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STRUCTURE PRESERVING GROUP ACTIONS ON STABLY ALMOST COMPLEX MANIFOLDS

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1. Introduction

Conner and Floyd in [1, 2] introduced the notion of periodic maps preserving a complex structure, applying bordism methods quite successfully. In a discussion with Gary Hamrick it became apparent that a somewhat weaker notion was also quite plausible, and the object of this note is to analyze this weaker structure.

Being given a manifold with boundary V and a differentiable action $\phi: G\times V\to V$, with G a finite group, the differential $d\phi: G\times\tau(V)\to\tau(V)$ induces a G action on the tangent bundle to V. Being given a real representation $\theta: G\times W\to W$ of G on a vector space W, one may form a G-bundle $W\times V\xrightarrow{\pi} V$, where G acts by $\theta\times\phi$ on $W\times V$. Then the Whitney sum of $\tau(V)$ and the bundle π has a G-action given by $d\phi$ and θ. Thinking of $E(\tau(V)\oplus\pi)$ as identified with $E(\tau(V))\times W$, the action is $d\phi\times\theta$.

A bundle map $J: \tau(V)\oplus\pi\to\tau(V)\oplus\pi$ which covers the identity map on V and such that $J^2=-1$ in the fibers gives $\tau(V)\oplus\pi$ a complex structure and if J commutes with the G action $d\phi\times\theta$, $\tau(V)\oplus\pi$ becomes a complex G-bundle over V.

If $\psi: G\times T\to T$ is a complex representation of G one may form the bundle $\bar{\pi}: T\times V\to V$ with G action given by $\psi\times\phi$, and if $i: T\to T$ is the function with $i^2=-1$giving the complex structure, $\tau(V)\oplus\pi\oplus\bar{\pi}$ is a complex G bundle if G acts by $d\phi\times\theta\times\psi$ and the complex structure is $J\times i$.

A stably almost complex structure on (V, ϕ) preserved by G would then be an equivalence class of systems (W, θ, J), where two such (W, θ, J) and (W', θ', J') are equivalent if there are complex representations (T, ψ, i) and (T', ψ', i') so that $\tau(V)\oplus\pi\oplus\bar{\pi}$ and $\tau(V)\oplus\pi'\oplus\bar{\pi}'$ are equivalent complex G-bundles.

The boundary of V inherits a stably almost complex structure preserved by G for $\tau(\partial V)\cong\tau(V)|_{\partial V}\oplus 1$ as G-bundles, where 1 is the trivial line bundle coming from the trivial representation of G.

It is clear that this differs from the Conner-Floyd approach in which (W, θ) and (T, ψ) are restricted to be trivial representations.

One may form bordism groups using the new structure preserving actions, which will be denoted $\omega_*^U(G, \mathcal{F}, \mathcal{F}')$ given by G actions preserving a complex structure which are $\mathcal{F}$-free and such that the boundary action is $\mathcal{F}'$-free, where, $\mathcal{F}, \mathcal{F}'$ are families in G as in Conner-Floyd [3]. The corresponding groups using the Conner-Floyd definition of "structure preserving" will be denoted $\Omega_*^U(G, \mathcal{F}, \mathcal{F}')$, and the forgetful homomorphism will be denoted by

$$\rho: \Omega_*^U(G, \mathcal{F}, \mathcal{F}') \to \omega_*^U(G, \mathcal{F}, \mathcal{F}').$$

The remainder of this paper will be devoted to analyzing $\omega_*^U(G, \mathcal{F}, \mathcal{F}')$ and ρ in the case when G is cyclic of prime order. Surprisingly, the cases $G=Z_2$ and $G=Z_p$ with p odd are considerably different, which is not the case for the Conner-Floyd groups.

2. Structure preserving involutions

Now consider the special case $G=Z_2$, writing (V, ϕ) as (V, t) where t is the involution generating the Z_2 action. There are three families for Z_2, the empty family ϕ, the family Free$=\{\{1\}\}$, and the family All of all subgroups. Letting $\omega_*^U(Z_2, \mathcal{F})=\omega_*^U(Z_2, \mathcal{F}, \phi)$, the groups of interest are related by an exact sequence

$$\begin{array}{ccc} \omega_*^U(Z_2, \text{Free}) & \xrightarrow{\;i\;} & \omega_*^U(Z_2, \text{All}) \\ & {}_{\partial}\nwarrow \quad \swarrow_{j} & \\ & \omega_*^U(Z_2, \text{All, Free}) & \end{array}$$

where i, j are induced by inclusion of families and ∂ by taking the boundary.

