Osaka Mathematical Journal
Vol. 8, No. 2, December, 1956.

Mass Distributions on the Ideal Boundaries of Abstract Riemann Surfaces, II[1)]

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The present article is concerned with the equilibrium potential on Riemann surfaces with positive boundary.

1. Let R^* be a Riemann surface with positive boundary and let $\{R_n\}$ $(n=0, 1, 2, \cdots)$ be its exhaustion with compact relative boundaries $\{\partial R_n\}$. Put $R=R^*-R_0$. Let $N_n(z, p)$ be a positive function in R_n-R_0 harmonic in R_n-R_0 except one point $p\in R$ such that $N_n(z, p)=0$ on ∂R_0, $\dfrac{\partial N_n(z, p)}{\partial n}=0$ on ∂R_n and $N_n(z, p)+\log|z-p|$ is harmonic in a neighbourhood of p. Then the $*$-Dirichlet integral of $N_n(z, p)$ taken over R_n-R_0 is $D^*(N_n(z, p))=U_n(p)$, where $U_n(p)=\lim\limits_{z\to p}(N_n(z, p)+\log|z-p|)$ and the $*$-Dirichlet integral is taken with respect to $N_n(z, p)+\log|z-p|$ in the neighbourhood of p. For $N_n(z, p)$ and $N_{n+i}(z, p)$, we have

$$\begin{aligned}
&D^*_{R_n-R_0}(N_n(z, p),\ N_{n+i}(z, p))=D^*_{R_{n+i}-R_0}(N_{n+i}(z, p))=2\pi U_{n+i}(p)^{2)},\\
&D_{R_n-R_0}(N_n(z, p)-N_{n+i}(z, p))=D^*_{R_n-R_0}(N_n(z, p))-2D^*_{R_n-R_0}(N_n(z, p),\ N_{n+i}(z, p))\\
&\quad+D^*_{R_n-R_0}(N_{n+i}(z, p))<D^*_{R_n-R_0}(N_n(z, p))-D^*_{R_{n+i}-R_0}(N_{n+i}(z, p))\\
&\quad=2\pi(U_n(p)-U_{n+i}(p))\,.
\end{aligned}$$

Hence $\{U_n(p)\}$ is decreasing with respect to n. Since $\int_{\partial R_0}\dfrac{\partial N_n(z, p)}{\partial n}ds=2\pi$ for every n, $\varliminf\limits_{n=\infty} U_n(p)>-\infty$, whence $\{U_n(p)\}$ converges. Therefore $D_{R_{n+i}-R_0}(N_{n+i}(z, p)-N_n(z, p))$ tends to zero if n and i tend to ∞, which implies that $\{N_n(z, p)\}$ converges in mean. Further $N_n(z, p)=0$ on ∂R_0 yields that $\{N_n(z, p)\}$ converges uniformly to a function $N(z, p)$, which clearly has the minimal $*$-Dirichlet integral over R, in every compact part of R. Clearly by the compactness of ∂R_0, we have $\int_{\partial R_0}\dfrac{\partial N(z, p)}{\partial n}ds=$

1) Resumé of this article appeared in Proc. Japan Acad. **32**, 1956.

2) Let $v_r(p)$ be a circular neighbourhood of p with respect to the local parameter: $v_r(p)=E[z\in R: |z-p|<r]$. Then $D^*(N_n(z, p),\ N_{n+i}(z, p))=\int_{\partial v(p)}(N_{n+i}(z, p)+\log|z-p|)\dfrac{\partial N_n(z, p)}{\partial n}ds$. By letting $r\to 0$, we have $D^*(N_{n+i}(z, p),\ N_n(z, p))=2\pi U_{n+i}(p)$. Clearly $*$-Dirichlet integral reduces to Dirichlet integral when the functions have no pole.

$\int\limits_{\partial R_0} \lim\limits_{n=\infty} \frac{\partial N_n(z,\ p)}{\partial n} ds = 2\pi$. We call $N(z,\ p)$ the $*$-Green's function of R with pole at p.

As in case of a Riemann surface with null-boundary, we define for R^* the ideal boundary point, by making use of $\{N(z,\ p_i)\}$, that is, if $\{p_i\}$ is a sequence of points in R having no point of accumulation in $R+\partial R_0$ for which the corresponding functions $N(z,\ p_i)$ $(i=1, 2, 3, \cdots)$ converge uniformly in every compact set of R, we say that $\{p_i\}$ is a fundamental sequence determining an *ideal boundary point*. The set of all the ideal boundary points will be denoted by B and the set $R+B$, by $\bar{R}$. The domain of definition of $N(z,\ p)$ may now be extended by writing $N(z,\ p) = \lim\limits_{i=\infty} N(z,\ p_i)$ $(z \in R$ and $p \in B)$, where $\{p_i\}$ is any fundamental sequence determining p. For p in B, the flux of $N(z,\ p)$ along ∂R_0 is also 2π. The distance between two points p_1 and p_2 of $\bar{R}$ is defined by

$$\delta(p_1,\ p_2) = \sup_{z \in R_1 - R_0} \left| \frac{N(z,\ p_1)}{1+N(z,\ p_1)} - \frac{N(z,\ p_2)}{1+N(z,\ p_2)} \right| .$$

The topology induced by this metric is homeomorphic to the original topology in R and we see easily that $R-R_1+\partial R_1+B$ and B are closed and compact. Evidently, if $\{p_i\}$ tends to p in δ-sense (with respect to δ-metric), then $N(z,\ p_i)$ tends to $N(z,\ p)$, that is $N(z,\ p)$ is continuous with respect to this metric and derivatives of $N(z,\ p_i)$ converges to those of $N(z,\ p)$ at every point z of R.

First, we shall prove the following

Lemma 1. *Let G be a compact or non-compact closed set containing a relatively closed set F and suppose that there exists at least one harmonic function $U(z)$ such that $U(z)=\varphi$ on $\partial R_0+\partial F$ and whose Dirichlet integral taken over $R-F$ is finite. Let $U_F(z)$ be the harmonic function in $R-F$ having the minimal Dirichlet integral over $R-F$ with boundary value φ on $\partial R_0+\partial F$ among all functioh $\{U_\alpha(z)\}$ having the same boundary value φ on $\partial R_0+\partial F$. Let $U_G(z)$ be a harmonic function in $R-G$ with the boundary value $U_F(z)$ on $\partial G+\partial R_0$ such that $U_G(z)$ has the minimal Dirichlet integral taken over $R-G$ among all functions with the boundary value $U_F(z)$ on $\partial G+\partial R_0$. Then*

$$U_G(z) = U_F(z) .$$

Proof. Let $U_n'(z)$ be a harmonic function in R_n-R_0-G such that $U_n'(z)=U_F(z)$ on $\partial G+\partial R_0$ and $\frac{\partial U_n'(z)}{\partial n}=0$ on ∂R_n-G. Then we see as in case of $N(z, p)$ that $\{U_n'(z)\}$ converges to a function $U'(z)$ in mean and that $U'(z)$ has the minimal Dirichlet integral (we shortly it denote by M.D.I) among all functions with boundary value $U_F(z)$ on $\partial R_0 + \partial G$. Assume $D_{R-G}(U'(z)) \leqq D_{R-G}(U_F(z)) - d$ $(d > 0)$. Then $D_{R_n - R_0 - G}(U_n'(z)) < D_{R-G}(U_F(z)) - d$ $(n = 1,2,3,\cdots)$. Now let $U_n''(z)$ be a harmonic function in $R_n - R_0 - F$ such that $U_n''(z) = U_F(z)$ on $\partial R_n \cap (G-F) + \partial R_0$ and $U_n''(z) = U'(z)$ on $\partial R_n - G$. Then by Dirichlet principle, $D_{R_n - R_0 - F}(U_n''(z)) \leqq D_{R_n - R_0 - G}(U_n'(z)) + D_{(R_n - R_0) \cap (G-F)}(U_F(z)) \leqq D_{R_n - R_0 - F}(U(z)) - d$.

Choose a subsequence $\{U_{n'}''(z)\}$ of $\{U_n''(z)\}$ which converges uniformly in every compact set of $R-F$ to a function $U^*(z)$. Then we have also $D_{R-F}(U^*(z)) \leqq \varliminf_{n'=\infty} D_{R_{n'} - R_0}(U_{n'}''(z)) \leqq D_{R-F}(U_F(z)) - d$. This contradicts the minimality of $D_{R-F}(U_F(z))$. Hence $D_{R-G}(U'(z)) = D_{R-G}(U_F(z))$ and $U'(z)$ is clearly the harmonic continuation of $U_F(z)$ by Dirichlet principle. On the other hand, it is clear that such $U'(z)$ is determined uniquely[3)] by the boundary value on $\partial R_0 + \partial G$. Hence $U_F(z) = U'(z) = U_G(z)$. Next, we consider the Dirichlet integral of $N(z, p)$.

Lemma 2. *Put* $N^M(z, p) = \min [M, N(z, p)]\, p \in \bar{R}$. *Then the Dirichlet integral of* $N^M(z, p)$ *over* R *satisfies*

$$D_R(N^M(z, p)) \leqq 2\pi M : M \geqq 0 .$$

Proof. We shall prove the lemma in three cases as follows:

Case 1. *$p \in R$ and the set $V_M(p) = E[z \in R : N(z, p) \geqq M]$ is compact.*

Case 2. *$p \in R$ and $V_M(p)$ is non-compact.*

Case 3. *$p \in B$.*

Case 1. *$p \in R$ and $V_M(p)$ is compact.* Let $N_n(z, p)$ be a function in $R_n - R_0$ such that $N_n(z, p)$ is harmonic in $R_n - R_0$ except p, $N_n(z, p) + \log|z-p|$ is harmonic in a neighbourhood of p, $N_n(z, p) = 0$ on ∂R_0 and $\dfrac{\partial N_n(z, p)}{\partial n} = 0$ on ∂R_n. Let $N_n'(z, p)$ be a harmonic function in $R_n - R_0 - V_M(p)$ such that $N_n'(z, p) = M$ on $\partial V_M(p)$, $N_n'(z, p) = 0$ on ∂R_0 and $\dfrac{\partial N_n'(z, p)}{\partial n} = 0$ on ∂R_n. Then the Dirichlet integral is $D_{R_n - R_0 - V_M(p)}(N_n'(z, p)) = \int_{\partial R_0} M \dfrac{\partial N_n'(z, p)}{\partial n} ds$. Clearly, $\{D_{R_n - R_0 - V_M(p)}(N_n'(z, p))\}$ is increasing with

3) Let $U_i(z)$ $(i = 1, 2)$ be a harmonic function in $R-G$ such that $U_1(z) = U_2(z)$ on $\partial G + \partial R_0$ and $U_i(z)$ has the finitely minimal Dirichlet integrals over $R-G$. Then by the minimality of $D(U_i(z))$, we have $D(U_i(z), V(z)) = 0$, where $V(z)$ is a harmonic function in $R-G$ such that $V(z) = 0$ on $\partial R_0 + \partial G$ and $D(V(z)) < \infty$. We can consider $U_1(z) - U_2(z)$ as $V(z)$. Hence

$$D(U_1(z) - U_2(z),\ U_1(z)) = D(U_1(z) - U_2(z),\ U_2(z)) = 0$$

whence $D(U_1(z) - U_2(z)) = 0$, i.e. $U_1(z) = U_2(z)$.

respect to n and $N_n'(z, p)$ converges in mean and also converges uniformly in every compact set of $R-V_M(p)$ to a function $N'(z, p)$ and $D_{R-V_M(p)}(N'(z, p))=2\pi M$ and further $N'(z, p)$ has M.D.I over $R-V_M(p)$ among all functions having the value M on $\partial V_M(p)$ and zero on ∂R_0. Let R' be a compact component of R bounded by ∂R_0 and a compact analytic curve γ which separates $V_M(p)$ from ∂R_0. Denote by $\omega^*(z)$ a harmonic function in R' such that $\omega^*(z)=0$ on ∂R_0 and $\omega^*(z)=1$ on γ and let $\omega_n(z)$ be a harmonic function in $R_n-R_0-V_M(p)$ such that $\omega_n(z)=1$ on $\partial V_M(p)$, $\omega_n(z)=0$ on ∂R_0 and $\frac{\partial \omega_n(z)}{\partial n}=0$ on ∂R_0. Then clearly, $D_{R_n-R_0-V_M(p)}(\omega_n(z))\leqq D_{R'}(\omega^*(z))$. On the other hand, by the maximum principle

$$|N_n(z, p)-N_n'(z, p)|<\delta_n\omega_n(z),$$

where $\delta_n=\max[|N_n(z, p)-M|]$ on $\partial V_M(p)$.
Let $n\to\infty$. Then $N_n(z, p)$ tends to $M(=N'(z, p))$ on $\partial V_M(p)$ and consequently $\delta_n\to 0$ as $n\to\infty$. Since $\delta_n\omega_n(z)\to 0$ as $n\to\infty$, we have $N(z, p)=N'(z, p)$ and $D_R(N^M(z, p))=D_{R-V_M(p)}(N(z, p))=\lim_{n=\infty} M\int_{\partial R_0}\frac{\partial N_n'(z, p)}{\partial n}ds=2\pi M$.

Case 2. *$p\in R$ and $V_M(p)$ is non-compact.* Take M' large so that $V_{M'}(p)$ is compact. Then since $N(z, p)\,(p\in R)$ has the M.D.I over $R-V_{M'}(p)$, $N(z, p)$ also has M.D.I over $R-V_M(p)$ by lemma 1. Therefore $N(z, p)=\lim_{n=\infty}N_n'(z, p)$ in $R-V_M(p)$, where $N_n'(z, p)$ is harmonic in $R-R_0-V_M(p)$, $N_n'(z, p)=0$ on ∂R_0, $N_n'(z, p)=M$ on $\partial V_M(p)$ and $\frac{\partial N_n'(z, p)}{\partial n}=0$ $\partial R_n-V_M(p)$. Hence

$$D_R(N^M(z, p))=D_{R-V_M(p)}(\lim_{n=\infty}N_n'(z, p))=\lim_{n=\infty}M\int_{\partial R_0}\frac{\partial N_n'(z, p)}{\partial n}ds=2\pi M.$$

Case 3. $p\in B$. Let $\{p_i\}$ be a fundamental sequence determining p. Then for any given positive number ε, we can find a narrow strip S[4] such that the interior of S contains $\partial V_M(p)\cap(R_n-R_0)$ and that $D_{R_n-R_0-V_M(p)-S}(N(z, p))\geqq D_{R_n-R_0-V_M(p)}(N(z, p))-\varepsilon$ and further $(V_M(p_i)\cap(R_n-R_0))\subset(S+V_M(p))$ for any $i\geqq i_0(S)$, where $V_M(p_i)=E[z\in R: N(z, p_i)\geqq M]$ and $i_0(S)$ is a suitable number depending on S and ε, because $N(z, p_i)$ converges uniformly in every compact part of R to $N(z, p)$. On the other hand, since the derivatives of $N(z, p_i)$ converge to those of $N(z, p)$ uniformly in R_n-R_0, we have

4) S may consist of a finite number of components.

$$D_{R_n-R_0-V_M(p)-S}(N(z,\ p)) \leqq \lim_{i=\infty} D_{R-V_M(p_i)}(N(z,\ p_i)) \leqq 2\pi M .$$

Hence, by letting $\varepsilon\to 0$ and then $n\to\infty$,

$$D_R(N^M(z,\ p)) = D_{R-V_M(p)}(N(z,\ p)) \leqq 2\pi M .$$

In the present part, we consider only positive continuous function $U(z)$ such that $U(z)=0$ on ∂R_0 and $D_R(U^M(z))<\infty$ for every M, where $U^M(z)=\min[M,\ U(z)]$. In what follows, in order to introduce the harmonicity or superharmonicity in $\bar{R}$, we make some preparations:

2. Capacity and the Equilibrium Potential of Relatively closed Sets in R.

Let F be a compact or non-compact relatively closed set in R having no common point with R_1. Denote by $\omega_n(z)$ a harmonic function in R_n-R_0-F such that $\omega_n(z)=0$ on ∂R_0, $\omega_n(z)=1$ on F except possibly a subset of capacity zero of F and $\frac{\partial\omega_n(z)}{\partial n}=0$ on ∂R_n-F. Then the Dirichlet integral of $\omega_n(z)$ and $\omega_{n+i}(z)$ taken over R_n-R_0-F is $D_{R_n-R_0-F}$ $(\omega_n(z)-\omega_{n+i}(z),\ \omega_n(z))=0$, whence

$$D_{R_n-R_0-F}(\omega_{n+i}(z)) = D_{R_n-R_0-F}(\omega_n(z)) + D_{R_n-R_0-F}(\omega_{n+i}(z)-\omega_n(z)) ,$$
$$D_{R_n-R_0-F}(\omega_n(z)) < D_{R_{n+i}-R_0-F}(\omega_{n+i}(z)) < D_{R_1-R_0}(\omega^*(z)) ,$$

where $\omega_*(z)$ is a harmonic function in R_1-R_0 such that $\omega^*(z)=0$ on ∂R_0 and $\omega^*(z)=1$ on ∂R_1. Hence $\{D_{R_n-R_0-F}(\omega_n(z))\}$ is convergent, which implies that

$$D_{R_n-R_0}(\omega_{n+i}(z)-\omega_n(z)) = D_{R_n-R_0}(\omega_{n+i}(z)) - D_{R_n-R_0}(\omega_n(z)) ,$$

tends to zero as n and i tend to ∞.
Hence $\omega_n(z)$ converges to a harmonic function $\omega_F(z)$ in mean. Since $\omega_n(z)=0$ on ∂R_0, $\omega_n(z)$ converges to $\omega_F(z)$ uniformly in every compact set of $R-F$. Evidently, $\omega_F(z)$ has M.D.I over $R-F$ among all functions having the value 1 on F except possibly a subset of capacity zero of F. We call such $\omega_F(z)$ the *equilibrium potential of* F and $D(\omega_F(z))=\int_{\partial R_0}\frac{\partial\omega_F(z)}{\partial n}ds$ the capacity of F. Then we have the following

Theorem 1.

1) *If* $F_n\uparrow F$, *then* $\omega_{F_n}(z)\uparrow\omega_F(z)$ *and* Cap $(F_n)\uparrow$ Cap (F).

2) *Let* G_ε *be the domain such that* $G_\varepsilon=E[z\in R\colon \omega_F(z)\geqq 1-\varepsilon]$ *and let* $\omega_{G_\varepsilon}(z)$ *be the equilibrium potential of* G_ε. *Then*

$$\omega_F(z) = (1-\varepsilon)\omega_{G_\varepsilon}(z) .$$

3) *Let ∂G_ε be the niveau curve of $\omega_F(z)$ with height $1-\varepsilon$. Then there exists a set H in the interval $(0,1)$ such that $\operatorname{mes} H=1$ and that $1-\varepsilon\in H$ implies*

$$\mathrm{Cap}\,(F)=\int_{\partial G_\varepsilon}\frac{\partial\omega_F(z)}{\partial n}ds=\int_{\partial R_0}\frac{\partial\omega_F(z)}{\partial n}ds\,.$$

Proof. Let $\omega_F(z)$ and $\omega_{F_n}(z)$ be the equilibrium potentials of F and F_n respectively. Then $\omega_F(z)\geqq\omega_{F_n}(z)$ and $D(\omega_F(z))\geqq D(\omega_{F_n}(z))$. On the other hand, clearly $\omega_{F_n}(z)$ is increasing with respect to n and $\lim\limits_{n=\infty}\omega_{F_n}(z)$ attains 1 on F except possibly a subst of F of capacity zero. Since $\omega_F(z)$ has the M.D.I, we have $D(\omega_F(z))=\lim\limits_{n=\infty}D(\omega_{F_n}(z))$ and $\omega_F(z)=\lim\limits_{n=\infty}\omega_{F_n}(z)$, because such a function is determined uniquely by its boundary value on F.

