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STRUCTURE PRESERVING GROUP ACTIONS ON STABLY ALMOST COMPLEX MANIFOLDS

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1. Introduction

Conner and Floyd in [1, 2] introduced the notion of periodic maps preserving a complex structure, applying bordism methods quite successfully. In a discussion with Gary Hamrick it became apparent that a somewhat weaker notion was also quite plausible, and the object of this note is to analyze this weaker structure.

Being given a manifold with boundary V and a differentiable action $\phi: G \times V \rightarrow V$, with G a finite group, the differential $d\phi: G \times \tau(V) \rightarrow \tau(V)$ induces a G action on the tangent bundle to V . Being given a real representation $\theta: G \times W \rightarrow W$ of G on a vector space W , one may form a G -bundle $W \times V \xrightarrow{\pi} V$, where G acts by $\theta \times \phi$ on $W \times V$. Then the Whitney sum of $\tau(V)$ and the bundle π has a G -action given by $d\phi$ and θ . Thinking of $E(\tau(V) \oplus \pi)$ as identified with $E(\tau(V)) \times W$, the action is $d\phi \times \theta$.

A bundle map $J: \tau(V) \oplus \pi \rightarrow \tau(V) \oplus \pi$ which covers the identity map on V and such that $J^2 = -1$ in the fibers gives $\tau(V) \oplus \pi$ a complex structure and if J commutes with the G action $d\phi \times \theta$, $\tau(V) \oplus \pi$ becomes a complex G -bundle over V .

If $\psi: G \times T \rightarrow T$ is a complex representation of G one may form the bundle $\bar{\pi}: T \times V \rightarrow V$ with G action given by $\psi \times \phi$, and if $i: T \rightarrow T$ is the function with $i^2 = -1$ giving the complex structure, $\tau(V) \oplus \pi \oplus \bar{\pi}$ is a complex G bundle if G acts by $d\phi \times \theta \times \psi$ and the complex structure is $J \times i$.

A stably almost complex structure on (V, ϕ) preserved by G would then be an equivalence class of systems (W, θ, J) , where two such (W, θ, J) and (W', θ', J') are equivalent if there are complex representations (T, ψ, i) and (T', ψ', i') so that $\tau(V) \oplus \pi \oplus \bar{\pi}$ and $\tau(V) \oplus \pi' \oplus \bar{\pi}'$ are equivalent complex G -bundles.

The boundary of V inherits a stably almost complex structure preserved by G for $\tau(\partial V) \cong \tau(V)|_{\partial V} \oplus 1$ as G -bundles, where 1 is the trivial line bundle coming from the trivial representation of G .

It is clear that this differs from the Conner-Floyd approach in which (W, θ) and (T, ψ) are restricted to be trivial representations.

One may form bordism groups using the new structure preserving actions, which will be denoted $\omega_*^U(G, \mathcal{F}, \mathcal{F}')$ given by G actions preserving a complex structure which are \mathcal{F} -free and such that the boundary action is \mathcal{F}' -free, where, $\mathcal{F}, \mathcal{F}'$ are families in G as in Conner-Floyd [3]. The corresponding groups using the Conner-Floyd definition of "structure preserving" will be denoted $\Omega_*^U(G, \mathcal{F}, \mathcal{F}')$, and the forgetful homomorphism will be denoted by

$$\rho: \Omega_*^U(G, \mathcal{F}, \mathcal{F}') \rightarrow \omega_*^U(G, \mathcal{F}, \mathcal{F}').$$

The remainder of this paper will be devoted to analyzing $\omega_*^U(G, \mathcal{F}, \mathcal{F}')$ and ρ in the case when G is cyclic of prime order. Surprisingly, the cases $G=Z_2$ and $G=Z_p$ with p odd are considerably different, which is not the case for the Conner-Floyd groups.