First, to analyze $\omega_*^U(Z_2, \text{All, Free})$, consider an involution (V, t) on an n-dimensional manifold, with t acting freely on ∂V with J the complex operator on $\tau(V)\oplus k\oplus l$ with involution $dt\oplus 1\oplus(-1)$, where k, l denote trivial bundles of dimensions k and l respectively.

The fixed point set of t in V is a disjoint union of closed submanifolds F^{n-q} of dimension $n-q$, with normal bundles ν_q. A neighborhood of the fixed set of t may be identified with the disjoint union of the disc bundles $D(\nu_q)$, and since t acts freely on the complement of this neighborhood, one may cut the remainder away up to cobordism.

Along F^{n-q}, the bundle $\tau(V)\oplus k\oplus l$ decomposes into the eigen-bundles of $dt\oplus 1\oplus(-1)$ which are preserved by J, so that $\tau(F^{n-q})\oplus k$, the $+1$ eigen-bundle, and $\nu^q\oplus l$, the (-1) eigen-bundle are complex bundles. Thus F^{n-q} is a stably almost complex manifold and ν_q is a q-plane bundle with a stable complex structure.

Letting B_q be the bundle over BO_q induced from the fibration $BU\to BO$, the bundle ν_q is induced by a map into B_q. Thus one has:

Proposition 2.1. $\omega_n^U(Z_2, \text{All, Free}) \cong \bigoplus_{q=0}^{n} \Omega_{n-q}^U(B_q)$.

The group $\Omega_n^U(Z_2, \text{All, Free}) \cong \bigoplus_{q=0}^{[n/2]} \Omega_{n-2q}^U(BU_q)$ and the restriction homomorphism ρ is induced by the obvious maps $BU_j \to B_{2j}$.

The homology of the space B_q was computed in [4], and is torsion free, so $\Omega_*^U(B_q)$ is computable explicitly. Since the homomorphism $\Omega_*^U(BU_j) \to \Omega_*^U(BU)$ is a monomorphism onto a direct summand, and factors through ρ, one has:

Proposition 2.2. *$\omega_*^U(Z_2, All, Free)$ is a free Ω_*^U module and the restriction*

$$\rho\colon \Omega_*^U(Z_2, All, Free) \to \omega_*^U(Z_2, All, Free)$$

is a monomorphism onto a direct summand.

Turning to $\omega_*^U(Z_2, \text{Free})$, consider an involution (V, t) on an n-dimensional manifold, with t acting freely and with J the complex operator on $\tau(V) \oplus k \oplus l$ with involution $dt \oplus 1 \oplus (-1)$. By identifying x and $t(x)$ in V, one obtains the orbit space V/t and a quotient map $\pi\colon V \to V/t$, with V/t also being an n-dimen -sional manifold. Since $dt \oplus 1 \oplus (-1)$ covers t which is free, $dt \oplus 1 \oplus (-1)$ is free and the orbit space $E(\tau(V) \oplus k \oplus l)/(dt \oplus 1 \oplus (-1))$ may be indentified with the total space of the bundle $\tau(V/t) \oplus k \oplus l\xi$ where ξ is the line bundle associated with the double cover $\pi\colon V \to V/t$. Since J commutes with $dt \oplus 1 \oplus (-1)$, one has induced a complex structure on $\tau(V/t) \oplus k \oplus l\xi$ and a complex structure on $\tau(V) \oplus k \oplus l\xi$ induces a complex structure on $\tau(V) \oplus k \oplus l$, which is the bundle induced by π, commuting with the action.

Now $2\xi \cong \xi \otimes_R C$ has a complex structure, so a complex structure on $\tau(V/t) \oplus k \oplus l\xi$ is equivalent to a stable complex structure on $\tau(V/t)$ if l is even, or to a stable complex structure on $\tau(V/t) \oplus \xi$ if l is odd. Since the parity of l for V and ∂V is the same, $\omega_*^U(Z_2, \text{Free})$ decomposes into two direct summands, $\omega_*^U(Z_2, \text{Free})^+$ and $\omega_*^U(Z_2, \text{Free})^-$ for l even and odd respectively.

First considering $\omega_*^U(Z_2, \text{Free})^+$, the class of V, if ∂V is empty, is completely determined by the stably almost complex manifold V/t with its double cover V. Hence $\omega_*^U(Z_2, \text{Free})^+ \cong \Omega_*^U(RP(\infty))$, by assigning to the class of V the class of the map $V/t \to RP(\infty)$ classifying the double cover. The homomorphism ρ sends $\Omega_*^U(Z_2, \text{Free})$ into $\omega_*^U(Z_2, \text{Free})^+$ and composing to $\Omega_*^U(RP(\infty))$ is the usual isomorphism for computing $\Omega_*^U(Z_2, \text{Free})$.