Proof of 2). If we replace $U_F(z)$ in lemma 1 by $\omega_F(z)$ in this Theorem, then we have at once 2).

Proof of 3). Let $\omega_n'(z)$ be a harmonic function in $R_n-R_0-G_\varepsilon$ such that $\omega_n'(z)=0$ on ∂R_0, $\omega_n'(z)=1-\varepsilon$ on ∂G_ε and $\frac{\partial\omega_n'(z)}{\partial n}=0$ on $\partial R_n-G_\varepsilon$. Then, since $\lim\limits_{n=\infty}\omega_n'(z)$ has M.D.I over $R-G_\varepsilon$, we have $\lim\limits_{n=\infty}\omega_n'(z)=\omega_F(z)$ by 2). On the other hand, since $\int\limits_{\partial G_\varepsilon\cap(R_n-R_0)}\frac{\partial\omega_n'(z)}{\partial n}ds=\int\limits_{\partial R_0}\frac{\partial\omega_n'(z)}{\partial n}ds$, $\frac{\partial\omega_n'(z)}{\partial n}\geqq 0$ on ∂G_ε and $\lim\limits_{n=\infty}\int\limits_{\partial R_0}\frac{\partial\omega_n'(z)}{\partial n}ds=\int\limits_{\partial R_0}\lim\limits_{n=\infty}\frac{\partial\omega_n'(z)}{\partial n}ds$, we have by Fatou's lemma

$$L_\varepsilon=\int\limits_{\partial G_\varepsilon}\frac{\partial\omega_F(z)}{\partial n}ds\leqq\lim_{n=\infty}\int\limits_{\partial G_\varepsilon}\frac{\partial\omega_n'(z)}{\partial n}ds=\int\limits_{\partial R_0}\frac{\partial\omega_F(z)}{\partial n}ds=L=D(\omega_F(z))\,.$$

Now we can take $p+iq=\omega_F(z)+i\bar{\omega}_F(z)$ as the local parameter at every point of $R-F$, where $\bar{\omega}_F(z)$ is the conjugate function of $\omega_F(z)$. Then $\frac{\partial\omega_F(z)}{\partial q}=0$ and $\frac{\partial\omega_F(z)}{\partial p}=1$ at every point of the niveau of $\omega_F(z)$ and the Dirichlet integral is

$$L=D(\omega_F(z))=\iint\limits_{R-F}\left\{\left(\frac{\partial\omega_F(z)}{\partial p}\right)^2+\left(\frac{\partial\omega_F(z)}{\partial q}\right)^2\right\}dp\,dq=\int_0^1 L_\varepsilon\,dp\,.$$

If there were a set E of positive measure in $(0,1)$ such that $1-\varepsilon\in E$ implies $L_\varepsilon<L$, we have $D(\omega_F(z))<L$. This is absurd. Hence we have 3).

Regular Domains. Let F be a compact or non-compact relatively closed domain in R and let $\omega_F(z)$ be its equilibrium potential of F. If

$\int_{\partial F} \frac{\partial \omega_F(z)}{\partial n} ds = \int_{\partial R_0} \frac{\partial \omega_F(z)}{\partial n} ds$, F is called a *regular domain*. We see at once by 3) of Theorem 1 that there exists a sequence of regular domains $G_\varepsilon = E[z \in R: \omega_F(z) \geq 1-\varepsilon]$ which we call the *regular domains generated by the equilibrium potential*, containing any closed set F of positive capacity and that any compact closed domain with analytic relative boundaries is always regular.

3. Definition of $U_D(z)$ for compact or non-compact Domain D.

Suppose a continuous function $U(z)$ in R such that $U(z)=0$ on ∂R_0, $D(U^M(z)) < \infty$ and a domain D. Let $U_D^M(z)$ be a harmonic function in $R-D$ such that $U_D^M(z) = U^M(z)$ on $\partial R_0 + \partial D$ and $U_D^M(z)$ has M.D.I over $R-D$. Then evidently, $U_D^M(z)$ is determined uniquely. We define $U_D(z)$ by $\lim_{M=\infty} U_D^M(z)$.

Theorem 3. *Let D be a regular domain and let $N^D(z, p)$ be a function in $R-D$ such that $N^D(z, p)$ is harmonic in $R-D$ except p where $N(z, p)+\log|z-p|$ is harmonic, $N^D(z, p)=0$ on $\partial R_0+\partial D$ and $N^D(z, p)$ has the minimal $*$-Dirichlet integral (it is taken with respect to $N(z, p)$ $+\log|z-p|$ in a neighbourhood of p). Then we have the following*

$$U_D(p) = \frac{1}{2\pi} \int_{\partial D} U(z) \frac{\partial N^D(z, p)}{\partial n} ds. \tag{1}$$

Proof. Let $\omega_n(z)$ be a harmonic functionin $R_n - R_0 - D$ such that $\omega_n(z)=0$ on ∂R_0, $\omega_n(z)=1$ on ∂D and $\frac{\partial \omega_n(z)}{\partial n}=0$ on $\partial R_n - D$ and let $N_n^D(z, p)$ be a harmonic function in $R_n - R_0 - D$ with one positive logarithmic singularity at p such that $N_n^D(z, p)=0$ on $\partial R_0 + \partial D \cap (R_n - R_0)$ and $\frac{\partial N_n^D(z, p)}{\partial n}=0$ on $\partial R_n - D$. Then by the maximum principle there exist constants M' and n such that $N_n^D(z, p) < M'$ for $n \geq n_0$ outside of a neighbourhood of p. Hence there exists a constant M'' such that $N_n^D(z, p) \leq M''(1-\omega_n(z))$ in $R_n - R_0$ outside of a neighbourhood of p for every $n \geq n_0$, whence $0 \leq \frac{\partial N_n^D(z, p)}{\partial n} < -M'' \frac{\partial \omega_n(z)}{\partial n}$ on $\partial D \cap (R_n - R_0)$. Now since D is regular, we have $\int_{\partial R_0} \frac{\partial \omega_D(z)}{\partial n} ds = \int_{\partial D} \frac{\partial \omega_D(z)}{\partial n} ds = \int_{\partial D} \lim_{n=\infty} \frac{\partial \omega_n(z)}{\partial n} ds$, where $\omega_0(z) = \lim_{n=\infty} \omega_n(z)$ is the equilibrium potential of D.

Assume that there exists a positive constant δ such that for infinitely many numbers m and $n (n > m)$ such that $\int_{\partial D \cap (R_n - R_m)} \frac{\partial \omega_n(z)}{\partial n} ds > \delta$. Then

$$\int_{\partial D\cap(R_m-R_0)}\frac{\partial\omega_n(z)}{\partial n}ds<\int_{\partial D\cap(R_n-R_0)}\frac{\partial\omega_n(z)}{\partial n}ds-\delta\,.$$

Let n tend to ∞. Then by Fatou's lemma

$$\int_{\partial D\cap(R_m-R_0)}\frac{\partial\omega_D(z)}{\partial n}ds\leq\varliminf_{n=\infty}\int_{\partial D\cap(R_n-R_0)}\frac{\partial\omega_n(z)}{\partial n}ds-\delta\leq\varliminf_{n=\infty}\int_{\partial R_0}\frac{\partial\omega_n(z)}{\partial n}ds-\delta\,.$$

Let m tend to ∞. Then $\int_{\partial D}\frac{\partial\omega_D(z)}{\partial n}ds\leq\int_{\partial R_0}\frac{\partial\omega_D(z)}{dn}ds-\delta$. This contradicts the regularity of D. Hence, for any given positive number ε, there exist numbers m and $n_0(\varepsilon, m)$ such that $0\leq\int_{\partial D\cap(R_n-R_m)}\frac{\partial\omega_n(z)}{\partial n}ds<\varepsilon$, for $n\geq n_0$. It follows that $\int_{\partial D\cap(R_n-R_0)}\frac{\partial N_n^D(z,\ p)}{\partial n}ds<M''\varepsilon$, for $n\geq n_0$. (2)

Let $U_n^M(z)$ be a harmonic function in R_n-R_0-D such that $U_n^M(z)=U^M(z)$ on $\partial R_0+\partial D$ and $\frac{\partial U_n^M(z)}{\partial n}=0$ on ∂R_n-D. Then by Green's formula

$$U_n^M(p)=\frac{1}{2\pi}\int_{\partial D\cap(R_n-R_0)}U^M(z)\frac{\partial N_n^D(z,\ p)}{\partial n}ds\,.$$

Let n tend to ∞. Then since $U_n^M(z)$ tends to $U_D^M(z)$ and by (2), we have

$$U_D^M(p)=\frac{1}{2\pi}\int_{\partial D}U^M(z)\frac{\partial N^D(z,\ p)}{\partial n}ds\,.$$

Hence by letting $M\to\infty$, we have $U_D(p)=\frac{1}{2\pi}\int_{\partial D}U(z)\frac{\partial N^D(z,\ p)}{\partial n}ds\,.$

5. Harmonicity and Superharmonicity in $\bar{R}$. If $U(z)$ is superharmonic in R and further, for any compact domain D, if $U(z)=U_D(z)$ or $U(z)>U_D(z)$, we say that $U(z)$ is *harmonic or superharmonic in* $\bar{R}$ respectively.

Theorem 3. *If $U(z)$ and $V(z)$ are positive, $U(z)=V(z)=0$ on ∂R_0 and harmonic in R and superharmonic in $\bar{R}$, then for a domain D*

1) $U_D(z)\leq U(z)$.
2) $U(z)\geq V(z)$ *implies* $U_D(z)\geq V_D(z)$.
3) $U_D(z)+V_D(z)={}_D(U+V)(z)$.
4) $(CU_D(z))={}_D(CU)(z)$ *for* $C\geq 0$.
5) $U_{D_1+D_2}(z)\leq U_{D_1}(z)+U_{D_2}(z)$ *for two domains D_1 and D_2.*
6) *If $D_1\supset D_2$, then* ${}_{D_1}(U_{D_2}(z))=U_{D_2}(z)$ *and* $U_{D_1}(z)\geq U_{D_2}(z)$.

The first five assertions are clear by definition. We shall prove 6). We see easily that $U^M(z)$ is superharmonic in $\bar{R}$ by the superharmonicity of $U(z)$ in $\bar{R}$. Assume $D_1 \supset D_2$. Then by lemma 1 $U_{D_1}^M(z) = {}_{D_2}(U_{D_1}^M(z))$. Hence by letting $M\to\infty$ $U_{D_1}(z) = {}_{D_2}(U_{D_1}(z)) \leqq {}_{D_2}(U(z)) = U_{D_2}(z)$.

Another Definition of $U_D(z)$. *If* $U(z)$ *is superharmonic in* $\bar{R}$, $U_D(z)$ *is given as follows*: Put $D_n = D\cap(R_n - R_0)$. Then

$$U_D(z) = \lim_{n=\infty} U_{D_n}(z)\,.$$

Proof. $U_{D_n}(z)$ is increasing with respect to n by 6) of the above Theorem. Hence $\{U_{D_n}(z)\}$ converge. Since $D(U_D^M(z)) \leqq D(U^M(z)) < \infty$, for any given positive number ε there exists a number n_0 such that $D_{D\cap(R-R_0)}(U_D^M(z)) < \varepsilon$ for $n \geqq n_0(M)$. On the other hand, since $U_{D_n}^M(z)$ has M.D.I over $R-D_n$ with boundary value $U^M(z) = U_{D_n}^M(z)$ on ∂D_n,

$$D_{R-D_n}(U_{D_n}^M(z)) \leqq D_{R-D_n}(U_D^M(z)) \leqq D_{R-D}(U^M(z)) + \varepsilon \quad \text{for } n \geqq n_0(M)\,.$$

Let $n\to\infty$ and then $\varepsilon\to 0$. Then

$$D_{R-D}(U_D^M(z)) \geqq \lim_{n=\infty}(D_{R-D_n}(U_{D_n}^M(z)) \geqq D_{R-D}(\lim_{n=\infty} U_{D_n}^M(z))\,.$$

Hence $\lim_{n=\infty} U_{D_n}^M(z)$ has M.D.I over $R-D$ with boundary value $U^M(z)$ on ∂D, whence $\lim_{n=\infty} U_{D_n}^M(z) = U_D^M(z)$ and $\lim_{n=\infty} U_{D_n}(z) \geqq U_D^M(z)$. Let $M\to\infty$. Then

$$\lim_{n=\infty} U_{D_n}(z) \geqq U_D(z)\,.$$

Next, put $M_n = \sup_{z\in R_n - R_0} U(z)$. Then clearly $U_{D_n}(z) = U_{D_n}^{M_n}(z) \leqq U_D^{M_n}(z)$. Let $n\to\infty$. Then $\lim_{n=\infty} U_{D_n}(z) \leqq U_D(z)$. Thus we have $\lim_{n=\infty} U_{D_n}(z) = U_D(z)$.

6. Equilibrium Potential of a closed subset A of B. Let A be a δ-closed set of B. Put $A_m = E\left[z\in\bar{R} : \delta(z, A) \leqq \frac{1}{m}\right]$. Then $R\cap A_m$ is a relatively closed set of R and $\bigcap_{m>0} A_m = A$. Let $\omega_{A_m,n}(z)$ be a harmonic function in $R_n - R_0 - A_m$ such that $\omega_{A_m,n}(z) = 0$ on ∂R_0, $\omega_{A_m,n}(z) = 1$ on ∂A_m and $\frac{\partial\omega_{A_m,n}(z)}{\partial n} = 0$ on $\partial R_n - A_m$. Then

$$\begin{aligned}D_{R_n-R_0-A_m}(\omega_{A_m,n}(z), \omega_{A_{m+i},n}(z)) &= \int_{\partial A_m\cap(R_n-R_0)} \frac{\partial\omega_{A_{m+i},n}(z)}{\partial n} ds \\ &= \int_{\partial A_{m+i}\cap(R_n-R_0)} \frac{\partial\omega_{A_{m+i},n}(z)}{\partial n} ds = D_{R_n-R_0-A_{m+i}}(\omega_{A_{m+i},n}(z))\,.\end{aligned}$$

Since $D(\omega_{A_m,n}(z))$ and $D(\omega_{A_{m+i}\,n}(z))$ converge as $n\to\infty$, we have $D_{R-R_0-A_m}(\omega_{A_{m+i},n}(z), \omega_{A_m}(z)) = D_{R-R_0-A_{m+i}}(\omega_{A_{m+i}}(z))$. Hence $D_{R-R_0-A_m}(\omega_{A_m}(z) - \omega_{A_{m+i}}(z)) = D_{R-R_0-A_m}(\omega_{A_m}(z)) - 2D_{R-R_0-A_m}(\omega_{A_m}(z), \omega_{A_{m+i}}(z)) + D_{R-R_0-A_m}$

$(\omega_{A_{m+i}}(z)) < D_{R-R_0-A_m}(\omega_{A_m}(z)) - D_{R-R_0-A_m}(\omega_{A_{m+i}}(z))$ and $D_{R-R_0-A_m}(\omega_{A_m}(z))$ is decreasing with respect to m. Therefore $\omega_{A_m}(z)$ converges to a function $\omega_A(z)$ in mean as $m\to\infty$. We call $\omega_A(z)=\lim_{m=\infty}\omega_{A_m}(z)$ the *equilibrium* potential of A. Suppose $\omega_A(z)>0$. Let $V(z)$ be a harmonic function in $R-G$ such that $V(z)=0$ on $\partial R_0+\partial G$ and $D(V(z))<\infty$, where G is a relatively closed set containing A. Then by lemma 1 $\omega_{A_m}(z)$ $(A_m\subset G)$ has M.D.I over $R-G$ among all functions having the boundary value $\omega_{A_m}(z)$ on ∂G. Hence

$$D(\omega_{A_n}(z)\pm\varepsilon V(z))\geqq D(\omega_{A_m}(z)),$$

for every small positive number ε. Since $\omega_{A_m}(z)$ converges to $\omega_A(z)$ in mean,

$$D(\omega_{A_m}(z)-\omega_A(z),\ V(z))\leqq\sqrt{D(\omega_{A_m}(z)-\omega_A(z))D(V(z))},$$

which implies $D(V(z),\ \omega_A(z))=0$. Since $V(z)$ is arbitrary, $\omega_A(z)$ has also M.D.I over $R-G$ among all functions having the boundary value $\omega_A(z)$ on ∂G. Therefore ${}_A\omega_A(z)=\omega_A(z)$. Hence if we take $G_\varepsilon=[z\in R:\ \omega_A(z)\geqq 1-\varepsilon]$, $\dfrac{\omega_A(z)}{1-\varepsilon}$ is *the equilibrium potential* of G_ε.

7. Integral Representation of Superharmonic Functions in $\bar{R}$.

Definition of $U_A(z)$ *for a* δ*-closed subset* A *of* B. $A_m=E\left[z\in R:\ \delta(z,A)\leqq\dfrac{1}{m}\right]$. Then A_m is relatively closed set and clearly $U_{A_m}(z)$ is decreasing as $m\to\infty$. We define $U_A(z)$ by $\lim_{m=\infty}U_{A_m}(z)$.

Theorem 4.
1) $N(z,\ p)$ $(p\in\bar{R})$ *is superharmonic in* R *and superharmonic in* $\bar{R}$, *more generally* $\int N(z,\ p)d\mu(p)$ *is superharmonic in* $\bar{R}$ *for* $\mu>0$.
2) $\omega_D(z)$ *and* $\omega_A(z)$ *are superharmonic in* $\bar{R}$.

Proof of 1). First, suppose $p\in R$. Since clearly $N(z,\ p)$ is superharmonic in R, it is sufficient to prove that $N(z,\ p)\geqq N_D(z,\ p)$ for every *compact* domain D. Since $N(z,\ p)$ has the minimal $*$-Dirichlet integral over R, we have by Green's formula and by Theorem 2

$$N(z,\ p)=\text{or}>\frac{1}{2\pi}\int_{\partial D}N(\zeta,\ p)\frac{\partial N^D(\zeta,\ z)}{\partial n}ds=N_D(z,\ p),$$

according as $p\in D$ or $p\notin D$.

Next, consider $p\in B$. Let $\{p_i\}$ be a fundamental sequence determining p. Then $N(z,\ p_i)$ tends to $N(z,\ p)$ on ∂D, hence

$$N(z,\ p)=\lim_{i=\infty} N(z,\ p_i)\geqq \frac{1}{2\pi}\int_{\partial D}\lim_{i=\infty} N(\zeta,\ p_i)\frac{\partial N^D(\zeta,\ z)}{\partial n}ds=N_D(z,\ p)\ .$$

Thus $N(z,\ p)(p\in \bar{R})$ is superharmonic in $\bar{R}$.