2. Structure preserving involutions

Now consider the special case $G=Z_2$, writing (V, ϕ) as (V, t) where t is the involution generating the Z_2 action. There are three families for Z_2 , the empty family ϕ , the family Free= $\{\{1\}\}$, and the family All of all subgroups. Letting $\omega_*^U(Z_2, \mathcal{F}) = \omega_*^U(Z_2, \mathcal{F}, \phi)$, the groups of interest are related by an exact sequence

$$\begin{array}{ccc} \omega_*^U(Z_2, \text{Free}) & \xrightarrow{i} & \omega_*^U(Z_2, \text{All}) \\ & \searrow \partial & \swarrow j \\ & & \omega_*^U(Z_2, \text{All}, \text{Free}) \end{array}$$

where i, j are induced by inclusion of families and ∂ by taking the boundary.

First, to analyze $\omega_*^U(Z_2, \text{All}, \text{Free})$, consider an involution (V, t) on an n -dimensional manifold, with t acting freely on ∂V with J the complex operator on $\tau(V) \oplus k \oplus l$ with involution $dt \oplus 1 \oplus (-1)$, where k, l denote trivial bundles of dimensions k and l respectively.

The fixed point set of t in V is a disjoint union of closed submanifolds F^{n-q} of dimension $n-q$, with normal bundles ν_q . A neighborhood of the fixed set of t may be identified with the disjoint union of the disc bundles $D(\nu_q)$, and since t acts freely on the complement of this neighborhood, one may cut the remainder away up to cobordism.

Along F^{n-q} , the bundle $\tau(V) \oplus k \oplus l$ decomposes into the eigen-bundles of $dt \oplus 1 \oplus (-1)$ which are preserved by J , so that $\tau(F^{n-q}) \oplus k$, the $+1$ eigen-bundle, and $\nu^q \oplus l$, the (-1) eigen-bundle are complex bundles. Thus F^{n-q} is a stably almost complex manifold and ν_q is a q -plane bundle with a stable complex structure.

Letting B_q be the bundle over BO_q induced from the fibration $BU \rightarrow BO$, the bundle ν_q is induced by a map into B_q . Thus one has:

Proposition 2.1. $\omega_n^U(Z_2, \text{All}, \text{Free}) \cong \bigoplus_{q=0}^n \Omega_{n-q}^U(B_q)$.

The group $\Omega_n^U(Z_2, \text{All}, \text{Free}) \cong \bigoplus_{q=0}^{\lfloor n/2 \rfloor} \Omega_{n-2q}^U(BU_q)$ and the restriction homomorphism ρ is induced by the obvious maps $BU_j \rightarrow B_{2j}$.

The homology of the space B_q was computed in [4], and is torsion free, so $\Omega_*^U(B_q)$ is computable explicitly. Since the homomorphism $\Omega_*^U(BU_j) \rightarrow \Omega_*^U(BU)$ is a monomorphism onto a direct summand, and factors through ρ , one has:

Proposition 2.2. $\omega_*^U(Z_2, \text{All}, \text{Free})$ is a free Ω_*^U module and the restriction

$$\rho: \Omega_*^U(Z_2, \text{All}, \text{Free}) \rightarrow \omega_*^U(Z_2, \text{All}, \text{Free})$$

is a monomorphism onto a direct summand.

Turning to $\omega_*^U(Z_2, \text{Free})$, consider an involution (V, t) on an n -dimensional manifold, with t acting freely and with J the complex operator on $\tau(V) \oplus k \oplus l$ with involution $dt \oplus 1 \oplus (-1)$. By identifying x and $t(x)$ in V , one obtains the orbit space V/t and a quotient map $\pi: V \rightarrow V/t$, with V/t also being an n -dimensional manifold. Since $dt \oplus 1 \oplus (-1)$ covers t which is free, $dt \oplus 1 \oplus (-1)$ is free and the orbit space $E(\tau(V) \oplus k \oplus l)/(dt \oplus 1 \oplus (-1))$ may be identified with the total space of the bundle $\tau(V/t) \oplus k \oplus l\xi$ where ξ is the line bundle associated with the double cover $\pi: V \rightarrow V/t$. Since J commutes with $dt \oplus 1 \oplus (-1)$, one has induced a complex structure on $\tau(V/t) \oplus k \oplus l\xi$ and a complex structure on $\tau(V) \oplus k \oplus l\xi$ induces a complex structure on $\tau(V) \oplus k \oplus l$, which is the bundle induced by π , commuting with the action.