For $\omega_*^U(Z_2, \text{Free})^-$, one has a classifying map $V/t \xrightarrow{f} RP(\infty)$ with ξ induced from the canonical bundle λ over $RP(\infty)$. The tangent bundle of $D(\xi)$, the disc bundle, is the pullback of $\tau(V/t) \oplus \xi$, so that $D(\xi)$ is a stably almost complex manifold. One then has the map $(D\xi, S\xi) \to (D\lambda, S\lambda) \to (T\lambda, *) \simeq (RP(\infty), *)$ where S is the sphere bundle and T is the Thom space, which defines a homomorphism from $\omega_*^U(Z_2, \text{Free})^-$ into the reduced bordism group $\tilde{\Omega}_{*+1}^U(RP(\infty))$. By applying transverse regularity arguments with $RP(\infty)$ considered as the Thom space of λ, one may reverse this process to recover V, so $\omega_*^U(Z_2, \text{Free})^- \cong \tilde{\Omega}_{*+1}^U(RP(\infty))$.

Combining these results gives:

Proposition 2.3. *$\omega_*^U(Z_2. \mathit{Free}) \cong \Omega_*^U(RP(\infty)) \oplus \tilde{\Omega}_{*+1}^U(RP(\infty))$ and ρ sends $\Omega_*^U(Z_2, \mathit{Free})$ isomorphically onto the first summand.*

Note. The Smith homomorphism is much more reasonably defined in $\omega_*^U(Z, \text{Free})$ than in Conner-Floyd's groups. Specifically, if (M, t) is a structure preserving involution, then splitting M gives a submanifold M' invariant under t whose normal bundle in M is the trivial line bundle of the non-trivial representation. Thus the Smith homomorphism maps the summands $\omega_*^U(Z_2, \text{Free})^+$ and $\omega_*^U(\mathrm{Z}_2, \text{Free})^-$ into each other. In particular

$$\Delta\colon \omega_n^U(\mathrm{Z}_2, \text{Free})^+ = \Omega_n^U(RP(\infty)) \to \omega_{n-1}^U(Z_2, \text{Free})^- = \tilde{\Omega}_n^U(RP(\infty))$$

is the reduction homomorphism, and

$$\Delta\colon \omega_n^U(Z_2, \text{Free})^- = \tilde{\Omega}_{n+1}^U(RP(\infty)) \to \omega_{n-1}^U(Z_2, \text{Free})^+ = \Omega_{n-1}^U(RP(\infty))$$

is obtained by dualizing $\xi \oplus \xi$.

To compute $\omega_*^U(\mathrm{Z}_2, \text{All})$, one makes use of the exact sequence of the families. Being given a map $F^{n-q} \to B_q$ representing an element of $\omega_n^U(Z_2, \text{All}, \text{Free})$, the bundle $\nu_q \oplus l$ is complex over F^{n-q} and hence $q+l$ is even. Thus along the boundary of $D(\nu_q)$, $q+l$ must also be even, and the homomorphism

$$\partial\colon \omega_n^U(\mathrm{Z}_2, \text{All}, \text{Free}) \to \omega_{n-1}^U(Z_2, \text{Free})$$

sends $\bigoplus_{q \text{ odd}} \Omega_{n-q}^U(B_q)$ into $\omega_{n-1}^U(Z_2, \text{Free})^-$ and $\bigoplus_{q \text{ even}} \Omega_{n-q}^U(B_q)$ into $\omega_{n-1}^U(Z_2, \text{Free})^+$.

The diagram

$$\begin{array}{ccc}
\Omega_n^U(Z_2, \text{All}, \text{Free}) & \xrightarrow{\ \partial\ } & \Omega_{n-1}^U(Z_2, \text{Free}) \\
\downarrow & & \cong \downarrow \rho \\
\bigoplus_{q \text{ even}} \Omega_{n-q}^U(BU_{q/2}) & & \omega_{n-1}^U(Z_2, \text{Free})^+ = \Omega_{n-1}^U(RP(\infty)) \\
\downarrow & \nearrow \partial & \\
\bigoplus_{q \text{ even}} \Omega_{n-q}^U(B_q) & &
\end{array}$$