The approximation to $V(z)=\int N(z,\ p)\,d\mu(p)$ by a sequence of functions $V_n(z)$ $(n=1,\ 2,\ \cdots)$ of the form $V_n(z)=\sum_{i=1}^{n} c_i N(z,\ p_i)$ can be done in every compact part of R. $V_n(z)=\frac{1}{2\pi}\int_{\partial D} V_n(\zeta)\frac{\partial N^D(\zeta,\ z)}{\partial n}ds$, which implies by letting $n\to\infty$ $V(z)=\frac{1}{2\pi}\int_{\partial D} V(\zeta)\frac{\partial N^D(\zeta,\ z)}{\partial n}ds=V_D(z)$. Therefore $V(z)$ is superharmonic in $\bar{R}$.

Proof of 2). Let G be a compact domain and let $\omega_D^n(z)$ be a harmonic function in $R-R_n-D$ such that $\omega_D^n(z)=0$ on ∂R_0, $\omega_D^n(z)=1$ on $\partial D\cap(R_n-R_0)$ and $\frac{\partial\omega_D^n(z)}{\partial n}=0$ on ∂R_n-D. Then

$$\omega_D^n(z)\geqq\frac{1}{2\pi}\int_{\partial G\cap(R_n-R_0)}\omega_D^n(\zeta)\frac{\partial N_n^G(\zeta,\ z)}{\partial n}ds\,,$$

where $N_n^G(\zeta,\ z)$ is the $*$-Green's function of R_n-R_0-G with pole at z. Let $n\to\infty$. Then

$$\omega_D(z)\geqq\frac{1}{2\pi}\int_{\partial G}\omega_D(\zeta)\frac{\partial N^G(\zeta,\ z)}{\partial n}ds={}_G\omega_D(z)\ .$$

Hence $\omega_D(z)$ is superharmonic in $\bar{R}$.

Put $G=A_m$. Then $\omega_{A_m}(z)\geqq\frac{1}{2\pi}\int_{\partial G}\omega_{A_m}(\zeta)\frac{\partial N^G(\zeta,\ z)}{\partial n}ds={}_G\omega_{A_m}(z)\ .$

Let $m\to\infty$. Then $\omega_A(z)\geqq\frac{1}{2\pi}\int_{\partial G}\omega_A(\zeta)\frac{\partial N^G(\zeta,\ z)}{\partial n}ds={}_G\omega_A(z)\ .$

Thus $\omega_A(z)$ is also superharmonic in $\bar{R}$.

Theorem 5. *If $U(z)$ is positive harmonicin R and superharmonic in $\bar{R}$, then for a δ-closed subset A of B, we have*

1) *There exists a mass distribution μ on A such that*

$$U_A(z)=\frac{1}{2\pi}\int_A N(z,\ p)\,d\mu(p)\,,$$

for all point z in R. The total mass $\mu(A)$ is given by $\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial U_A(z)}{\partial n}ds$.

2) ${}_A\omega_A(z)=\omega_A(z)=\frac{1}{2\pi}\int_A N(z,\ p)\,d\mu(p)$ for $\omega_A(z)>0$.

2′) *If p is an ideal boundary point such that $\omega_p(z)>0$, then*

$$\omega_p(z)=KN(z,\ p)\,,\qquad K>0\,.$$

3) $$U(z)=\frac{1}{2\pi}\int_B N(z,\ p)\,d\mu(p)\,.$$

Proof. Put $A_m=E\left[z\in R:\ \delta(z,\ A)\leqq\frac{1}{m}\right]$ and $A_{m,n}=A_m\cap(R_n-R_0)$. Then by 5. $U_A(z)=\lim_{m=\infty}\lim_{n=\infty}U_{A_{m,n}}(z)$. Now $U(z)\geqq U_{A_m}(z)\geqq U_{A_{m,n}}(z)$ for $z\notin A_{m,n}$, $U(z)=U_{A_{m,n}}(z)$ for $z\in A_{m,n}(z)$ is continuous on $A_{m,n}(z)$, whence $U_{A_{m,n}}(z)$ is superharmonic at every point of $A_{m,n}$. Hence it can be proved by the method of F. Riesz-Frostmann that the functional

$$J(\mu)=\frac{1}{2}\frac{1}{4\pi^2}\int\int_{A_{m,n}}N(z,\ p)\,d\mu(p)\,d\mu(z)-\frac{1}{2\pi}\int_{A_{m,n}}U_{A_{m,n}}(z)\,d\mu(z)\,,$$

is minimized by a unique mass distribution on $\mu(A_{m,n})$ on $A_{m,n}$ among all non negative mass distributions. The function $V(z)$ given by $\frac{1}{2\pi}\int_{A_{m,n}}N(z,\ p)\,d\mu(p)$ is equal to $U(z)$ on $A_{m,n}$ except possibly a subset of capacity zero of $A_{m,n}$ and has the M.D.I, because $V(z)$ is a linear form of $N(z,\ p)$ $(p\in R)$. Therefore $U_{A_{m,n}}(z)=V(z)$, where the total mass is given by $\frac{1}{2\pi}\int\frac{\partial U_{A_{m,n}}(z)}{\partial n}ds$ for every n and m. Since $N(z,\ p)$ is a δ-continuous function of p for fixed z and the total mass is less than $\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds$, $\mu(A_{m,n})$ has an weak limit $\mu(A_m)$ on A_m as $n\to\infty$. Hence $U_{A_m}(z)=\frac{1}{2\pi}\int_{A_m}N(z,\ p)d\mu(p)$ and by letting $m\to\infty$, $U_A(z)=\frac{1}{2\pi}\int_A N(z,p)d\mu(p)$. 2) and 2′) are clear by the property of $\omega_A(z)$ and 3) is also clear, if we consider B as A.

8. Classifications of the Ideal Boundary Points.

Regular or Singular ideal Boundary Point. Take an ideal boundary point p as a closed subst A of B. Then we call p a *regular or singular* ideal boundary point according as $\omega_p(z)=0$ or $\omega_p(z)>0$.

In what follows, we shall consider another classification. We shall prove the following

Theorem 6. *Let $U(z)$ be a harmonic in R and superharmonic function in $\bar{R}$ and let A be a closed subset of capacity zero of $\bar{R}$. Then*

$$_AU_A(z)=U_A(z)\,.$$

Proof. Let G be a compact domain in R. Then

$$U(z)=V_G(z)+U'(z)\qquad\text{for}\quad z\in R-G\,,\tag{a}$$

where $V_G(z)$ is a harmonic function in $R-G$ such that $V_G(z)=U(z)$ on

$\partial G+\partial R_0$ and $V_G(z)$ has M.D.I over $R-G$ and $U'(z)$ is a harmonic function in $R--G$ such that $U'(z)=0$ on $\partial G+\partial R_0$ and $U'(z)$ is superharmonic in $\overline{R-G}$. In fact, let D be a domain in R. Then since $D+G\supset G$, by Lemma 1, $V_G(z)=V_{D+G}(z)$, where $V_{D+G}(z)$ is a harmonic function in $R-G-D$ such that $V_{D+G}(z)=V_G(z)$ on $\partial D+\partial G+\partial R_0$ and $V_{D+G}(z)$ has M.D.I over $R-G-D$. Now, since $U(z)$ is superharmonic in $\bar{R}$ and $V_G(z)=V_{G+D}(z)$,

$$U(z)=U'(z)+V_G(z)\geqq\frac{1}{2\pi}\int_{(\partial G-D)+(\partial D-G)}U(\zeta)\frac{\partial N^{D+G}(\eta,z)}{\partial n}ds$$

$$=\frac{1}{2\pi}\int_{(\partial G-D)+(\partial D-G)}(V_G(z)+U'(z))\frac{\partial N^{D+G}(\zeta,z)}{\partial n}ds=V_{G+D}(z)+U_D'(z).$$

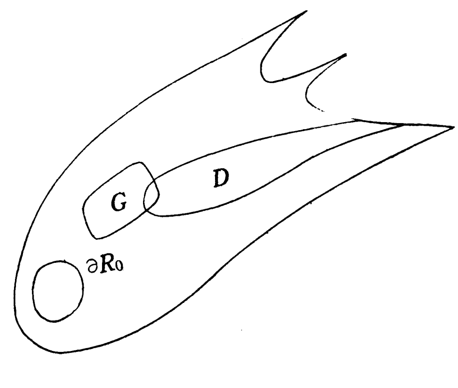
Hence $$U'(z)\geqq U_D'(z), \tag{b}$$

where $U_D'(z)$ is a harmonic function in $R-G-D$ such that $U_D'(z)=0$ $=U'(z)$ on $\partial G+\partial R_0-D$, $U_D'(z)=U'(z)$ on $\partial D-G$ and $U'(z)$ has M.D.I over $R-G-D$. This means that $U'(z)$ is superharmonic in $\overline{R-G}$.

Consider $A_{m,n}=A_m\cap(R_n-R_0)$ as D in (a). Then by (a)

$$U_{A_{m,n}}(z)=V_{A_{m,n}}(z)+U'_{A_{m,n}}(z)+(V_G-V_{A_{m,n}})(z)\quad\text{for } z\in R-A_{m,n}-G, \tag{c}$$

where $V_{A_{m,n}}(z)$ is a harmonic function in $R-G$ such that $V_{A_{m,n}}(z)$ $=U_{A_{m,n}}(z)$ on $\partial R_0+\partial G$ and $V_{A_{m,n}}(z)$ has M.D.I over $R-G$ and $U'_{A_{m,n}}(z)$ is a harmonic function in $R-G-A_{m,n}$ such that $U'_{A_{m,n}}(z)=0$ on $\partial R_0+\partial G$ $-A_{m,n}$, $U'_{A_{m,n}}(z)=U'(z)$ on $\partial A_{m,n}-G$ and $U'_{A_{m,n}}(z)$ has M.D.I over $R-G$ $-A_{m,n}$. Hence by (b) $U'_{A_{m,n}}(z)\leqq$ $U'(z)$.



And $(V_G-V_{A_{m,n}})(z)$ is a harmonic function in $R-G-A_{m,n}$ such that $(V_G-V_{A_{m,n}})(z)=0$ on $\partial R_0+\partial G-D$, $(V_G-V_{A_{m,n}})(z)=V_G(z)-V_{A_{m,n}}(z)$ ($V_G(z)=U(z)$ and ∂G) on $\partial A_{m,n}$ and $(V_G-V_{A_{m,n}})(z)$ has M.D.I over $R-G-A_{m,n}$. Clearly since $U(z)\geqq U_{A_{m,n}}(z)$, $0\leqq(V_G-V_{A_{m,n}}(z)\leqq M\omega'_{A_{m,n}}(z)$, where $M=\max_{z\in\partial G}V_G(z)$ and $\omega'_{A_{m,n}}(z)$ is the equilibrium potential of $A_{m,n}$ with respect to $R-G$.

Let $n\to\infty$. Then $U'_{A_{m,n}}(z)\uparrow U'_{A_m}(z)$, since $U'(z)$ is superharmonic in $R-G$. $U_{A_{m,n}}(z)\uparrow U_{A_m}(z)$ implies $V_{A_{m,n}}(z)\uparrow V_{A_m}(z)$. $(V_G-V_{A_{m,n}})(z)\to(V_G-V_{A_m})(z)$. Here $V_{A_{m,n}}(z)$ converges to $V_{A_m}(z)$ in mean, because $D_{R-G}(V_{A_{m,n}}(z))=\int_{\partial G}V_{A_{m,n}}(z)\frac{\partial V_{A_{m,n}}(z)}{\partial n}ds$ and ∂G is compact. Hence

$V_{A_m}(z)$ has also M.D.I over $R-G$ with boundary value $U_{A_m}(z)$ on ∂G and 0 on ∂R_0. Therefore

$$U_{A_m}(z) = V_{A_m}(z) + U'_{A_m}(z) + (V_G - V_{A_m})(z) . \tag{d}$$

Let $m \to \infty$. Then $V_{A_m}(z) \downarrow U_A(z)$, $V_{A_m}(z) \downarrow V_A(z)$, $U'_{A_m}(z) \downarrow U_A(z)$ and $0 = \lim_{m=\infty} (V_G - V_{A_m})(z) \leqq M\omega_A'(z) = 0$. Hence

$$U_A(z) = V_A(z) + U_A'(z) . \tag{e}$$

By (d) and (e), we have

$$U_{A_m}(z) - U_A(z) = V_{A_m}(z) - V_A(z) + (U'_{A_m}(z) - U_A'(z)) + (V_G - V_{A_m})(z) ,$$

where $V_{A_m}(z) = {}_G U_{A_m}(z)$ and $V_A(z) = {}_G U_A(z)$ by definition and the last two terms on the right hand side are non negative. Hence

$$U_{A_m}(z) - U_A(z) \geqq {}_G(U_{A_m}(z) - U_A(z)) .$$

Suppose $G = A_{m',n'}$ $(n' < m)$. Then by letting $n' \to \infty$, we have

$$U_{A_m}(z) - U_A(z) \geqq {}_{A_{m'}}(U_{A_m}(z) - U_A(z)) . \tag{f}$$

Proof of the theorem. Since $U_A(z)$ is representable in the form (e) for any compact domain G, $U_A(z)$ is clearly superharmonic in $\bar{R}$, that is $U_A(z) \geqq {}_G U_A(z) = V_A(z)$ for domain G. Hence ${}_{A_{m'}} U_A(z) \leqq U_A(z)$ for every m' and ${}_A U_A(z) \leqq U_A(z)$.

Let z be a point of R. Then, since $U_{A_m}(z) \downarrow U_A(z)$ as $m \to \infty$, for any given positive number ε, there exists a number m_0 depending on z such that

$$\varepsilon > U_{A_{m+i}}(z) - U_A(z) > 0 \qquad \text{for} \quad m+i \geqq m_0 .$$

Then by (f)

$$0 < {}_{A_{m'}}(U_{A_{m+i}}(z)) - U_A(z)) < U_{A_{m+i}}(z) - U_A(z) < \varepsilon .$$

On the other hand, by 6) of Theorem 3 ${}_{A_{m'}}(U_{A_{m+i}}(z)) = U_{A_{m+i}}(z)$ for $m+i \geqq m'$. Hence

$${}_{A_{m'}}(U_{A_{m+i}}(z) + U_A(z) - U_{A_{m+i}}(z)) \geqq U_{A_{m+i}}(z) - \varepsilon .$$

Thus by letting $\varepsilon \to 0$, ${}_{A_{m'}}(U_A(z)) \geqq U_A(z)$. Therefore ${}_A U_A(z) = U_A(z)$.

Putting $A = q$, we define the function $\Psi(q)$ of q in B as $\frac{1}{2\pi} \int_{\partial R_0} \frac{\partial N_q(z, q)}{\partial n} ds$. Then we have

Theorem 7.

1) *$\Phi(q)$ has only two possible values* 1 *and* 0.

2) *Denote by B_0 and B_1 the sets of points of B for which $\Psi(q)=0$ and $\Psi(q)=1$ respectively. Then $B=B_0+B_1$ and B_0 is void or an F_σ.*

We shall prove 1) in two cases as follows:

Case 1. *q is regular ideal boundary point,* i.e. $\omega_q(z)=0$.

Case 2. *q is a singular ideal boundary point,* i.e. $\omega_q(z)>0$.

Case 1. $\omega_q(z)=0$. We have $N_q(z, q)=\Psi(q)N(z, q)$ by 2) of Theorem 5 and ${}_qN_q(z, q)=\Psi^2(q)N(z, q)=\Psi(q)N(z, q)=N_q(z, q)$ by Theorem 6. Hence we have $\Psi(q)=0$ or 1.

Case 2. $\omega_q(z)>0$. In this case we have $N(z, q)=K\omega_q(z)=N_q(z, q)$ $=K_q\omega_q(z)=K\Psi(q)N_q(z, q)$ by 2′) of Theorem 5. Hence $\omega_q(z)>0$ implies $\Psi(q)=1$.

Proof of 2). The set Γ_m is defined as the set (possible void) of all points q of B such that $\Psi(A_m(q))=\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial N_{A_m(q)}(z, q)}{\partial n}ds\leq\frac{1}{2}$ (this means $\Psi(q)=0$), where $A_m(q)=E\left[z\in\bar{R}: \delta(z, q)\leq\frac{1}{m}\right]$. Then clearly $B_0=\bigcup_{m>1}\Gamma_m$. We shall show that $\Psi(A_m(q))$ is a lower semicontinuous function of q.

By definition $N_{A_m(q)}(z, q)=\lim_{n=\infty}N_{A_{m,n}(q)}(z, q)$, where $A_{m,n}(q)=A_m(q)\cap(R_n-R_0)$. Hence, for any given positive number ε, there exists a number n such that $\Psi(A_{m,n}(q))=\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial N_{A_{m,n}(q)}(z, q)}{\partial n}ds\geq\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial N_{A_m(q)}(z, q)}{\partial n}ds$ $-\varepsilon=\Psi(A_m(q))-\varepsilon$. Suppose $q_i\to q$. Then $A_{m,n}(q_i)\to A_{m,n}(q)$. Hence by the compactness of $A_{m,n}(q)$

$$\varliminf_{i=\infty}N_{A_m(q_i)}(z, q_i)\geq\varliminf_{i=\infty}\int_{\partial A_{m,n}(q_i)}N(\zeta, q_i)\frac{\partial N^{A_{m,n}(q_i)}(\zeta, z)}{\partial n}ds$$

$$=\int_{\partial A_{m,n}(q)}N(\zeta, q)\frac{\partial N^{A_{m,n}(q)}(\zeta, z)}{\partial n}ds=N_{A_{m,n}(q)}(z, q)\,.$$

Consequently $\varliminf_{i=\infty}\Psi(A_m(q_i))\geq\Psi(A_m(q))-\varepsilon$, whence by letting $\varepsilon\to0$

$$\varliminf_{i=\infty}\Psi(A_m(q_i))\geq\Psi(A_m(q))\,.$$

Therefore $\Psi(A_m(q))$ is lower semicontinuous with respect to q, whence Γ_m is closed and B_0 is an F_σ.

9. Canonical Distributions. We shall consider properties of B_0 and B_1.

Theorem 8.

1) *Cap* $(B_0)=0$.

2) *If $U(z)$ is given by* $\frac{1}{2\pi}\int_{B_0}N(z, p)\,d\mu(p)$, $U_{B_0}(z)=0$

3) *Every function* $U(z)$ *which is harmonic in* R *and superharmonic in* $\bar{R}$ *is representable by a mass distribution on* B_1 *such that*

$$U(z)=\frac{1}{2\pi}\int_{B_1} N(z,p)\,d\mu(p)\,.$$

Proof of 1). The set Γ_m, being closed and compact, may be covered by a finite number of its closed subsets whose diameters are less than $\frac{1}{m}$. It is sufficient, by 5) of Theorem 4, to prove 1) for any closed subset A whose diameter is less than $\frac{1}{m}$, of Γ_m. Assume $\mathrm{Cap}(A)>0$. Then $0<{}_A\omega_A(z)=\omega_A(z)=\frac{1}{2\pi}\int_A N(z,p)\,d\mu(p)$ by 1) of Theorem 5. On the other hand, since ${}_A\omega_A(z)=\lim_{m=\infty}\lim_{n=\infty} {}_{A_{m,n}}\omega_A(z)$, for any given positive number ε, there exist numbers m and n such that

$$\mathrm{Cap}\,(A)=\int_{\partial R_0}\frac{\partial\omega_A(z)}{\partial n}ds\leqq\int_{\partial R_0}\frac{\partial({}_{A_{m,n}}\omega_A(z))}{\partial n}ds+\varepsilon\,,$$

where $A_m=E\left[z\in\bar{R}:\delta(z,A)\leqq\frac{1}{m}\right]$ and $A_{m,n}=A_m\cap(R_n-R_0)$.