Now $2\xi \cong \xi \otimes_R \mathbb{C}$ has a complex structure, so a complex structure on $\tau(V/t) \oplus k \oplus l\xi$ is equivalent to a stable complex structure on $\tau(V/t)$ if l is even, or to a stable complex structure on $\tau(V/t) \oplus \xi$ if l is odd. Since the parity of l for V and ∂V is the same, $\omega_*^U(Z_2, \text{Free})$ decomposes into two direct summands, $\omega_*^U(Z_2, \text{Free})^+$ and $\omega_*^U(Z_2, \text{Free})^-$ for l even and odd respectively.

First considering $\omega_*^U(Z_2, \text{Free})^+$, the class of V , if ∂V is empty, is completely determined by the stably almost complex manifold V/t with its double cover V . Hence $\omega_*^U(Z_2, \text{Free})^+ \cong \Omega_*^U(RP(\infty))$, by assigning to the class of V the class of the map $V/t \rightarrow RP(\infty)$ classifying the double cover. The homomorphism ρ sends $\Omega_*^U(Z_2, \text{Free})$ into $\omega_*^U(Z_2, \text{Free})^+$ and composing to $\Omega_*^U(RP(\infty))$ is the usual isomorphism for computing $\Omega_*^U(Z_2, \text{Free})$.

For $\omega_*^U(Z_2, \text{Free})^-$, one has a classifying map $V/t \xrightarrow{f} RP(\infty)$ with ξ induced from the canonical bundle λ over $RP(\infty)$. The tangent bundle of $D(\xi)$, the disc bundle, is the pullback of $\tau(V/t) \oplus \xi$, so that $D(\xi)$ is a stably almost complex manifold. One then has the map $(D\xi, S\xi) \rightarrow (D\lambda, S\lambda) \rightarrow (T\lambda, *) \simeq (RP(\infty), *)$ where S is the sphere bundle and T is the Thom space, which defines a homomorphism from $\omega_*^U(Z_2, \text{Free})^-$ into the reduced bordism group $\tilde{\Omega}_{*+1}^U(RP(\infty))$. By

applying transverse regularity arguments with $RP(\infty)$ considered as the Thom space of λ , one may reverse this process to recover V , so $\omega_*^U(Z_2, \text{Free})^- \cong \tilde{\Omega}_{*+1}^U(RP(\infty))$.

Combining these results gives:

Proposition 2.3. $\omega_*^U(Z_2, \text{Free}) \cong \Omega_*^U(RP(\infty)) \oplus \tilde{\Omega}_{*+1}^U(RP(\infty))$ and ρ sends $\Omega_*^U(Z_2, \text{Free})$ isomorphically onto the first summand.

Note. The Smith homomorphism is much more reasonably defined in $\omega_*^U(Z, \text{Free})$ than in Conner-Floyd's groups. Specifically, if (M, t) is a structure preserving involution, then splitting M gives a submanifold M' invariant under t whose normal bundle in M is the trivial line bundle of the non-trivial representation. Thus the Smith homomorphism maps the summands $\omega_*^U(Z_2, \text{Free})^+$ and $\omega_*^U(Z_2, \text{Free})^-$ into each other. In particular

$$\Delta: \omega_n^U(Z_2, \text{Free})^+ = \Omega_n^U(RP(\infty)) \rightarrow \omega_{n-1}^U(Z_2, \text{Free})^- = \tilde{\Omega}_n^U(RP(\infty))$$

is the reduction homomorphism, and

$$\Delta: \omega_n^U(Z_2, \text{Free})^- = \tilde{\Omega}_{n+1}^U(RP(\infty)) \rightarrow \omega_{n-1}^U(Z_2, \text{Free})^+ = \Omega_{n-1}^U(RP(\infty))$$

is obtained by dualizing $\xi \oplus \xi$.

