 *that* $F \subset R + B_1$ *is necessary.*

(Received September 19, 1956)