Now $\omega_A(z)$ can be approximated on $A_{m,n}$ by a sequence of functions $V_l(z)=\sum_{i=1}^{l}c_iN(z,q_i)\,(q_i\in A)$ $(l=1,2,\cdots)$. Then by Fatou's lemma

$$\begin{aligned}\mathrm{Cap}\,(A)&=\int_{\partial R_0}\frac{\partial\omega_A(z)}{\partial n}ds\leqq\lim_{l=\infty}\int_{\partial R_0}\frac{\partial V_l(z)}{\partial n}ds\leqq\frac{1}{2}\int_{\partial R_0}\frac{\partial\omega_A(z)}{\partial n}ds+\varepsilon\\&=\frac{1}{2}\mathrm{Cap}\,(A)+\varepsilon\,,\end{aligned}$$

because $A_m\subset v_m(q_i)=E\left[z\in\bar{R}:\delta(z,q_i)\leqq\frac{1}{m}\right]$ for every $q_i\in A$ implies $\int_{\partial R_0}\frac{\partial N_{v_m(q_i)}(z,q_i)}{\partial n}ds\leqq\frac{1}{2}\int_{\partial R_0}\frac{\partial N(z,q_i)}{\partial n}ds$. This is absurd. Hence $\mathrm{Cap}\,(A)=0$, $\mathrm{Cap}\,(\Gamma_m)=0$ and $\mathrm{Cap}\,(B_0)=0$.

Proof of 2). As above, we have for $A\subset\Gamma_m$, $U_A(z)\leqq U_{A_m}(z)$ and $\int_{\partial R_0}\frac{\partial U_A(z)}{\partial n}ds\leqq\int_{\partial R_0}\frac{\partial U_{A_m}(z)}{\partial n}ds\leqq\frac{1}{2}\int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds$, whence *mass of* $U_A(z)\leqq\frac{1}{2}$ *mass of* $U(z)$ and *mass of* ${}_AU_A(z)\leqq\frac{1}{2}$ *mass of* $U_A(z)$. On the other hand, since $\mathrm{Cap}\,(A)=0$, we have by Theorem 6 ${}_AU_A(z)=U_A(z)$. Hence $U_A(z)=0$, $U_{\Gamma_m}(z)=0$ and $U_{B_0}(z)=0$.

Proof of 3. Suppose $U(z)=\frac{1}{2\pi}\int_{B_0}N(z,p)\,d\mu(p)$. Put $\Gamma_{m,n}=E\Big[z\in B:$ $\delta(z,\Gamma_m)\leqq\frac{1}{n}\Big]$. Let z be a point R. Since $U_{\Gamma_m}(z)=0$, for any given positive number ε, there exists a number $n(m)$ such that $U_{\Gamma_{m,n}}(z) \leqq \frac{\varepsilon}{2^m}$ for $n \geqq n(m)$. For each m select $\Gamma_m'(=\Gamma_{m,n})$ in this fashion. Put $C_m = \sum_{i=1}^{m} \Gamma_i'$. Then C_m are closed and form a increasing sequence as $m \to \infty$. Denote by $\tilde{A}_m$ the closure of the complement of C_m in B. Then the distance between $\tilde{A}_m$ and Γ_m is at least $\frac{1}{n(m)}$. Thus $\{\tilde{A}_m\}$, which forms a descending sequence, has an intersection $\tilde{A}$ which is closed and, having no point in common with any Γ_m, is a subset of B_1.

Now $U_{C_m}(z) \leqq \sum_{i=1}^{m} U_{\Gamma_i'}(z) \leqq \sum_{i=1}^{m} 2^{-i}\varepsilon \leqq \varepsilon$. Observing $\tilde{A}_m + C_m = B$, we obtain

$$U(z) = U_B(z) = U_{\tilde{A}_m + C_m}(z) \leqq U_{\tilde{A}_m}(z) + U_{C_m}(z) \leqq U_{\tilde{A}_m}(z) + \varepsilon .$$

Let $m \to \infty$ and then $\varepsilon \to 0$. Then $\bigcap_{m>1} \tilde{A}_m \subset B_1$ and $U(z) = U_{B_1}(z) = \frac{1}{2\pi} \int_{B_1} N(z, p)\, d\mu(p)$. Thus $U(z)$ is representable by a mass distribution on B_1 without any change of $U(z)$.

Proof of 3). Suppose that $U(z)$ is harmonic in R and superharmonic in $\bar{R}$. Then $U(z) = \frac{1}{2\pi} \int_B N(z, p)\, d\mu(p) = \frac{1}{2\pi} \int_{B_1} N(z, p)\, d\mu_1(p) + \frac{1}{2\pi} \int_{B_0} N(z, p)\, d\mu_0(p)$ by 3) of Theorem 5. As above $\int_{B_0} N(z, p)\, d\mu_0(p) = \int_{B_1} N(z, p)\, d\mu_1'(p)$. Then $U(z) = \frac{1}{2\pi} \int_{B_1} N(z, p)\, d(\mu_1 + \mu_1')(p)$. Thus we have 3). We call such distribution on B_1 *canonical.*

10. Minimal Functions. Let $U(z)$ be a function which is harmonic in R and superharmonic in $\bar{R}$. If $U(z) \geqq V(z) \geqq 0$ implies $V(z) = KU(z)$ $(0 \leqq K \leqq 1)$ for every function $V(z)$ such that both $U(z) - V(z)$ and $V(z)$ are harmonic in R and superharmonic in $\bar{R}$, $U(z)$ is called a *minimal function.*

Theorem 9.

1) *Let $U(z)$ be a minimal function such that $U_A(z) > 0$ and $U(z) - U_A(z)$ are superharmonic function in $\bar{R}$. Then* $U(z) = \left(\frac{1}{2\pi} \int_{\partial R_0} \frac{\partial U(z)}{\partial n} ds\right) N(z, p)$ $(p \in A)$.

2) *Every minimal function is a multiple of some $N(z, p)$ $(p \in B_1)$.*

3) *$N(z, p)$ is minimal or not according as $\Psi(p) = 1$ or $= 0$.*

Proof of 1). $U_A(z) = \frac{1}{2\pi} \int_A N(z, p)\, d\mu(p) > 0$ implies $\mu(A) > 0$ and $A \cap B_1 \neq 0$. Hence A has a closed subset A_1 for which $\mu(A_1) > 0$. A_1, being compact, can be covered by a finite number of its closed subsets, all of them having diameters less than some selected positive number. At least one such subset has a positive μ mass. We select a particular such and call it A_2. By proceeding in this way inductively, it is possible to construct a descending sequence $A_1, A_2, \cdots$, of closed sets of A whose diameters approach zero and each of which has a positive μ mass. Let p be the unique point common to all A_n and B_1. Now since $\mu(A_n)>0$, the integral $\frac{1}{2\pi}\int_{A_n} N(z,p)\,d\mu(A_n)$ extended over A_n instead of A represents a superharmonic function $U_n(z)$ such that mass of $U(z) \geqq$ mass of $U_A(z) \geqq$ mass of $U_n(z)$, because $U(z)-U_A(z)$ is superharmonic in $\bar{R}$, i.e. $U(z)-U_A(z)$ is represented by a positive mass distribution. Hence the minimality of $U(z)$ implies $U_n(z)=C_nU(z)$ $(0<C_n\leqq 1)$. If we write $\mu_n'(e)=\mu\cdot\frac{1}{C_n}$, $\{\mu_n'(e)\}$ has as an weak limit a point mass of amount $\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds$ located at p. Thus we have $U(z)=\left(\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds\right)N(z,p)$ $(p\in A)$.

Proof of 2). Take B as A. Then we have at once 2).

Proof of 3). Suppose $p\in B_1$ and a function $U(z)$ such that both $U(z)$ and $0<N(z,p)-U(z)=V(z)$ are harmonic in R and superharmonic in $\bar{R}$. Then

$$N_p(z,p)=U_p(z)+V_p(z)=U(z)+V(z)=N(z,p)\,,$$
$$U_p(z)\leqq U(z),\quad V_p(z)\leqq V(z),\quad \text{whence}\quad U_p(z)=U(z)\quad\text{and}\quad V_p(z)=V(z)\,.$$

Hence by 1) of Theorem 5 $U(z)=U_p(z)=K_1V(z,p)$ and $V(z)=V_p(z)=K_2N(z,p)$. Thus $N(z,p)$ is minimal.

Next, suppose that $p\in B_0$ and $N(z,p)$ is minimal. Then $N(z,p)$ is representable by 3) of Theorem 8 by a mass distribution on B_1, that is $N(z,p)=\int_{B_1}N(z,p)\,d\mu(p)$. If μ is a point mass at $q\in B_1$, $N(z,p)=N(z,q)$. This implies $p=q\in B_1$. This is absurd. Hence μ is not a point mass. As 1) of this Theorem we can select a decesending sequence of closed subsets $\{A_n\}$ of B_1 such that $\mu(A_n)>0$ and diameters of $\{A_n\}$ tend to zero as $n\to\infty$. Then the restriction of μ mass on A_n represents a superharmonic function $V_n(z)$ such that $N(z,p)-V_n(z)$ is superharmonic in $\bar{R}$. Hence as 1) we have $N(z,p)=N(z,p^*)$, i.e. $p^*=p$, where $p^*=\bigcap_{n>0}A_n\subset B_1$. This contradicts $p\in B_0$. Hence $N(z,p)$ is non-minial.

By preceeding paragraphs we have the shema as follows:

$$\text{Ideal boundary point}\begin{array}{l}\nearrow \text{Regular I.B.P} \longrightarrow B_0 \text{ (non-minim point)}\\ \qquad\qquad\qquad\quad\searrow\\ \searrow \text{Singular I.B.P} \longrightarrow B_1 \text{ (minimal point)}\end{array}$$

We see easily that if $R \notin 0_{AD}$,[5] R has no singular ideal boundary point and if R is a Riemann surface of finite connectivity, R has no point of B_0.

In what follow, we shall prove useful properties of points of $R+B_1$.

Theorem 10.

1) *Let* $V_m(p)=E[z \in R: N(z, p \geqq m]$ *and* $v_n(p)=E\left[z \in \bar{R}: \delta(z, p) \leqq \frac{1}{n}\right]$ *and suppose* $p \in R+B_1$. *Then*

$$V_{V_m(p)}(z, p)=N(z, p) \textit{ for very } m \textit{ less than } M^*=\sup_{z \in R} N(z, p).$$

Hence $N(z, p)=m\omega_{V_m(p)}(z)$.

2) *For every* $V_m(p)$ $p \in R+B_1$ *there exists a number* n *such that*

$$V_m(p) \supset (R \cap v_n(p)) .$$

Proof. Since $N(z, p)$ $p \in R$ has the minimal $*$-Dirichlet integral over R, 1) is clear for $p \in R$ and since $N(z, p)$ has its pole at p, 2) is also evident for $p \in R$. Hence we have only to prove for $N(z, p)$ $p \in B_1$.

Proof of 1). First we remark that $p \in B_1$ and $\omega_p(z)=0$ imply $\sup\limits_{z \in R} N(z, p)=M^*=\infty$. In fact, suppose $N(z, p) \leqq M<\infty$ and $\omega_p(z)=0$. Then $N_p(z, p) \leqq M\omega_p(z)=0$, whence $p \in B_0$.

Therefore we shall prove 1) in two cases as follows:

Case 1. $p \in B_1$, $\omega_p(z)=0$ *and* $\sup\limits_{z \in R} N(z, p)=\infty$.

Case 2. $p \in B_1$ *and* $\omega_p(z)>0$.

Case 1. $p \in B_1$, $\omega_p(z)=0$ *and* $\sup\limits_{z \in R} N(z, p)=\infty$. Put $\lim\limits_{n=\infty} N_{v_n(p)-V_m(p)}(z, p)$ $=N'(z, p)$. Then, since $v_n(p) \supset v_n(p)-V_m(p)$, $N'(z, p)$ has no mass excep p. Hence $N'(z, p)=KN(z, p)$ $(0 \leqq K<1)$. But $\sup\limits_{z \in R} N(z, p)=\infty$ and $\sup\limits_{z \in R} N'(z, p) \leqq m$ implies $N'(z, p)=0$. On the other hand, $N(z, p)$ $=N_p(z, p) \leqq \lim\limits_{n=\infty} N_{v_n(p) \cap V_m(p)}(z, p)+N'(z, p) \leqq N(z, p)$. Therefore

$$N(z, p) \geqq N_{V_m(p)}(z, p) \geqq \lim_{n=\infty} N_{v_n(p) \cap V_m(p)}(z, p) \geqq N(z, p) ,$$

whence $N(z, p)=N_{V_m(p)}(z, p)$.

Case 2. $p \in B_1$ *and* $\omega_p(z)>0$. In this case $N(z, p)=K\omega_p(z)$. Hence our assertion is evident.

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5) O_{AD} is the class of Riemann surfaces on which no non constant Dirichlet Bounded analytic function exists.

Proof of 2). Since $N_{V_m(p)}(z, p)=N(z, p)$ has M.D.I over $R-V_m(p)$, $\frac{N(z, p)}{m}$ can be considerd as the equilibrium potential of $V_m(p)$. Hence we can suppose by 1) of Theorem 1 that $V_m(p)$ is regular, that is, $\int_{\partial V_m(p)} \frac{\partial N(z, p)}{\partial n} ds=2\pi$.

Let q be a point R not contained in $V_m(p)$. Let $N_n(z, p)$ be a harmonic function in $R_n-R_0-V_m(p)$ such that $N_n(z, p)=0$ on ∂R_0, $N_n(z, p)=m$ on $\partial V_m(p)$ and $\frac{\partial N_n(z, p)}{\partial n}=0$ on $\partial R_n-V_m(p)$. Let $N_n(z, q)$ be a function in R_n-R_0 such that $N_n(z, q)=0$ on ∂R_0, $\frac{\partial N_n(z, q)}{\partial n}=0$ on ∂R_n and $N_n(z, q)$ is harmonic in R_n-R_0 except q where $N_n(z, q)$ has a logarithmic singularity. Then clearly $\lim_{n=\infty} N_n(z, p)=N(z, p)$, because $\frac{N(z, p)}{m}=\omega_{V_m(p)}(z)$. $N_n(z, q)$ converges to a function $N(z, q)$.

By Green's formula

$$\int_{\partial V_m(p)\cap(R_n-R_0)} N_n(z, q)\frac{\partial N_n(z, p)}{\partial n} ds=2\pi N_n(q, p)\,.$$

Since $V_m(p)$ is regular and $N_n(z, q)$ is uniformly bounded on $\partial V_m(p)$, we have by letting $n\to\infty$

$$\begin{aligned}\frac{1}{2\pi}\lim_{n=\infty}\int_{\partial V_m(p)\cap(R_n-R_0)} N_n(z, p)\frac{\partial N_n(z, p)}{\partial n} ds &= \frac{1}{2\pi}\int_{\partial V_m(p)} N(z, q)\frac{\partial N(z, p)}{\partial n} ds\\ &= N(q, p)\,. \end{aligned}\tag{5}$$

Assume that 2) is fales. Then there exists a sequence of point $\{q_i\}$ such that $q_i\notin V_m(p)$ and $\lim_{i=\infty}\delta(q_i, p)=0$. If $M^*=\infty$ (resp. $M^*<\infty$), let $m'=2m$ $\left(\text{resp. } m'=m^*: M^*-\frac{\delta}{2}>m^*>m+\frac{\delta}{2},\ \text{where } \delta=\frac{M^*-m}{2}\right)$ and suppose that $V_{m'}(p)$ is regular. Then $V_m(p)\supset V_{m'}(p)\not\ni q_i$. Since $\int_{\partial V_{m'}(p)}\frac{\partial N(z, p)}{\partial n} ds=2\pi$, there exists a number n_0 such that

$$\int_{\partial V_{m'}(p)\cap(R_n-R_0)}\frac{\partial N(z, p)}{\partial n} ds>\frac{3\pi}{2}\left(\text{resp. } 2\pi-\varepsilon_0,\ \text{where } 0<\varepsilon_0<\frac{\pi\delta}{2\left(m+\frac{\delta}{4}\right)}\right)\quad \text{for}\quad n\geqq n_0,$$

Now by 5)

$$\int_{\partial V_{m'}(p)\cap(R_{n_0}-R)} N(z, q_i)\frac{\partial N(z, p)}{\partial n} ds<\int_{\partial V_{m'}(p)} N(z, q_i)\frac{\partial N(z, p)}{\partial n} ds=N(q_i, p)<m\,.$$

Hence there exists at least one point z_i on $\partial V_{m'}(p)\cap(R_{n_0}-R_0)$ such that $N(z, q_i)<\frac{4m}{3}\left(\text{resp.} <m\left(\frac{2\pi}{2\pi-\varepsilon_0}\right)\leqq m+\frac{\delta}{4}\right)$. Let i tend ∞. Then we

have $N(z_0, p) < \frac{4m}{3}\left(\text{resp.} < m + \frac{\delta}{4}\right)$, where z_0 is one of the limiting points of $\{z_i\}$. This contradicts $N(z_0, p) = m'$. Hence we have 2).