To compute $\omega_*^U(Z_2, \text{All})$, one makes use of the exact sequence of the families. Being given a map $F^{n-q} \rightarrow B_q$ representing an element of $\omega_n^U(Z_2, \text{All}, \text{Free})$, the bundle $\nu_q \oplus l$ is complex over F^{n-q} and hence $q+l$ is even. Thus along the boundary of $D(\nu_q)$, $q+l$ must also be even, and the homomorphism

$$\partial: \omega_n^U(Z_2, \text{All}, \text{Free}) \rightarrow \omega_{n-1}^U(Z_2, \text{Free})$$

sends $\bigoplus_{q \text{ odd}} \Omega_{n-q}^U(B_q)$ into $\omega_{n-1}^U(Z_2, \text{Free})^-$ and $\bigoplus_{q \text{ even}} \Omega_{n-q}^U(B_q)$ into $\omega_{n-1}^U(Z_2, \text{Free})^+$.

The diagram

$$\begin{array}{ccc} \Omega_n^U(Z_2, \text{All}, \text{Free}) & \xrightarrow{\partial} & \Omega_{n-1}^U(Z_2, \text{Free}) \\ \downarrow & & \cong \downarrow \rho \\ \bigoplus_{q \text{ even}} \Omega_{n-q}^U(BU_{q/2}) & & \omega_{n-1}^U(Z_2, \text{Free})^+ = \Omega_{n-1}^U(RP(\infty)) \\ \downarrow & \nearrow \partial & \\ \bigoplus_{q \text{ even}} \Omega_{n-q}^U(B_q) & & \end{array}$$

commutes, and $\rho\partial$ is known to map onto $\tilde{\Omega}_{n-1}^U(RP(\infty))$. The summand Ω_{n-1}^U complementary to $\tilde{\Omega}_{n-1}^U(RP(\infty))$ is realized as the manifolds $M \times Z_2$ with M stably almost complex and t interchanging the two copies of M . Applying i and the augmentation $\varepsilon: \omega_{n-1}^U(Z_2, \text{All}) \rightarrow \Omega_{n-1}^U$ which takes the cobordism class of the underlying manifold, one obtains $2[M]$. Thus i is monic on this summand and

the image of ∂ in $\omega_{n-1}^U(Z_2, \text{Free})^+$ is precisely $\tilde{\Omega}_{n-1}^U(RP(\infty))$.

Now considering $\omega_{*}^U(Z_2, \text{Free})^- \cong \tilde{\Omega}_{*+1}^U(RP(\infty))$, one notes that $\tilde{\Omega}_{*}^U(RP(\infty))$ is generated as Ω_{*}^U module by the inclusion maps $RP(2i+1) \rightarrow RP(\infty)$ which are obtained by Thomifying the inclusion $RP(2i) \rightarrow RP(\infty)$, for which the induced double cover is the antipodal involution on S^{2i} . The complex structure imparted may be considered as that given by considering $S^{2i} \subset C^{i+1}$, where C^{i+1} has the involution given by multiplication by -1 , and the complex structure given by multiplication by $\sqrt{-1}$, imparting the appropriate structure to $\tau(S^{2i}) \oplus 1 \oplus 1$. The same construction gives an involution on $D^{2i+1} \subset C^{i+1}$ with appropriate structure on $\tau(D^{2i+1}) \oplus 0 \oplus 1$. Thus these classes are in the image of ∂ , and since ∂ is a Ω_{*}^U module homomorphism, $\omega_{*}^U(Z_2, \text{Free})^-$ is contained in the image of ∂ .