commutes, and $\rho\partial$ is known to map onto $\tilde{\Omega}_{n-1}^U(RP(\infty))$. The summand Ω_{n-1}^U complementary to $\tilde{\Omega}_{n-1}^U(RP(\infty))$ is realized as the manifolds $M \times Z_2$ with M stably almost complex and t interchanging the two copies of M. Applying i and the augmentation $\varepsilon\colon \omega_{n-1}^U(Z_2, \text{All}) \to \Omega_{n-1}^U$ which takes the cobordism class of the underlying manifold, one obtains $2[M]$. Thus i is monic on this summand and the image of ∂ in $\omega^U_{n-1}(Z_2, \text{Free})^+$ is precisely $\tilde{\Omega}^U_{n-1}(RP(\infty))$.

Now considering $\omega^U_*(Z_2, \text{Free})^- \cong \tilde{\Omega}^U_{*+1}(RP(\infty))$, one notes that $\tilde{\Omega}^U_*(RP(\infty))$ is generated as Ω^U_* module by the inclusion maps $RP(2i+1) \to RP(\infty)$ which are obtained by Thomifying the inclusion $RP(2i) \to RP(\infty)$, for which the induced double cover is the antipodal involution on S^{2i}. The complex structure imparted may be considered as that given by considering $S^{2i} \subset C^{i+1}$, where C^{i+1} has the involution given by multiplication by -1, and the complex stucture given by multiplication by $\sqrt{-1}$, imparting the appropriate structure to $\tau(S^{2i}) \oplus 1 \oplus 1$. The same construction gives an involution on $D^{2i+1} \subset C^{i+1}$ with appropriate structure on $\tau(D^{2i+1}) \oplus 0 \oplus 1$. Thus these classes are in the image of ∂, and since ∂ is a Ω^U_* module homomorphism, $\omega^U_*(Z_2, \text{Free})^-$ is contained in the image of ∂.

Thus one has compatible splittings for the sequences to obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \Omega^U_n & \to & \Omega^U_n(Z_2, \text{All}) & \to & \bigoplus_{q \text{ even}} \Omega^U_{n-q}(BU_{q/2}) & \to & \tilde{\Omega}^U_{n-1}(RP(\infty)) & \to 0 \\
 & 1\downarrow & & \rho\downarrow & & \rho'\downarrow & & \rho''\downarrow & \\
0 \to & \Omega^U_n & \to & \omega^U_n(Z_2, \text{All}) & \to & \bigoplus_{q} \Omega^U_{n-q}(B_q) & \to & \tilde{\Omega}^U_{n-1}(RP(\infty)) \oplus \tilde{\Omega}^U_n(RP(\infty)) & \to 0
\end{array}$$

in which both ρ' and ρ'' are monomorphisms onto direct summands, and 1 is the identity.

Rather than belabor the point further, one has:

Proposition 2.4. ρ: $\Omega^U_*(Z_2, All) \to \omega^U_*(Z_2, All)$ *is a monomorphism.*

3. Maps of odd prime period

Now consider the case $G = Z_p$ with p an odd prime, again writing (V, ϕ) as (V, t) where t is a diffeomorphism of period p. Again there are three families: ϕ, Free, and All and one has an exact sequence

$$\begin{array}{ccc}
\omega^U_*(Z_p, \text{Free}) & \xrightarrow{i} & \omega^U_*(Z_p, \text{All}) \\
 & \nwarrow \quad \swarrow & \\
 & \omega^U_*(Z_p, \text{All}, \text{Free}) . &
\end{array}$$

To begin, consider $\omega^U_*(Z_p, \text{Free})$. If (V, t) is a free action of Z_p on an n-manifold with $dt \times s$ acting on $\tau(V) \oplus \pi$, where π is given by the representation (W, θ), then one may form the orbit space V/Z_p which is an n-manifold with pr: $V \to V/Z_p$ the projection. Since $dt \times s$ acts freely on $E(\tau(V) \oplus \pi)$, $E(\tau(V) \oplus \pi)/Z_p \to V/Z_p$ is a vector bundle and complex structures preserved by $dt \times s$ are given by complex structures on the quotient bundle.