11. The $*$–Green's Function $N(z, q)$ in $\bar{R}$.

We give definition of $N(p, q)$ in three cases as follows:

Case 1. *$N(p, q)$ when p or $q \in R$.*

Case 2. *$N(p, q)$ for $p \in (R+B_1)$ and $q \in \bar{R}$.*

Case 3. *$N(p, q)$ for $p \in B_0$ and $q \in \bar{R}$.*

Definition of $N(p, q)$ in case 1: p or q *is contained* in R. If two points p and q are contained in R, we have by definition $N_n(p, q) = N_n(q, p)$, where $N_n(z, p)$ and $N_n(z, q)$ are $*$–Green's functions of $R_n - R_0$ with poles at p and q respectively. Hence, by letting $n \to \infty$, $N(p, q) = N(q, p)$. Next, suppose $p \in B$ and $q \in R$. Let $\{p_i\}$ be one of fundamental sequences determining p. Then, since $N(p_i, q) = N(q, p_i)$ and since $N(z, p_i)$ converges to $N(z, p)$ uniformly in every compact set of R, $N(p_i, q)$ has a limit denoted by $N(p, q)$ as $p_i \to p$. More generally, suppose that a sequence $\{p_i\}$ of $\bar{R}$ tends to p with respect to δ–metric and that q belongs to R. Then we have $N(q, p) = \lim_{i=\infty} N(q, p_i) = \lim_{i=\infty} N(p_i, q)$. Hence $N(z, q)$ $(q \in R)$ has a limit when z tends to $p \in \bar{R}$. In this case we define the value of $N(z, q)$ at p by this limit denoted by $N(p, q)$. Thus we have the following

Lemma 1. *If at least one of two points p and q is contained in R, then*

$$N(p, q) = N(q, p)\,.$$

$N(z, q)$ is defined in $\bar{R}$ for $q \in R$ but $N(z, q)$ has been defined only in R for $q \in B$. In the sequel, we shall define $N(z, q)$ in $\bar{R}$ for $q \in B$. At first, consider case 2. For this purpose, we shall prove the following

Lemma. 2. *Let $V_m(p)$ be the set $E[z \in R: N(z, p) \geqq m]$ for $p \in B_1$. Then $V_m(p)$ may consist of at most enumerably infinite number of domains D_l $(l = 1, 2, \cdots)$. Then*

1) *The Dirichlet integral of $N(z, p)$ taken over $R - V_m(p)$ is $2\pi m$ for every $m < M^* = \sup_{z \in R} N(z, p)$.*

2) *Let D_l be a component of $V_m(p)$. Then D_l contains a subset D_l' of $V_{m'}(p)$ for $m': m < m' < M^*$.*

3) *For D_l of regular domain $V_m(p)$, the Dirichlet integral of $N(z, p)$ taken over $D_l - D_l'$ is $2\pi(m'-m) \int_{\partial D_l} \frac{\partial N(z, p)}{\partial n} ds$ and*

$$\lim_{n=\infty} \int_{\partial D_l \cap (R_n - R_0)} \frac{\partial U_n(z)}{\partial n} ds = \lim_{n=\infty} \int_{\partial D_l' \cap (R_n - R_0)} \frac{\partial U_n(z)}{\partial n} ds = \int_{\partial D_l} \lim_{n=\infty} \frac{\partial U_n(z)}{\partial n} ds$$
$$= \int_{\partial D_l'} \lim_{n=\infty} \frac{\partial U_n(z)}{\partial n} ds = \int_{\partial D_l} \frac{\partial N(z, p)}{\partial n} ds,$$

where $U_n(z)$ is a harmonic function in $(D_l - D_l') \cap (R_n - R_0)$ such that $U_n(z) = m$ on ∂D_l, $U_n(z) = m'$ on $\partial D_l'$ and $\frac{\partial U_n(z)}{\partial n} = 0$ on $\partial R_n \cap (D_l - D_l')$.

Proof of 1). $p \in B_1$ implies by 1) of Theorem 10, that $N_{V_m(p)}(z, p) = N(z, p)$. Hence $\frac{N(z, p)}{m}$ is the equilibrium potential of $V_m(p)$. Therefore, $N(z, p) = \lim_{n=\infty} U_n'(z)$, where $U_n'(z)$ is a harmonic function in $R_n - R_0 - V_m(p)$ such that $U_n'(z) = 0$ on ∂R_0, $U_n'(z) = m$ on $\partial V_m(p)$ and $\frac{\partial U_n'(z)}{\partial n} = 0$ on $\partial R_0 \cap (R - V_m(p))$. The Dirichlet integral of $U_n'(z)$ over $R_n - R_0 - V_m(p)$ is $m \int_{\partial V_m(p) \cap (R_n - R_0)} \frac{\partial U_n(z)}{\partial n} ds$. Since $D(U_n'(z))$ is increasing with respect to n and $U_n'(z)$ tends to $N(z, p)$ as $n \to \infty$,

$$\lim_{n=\infty} D_{R_n - R_0 - V_m(p)}(U_n'(z)) = D_{R - V_m(p)}(N(z, p)) = \lim_{n=\infty} m \int_{\partial R_0} \frac{\partial U_n'(z)}{\partial n} ds$$
$$= m \int_{\partial R_0} \frac{\partial N(z, p)}{\partial n} ds = 2\pi m .$$

Proof of 2). Assume that D_l has no point of $V_{m'}(p)$ $(m' > m)$. Put $N'(z, p) \equiv m$ in D_l and $N'(z, p) = N(z, p)$ for $z \in (R - D_l)$. Then $D(N'(z, p)) < D(N(z, p))$. This contradicts that $\frac{N(z, p)}{m}$ is the equilibrium potential of $V_m(p)$. Hence we have 2).

Proof of 3). Since $\frac{N(z, p)}{m}$ can be considered as the equilibrium potential of $V_m(p)$, $N(z, p)$ has M.D.I over $V_m(p) - V_{m'}(p)$ among all functions having the boundary values m on $\partial V_m(p)$ and m' on $\partial V_{m'}(p)$ respectively, whence $N(z, p)$ has also M.D.I over $D_l - D_l'$ among all functions with values m on ∂D_l and m' on $\partial D_l'$. Hence $U_n(z) \to N(z, p)$ as $n \to \infty$. Since $D(U_n(z))$ is increasing with respect to n and by Fatou's lemma, we have

$$D_{D_l - D_l'}(N(z, p)) = \lim_{n=\infty} D_{(D_l - D_l') \cap (R_n - R_0)}(U_n(z)) = (m' - m) \lim_{n=\infty} \int_{\partial D_l \cap (R_n - R_0)} \frac{\partial U_n(z)}{\partial n} ds$$

$$= (m'-m) \lim_{n=\infty} \int_{\partial D_l' \cap (R_n - R_0)} \frac{\partial U_n(z)}{\partial n} ds \geqq (m'-m) \int_{\partial D_l} \lim_{n=\infty} \frac{\partial U_n(z)}{\partial n} ds$$

$$= (m'-m) \int_{\partial D_l} \frac{\partial N(z, p)}{\partial n} ds . \tag{6}$$

$$D_{D_l - D_l'}(N(z, p)) \geqq (m'-m) \int_{\partial D_l} \lim_{n=\infty} \frac{\partial N(z, p)}{\partial n} ds = (m'-m) \int_{\partial D_l'} \frac{\partial N(z, p)}{\partial n} ds .$$

On the other hand, by 1) and by the regularity of $V_m(p)$ and $V_{m'}(p)$

$$\sum_l D_{D_l - D_l'}(N(z, p)) = D_{V_m(p) - V_{m'}(p)}(N(z, p)) = 2\pi(m'-m)$$

$$= (m'-m) \int_{\partial R_0} \frac{\partial N(z, p)}{\partial n} ds = (m'-m) \int_{\partial V_m(p)} \frac{\partial N(z, p)}{\partial n} ds$$

$$= (m'-m) \int_{\partial V_{m'}(p)} \frac{\partial N(z, p)}{\partial n} ds = (m-m') \sum_l \int_{\partial D_l} \frac{\partial N(z, p)}{\partial n} ds$$

$$= (m'-m) \sum_l \int_{\partial D_l'} \frac{\partial N(z, p)}{\partial n} ds . \tag{7}$$

If $D_{D_l - D_l'}(N(z, p)) > (m'-m) \int_{\partial D_l} \frac{\partial N(z, p)}{\partial n} ds$ or $(m'-m) \int_{\partial D_l'} \frac{\partial N(z, p)}{\partial n} ds$ for at least one D_l or D_l' respectively, (6) will be a contradiction. Hence

$$D_{D_l - D_l'}(N(z, p)) = (m'-m) \int_{\partial D_l} \frac{\partial N(z, p)}{\partial n} ds = (m'-m) \int_{\partial D_l'} \frac{\partial N(z, p)}{\partial n} ds$$

for every D_l and D_l'. Therefore

$$D_{D_l - D_l'}(N(z, p)) = \lim_{n=\infty} D_{D_l - D_l'}(U_n(z)) = (m'-m) \lim_{n=\infty} \int_{\partial D_l \cap (R_n - R_0)} \frac{\partial U_n(z)}{\partial n} ds$$

$$= (m'-m) \lim_{n=\infty} \int_{\partial D_l' \cap (R_n - R_0)} \frac{\partial U_n(z)}{\partial n} ds = (m'-m) \int_{\partial D_l} \frac{\partial N(z, p)}{\partial n} ds$$

$$= (m'-m) \int_{\partial D_l} \lim_{n=\infty} \frac{\partial U_n(z)}{\partial n} ds = (m'-m) \int_{\partial D_l'} \lim_{n=\infty} \frac{\partial U_n(z)}{\partial n} ds .$$

Thus we have 3).

Lemma. 3. *Suppose $p \in B_1$ and $q \in \bar{R}$. Let $V_m(p)$ and $V_{m'}(p)$ be regular domains with m and m' such that $\sup_{z \in R} N(z, p) > m' > m$, i.e. $V_m(p) \supset V_{m'}(p)$. Then*

$$2\pi N^{V_{m'}(p)}(p, q) = \int_{\partial V_{m'}(p)} N(z, q) \frac{\partial N(z, p)}{\partial n} ds \geqq \int_{\partial V_m(p)} N(z, q) \frac{\partial N(z, p)}{\partial n} ds$$

$$= 2\pi N^{V_m(p)}(p, q) .$$

Proof. Let D be one of D_l which is a component of $V_m(p)$ and D' be the set of $V_m'(p)$ contained in D. Let $N_n^D(\zeta, z)$ be the $*$-Green's function of $D\frown(R_n-R)$, that is, $N_n^D(\zeta, z)=0$ on $\partial D\frown(R_n-R_0)$, $\dfrac{\partial N_n^D(\zeta, z)}{\partial n}ds=0$ on ∂R_n-D and $N_n^D(\zeta, z)$ is harmonic in $D\frown(R_n-R_0)$ except a logarithmic singularity at z. Then for given n_0 there exist constants L and n_1 such that $N_n^D(\zeta, z)\leqq L$ in $D\frown(R_{n_0}-R_0)$ for $n\geqq n_1$.

Let $U_n(\zeta)$ be the function defined in 3) of lemma e, i.e. $U_n(z)=m$ on ∂D, $U_n(\zeta)=m'$ on $\partial D'$ and $\dfrac{\partial U_n(\zeta)}{\partial n}=0$ on $\partial R_n\frown(D-D')$. Then, since $U_n(\zeta)-m=m'-m$ on $\partial D'$ and $\dfrac{\partial N_n^D(\zeta, z)}{\partial n}ds=\dfrac{\partial U_n(\zeta)}{\partial n}=0$ on $\partial R_n\frown(D-D')$, there exist suitable constants δ, and n_1' by the maximum principle such that

$$N_n^D(\zeta, z)<\frac{L}{\delta}(U_n(\zeta)-m) \qquad \text{in} \quad D\subset(R_{n_0}-R_0) \quad \text{for } n\geqq n_1'.$$

Hence

$$0\leqq\frac{\partial N_n^D(\zeta, z)}{\partial n}<\frac{L}{\delta}\frac{\partial U_n(\zeta)}{\partial n} \qquad \text{on} \quad \partial D\frown(R_{n_0}-R_0) \quad \text{for } n\geqq n_1'.$$

Therefore by 3) of lemma 2

$$\lim_{n=\infty}\int_{\partial D\frown(R_n-R_0)}\frac{\partial N_n^D(\zeta, z)}{\partial n}ds=\int_{\partial D}\lim_{n=\infty}\frac{\partial N_n^D(\zeta, z)}{\partial n}ds. \tag{8}$$

Suppose $q\in R$ and let $N_{D,n}(z, q)$ be a harmonic function in $D\frown(R_n-R_0)$ such that $N_{D,n}(z, q)=N(z, q)$ $\partial D\frown(R_n-R_0)+\partial R_0$ and $\dfrac{\partial N_{D,n}(z, q)}{\partial n}=0$ on $\partial R_n\frown D$. Then $N_{D,n}(z, q)$ converges (converges in mean) to a function $N_D(z, q)$ which is called *the solution of the $*$-Dirichlet problem with boundary value $N(z, q)$ on ∂D.*

Since $N(z, q)$ is uniformly bounded on $\partial D\frown(R-R_{n''})$, where n'' is a suitable number, it can be proved in the same manner as Theorem 2, by (8) that

$$\begin{aligned}\lim_{n=\infty}N_{D,n}(z, q)&=\lim_{n=\infty}\frac{1}{2\pi}\int_{\partial D\frown(R_n-R_0)}N(\zeta, q)\frac{\partial N_n^D(\zeta, z)}{\partial n}ds\\ &=\frac{1}{2\pi}\int_{\partial D}N(\zeta, q)\frac{\partial N^D(\zeta, z)}{\partial n}ds=N_D(z, q).\end{aligned}$$

where $N^D(\zeta, z)=\lim\limits_{n=\infty}N_n^D(\zeta, z)$.

Now, since $N(z, q)$ has M.D.I or minimal $*$-Dirichlet integral over D according as $q\notin D$ or $q\in D$, $N(z, q)=\lim\limits_{n=\infty}N_n'(z, q)$, where $N_n'(z, q)$ is a harmonic function in $D\frown(R_n-R_0)$ or harmonic except a logarithmic singularity at q such that $N_n'(z, q)=N(z, q)$ on $\partial D\frown(R_n-R_0)$ and $\dfrac{\partial N_n'(z, q)}{\partial n}$

$=0$ on $\partial R_n \cap D$. Hence $N_D(z, q) = \lim_{n=\infty} N_{D,n}(z, q) = \lim_{n=\infty} N_n'(z, q) = N(z, q)$ or $< N(z, q) = \lim_{n=\infty} N_n'(z, q)$ according as $q \notin D$ or not. Thus

$$N(z, q) \geqq N_D(z, q) = \frac{1}{2\pi} \int_{\partial D} N(\zeta, q) \frac{\partial N^D(\zeta, z)}{\partial n} ds. \tag{9}$$

Let $\{q_i\}$ be a fundamental sequence determining a point $q \in B$. Then, since $N(\zeta, q_i)$ tends to $N(\zeta, q)$ as $i \to \infty$, by Fatou's lemma and by (9)

$$N(z, q) \geqq N_D(z, q) = \frac{1}{2\pi} \int_{\partial D} N(\zeta, q) \frac{\partial N^D(\zeta, z)}{\partial n} ds, \tag{9'}$$

where $N_D(z, q)$ is the solution of $*$-Dirichlet problem in D with boundary value $N(z, q)$ on ∂D.

$N^M_{D,n}(z, q)$ be a harmonic function in $D \cap (R_n - R_0)$ such that $N^M_{D,n}(z, q) = N^M(z, q)$ on $\partial R_0 + \partial D \cap (R_n - R_0)$ and $\frac{\partial N^M_{D,n}(z, q)}{\partial n} = 0$ $\partial R_n \cap D$. Then $N^M_{D,n}(z, q)$ converges to a function $N^M_D(z, q)$ as $n \to \infty$. Clearly, as in case of $N_D(z, q)$, $N^M_D(z, q)$ is given by

$$N^M_D(z, q) = \frac{1}{2\pi} \int_{\partial D} N^M(\zeta, z) \frac{\partial N^D(\zeta, z)}{\partial n} ds,$$

i.e. $N^M_D(z, q)$ is the solution of $*$-Dirichlet problem in D with boundary value $N^M(z, q)$, whence $\lim_{M=\infty} N^M_D(z, q) = N_D(z, q)$.

The Dirichlet integral $\sum_l D_{D_l}(N^M_{D_l, n}(z, q)) \leqq \sum_l D_{D_l}(N^M(z, q)) \leqq 2\pi M$. Hence by letting $n \to \infty$ $\sum_l D_{D_l}(N^M_D(z, q)) \leqq 2\pi M$. For simplicity, we denote by $N^M_{V_m(p)}(z, q)$ the function being equal to $N^M_{D_l}(z, q)$ (solution of $*$-Dirichlet problem in D_l) in every domain D_l with boundary value $N^M(z, q)$.

Next, as in 3) of Lemma 2, it is proved that $N(z, p) = \lim_{n=\infty} U_n(z)$ in $V_m(p) - V_{m'}(p)$, $\frac{\partial U_n(z)}{\partial n} \to \frac{\partial N(z, p)}{\partial n}$ on $\partial V_m(p) + \partial V_{m'}(p)$

$$\int_{\partial V_m(p)} \frac{\partial N(z, p)}{\partial n} ds = \lim_{n=\infty} \int_{\partial V_m(p)} \frac{\partial U_n(z)}{\partial n} ds \qquad \text{and}$$

$$\int_{\partial V_{m'}(p)} \frac{\partial N(z, p)}{\partial n} ds = \lim_{n=\infty} \int_{\partial V_{m'}(p)} \frac{\partial U_n(z)}{\partial n} ds, \tag{10}$$

where $U_n(z)$ is a harmonic function in $(V_m(p) - V_{m'}(p)) \cap (R_n - R_0)$ such that $U_n(z) = m$ on $\partial V_m(p)$, $U_n(z) = m'$ on $\partial V_{m'}(p)$ and $\frac{\partial U_n(z)}{\partial n} = 0$ on $\partial R_n \cap (V_m(p) - V_{m'}(p))$.

Let $N^M_{V_m(p),n}(z, q)=N^M_{D_l,n}(z, q)$ in every domain $D_l\cap(R_n-R_0)$. Then we have by Green's formula

$$\int_{\partial V_m(p)\cap(R_n-R_0)} N^M_{V_m(p),n}(z, q)\frac{\partial U_n(z)}{\partial n}ds=\int_{\partial V_{m'}(p)\cap(R_n-R_0)} N^M_{V_m(p),n}(z, q)\frac{\partial U_n(z)}{\partial n}ds,$$

because $U_n(z)=m$ and m' on $\partial V_m(p)$ and $\partial V_m'(p)$ respectively and $\int_{\partial V_m(p)\cap(R_n-R_0)}\frac{\partial N_{V_m(p),n}(z, q)}{\partial n}ds=\int_{\partial R_n\cap V_m(p)}\frac{\partial N^M_{V_m(p),n}(z, q)}{\partial n}ds=0$ and $\int_{\partial V_{m'}(p)\cap(R_n-R_0)}\frac{\partial N^M_{V_m(p),n}(z, q)}{\partial n}ds=\int_{\partial R_nV_{m'}(p)}\frac{\partial N^M_{V_m(p),n}(z, q)}{\partial n}ds=0$. Let $n\to\infty$. Then by (10)

$$\int_{\partial V_m(p)} N^M_{V_m(p)}(z, q)\frac{\partial N(z, p)}{\partial n}ds=\int_{\partial V_{m'}(p)} N^M_{V_m(p)}(z, q)\frac{\partial N(z, p)}{\partial n}ds. \tag{11}$$

Therefore by letting $M\to M^*$, by (9′) and (11) we have

$$\begin{aligned}2\pi N^{V_{m'}(p)}(p, q)&=\int_{\partial V_{m'}(p)} N(z, p)\frac{\partial N(z, p)}{\partial n}ds\\ &\geqq\int_{\partial V_m(p)} N(z, p)\frac{\partial N(z, p)}{\partial n}ds=2\pi N^{V_m(p)}(p, q).\end{aligned}$$

Definition of $N(p, q)$ in Case 2: for $p\in R+B_1$ and $q\in\bar{R}$. Since $N^{V_m(p)}(p, q)$ is increasing with respect to m, $N^{V_m(p)}(p, q)$ has a limit denoted by $N(p, q)$ as $m\uparrow M^*=\sup_{z\in R}N(z, p)$. We define the value of $N(z, q)$ at $p\in B_1$ by this limit. It is easily proved that, in case 1) this definition of $N(p, q)$ coincides with what has been given previously. In fact, it is evident that $N(p, q)=\frac{1}{2\pi}\int_{\partial V_m(p)} N(z, p)\frac{\partial N(z, p)}{\partial n}ds$ for $p\in R$ and $V_m(p)\ni q$ and that, by (5) $N^{V_m(p)}(p, q)=\frac{1}{2\pi}\int_{\partial V_m(p)} N(z, q)\frac{\partial N(z, p)}{\partial n}ds$ $=N(q, p)=\lim_{i=\infty}N(q, p_i)=\lim_{i=\infty}N(p_i, q)=N(p, q)$ for $p\in B$ and $q\in R$, where $\{p_i\}$ is a fundamental sequence determining p.