Thus one has compatible splittings for the sequences to obtain a commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow \Omega_n^U \rightarrow \Omega_n^U(Z_2, \text{All}) \rightarrow \bigoplus_{q \text{ even}} \Omega_{n-q}^U(BU_{q/2}) \rightarrow \tilde{\Omega}_{n-1}^U(RP(\infty)) \rightarrow 0 \\ \quad \quad \quad \downarrow 1 \quad \quad \quad \downarrow \rho \quad \quad \quad \downarrow \rho' \quad \quad \quad \downarrow \rho'' \\ 0 \rightarrow \Omega_n^U \rightarrow \omega_n^U(Z_2, \text{All}) \rightarrow \bigoplus_q \Omega_{n-q}^U(B_q) \rightarrow \tilde{\Omega}_{n-1}^U(RP(\infty)) \oplus \tilde{\Omega}_n^U(RP(\infty)) \rightarrow 0 \end{array}$$

in which both ρ' and ρ'' are monomorphisms onto direct summands, and 1 is the identity.

Rather than belabor the point further, one has:

Proposition 2.4. $\rho: \Omega_{*}^U(Z_2, \text{All}) \rightarrow \omega_{*}^U(Z_2, \text{All})$ is a monomorphism.

3. Maps of odd prime period

Now consider the case $G=Z_p$ with p an odd prime, again writing (V, ϕ) as (V, t) where t is a diffeomorphism of period p . Again there are three families: ϕ , Free, and All and one has an exact sequence

$$\begin{array}{ccc} \omega_{*}^U(Z_p, \text{Free}) & \xrightarrow{i} & \omega_{*}^U(Z_p, \text{All}) \\ & \swarrow & \searrow \\ & \omega_{*}^U(Z_p, \text{All, Free}) & \end{array}$$

To begin, consider $\omega_{*}^U(Z_p, \text{Free})$. If (V, t) is a free action of Z_p on an n -manifold with $dt \times s$ acting on $\tau(V) \oplus \pi$, where π is given by the representation (W, θ) , then one may form the orbit space V/Z_p which is an n -manifold with $pr: V \rightarrow V/Z_p$ the projection. Since $dt \times s$ acts freely on $E(\tau(V) \oplus \pi)$, $E(\tau(V) \oplus \pi)/Z_p \rightarrow V/Z_p$ is a vector bundle and complex structures preserved by $dt \times s$ are given by complex structures on the quotient bundle.

Now (W, θ) may be decomposed by means of the irreducible representations

into a direct sum of subrepresentations W_0 , which is trivial, and W_k for $1 \leq k \leq (p-1)/2$ where W_k is a complex vector space in which s acts as multiplication by $\exp\left(\frac{2\pi ik}{p}\right)$. In particular, $E(\pi)/Z_p \rightarrow V/Z_p$ is then the Whitney sum of a trivial bundle ξ_0 with fiber W_0 and the complex vector bundles ξ_k with fiber W_k associated with the p -fold cover $V \rightarrow V/Z_p$. Thus $E(\tau(V) \oplus \pi)/Z_p$ is the total space of the bundle $\tau(V/Z_p) \oplus \xi_0 \oplus (\oplus \xi_k)$. Since $(\oplus \xi_k)$ has been given a complex structure, the complex structures on $\tau(V)$ preserved under the action are given precisely by stably almost complex structures on V/Z_p . Thus a structure preserving Z_p action is just a principal Z_p bundle over a stably almost complex manifold. Assigning to (V, t) the map $V/Z_p \rightarrow BZ_p$ classifying the cover then defines an isomorphism of $\omega_*^U(Z_p, \text{Free})$ with $\Omega_*^U(BZ_p)$. When applied to structure preserving actions of Z_p in the sense of Conner and Floyd, one also obtains an isomorphism and so one obtains:

Proposition 3.1 *The restriction homomorphism $\rho: \Omega_*^U(Z_p, \text{Free}) \rightarrow \omega_*^U(Z_p, \text{Free})$ is an isomorphism.*