Now (W, θ) may be decomposed by means of the irreducible representations into a direct sum of subrepresentations W_0, which is trivial, and W_k for $1 \leq k \leq (p-1)/2$ where W_k is a complex vector space in which s acts as multiplication by $\exp\left(\frac{2\pi ik}{p}\right)$. In particular, $E(\pi)/Z_p \to V/Z_p$ is then the Whitney sum of a trivial bundle ξ_0 with fiber W_0 and the complex vector bundles ξ_k with fiber W_k associated with the p-fold cover $V \to V/Z_p$. Thus $E(\tau(V)\oplus\pi)/Z_p$ is the total space of the bundle $\tau(V/Z_p)\oplus\xi_0\oplus(\oplus\xi_k)$. Since $(\oplus\xi_k)$ has been given a complex structure, the complex structures on $\tau(V)$ preserved under the action are given precisely by stably almost complex structures on V/Z_p. Thus a structure preserving Z_p action is just a principal Z_p bundle over a stably almost complex manifold. Assigning to (V, t) the map $V/Z_p \to BZ_p$ classifying the cover then defines an isomorphism of $\omega_*^U(Z_p, \text{Free})$ with $\Omega_*^U(BZ_p)$. When applied to structure preserving actions of Z_p in the sense of Conner and Floyd, one also obtains an isomorphism and so one obtains:

Proposition 3.1 *The restriction homomorphism* $\rho\colon \Omega_*^U(Z_p, \mathit{Free}) \to \omega_*^U(Z_p, \mathit{Free})$ *is an isomorphism.*

In the commutative diagram

$$\begin{array}{ccc} \Omega_*^U(Z_p, \text{All, Free}) & \xrightarrow{\partial'} & \Omega_*^U(Z_p, \text{Free}) \\ \rho\downarrow & & \cong\downarrow\rho \\ \omega_*^U(Z_p, \text{All, Free}) & \xrightarrow{\partial} & \omega_*^U(Z_p, \text{Free}) \end{array}$$

it is known that the image of ∂' is $\tilde{\Omega}_*^U(BZ_p)$, and the composite

$$\Omega_*^U \to \omega_*^U(Z_p, \text{Free}) \xrightarrow{i} \omega_*^U(Z_p, \text{All}) \xrightarrow{\varepsilon} \Omega_*^U$$

is multiplication by p on the complementary summand, so the image of ∂ is precisely $\tilde{\Omega}_*^U(BZ_p)$.

Thus one has a splitting, giving the diagram

$$\begin{array}{ccccccccc} 0 \to & \Omega_*^U & \to & \Omega_*^U(Z_p, \text{All}) & \to & \Omega_*^U(Z_p, \text{All, Free}) & \to & \tilde{\Omega}_*^U(BZ_p) & \to 0 \\ & \downarrow 1 & & \rho\downarrow & & \rho\downarrow & & \downarrow 1 & \\ 0 \to & \Omega_*^U & \to & \omega_*^U(Z_p, \text{All}) & \to & \omega_*^U(Z_p, \text{All, Free}) & \to & \tilde{\Omega}_*^U(BZ_p) & \to 0 . \end{array}$$

Now consider the group $\omega_*^U(Z_p, \text{All, Free})$. Letting (V, t) be an action which is free on ∂V, the fixed point set of V is a disjoint union of closed submanifolds F^{n-q} with normal bundles ν_q and V may be replaced by the disc bundles of the ν_q. At points of F^{n-q}, the bundle $\tau\oplus\pi$ decomposes into $\tau(F^{n-q})\oplus\xi_0$, where ξ_0 is the trivial bundle of W_0, which is the trivial eigen-bundle, and bundles $(\nu_q)_k\oplus\xi_k$, where ξ_k is the trivial bundle with fiber W_k and $(\nu_q)_k$ is a sub-bundle of $\nu_q | F^{n-q}$, giving the eigen-bundle corresponding to multiplication by $\exp\left(\frac{2\pi ik}{p}\right)$ for $1 \leq k \leq (p-1)/2$. Considered as a complex Z_p bundle, the bundle $\tau \oplus \pi$ decomposes into complex sub-bundles η_0, the trivial eigen-bundle, and η_j, $1 \leq j \leq p-1$ on which $dt \times s$ acts as multpilication by $\exp\left(\frac{2\pi ij}{p}\right)$. Taking the parts of the complex decomposition which give the real decomposition, one has $\eta_0 \cong \tau(F^{n-q}) \oplus \xi_0$. so F^{n+q} is stably almost complex, and $(\nu_q)_k \oplus \xi_k \cong \eta_k \oplus \eta_j$ where $(p-1)/2 \leq j \leq p-1$ and $\exp\left(\frac{2\pi ij}{p}\right)$ is the complex conjugate of $\exp\left(\frac{2\pi ik}{p}\right)$, or $j=p-k$.