Remark. *Let $V_m(p)$ be a regular domain and let $\{V_{m_i}(p)\}$ be a sequence of regular domain with $m_i\uparrow m$. Then $N^{V_m(p)}(p, q)=\lim_{i=\infty}N^{V_{m_i}(p)}(p, q)$.*

In fact, there exists a number n, for any given positive number ε, such that

$$\int_{\partial V_m(p)\cap(R_n-R_0)} N(z, q)\frac{\partial N(z, p)}{\partial n}ds\geqq\int_{\partial V_m(p)} N(z, q)\frac{\partial N(z, p)}{\partial n}ds-\varepsilon.$$

On the other hand, suppose $z_i\in\partial V_{m_i}(p)$, $z_0\in\partial V_m(p)$ and $z_i\to z$. Then

$\frac{\partial N(z_i, p)}{\partial n} ds \to \frac{\partial N(z_0, p)}{\partial n} ds$ and $N(z_i, q) \to N(z_0, q)$, hence

$$\lim_{i=\infty} \int_{\partial V_{m_i}(p)} N(z, q) \frac{\partial N(z, p)}{\partial n} ds \geq \lim_{i=\infty} \int_{\partial V_{m_i}(p) \cap (R_n - R_0)} N(z, q) \frac{\partial N(z, p)}{\partial n} ds$$

$$= \int_{\partial V_m(p) \cap (R_n - R_0)} N(z, q) \frac{\partial N(z, p)}{\partial n} ds \geq \int_{\partial V_m} N(z, p) \frac{\partial N(z, p)}{\partial n} ds - \varepsilon .$$

By letting $\varepsilon \to 0$, $\lim_{i=\infty} N^{V_{m_i}(p)}(p, q) \geq N^{V_m(p)}(p, q)$. Next, $m_i < m$ implies $N^{V_{m_i}(p)}(p, q) \leq N^{V_m(p)}(p, q)$ and $\overline{\lim_{i=\infty}} N^{V_{m_i}(p)}(p, q) \leq N^{V_m(p)}(p, q)$. Thus we have $N^{V_m(p)}(p, q) = \lim_{i=\infty} N^{V_{m_i}(p)}(p, q)$.

We define $N^{V_m(p)}(p, q)$ for any domain $V_m(p)$ by $\lim_{i=\infty} N^{V_{m_i}(p)}(z, p)$ as above. This definition coincides with what has been defined previousely for regular domain $V_n(p)$. Hence $N^{V_m(p)}(p, q)$ is defined for every $m < \sup_{z \in R} N(z, p)$.

Definition of Superharmonicity at a point $p \in R + B_1$. Suppose a function $U(z)$ in $\bar{R}$. If $U(p) \geq \frac{1}{2\pi} \int_{\partial V_m(p)} U(z) \frac{\partial N(z, p)}{\partial n} ds$ holds for regular $V_m(p)$ of $N(z, p)$, we say that $U(z)$ is *superharmonic in the weak sense at a point* p. Thus we shall have the following

Theorem 11.

1). $N(p, p) = \sup_{z \in R} N(z, p)$ *for* $p \in R + B_1$.

2). $N(z, q) (q \in \bar{R})$ *is* δ*-lower semicontinuous in* $R + B_1$.

3). $N(z, q)$ *is superharmonic in the weak sense at every point of* $R + B_1$.

4). $N(p, q) = N(q, p)$ *for two points* p *and* q *belonging to* $R + B_1$.

Proof. 1) and 3) are clear by definition.

Proof of 2). Let $\{p_i\}$ be a sequence of points of $R + B_1$ tending to p. Since by the above remark $N^{V_m(p)}(p, q) = \lim_{m \to m'} N^{V_{m'}(p)}(p, q) (m' \uparrow m)$, there exists a number m', for any given positive number ε, such that $V_{m'}(p)$ is regular and $N^{V_m(p)}(p, q) \leq N^{V_{m'}(p)}(p, q) + \varepsilon$. Hence there exists a number n_0 such that

$$N^{V_m(p)}(p, q) \leq \frac{1}{2\pi} \int_{\partial V_{m'}(p) \cap (R_n - R_0)} N(z, q) \frac{\partial N(z, p)}{\partial n} ds + 2\varepsilon \qquad \text{for} \quad n \geq n_0 .$$

Let $V_{m''}(p_i)$ be a sequence of regular domains such that $p_i \to p$ and $m'' \uparrow m$. Replace $G_{V_m(p)}(p, q)$ by $N^{V_m(p)}(p, q)$ in 3) of Theorem 1 of Part I.

Then $N(\alpha_i, q)$ on $\partial V_{m''}(p_i)$ tends to $N(\alpha, q)$ on $\partial V_m(p)$ and $\frac{\partial N(\alpha_i, p)}{\partial n} ds$ tends to $\frac{\partial N(\alpha, p)}{\partial n} ds$, whence $\varliminf_{i=\infty} N^{V_m(p)}(p_i, q) \geqq \varliminf_{i=\infty} N^{V_{m''}(p_i)}(p_i, q) \geqq N^{V_m(p)}(p, q) - 2\varepsilon$ and $\varliminf_{i=\infty} N(p_i, q) \geqq N(p, q)$. Hence we have 2).

Proof of 4). Replace $G_{V_m(q)}(p, q)$ and $G_{V_n(p)}(q, p)$ by $N^{V_m(p)}(p, q)$ and $N^{V_n(q)}(q, p)$ respectively and consider that $\{V_m(p)\}$ clusters at B as $m \uparrow M^* = \sup_{z \in R} N(z, p)$. Then we at once 4), where $V_m(p)$ and $V_n(q)$ are regular. Now we define $N(z, q)$ not only in $R+B_1$ but also in B_0.

Definition of $N(z, q)$ in Case 3: for $p \in B_0$ and $q \in \bar{R}$. At first, if $p \in B_0$, $N(z, p)$ is represented by $\int_{B_1} N(z, p_\alpha)\, d\mu(p_\alpha) (p_\alpha \in B_1)$ by Theorem 8 for $z \in R$, where $\mu(p_\alpha)$ is an weak limit and its uniqueness cannot be proved by the present author.

Let $p_{\alpha_i} (\in \bar{R})$ $(i = 1, 2, \cdots)$ tend to p_α with respect to δ-metric. Then, since $N(z, p_{\alpha_i}) \to N(z, p_\alpha)$ on $\partial V_m(q)$ for $q \in R+B_1$. Hence, by Fatou's lemma

$$N^{V_m(q)}(q, p_\alpha) = \frac{1}{2\pi} \int_{\partial V_m(q)} N(z, p_\alpha) \frac{\partial N(z, q)}{\partial n} ds$$

$$\leqq \varliminf_{i=\infty} \int_{\partial V_m(q)} N(z, p_{\alpha_i}) \frac{\partial N(z, q)}{\partial n} ds = \varliminf_{i=\infty} N^{V_m(q)}(q, p_{\alpha_i}) .$$

Hence $N^{V_m(q)}(q, p_\alpha)$ is lower semicontinuous with respect to p_α for fixed $q \in R+B_1$. Since $N^{V_m(q)}(q, p) \uparrow N(q, p)$ at every point p, $\lim_{m \to M^*} \int N^{V_m(q)}(q, p_\alpha)\, d\mu(p_\alpha) = \int N(q, p_\alpha)\, d\mu(p_\alpha)$ $(M^* = \sup_{z \in R} (z, q))$, whence

$$N(q, p) = \lim_{m \to M^*} N^{V_m(q)}(q, p) = \lim_{m \to M^*} \frac{1}{2\pi} \int_{\partial V_m(q)} \left(\int_{B_1} N(z, p) \frac{\partial N(z, q)}{\partial n} d\mu(p_\alpha) \right) ds$$

$$= \frac{1}{2\pi} \int_{B_1} \left(\lim_{m \to M^*} \int_{\partial V_m(q)} N(z, p_\alpha) \frac{\partial N(z, q)}{\partial n} ds \right) d\mu(p_\alpha) = \int_{B_1} N(q, p_\alpha)\, d\mu(p_\alpha) . \quad (13)$$

Hence the representation

$$N(z, p) = \int_{B_1} N(z, p_\alpha)\, d\mu(p_\alpha) \qquad (14)$$

is valid not only in R but also in B_1.

The value of $N(q, p)$ $(q \in R+B_1$ and $p \in B_0)$ does not depend on a particular choice of distribution $\mu(p_\alpha)$, because the left hand side of (13) is given by $\lim_{m \to M^*} N^{V_m(q)}(q, p)$, that is $N(q, p)$ depends only on the value of $N(z, p)$ in R. Now (14) means that the potential of a unit mass on $p \in B_0$ has the same behaviour in $R+B_1$ as the potential of mass distribution $\int_{B_1} d\mu(p_\alpha)$. From this point of view, we may consider that a point $p \in B_0$ is spanned by points $p_\alpha \in B_1$ with weight $\mu(p_\alpha)$. Hence it is natural to define the value of $N(z, q)(q \in \bar{R})$ at $z = p \in B_0$ by

$$\int_{B_1} N(p_\alpha, q)\, d\mu(p_\alpha)\,. \tag{15}$$

we shall prove the following

Theorem 12.

1). *$N(p, q) = N(q, p)$ for $p \in \bar{R}$ and $q \in R + B_1$. Hence $N(q, p)$ and $N(p, q)$ does not depend on a particular choice of distribution $\mu(p_\alpha)$.*

2). *$N(q, z)(q \in R + B_1)$ is δ-lower semicontinuous in $\bar{R}$.*

1′). *$N(p, q) = N(q, p)$ for p and q belonging to $\bar{R}$.*

2′) *$N(z, q)(q \in \bar{R})$ is δ-lower semicontinuous in $\bar{R}$.*

Proof of 1). For $p \in R + B_1$ our assertion is evident by 4) of Theorem 11. We show for $p \in B_0$. In this case, since $N(p_\alpha, q) = N(q, p_\alpha)$ by 4) of Theorem 11, we have by (14) and (15)

$$N(q, p) = \int_{B_1} N(q, p_0)\, d\mu(p_\alpha) = \int_{B_1} N(p_\alpha, q)\, d\mu(p_\alpha) = N(p, q)\,.$$

Since $N(q, p)$ does not depend on a particular distribution, $N(p, q)$ also does not depend on it.

Proof of 2). If $p \in R + B_1$, is clear by Theorem 11. Let $\{p_i\}$ be a sequence of points tending to $p \in B_0$. They by 1) of this theorem $N(q, p_i) = N(p_i, q)$ and $N(p, q) = N(p, q)$. On the other hand, by Fatou's lemma $\varliminf_{i=\infty} N^{V_m(q)}(q, p_i) \geqq N^{V_m(q)}(q, p)$, which implies $\varliminf_{i=\infty} N(q, p_i) \geqq N(q, p)$. Hence

$$\varliminf_{i=\infty} N(p_i, q) = \varliminf_{i=\infty} N(q, p_i) \geqq N(q, p) = N(p, q)\,.$$

This completes the proof of 2).

Proof of 1′). If at least one of p and q belongs to $R + B_1$, our assertion is 1). Suppose that both p and q belong to B_0. In this case

$$N(z, p) = \int_{B_1} N(z, p)\, d\mu(p_\alpha) \quad \text{and} \quad N(z, q) = \int_{B_1} N(z, q_\beta)\, d\mu(q_\beta) \ \ (p_\alpha \text{ and } q_\beta \in B_1).$$

Hence by (14) and by 1) of the this theorem

$$\begin{aligned} N(q, p) &= \int N(q_\beta, p)\, d\mu(q_\beta) \\ &= \int \left(\int (N(p_\alpha, q_\beta))\, d\mu(p_\alpha)\right) d\mu(q_\beta) = \int N(p, q_\beta)\, d\mu(q_\beta) = N(p, q)\,. \end{aligned}$$

It is proved as in 1) that $N(p, q)$ does not depend on particular distributions

$$\mu(p_\alpha) \qquad \text{and} \qquad \mu(q_\beta)\,.$$

Proof of 2′). Let $\{p_i\}$ be a sequence tending to p. Then for every point q_β, $\varliminf_{i=\infty} N(p_i, q_\beta) \geqq N(p, q_\beta)$, which yields at once by Fatou's lemma

$$\varliminf_{i=\infty} N(p_i, q) = \varliminf_{i=\infty} \int N(p_i, q_\beta)\, d\mu(q_\beta) \geqq \int N(p, q_\beta)\, d\mu(q_\beta) = N(p, q)\,.$$

Remark. Let $U(z)$ be a function given by $\int N(z, p)\, d\mu(p) (\mu > 0)$. Then $U(z)$ is lower semicontinuous in $\bar{R}$.

12. Mass Distributions on $\bar{R}$.

We have seen that $N(z, p)$ has the essential properties of the logarithmic potential: lower semicontinuity on $\bar{R}$, symmetry and superharmonicity in the weak sense on $R+B_1$. But there exists a fatal difference between our space and the euclidean space, that is, in our space there may exist points of B_0 where we cannot distribute any *true mass*. A distribution μ on B_0 may be called a *pseudo distribution* in the sense that $U_{B_0}(z)=0$ and μ can be replaced, by Theorem 8, by a distribution on B_1, where $U(z) = \int_{B_0} N(z, p)\, d\mu(p)$. In other words, even when B_0 is not empty, B_0 behaves as an empty set for mass distributions.

Mass Distributions on $\boldsymbol{R}+\boldsymbol{B}_1$. Since $N(z, p)$ has the above properties, it is easy to construct the potential theory on $R+B_1$.

The energy integral $I(\mu)$ of a mass distribution μ on a closed subset F of $R+B_1$ is defined as

$$I(\mu) = \iint_F N(q, p)\, d\mu(p)\, d\mu(q)\,.$$

The *$*$-Capacity* $*\mathrm{Cap}\,(F)$ and the *transfinite diameter* D_F of F are defined as follows: $\dfrac{*\mathrm{Cap}\,(F)}{2\pi}$ is defined as the least upper bound of total mass of μ on F whose potential is not greater than 1 on F.
$D_F = \lim_{n=\infty} {}_nD_F$, where

$$\frac{1}{{}_nD_F} = \frac{1}{2\pi\, {}_nC_2}\, \Big(\inf_{p_i, p_j \in F} \sum_{\substack{i<j \\ i=1}}^{n,\, n} N(p_i, p_j) \Big)\,.$$

We see easily the following

Lemma. $\mathrm{Cap}\,(F) > 0$ *implies* $*\mathrm{Cap}\,(F) > 0$ *for a closed susbet F of* $R+B_1$.

In fact, if $\mathrm{Cap}\,(F) > 0$, $\omega_F(z) = {}_F\omega_F(z) > 0$ and $\omega_F(z) = \int_F N(z, p)\, d\mu(p)$. Now the total mass of μ is given by $\dfrac{1}{2\pi} \displaystyle\int_{\partial R_0} \frac{\partial \omega_F(z)}{\partial n}\, ds$ and $\omega_F(z) \leqq 1$, whence $*\text{Cap}(F)>0$.

Then we have as in space the following

Theorem 13. *Let F be a closed subset of positive $*$Capacity of $R+B_1$. Then there exists a unit mass distribution μ on F whose energy integral is minimal and its potential $U(z)$ satisfies the following conditions*:

1). *$U(z)$ is a constant C on the kernel of the distribution, whence $I(\mu)=D(U(z))=2\pi C$.*

2). *$U(z)=U_F(z)$.*

3). *$U(z)=C$ on F except possibly a subset of $*$-Capacity zero of F.*

4). *$U(z)=C\omega_F(z)$.*

Proof. 1) and 3) can be proved as in space.

Proof of 2). Since $p\in R+B_1$, $N(z,p)=N_{v_m(p)}(z,p)$ for every point of $R+B_1$, where $v_m(p)=E[z\in\bar{R}:\ \delta(z,p)\leq\frac{1}{m}]$. This implies $N_{F_m}(z,p)=N(z,p)$, where $F_m=E[z\in\bar{R}:\ \delta(z,F)\leq\frac{1}{m}]$, because $F_n\supset v_m(p)$. Hence we have $U_F(z)=U(z)$.

Proof of 4). Put $U(z)=C\omega^*(z)$. Then by 2) ${}_F(\omega^*(z))=\omega^*(z)$ and not greater than 1 on F. Hence $\omega^*(z)=\lim_{m=\infty}\lim_{n=\infty}\omega^*_{m,n}(z)$, where $\omega^*_{m,n}(z)$ is a harmonic function in $R_n-R_0-F_m$ such that $\omega^*_{m,n}(z)=\omega^*(z)$ on $\partial F_m\cap(R_n-R_0)$, $\omega^*_{m,n}(z)=0$ on ∂R_0 and $\frac{\partial\omega^*_{m,n}(z)}{\partial n}=0$ on ∂R_n-F_m. On the other hand, $\omega_F(z)=\lim_{m=\infty}\lim_{n=\infty}\omega_{m,n}(z)$, where $\omega_{m,n}(z)$ is a harmonic function in $R_n-R_0-F_m$ such that $\omega_{m,n}(z)=1$ on $\partial F_m\cap(R_n-R_0)$, $\omega_{m,n}(z)=0$ on ∂R_0 and $\frac{\partial\omega_{m,n}(z)}{\partial n}=0$ on ∂R_n-F_n. Hence $\omega_{m,n}(z)\geqq\omega^*_{m,n}(z)$, whence by letting $n\to\infty$ and then $m\to\infty$, $\omega_F(z)\geqq\omega^*(z)$. Next, the set $A_\lambda=E[z\in\bar{R}:\ \omega^*(z)\leqq1-\lambda]\cap F$ is clearly closed by the lower semicontinuity of $\omega^*(z)$. $*\text{Cap}(A_\lambda)=0$ implies $\text{Cap}(A_\lambda)=0$ by Lemma. Hence $0=\omega_{A_\lambda}(z)=\lim_{m=\infty}\lim_{n=\infty}\omega_{A_\lambda,m,n}(z)$, where $A_{\lambda,m}=E[z\in\bar{R}:\ \delta(z,A_\lambda)\leqq\frac{1}{m}]$ and $\omega_{A_\lambda,m,n}(z)$ is a harmonic function in $R_n-R_0-A_{\lambda,m}$ such that $\omega_{A_\lambda,m,n}(z)=1$ on $\partial A_{\lambda,m}$, $\omega_{A_\lambda,m,n}(z)=0$ on ∂R_0 and $\frac{\partial\omega_{A_\lambda,m,n}(z)}{\partial n}=0$ on $\partial R_n-A_{\lambda,m}$. Let $\{\lambda_i\}$ be a sequence such that $\lambda_i\downarrow0$. Then

$$\omega^*_{m,n}(z)+\sum_{\lambda_i}\omega_{A_{\lambda_i},m,n}(z)\geqq\omega_{m,n}(z)\,.$$

Hence by letting $n\to\infty$ and then $m\to\infty$, $\omega^*(z)\geqq\omega_F(z)$. Then $\omega^*(z)=\omega_F(z)$.