In the commutative diagram

$$\begin{array}{ccc} \Omega_*^U(Z_p, \text{All}, \text{Free}) & \xrightarrow{\partial'} & \Omega_*^U(Z_p, \text{Free}) \\ \rho \downarrow & & \cong \downarrow \rho \\ \omega_*^U(Z_p, \text{All}, \text{Free}) & \xrightarrow{\partial} & \omega_*^U(Z_p, \text{Free}) \end{array}$$

it is known that the image of ∂' is $\tilde{\Omega}_*^U(BZ_p)$, and the composite

$$\Omega_*^U \rightarrow \omega_*^U(Z_p, \text{Free}) \xrightarrow{i} \omega_*^U(Z_p, \text{All}) \xrightarrow{\varepsilon} \Omega_*^U$$

is multiplication by p on the complementary summand, so the image of ∂ is precisely $\tilde{\Omega}_*^U(BZ_p)$.

Thus one has a splitting, giving the diagram

$$\begin{array}{ccccccc} 0 \rightarrow & \Omega_*^U & \rightarrow & \Omega_*^U(Z_p, \text{All}) & \rightarrow & \Omega_*^U(Z_p, \text{All}, \text{Free}) & \rightarrow & \tilde{\Omega}_*^U(BZ_p) & \rightarrow & 0 \\ & \downarrow 1 & & \rho \downarrow & & \rho \downarrow & & \downarrow 1 & & \\ 0 \rightarrow & \Omega_*^U & \rightarrow & \omega_*^U(Z_p, \text{All}) & \rightarrow & \omega_*^U(Z_p, \text{All}, \text{Free}) & \rightarrow & \tilde{\Omega}_*^U(BZ_p) & \rightarrow & 0. \end{array}$$

Now consider the group $\omega_*^U(Z_p, \text{All}, \text{Free})$. Letting (V, t) be an action which is free on ∂V , the fixed point set of V is a disjoint union of closed submanifolds F^{n-q} with normal bundles ν_q and V may be replaced by the disc bundles of the ν_q . At points of F^{n-q} , the bundle $\tau \oplus \pi$ decomposes into $\tau(F^{n-q}) \oplus \xi_0$, where ξ_0 is the trivial bundle of W_0 , which is the trivial eigen-bundle, and bundles $(\nu_q)_k \oplus \xi_k$, where ξ_k is the trivial bundle with fiber W_k and $(\nu_q)_k$ is a sub-bundle

of $\nu_q|F^{n-q}$, giving the eigen-bundle corresponding to multiplication by $\exp\left(\frac{2\pi ik}{p}\right)$ for $1 \leq k \leq (p-1)/2$. Considered as a complex Z_p bundle, the bundle $\tau \oplus \pi$ decomposes into complex sub-bundles η_0 , the trivial eigen-bundle, and η_j , $1 \leq j \leq p-1$ on which $dt \times s$ acts as multiplication by $\exp\left(\frac{2\pi ij}{p}\right)$. Taking the parts of the complex decomposition which give the real decomposition, one has $\eta_0 \cong \tau(F^{n-q}) \oplus \xi_0$, so F^{n+q} is stably almost complex, and $(\nu_q)_k \oplus \xi_k \cong \eta_k \oplus \eta_j$ where $(p-1)/2 \leq j \leq p-1$ and $\exp\left(\frac{2\pi ij}{p}\right)$ is the complex conjugate of $\exp\left(\frac{2\pi ik}{p}\right)$, or $j = p-k$.