After stabilization, the bundles η_k and η_{p-k} are stable complex bundles subject only to the condition that $\eta_k \oplus \eta_{p-k}$ should be stably isomorphic as complex bundle with $(\nu_q)_k$. Thus, the class of (V, t) is completerly determined by the bordism classes $F^{n-q}_{(r)} \to BU_{r_1} \times BU \times \cdots \times BU_{r_{(p-1/2)}} \times BU$ where $r_1 + \cdots + r_{(p-1/2)} = q/2$, where $F^{n-q}_{(r)}$ are the portions of F^{n-q} over which $(\nu_q)_k$ has real dimension $2r_k$, the map into BU_{r_k} classifying $(\nu_q)_k$, and that into the k-th BU factor classifying η_k. Thus, one has

Proposition 3.2 *$\omega^U_n(Z_p, All, Free)$ is isomorphic to*

$$\bigoplus_{(r)} \Omega^U_{n-2r}(BU_{r_1} \times BU \times \cdots \times BUr_{(p-1/2)} \times BU)\,,$$

the sum being over all sequences $(r)=(r_1, \cdots, r_{(p-1/2)})$ of non-negative integers, and with $r=r_1+\cdots+r_{(p-1/2)}$.

In order to analyze $\rho\colon \Omega^U_n(Z_p$, All Free$) \to \omega^U_*(Z_p$, All, Free),one may simply note that analogously $\Omega^U_n(Z_p$, All, Free) is isomorphic to

$$\bigoplus_{(s,t)} \Omega^U_{n-2r}(BU_{s_1} \times BU_{t_1} \times \cdots \times BU_{s(p-1/2)} \times BU_{t(p-1/2)})$$

where $\frac{q}{2} = r = s_1 + \cdots + s_{(p-1/2)} + t_1 + \cdots + t_{(p-1/2)}$ and the map of $F^{n-q}_{(s,t)}$ into BU_{sk} classifies η_k and into BU_{tk} classifies η_{p-k}, with $(\nu_q)_k \cong \eta_k \oplus \eta_{p-k}$ in this case. The map ρ is then induced by the maps $\bigcup_{s_k+t_k=r_k} BU_{sk} \times BU_{tk} \to BU_{r_k} \times BU$ given by the Whitney sum map $BU_{s_k} \times BU_{tk} \to BU_{rk}$ and by $BU_{s_k} \times BU_{tk} \xrightarrow{pr} BU_{s_k} \xrightarrow{\sigma} BU$ where pr is the projection and σ is stabilization.

One may then observe that ρ is anything but monic, for many summands in $\Omega^U_n(Z_p$, All, Free) map to the same summand in $\omega^U_n(Z_p$, All, Free). (One need only look at the terms with $n=2r$ in which many copies of Z map to a single copy of Z). Since, by the commutative diagram, the kernels of the homomorphisms $\rho\colon \Omega^U_n(Z_p$, All, Free$) \to \omega^U_n(Z_p$, All, Free) and $\rho\colon \Omega^U_n(Z_p$, All$) \to \omega^U_n(Z_p$, All) are isomorphic, one sees that $\rho\colon \Omega^U_n(Z_p$, All$) \to \omega^U_n(Z_p$, All) is also not monic.

The homomorphism ρ is also not epic, for the map

$\bigcup_{s_k+t_k=r_k} BU_{s_k}\times BU_{t_k}\to BU_{r_k}\times BU$ factors through $BU_{r_k}\times BU_{r_k}$. One can, of course, compute ρ: $\Omega_n^U(Z_p$, All, Free$)\to\omega_n^U(Z_p$, All, Free) explicitly since the groups and map are completely known, but it hardly seems worthwhile,.

As a final note, one should consider the reason why the Z_2 and Z_p cases, p odd, are so different. Clearly the problem is the dissimilarity between the nature of real representations in the two cases. In studying $\Omega_*^U(G, *, *)$ only the complex representations really play a role, while in $\omega_*^U(G, *, *)$ both types enter.

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References

[1] P.E. Conner and E.E. Floyd: Cobordism theories, (mimeographed), A.M.S. Seattle Conference, 1963.

[2] P.E. Conner and E.E. Floyd: *Periodic maps which preserve a complex structure,* Bull. Amer. Math. Soc. **70** (1964), 574–579.

[3] P.E. Conner and E.E. Floyd: *Maps of odd period,* Ann. of Math. **84** (1966), 132–156.

[4] R.E. Stong: *On the cobordism of pairs*, Pacific J. Math. **38** (1971), 803–816.