Corollary. Cap $(F)=$*Cap (F) *for a closed subset of* $R+B_1$.

In fact, since $\omega_F(z)=\frac{U(z)}{C}$, *Cap $(F)=2\pi\frac{1}{2\pi C}\int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds=\frac{2\pi}{C}$ $=\frac{4\pi^2}{I(\mu)}=\int_{\partial R_0}\frac{\partial\omega_F(z)}{\partial n}ds$. Hence *Cap $(F)=$Cap (F) and Cap $(F)=1/I(\mu)$, where μ is the equilibrium distribution of total mass unity on F.

Theorem 14. (*Extension of Evans-Selberg's Theorem*). *Let F be a closed subset of $R+B_1$. Then* Cap $(F)=0$, *if and only if there exists a unit mass distribution on F whose potential $U(z)$ satisfies the following conditions*:

1). $U(z)=0$ *on* ∂R_0.

2). $U(z)=\infty$ *at every point of* F.

3). $U(z)=U_F(z)$ *and* $\frac{U(z)}{m}$ *is the equilibrium potential of the set* $G_n=E[z\in R:\ U(z)\geqq m]$ *for every* m.

Proof. If such $U(z)$ exists, clearly Cap $(F)=0$. Next Cap (F) $=$*Cap $(F)=0$ implies by 1) of Theorem 12 $D_F=0$. Replace $G(p_i, p_j)$ by $N(p_i, p_j)$ in Part I. Then we have 1) and 2). Since every point mass of $V^m(z)=\frac{1}{2\pi m}(\sum_{i=1}^{m}N(z, p_i))$ is contained in F, $V_F^m(z)=V^m(z)$. This implies $U(z)=(\sum_{i=1}^{\infty}\frac{V^i(z)}{2^i})=U_F(z)$. Hence $\frac{U(z)}{m}$ is the equilibrium potential of $G_m=E[z\in R:\ U(z)\geqq m]$.

Remark 1. *Let p be a point in B_0. Then $N(z, p)=\int_{B_1}N(z, p_\alpha)\,d\mu(p_\alpha)$ and $U(p)=\int U(p_\alpha)\,d\mu(p_\alpha)$. Hence $U(z)$ may be infinite on a larger set F' containing F.*

Remark 2. *Theorem 14 holds for an F_σ of $R+B_1$ of capacity zero.*

Remark 3. *We cannot omit the condition that $F\in R+B_1$, (See an example).*

Mass Distribution on $\bar{R}$. Definition of *Cap (F) and D_F for closed subset F of $\bar{R}$. Let F be a closed set of $\bar{R}$. Then $F\cap(R+B_1)$ is a G_δ, since B_0 is an F_σ. We define *Capacity and the transfinite diameter of F as follows: Put $F_m=E[z\in\bar{R}:\ \delta(z, F)\leqq\frac{1}{m}]$ and put *Cap (F_m) $=\sup_\alpha$ *Cap (F_α) and $D_{F_m}=\sup_\alpha D_{F_\alpha}$, where F_α is a closed subset of $R+B_1$ contained in F_m. Since clearly *Cap (F_m) and D_{F_m} are decreasing with respect to m. Put *Cap $(F)=\lim_{m=\infty}$ *Cap (F_m) and $D_F=\lim_{m=\infty}D_{F_m}$. Then we have the following

Theorem 15. $^*\mathrm{Cap}\,(F)=\mathrm{Cap}\,(F)=4\pi^2 D_F$ *for a closed set F of $\bar{R}$.*

In fact, let $\omega_\alpha(z)$ be the equilibrium potential of F_α. Since $F_\alpha \subset F \cap (R+B_1)$, ${}_{F_m}\omega_\alpha(z)=\omega_\alpha(z)$ for every F_α. We assume $F_\alpha \uparrow$. Then $\omega_{F_\alpha}(z)$ converges to a function $\hat{\omega}(z)$. Then ${}_{F_m}(\hat{\omega}(z)) \geqq {}_{F_m}\omega_{F_\alpha}(z)=\omega_{F_\alpha}(z)$ for every α. On the other hand, clearly ${}_{F_m}(\hat{\omega}(z)) \leqq \hat{\omega}(z)$, because $\hat{\omega}(z)$ is superharmonic in $\bar{R}$. Therefore ${}_{F_m}(\hat{\omega}(z)) \leqq \omega(z)$. This implies that $\hat{\omega}(z)$ has has M.D.I. over $R-F$. Hence $\hat{\omega}(z)=\omega_{F_m}(z)$, since $\hat{\omega}(z)=1$ on $F_m \cap R$. Hence $\mathrm{Cap}\,(F_m)={}^*\mathrm{Cap}\,(F_m)$, whence $4\pi^2 D_F={}^*\mathrm{Cap}\,(F)=\mathrm{Cap}\,(F)$. Particularly $\mathrm{Cap}\,(B_0)={}^*\mathrm{Cap}\,(B_0)=0$. Thus two capacities coincide each other. We call them capacity. Since $\omega_F(z)={}_F\omega_F(z)$ and $\omega_F(z)$ is lower semicontinuous, we can prove as 3) of Theorem 13 the following

Corollary. *If $\omega_F(z) \neq 0$, $\omega_F(z)=1$ except possibly a subset of capacity zero of F.*

Hence $\omega_F(z)$ has the characteristic property of the equilibrium potential in space. The capacity of Borel sets of $\bar{R}$ is defined as usual.

An Example

We shall construct a Riemann surface with singular ideal boundary points and points of B_0 and further we show that the condition of theorem 13 is necessary.

Let r_n be a circle: $|z|=r_n$ $(n=1, 2, \cdots)$, where $r_1<r_2<r_3, \cdots, r_1=1$ and $\lim\limits_{n=\infty} r_n=2$. Denote by $\check{R}_n$ a ring domain: $r_n<|z|<r_{n+1}$ and let A_n, B_n, C_n ring domains such that $A_n: r_{n+1}>|z|> r_{n,\alpha}$, $B_n: r_{n,\alpha}>|z|> r_{n,\beta}$, $C_n: r_{n,\beta}>|z|>r_n$ with $r_n<r_{n,\beta}<r_{n,\alpha}<r_{n+1}$.

$\{A_n\}$. Let $\Gamma_{A,n}$ be a circle: $|z|=\sqrt{r_{n+1}, r_{n,\alpha}}$. Then there exists a constant Q_n depending only on the modulus of A_n, i.e. $\log \dfrac{r_{n+1}}{r_{n,\alpha}}$ such that $\max\limits_{z\in\Gamma_{A,n}} U(z) \leqq Q_n \min\limits_{z\in\Gamma_{A,n}} U(z)$ for any positive harmonic function $U(z)$ in A_n. Choose a sequence P_n such that $\lim\limits_{n=\infty} \dfrac{P_n}{Q_n}=\infty$, (Fig. 1).

$\{B_n^*\}$. In B_n we make so many radial slits and connect them so that every harmonic function $|U(z)| \leqq P_n$ in B_n satisfies the condition that the oscillation of $U(z)$ on $\Gamma_{B,n}$ is less than $\dfrac{1}{n}$, where $\Gamma_{B,n}$ is a circle in B_n such that $\Gamma_{B,n}: |z|=\sqrt{r_{n,\alpha}, r_{n,\beta}}$. We make the above slits as follows.

Put $B_n=B$, $\alpha=\log r_{n,\alpha}$ and $B=\log r_{n,\beta}$. Let $J(\supset \Gamma_{B,n})$ be a ring domain such that

$$J:\ \beta+\frac{\alpha-\beta}{3}<\log |z|<\beta+\frac{2(\alpha-\beta)}{3}.$$

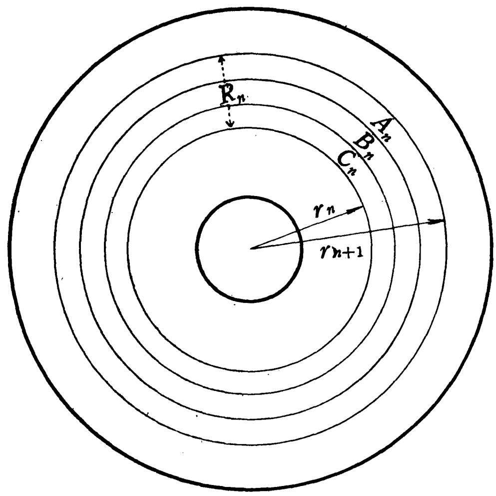


Fig. 1

Let $U(z)$ be a harmonic function in J such that $|U(z)| \leqq P_n$. Then $U(z) = \frac{1}{2\pi} \int_{\partial J} U(\zeta) \frac{\partial G(\zeta, z)}{\partial n} ds$, where $G(\zeta, z)$ is the Green's function of J with pole at z. Since $\frac{\partial G(\zeta, z)}{\partial n}$ is a continuous function of z in J for fixed ζ and since $U(z_1) - U(z_2) = \frac{1}{2\pi} \int_{\partial r} U(\zeta) \left(\frac{\partial G(\zeta, z_1)}{\partial n} - \frac{\partial G(\zeta, z_2)}{\partial n} \right) ds$, there exists a number m depending only on the modulus of J but on $U(z)$ such that $|\arg z_1 - \arg z_2| \leqq \frac{2\pi}{2^m}$ implies $|U(z_1) - U(z_2)| < \frac{1}{2n}$ for every pair of points z_1 and z_2 on the circle $\Gamma_{B,n}$.

Let H_i and H_i' $(i=1, 2, 3, \cdots, m)$ be ring domains as follows:

$$H_i: \alpha - (2i-1)s > \log|z| > \alpha - 2is,$$

$$H_i': \beta + (2i-1)s < \log|z| < \beta + 2is, \quad \text{where} \quad s = \frac{(\alpha - \beta)}{3 \cdot 2m}.$$

Let S_i^j and $\acute{S}_i^j$ $(j=1, 2, 3, \cdots, 2^{ml})$ slits in H_i and H_i' respectively as follows:

$$S_i^j: \alpha - (2i-1)s > \log|z| > -2is, \ \arg z = \frac{2\pi j}{2^{ml}}.$$

$$\acute{S}_i^j: \beta + (2i-1)s > \log|z| < \beta + 2is, \ \arg z = \frac{2\pi j}{2^{ml}}.$$

where l is a large integer so that $|U(z)| \leqq P_n$ and $U(z) = 0$ on $\sum_j S_i^j$ imply $|U(z)| < \frac{1}{2n \cdot m!}$ on a circle Γ_i for every harmonic function in $H_i - \sum_j S_i^j$.

Clearly $H_i - \sum_j S_i^j$ and $H_i' - \sum_j \acute{S}_i^j$ $(i=1, 2, \cdots, m)$ are conformally equivalent. Hence $|U(z)| \leqq P_n$ in H_i or H_i' and $U(z)=0$ on $\sum_i S_i^j$ or $\sum_j \acute{S}_i^j$ imply $|U(z)| < \frac{1}{2nm!}$ on Γ_i and Γ_i' respectively, where Γ_i and Γ_i' are circles as follows:

$$\Gamma_i : \log|z| = \alpha - (2i-1)\, s - \frac{s}{2},$$

$$\Gamma_i' : \log|z| = \beta + (2i-1)\, s + \frac{s}{2}.$$

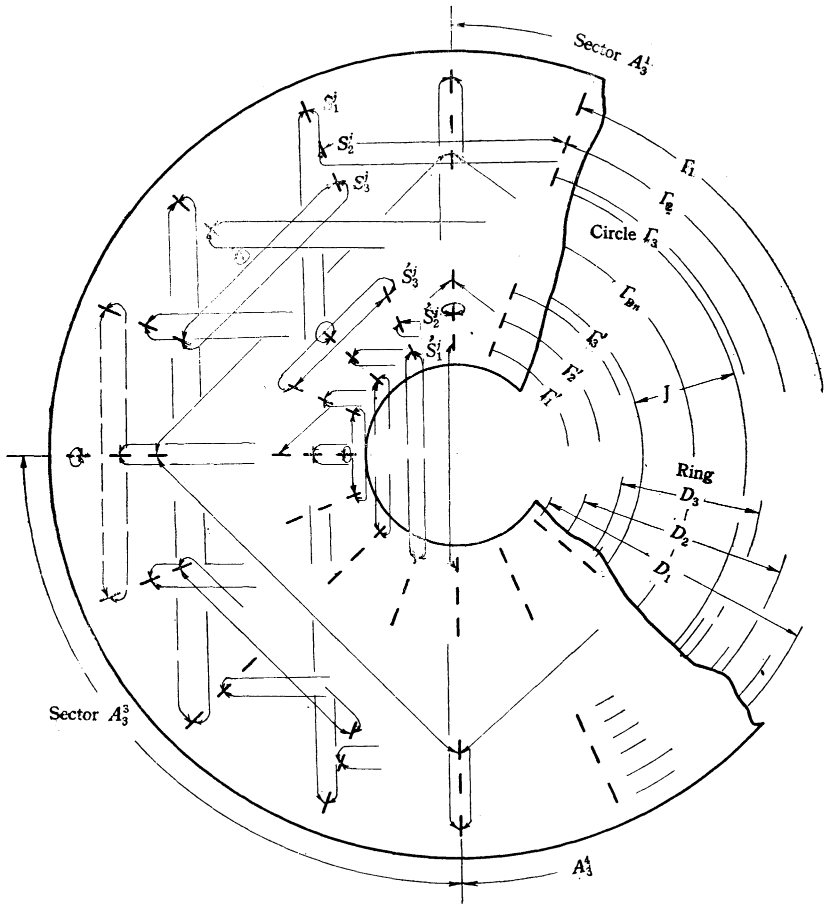
In H_1 and H_1' identify the two edges of the slits S_1^j and $\acute{S}_1^j$ $(j=1, 2, 3, \cdots, 2^{m_l})$ lying symmetrically with respect to the real axis. Next, in H_2 and H_2' identify the two edges of S_2^j and $\acute{S}_2^j$ lying symmetrically with respect ta the imaginary axis. In H_3 and H_3', in every sector A_3^t: $\frac{(t-1)\,\pi}{2} < \text{agr}\, z < \frac{t\pi}{2}$ identify two edges of slits S_3^j and $\acute{S}_3^j$ lying symmetrically with respect to the radius: $\arg z = \frac{(t-1)\,\pi}{2} + \frac{\pi}{4}$ $(t=1, 2, 3, 4)$. Generally speaking, let A_i^t be a sector as follows:

$$A_i^t : \frac{(t-1)\,\pi}{2^{i-2}} < \arg z < \frac{t\pi}{2^{i-2}}, \quad t=1, 2, 3, \cdots, 2^{i-1}.$$

In every A_i^t identify the two edgds of S_i^j and $\acute{S}_i^j$ lying symmetrically with respect to the radius: $\arg z = \frac{(t-1)\,\pi}{2^{i-2}} + \frac{\pi}{2^{i-1}}$. Then we have a Riemann surface $\{B_n^*\}$ with only two boundary components lying on $\log|z| = \alpha$ and $\log|z| = \beta$.

We shall show that $\{B_n^*\}$ has the property above stated. (Fig. 2).

Suppose a positive harmonic function $|U(z)| \leqq P_n$. Let $T_1(z)$ be a transformation such that $T_1(z)$ is the symmetric point of z with respect to the real axis. Then $U(z) - U(T_1(z))$ is harmonic in B_n^* and vanishes on $\sum_j (S_j' + \acute{S}_1^j)$, whence $|U(z) - U(T_1(z))| < \frac{1}{2n \cdot m!}$ on circles Γ_1 and Γ_1'. Hence by the maximum principle $|U(z) - U(T_1(z))| < \frac{1}{2n \cdot m!}$ in the ring domain bounded by Γ_1 and Γ_1'. Let $T_2(z)$ be a transformation such that $T_2(z)$ is the symmetric point of z with respect to the imaginary axis. Then as above $|U(z) - U(T_2(z))| < \frac{1}{2n \cdot m!}$ in the domain bounded by Γ_2 and Γ_2'. Next, consider $U(z)$ in a ring domain $\Gamma_3 : \beta + 5s < \log|z| < \alpha - 5s$. Let T_3^1 be a transformation such that $T_3^1(z)$ is the symmetric point of z with respect to the radius: $\arg z = \frac{\pi}{4}$. Then $U(z) - U(T_3^1(z))$ is har-



B_n $(m=3)$

Fig. 2

monic in D_3 and $U(z)-U(T_3^1(z))=0$ on $\sum_j (S_3^j+S_3^j)\cap(A_3^1+A_3^3)$. Hence $|U(z)-U(T_3^1(z))|<\dfrac{1}{2n\cdot m!}$ for $z\in(A_3^1+A_3^3)\cap(\Gamma_3+\Gamma_3')$, similarly $|U(z)-U(T_j^2(z))|<\dfrac{1}{2n\cdot m!}$ for $z\in(A_3^2+A_3^4)\cap(\Gamma_3+\Gamma_3')$, where T_3^2 is a transformation with respect to $\arg z=\dfrac{3\pi}{4}$. Let z_1 and z_2 be two points in A_3^2 and A_3^4 such that $z_2=T_3^1(z_1)$. Then $z_2=T_3^2\cdot T_1\cdot T_2(z_1)$, where $T_3^2\cdot T_1\cdot T_2(z_1)$ and z_2 are contained in A_3^4. Hence by the property of T_1, T_2 and T_3^2 $|U((z_1)-U(z_2)|<\dfrac{3}{2n\cdot m!}$ on $\Gamma_3+\Gamma_3'$, whence by the maximum principle

$$|U(z)-U(T_3^1(z))|<\frac{3}{2n\cdot m!}<\frac{3!}{2n\cdot m!}$$

in the domain bounded by Γ_3 and Γ_3'. In the sequel, we say that T_3^1 has the *deviation* $<\frac{3!}{2n\cdot m!}$.

For every i, consider $U(z)$ in a ring domain D_i:

$$D_i:\ \beta+(2i-1)\,s<\log|z|<\alpha-(2i-1)\,s\,.$$

Let $T_i^t(z)$ $(t=1, 2, \cdots, 2)$ be a transformation such that $T_i^t(z)$ is the symmetric point of z with respect to the radius: $\arg z=\frac{2\pi(t-1)}{2^{i-1}}+\frac{\pi}{2^{i-1}}$. Then $U(z)-U(T_i^t(z))$ is harmonic in D_i. On the other hand, we have as above cases $|U(z)-U(T_i^t(z))|<\frac{1}{2n\cdot m!}$ on $A_i^t\cap(\Gamma_i+\Gamma_i')$ for every t. Now let z_1 and z_2 be two points not contained in A_i^t such that $T_i^t(z_1)=z_2$. Then there exists a system S_{z_1, z_2} of transformations satisfying the following conditions:

1°. S_{z_1, z_2} is composed of at most $i-1$ transformations contained in T_1, T_2, $\{T_3^t\}$, $\cdots$, $\{T_i^t\}$.