After stabilization, the bundles η_k and η_{p-k} are stable complex bundles subject only to the condition that $\eta_k \oplus \eta_{p-k}$ should be stably isomorphic as complex bundle with $(\nu_q)_k$. Thus, the class of (V, t) is completely determined by the bordism classes $F_{(r)}^{n-q} \rightarrow BU_{r_1} \times BU \times \cdots \times BU_{r_{(p-1)/2}} \times BU$ where $r_1 + \cdots + r_{(p-1)/2} = q/2$, where $F_{(r)}^{n-q}$ are the portions of F^{n-q} over which $(\nu_q)_k$ has real dimension $2r_k$, the map into BU_{r_k} classifying $(\nu_q)_k$, and that into the k -th BU factor classifying η_k . Thus, one has

Proposition 3.2 $\omega_n^U(Z_p, \text{All}, \text{Free})$ is isomorphic to

$$\bigoplus_{(r)} \Omega_{n-2r}^U(BU_{r_1} \times BU \times \cdots \times BU_{r_{(p-1)/2}} \times BU),$$

the sum being over all sequences $(r) = (r_1, \dots, r_{(p-1)/2})$ of non-negative integers, and with $r = r_1 + \cdots + r_{(p-1)/2}$.

In order to analyze $\rho: \Omega_n^U(Z_p, \text{All}, \text{Free}) \rightarrow \omega_n^U(Z_p, \text{All}, \text{Free})$, one may simply note that analogously $\Omega_n^U(Z_p, \text{All}, \text{Free})$ is isomorphic to

$$\bigoplus_{(s,t)} \Omega_{n-2r}^U(BU_{s_1} \times BU_{t_1} \times \cdots \times BU_{s_{(p-1)/2}} \times BU_{t_{(p-1)/2}})$$

where $\frac{q}{2} = r = s_1 + \cdots + s_{(p-1)/2} + t_1 + \cdots + t_{(p-1)/2}$ and the map of $F_{(s,t)}^{n-q}$ into BU_{s_k} classifies η_k and into BU_{t_k} classifies η_{p-k} , with $(\nu_q)_k \cong \eta_k \oplus \eta_{p-k}$ in this case. The map ρ is then induced by the maps $\cup BU_{s_k} \times BU_{t_k} \rightarrow BU_{r_k} \times BU$ given by the Whitney sum map $BU_{s_k} \times BU_{t_k} \xrightarrow{s_k + t_k = r_k} BU_{r_k}$ and by $BU_{s_k} \times BU_{t_k} \xrightarrow{pr} BU_{s_k} \xrightarrow{\sigma} BU$ where pr is the projection and σ is stabilization.

One may then observe that ρ is anything but monic, for many summands in $\Omega_n^U(Z_p, \text{All}, \text{Free})$ map to the same summand in $\omega_n^U(Z_p, \text{All}, \text{Free})$. (One need only look at the terms with $n=2r$ in which many copies of Z map to a single copy of Z). Since, by the commutative diagram, the kernels of the homomorphisms $\rho: \Omega_n^U(Z_p, \text{All}, \text{Free}) \rightarrow \omega_n^U(Z_p, \text{All}, \text{Free})$ and $\rho: \Omega_n^U(Z_p, \text{All}) \rightarrow \omega_n^U(Z_p, \text{All})$ are isomorphic, one sees that $\rho: \Omega_n^U(Z_p, \text{All}) \rightarrow \omega_n^U(Z_p, \text{All})$ is also not monic.

The homomorphism ρ is also not epic, for the map

$\prod_{s_k+t_k=r_k} BU_{s_k} \times BU_{t_k} \rightarrow BU_{r_k} \times BU$ factors through $BU_{r_k} \times BU_{r_k}$. One can, of course, compute $\rho: \Omega_n^U(Z_p, \text{All, Free}) \rightarrow \omega_n^U(Z_p, \text{All, Free})$ explicitly since the groups and map are completely known, but it hardly seems worthwhile.

As a final note, one should consider the reason why the Z_2 and Z_p cases, p odd, are so different. Clearly the problem is the dissimilarity between the nature of real representations in the two cases. In studying $\Omega_*^U(G, *, *)$ only the complex representations really play a role, while in $\omega_*^U(G, *, *)$ both types enter.

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