2°. S_{z_1, z_2} has the form $z_2=T_{n_1}^{s_1}T_{n_2}^{s_2}, \cdots, T_{n_k}^{s_k}(T_i^{s_i})\,T_{n_{k+1}}^{s_{k+2}}, \cdots, T_{n_L}^{s_L}$,

$$L\leqq i-1 \quad\text{and}\quad n_p\neq i \ \text{ for } p=1, 2, \cdots, k,\ k+2, \cdots, L)$$

3. $T_{n_{k+2}}^{s_{k+2}}T_{n_{k+3}}^{s_{k+3}}, \cdots, T_{n_L}^{s_L}(z_1)$ is contained in $A_i^{s_i}$ with the same index s_i as that of $T_i^{s_i}$. Now suppose that the deviation of T_j^t is less than $\frac{j!}{2n\cdot m!}$ for every $j\leqq i-1$ (this is clear for $j=1, 2, 3$). But the deviation of of S_{z_1, z_2} is less than the sum of deviations of $\{T_j\}$ contained in S_{z_1, z_2}. Hence the deviation of T_i^t is less than $\frac{i!}{2n\cdot m!}$, that is $|U(z)-U(T_i^t(z)|<\frac{i!}{2n\cdot m!}$ on $\Gamma_i+\Gamma_i'$ for every t. This implies $|U(z)-U(T_i^t(z))|<\frac{i!}{2n\cdot m!}$ in the ring domain bounded by $\Gamma_i+\Gamma_i'$. Hence the deviation of T_i^t is less than $\frac{i!}{2n\cdot m!}$ in J for every i and t. On the other hand, $|U(z_1)-U(z_2)|<\frac{1}{2n}$ for z_1 and z_2 on $\Gamma_{B,n}$ with $|\arg z_1-\arg z_2|<\frac{2\pi}{2^m}$. Therefore the oscillation of $U(z)$ on $\Gamma_{B,n}$ is less than $\frac{1}{n}$.

Let R_n be a domain bounded by Γ $(|z|=1)$ and $\Gamma_{B,n}$. Then $\bigcap_{n\geqq 1} R_n$ is a Riemann surface with one compact boundary component Γ and one ideal boundary camponent.

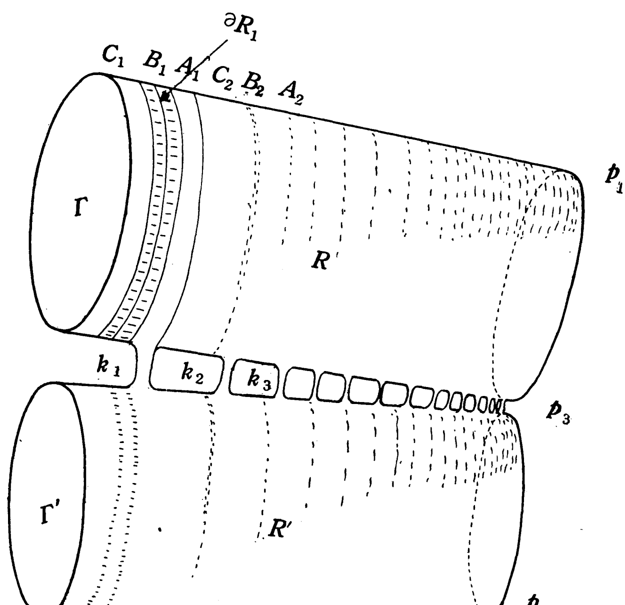
Let $\{k_n\}$ be slits on the radius: $\arg z=0$ in C_n and let $w_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-k_n$ such that $w_{n,n+i}(z)=0$ on $\Gamma+\partial R_{n+i}$ $(=\Gamma_{B,n+i})$, $w_{n,n+i}(z)=1$ on k_n. Put $w_n(z)=\lim_{i=\infty} w_{n,n+i}(z)$. Let $w^*_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-k_n$ such that $w^*_{n,n+i}(z)=0$ on Γ, $w^*_{n,n+i}(z)=1$ on k_n and $\frac{\partial \omega^*_{n+i}(z)}{\partial n}=0$ on ∂R_{n+i}. Put $\lim_{i=\infty} w^*_{n,n+i}(z)=w^*_n(z)$. If we make every k_n sufficiently small, we have

$$\lim_{n=\infty}\left(\max_{z\in\Gamma_{B,n}}\sum_n^\infty w_n(z)\right)=0\,, \tag{1}$$

$$\overline{\lim_{n=\infty}}\left(\max_{z\in\Gamma_{B,n}}\sum_n^\infty w^*_n(z)\right)\leqq\frac{1}{4}\,. \tag{2}$$

Therefore we can suppose that $\{k_n\}$ have been chosen small so that the above conditions are satisfied.

Riemann surface $\tilde{R}$. Let R' be one more Riemann surface which is identical to R. From now, we denote by $V'(z), k', \cdots$ the function, figure, $\cdots$, on R' which corresponds to the function $V(z)$, figure $k, \cdots$ on R respectively. Identify k_n and k_n' for every n. Put $R+\tilde{R}'=\tilde{R}$. Then $\tilde{R}$ is a Riemann surface with two compact boundary compenent Γ and Γ' and has only one ideal boundary component. In what follows, we show that $\tilde{R}$ has the following properties, (Fig. 3).



Riemann surface $\tilde{R}$.

Fig. 3

1). *$\tilde{R}$ has no unbounded positive harmonic functions.*

Let R_n^A be the compact surface of R bounded by Γ and $\Gamma_{A,n}$. Clearly $\bigvee_n R_n^A = R$. Let $\hat{V}_n^A(z)$ be a harmonic function in $R_n^A + R_n^{A\prime}$ such that $\hat{V}_n^A(z)=0$ on $\Gamma+\Gamma'$ and $V_n^A(z)=1$ on $\Gamma_{A,n}+\Gamma'_{A,n}$. Then $\lim_{n=\infty} \hat{V}_n^A(z)=\hat{V}(z)$ $=\frac{\log|z|}{\log 2}$ in the ring domain: $1<|z|<2$. Hence $V(z)$ tends to 1 as z converges to the ideal boundary of R. Let $V_{n,n+i}^A(z)$ be a harmonic function in $R_n^A+R_{n+i}^{A\prime}-\sum_{n+1}^{n+i} k_j$ such that $V_{n,n+i}^A(z)=0$ on $\Gamma+\Gamma'+\sum_{n+1}^{n+i} k_j$ $+\Gamma_{A,n+i}$ and $V_{n,n+i}^A(z)=1$ on $\Gamma_{A,n}$. Consider $V_{n,n+i}^A(z)$ in $R-\sum_{j=1}^{\infty} k_j$. Then $V_{n,n+i}^A(z) \geqq \hat{V}_n^A(z)-\sum_{j=1}^{\infty} w_j(z)$. Hence by letting $i\to\infty$ and then $n\to\infty$ we have $\lim_{n=\infty}\lim_{i=\infty} \hat{V}_{n,n+i}^A(z)=V^A(z)\geqq \hat{V}(z)-\sum_{j=1}^{\infty} w_j(z)$ in $R-\sum_{j=1}^{\infty} k_j$. Therefore by (1) $V^A(z)>0$. (Fig. 4)

Consider a positive harmonic function $U(z)$ in $\tilde{R}$ vanishing on $\Gamma+\Gamma'$. Assume $\max_{z\in\Gamma_{A,n}} U(z)\geqq P_n$ for infinitely many numbers n. Then $\min U(z)$ $\geqq \frac{P_n}{Q_n}$. Hence by the maximum principle $U(z)\geqq\frac{P_n}{Q_n}\,(V_{n,n+i}^A(z))$ in $R-\sum k_i$. Thus we have by letting $i\to\infty$ and then $n\to\infty$, $U(z)=\infty$. This is absurd. Hence by the maximum principle $U(z)\leqq \max_{z\in B_n+B_n'} U(z)$ $\leqq \max_{z\in\Gamma_{A,n}+\Gamma_{A,n}} U(z)\geqq P_n$ except for finitely many numbers. This implies by the property of B_n^* and $B_n^{*\prime}$ the oscillations of $U(z)$ on $\Gamma_{B,n}$ and $\Gamma'_{B,n}$ tend to zero as $n\to\infty$.

Let $\hat{V}_n(z)$ be a harmonic function in R_n+R_n' such that $\hat{V}_n(z)=0$ on $\Gamma+\Gamma'$ and $\hat{V}_n(z)=1$ on $\partial R_n+\partial R_n'$. Then $\lim_{n=\infty}\hat{V}_n(z)=\hat{V}(z)=\lim_{n=\infty}\hat{V}_n^A(z)$. Let $V_{n,n+i}(z)$ be a harmonic function in $R_n+R'_{n+i}-\sum_{n+1}^{n+i} k_j'$ such that $V_{n,n+i}(z)=0$ on $\Gamma+\Gamma'+\partial R'_{n+i}+\sum_{n+1}^{n+i} k_j$ and $V_{n,n+i}(z)=1$ on ∂R_n. Consider $V_{n,n+i}(z)$ in $R-\sum_1^{\infty} k_j$. Then as above, we have $V(z)=\lim_{n=\infty}\lim_{i=\infty} V_{n,n+i}(z)$ $\geqq \hat{V}(z)-\sum^{\infty} w_j(z)$ in $R-\sum_1^{\infty} k_j$. Therefore by (1)

$$\lim_{n=\infty}\big(\min_{z\in\Gamma_{B,n}} V(z)\big)=1\,. \tag{3}$$

Next, consider $V(z)$ in $R'-\sum_1^{\infty} k_j$. Then also we have $V(z)=\lim_n\lim_i V_{n+i}(z)$ $\leqq\sum_1^{\infty} w_j'(z)$ in $R'-\sum_1^{\infty} k_j'$. Hence by (1)

$$\overline{\lim_{n=\infty}}\big(\max_{z\in\Gamma_{B,n}} V(z)\big)=0\,. \tag{4}$$

We call such $V(z)$ the harmonic measure of the ideal boundary determined by a non-compact domain $G=R-\sum_1^\infty k_j$, (Fig. 5).

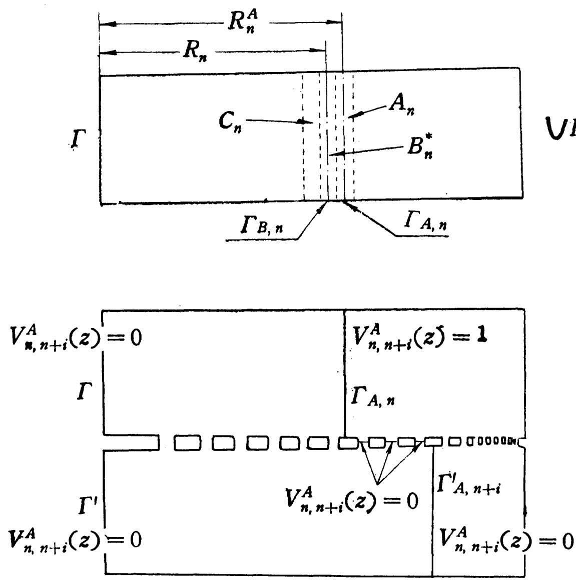


Fig. 4

If $\sup_{z\in\tilde{R}}(U(z)=\infty$, $\max_{z\in\Gamma_{B,n}+\Gamma'_{B,n}} U(z)$ tends to ∞ as $n\to\infty$. This implies by property of B_n^* and $B_n^{*\prime}$ that at least one of $M_n=\min_{z\in\Gamma_{B,n}} U(z)$ and $M_n'=\min_{z\in\Gamma'_{B,n}} U(z)$ tends to ∞ as $n\to\infty$. Assume $M_n\uparrow\infty$. Then clearly

$$U(z)\geqq M_n(V_{n,n+i}(z))-\sum_1^\infty w_j(z)) \qquad \text{in } R-\sum_1^\infty k_j,$$

whence we have by letting $i\to\infty$ and then $n\to\infty$, $U(z)=\infty$. Therefore $U(z)$ is bounded $\leqq M$ in $\tilde{R}$.

2) *There exist only two linearly independint positive harmonic functions vanishing on* $\Gamma+\Gamma'$. Consider $U(z)$ in $R-\sum_1^\infty k_j$. Put $L=\overline{\lim_{n=\infty}}(\max_{z\in\Gamma_{B,n}} U(z)) =\overline{\lim_{n=\infty}}(\min_{z\in\Gamma_{B,n}} U(z))$. Then for any given positive number ε, there exist infinitely many numbers n such that

$$L+\varepsilon\geqq\max_{z\in\Gamma_{B,n}} U(z)\geqq\min_{z\in\Gamma_{B,n}} U(z)\geqq L-\varepsilon.$$

Since $U(z)>0$, $(L+\varepsilon)(V_{n,n+i}(z)+\sum_1^\infty w_j(z))\geqq U(z)\geqq(L-\varepsilon)(V_{n,n+i}(z)-\sum_1^\infty w_j(z))$ in R. Let $i\to\infty$ and then $n\to\infty$ and further let $\varepsilon\to 0$. Then

$$L(V(z)+\sum_1^\infty w_j(z)) \geqq U(z) \geqq L(V(z)-\sum_1^\infty w_j(z)) \,.$$

Hence by (1) and (3) we have $\lim_{n=\infty}(\max_{z\in\Gamma_{B,n}} U(z))=\lim_{n=\infty}(\min_{z\in\Gamma_{B,n}} U(z))=L$. Similarly we have $\lim_{n=\infty}(\max_{z\in'\Gamma_{B,n}} U(z))=\lim_{n\infty z}(\min_{\in\Gamma_{B,n}} U(z))=L'$.

Consider $U(z)$ in $\tilde{R}$. Then by (1), (3) and (4) we have as above, for any given positive number ε,

$$(L+\varepsilon)V(z)+(L'+\varepsilon)V'(z) \geqq U(z) \geqq (L-\varepsilon)V(z)+(L'-\varepsilon)V'(z)\,,$$

where $V'(z)$ is the harmonic measure of the ideal boundary determined by G'. Hence $U(z)=LV(z)+L'V'(z)$. *Thus we have*

3) *There exists no function* $N(z, p)$ *such that* $\sup_{z\in R} N(z, p)=\infty$.

4) *There exists at least two singular ideal boundary points* $\in B_1$. Let $V^*_{n,n+i}(z)$ be a harmonic function in $R'_{n+i}+R_n-\sum_{n+1}^{n+i} k_j$ such that $V^*_{n,n+i}(z)=0$ on $\Gamma+\Gamma'$, $V^*_{n,n+i}(z)=1$ on ∂R_n and $\frac{\partial V^*_{n,n+i}(z)}{\partial n}=0$ $\sum_{n+1}^{n+i} k_j+\partial R'_{n+i}$. Put $V^*(z)=\lim_{i=\infty}\lim_{n=\infty} V^*_{n,n+i}(z)$. $V^*(z)$ is called the equilibrium potential of the ideal boundary determined by non-compact domain G and it is proved as $\omega_F(z)$ is superharmonic in R $(\tilde{R}+B)$. Clearly $V^*(z)\geqq V(z)$,, whence $\lim_{n=\infty}(\min_{z\in\Gamma_{B,n}} V(z))=1$. On the other hand, since $V^*(z)\leqq\sum_1^\infty w^*_n(z)$ in $R'-\sum_1^\infty k_j'$, we have by (4) $\lim_{n=\infty}(\max_{z\in\Gamma_{B,n}} V^*(z))\leqq\frac{1}{4}$. Hence $V^*(z)\neq V'^*(z)$, (Fig. 5). Now $V^*(z)$ and $V^{*\prime}(z)$ are superharmonic in $\tilde{\tilde{R}}$, that is $V^*(z)$

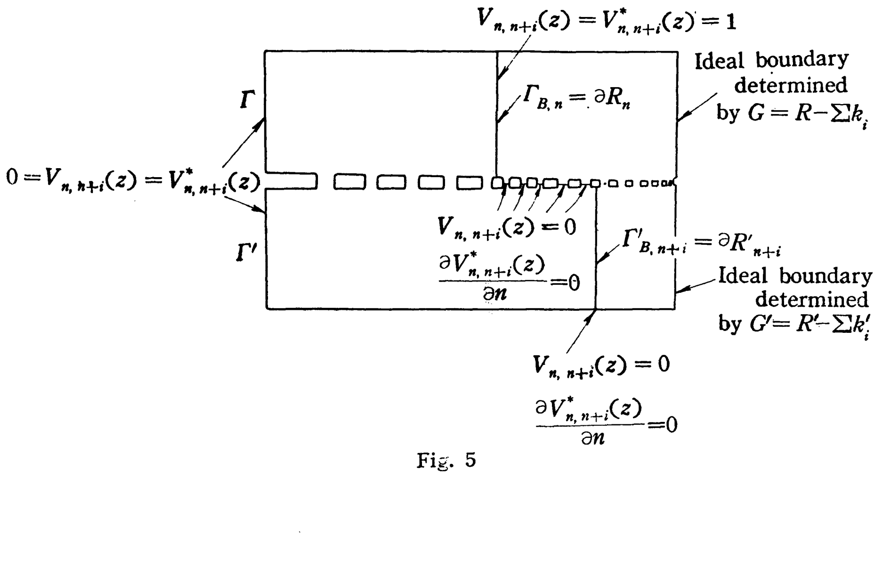


Fig. 5

$= \int_{B_1} N(z, p_\alpha)\, d\mu(p_\alpha)\;\; V'^*(z) = \int_{B_1} N(z, p_\alpha')\, d\mu'(p_\alpha)$. Hence by the symmetric structure of $\tilde{R}$ there must exist at least two singular points p_1 and p_2 in B_1 such that $N(z, p_1) \neq N(z, p_2)$ and $N(z, p_1) = N(T(z), p_2)$, where $T(z)$ is the symmetric point of z with respect to $\sum_1^\infty k_j$. On the other hand, by 2), $N(z, p_1) = N(z, p_1)) = \lambda V(z) + \mu V'(z)$ and $N(z, p_2) = \mu V(z) + \lambda V'(z)$ $(\lambda \neq \mu,\ \mu \geqq 0,\ \lambda \geqq 0)$.

5) *There exists at least one ideal boundary point belonging to* B_0. Let $\{p_1^i\}$ and $\{p_2^i\}$ be fundamental sequences determining p_1 and p_2 respectively. Then $\{p_1^i\}$ and $\{p_2^i\}$ are not contained in $\sum^\infty k_i$, because the symmetric structure of R implies $N(z, p_1) = N(z, p_2)$. Connect p_1^i and p_1^i with a curve C^i. Then there exists a point p_3^i on k_i. Choose a subsequence $\{p_3^i\}$ for which $N(z, p_3^i)$ converges to a function $N(z, p_3)$. Then $N(z, p_3) = \frac{1}{2}(N(z, p_1) + N(z, p_2))$, because $N(z, p_1) = N(T(z), p_2)$, i.e. $N(z, p_3) = K(V(z) + V'(z))$ and $\int_{\partial R_0} \frac{\partial N(z, p_i)}{\partial n} ds = 2\pi$ $(i = 1, 2, 3)$. Then $N(z, p_3)$ and $N(z, p_3) - \frac{1}{2}N(z, p_1) = \frac{1}{2}N(z, p_2)$ are superharmonic and $N(z, p_1)$ is not a multiple of $N(z, p_3)$. Hence $N(z, p_3)$ is not minimal, i.e. $p_3 \in B_0$.

$0 = \mathrm{Cap}\,(B_0) = \mathrm{Cap}\,(p_3)$ and p_3 is a closed set. But there exists no unbounded positive superharmonic function in $\tilde{R}$. *Therefore the condition of Theorem* 14 *that* $F \subset R + B_1$ *is necessary.*

(Received September 19, 1956)