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## HAUSDORFF MEASURE OF THE SAMPLE PATHS OF GAUSSIAN RANDOM FIELDS

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### 1. Introduction

Let  $Y(t)$  ( $t \in \mathbf{R}^N$ ) be a real-valued, centered Gaussian random field with  $Y(0)=0$ . We assume that  $Y(t)$  ( $t \in \mathbf{R}^N$ ) has stationary increments and continuous covariance function  $R(t,s)=EY(t)Y(s)$  given by

$$(1.1) \quad R(t,s) = \int_{\mathbf{R}^N} (e^{i\langle t,\lambda \rangle} - 1)(e^{-i\langle s,\lambda \rangle} - 1)\Delta(d\lambda),$$

where  $\langle x,y \rangle$  is the ordinary scalar product in  $\mathbf{R}^N$  and  $\Delta(d\lambda)$  is a nonnegative symmetric measure on  $\mathbf{R}^N \setminus \{0\}$  satisfying

$$(1.2) \quad \int_{\mathbf{R}^N} \frac{|\lambda|^2}{1+|\lambda|^2} \Delta(d\lambda) < \infty.$$

Then there exists a centered complex-valued Gaussian random measure  $W(d\lambda)$  such that

$$(1.3) \quad Y(t) = \int_{\mathbf{R}^N} (e^{i\langle t,\lambda \rangle} - 1)W(d\lambda)$$

and for any Borel sets  $A, B \subseteq \mathbf{R}^N$

$$E(W(A)\overline{W(B)}) = \Delta(A \cap B) \quad \text{and} \quad W(-A) = \overline{W(A)}.$$

It follows from (1.3) that

$$(1.4) \quad E[(Y(t+h) - Y(t))^2] = 2 \int_{\mathbf{R}^N} (1 - \cos\langle h,\lambda \rangle)\Delta(d\lambda).$$

We assume that there exist constants  $\delta_0 > 0$ ,  $0 < c_1 \leq c_2 < \infty$  and a non-decreasing, continuous function  $\sigma: [0, \delta_0) \rightarrow [0, \infty)$  which is regularly varying at the origin with index  $\alpha$  ( $0 < \alpha < 1$ ) such that for any  $t \in \mathbf{R}^N$  and  $h \in \mathbf{R}^N$  with  $|h| \leq \delta_0$

$$(1.5) \quad E[(Y(t+h) - Y(t))^2] \leq c_1 \sigma^2(|h|).$$

and for all  $t \in \mathbf{R}^N$  and any  $0 < r \leq \min\{|t|, \delta_0\}$

$$(1.6) \quad \text{Var}(Y(t) | Y(s): r \leq |s-t| \leq \delta_0) \geq c_2 \sigma^2(r).$$

If (1.5) and (1.6) hold, we shall say that  $Y(t)$  ( $t \in \mathbf{R}^N$ ) is strongly locally  $\sigma$ -nondeterministic. We refer to Monrad and Pitt [14], Berman [4] [5] and Cuzick and Du Pez [6] for more information on (strongly) locally nondeterminism.

We associate with  $Y(t)$  ( $t \in \mathbf{R}^N$ ) a Gaussian random field  $X(t)$  ( $t \in \mathbf{R}^N$ ) in  $\mathbf{R}^d$  by

$$(1.7) \quad X(t) = (X_1(t), \dots, X_d(t)),$$

where  $X_1, \dots, X_d$  are independent copies of  $Y$ . The most important example of such Gaussian random fields is the fractional Brownian motion of index  $\alpha$  (see Example 4.1 below).

It is well known (see [1], Chapter 8) that with probability 1

$$\dim X([0, 1]^N) = \min\left(d, \frac{N}{\alpha}\right).$$

The objective of this paper is to consider the exact Hausdorff measure of the image set  $X([0, 1]^N)$ . The main result is the following theorem, which generalizes a theorem of Talagrand [22].

**Theorem 1.1.** *If  $N < \alpha d$ , then with probability 1*

$$(1.8) \quad 0 < \phi\text{-}m(X([0, 1]^N)) < \infty,$$

where  $\phi(s) = \psi(s)^N \log \log \frac{1}{s}$ ,  $\psi$  is the inverse function of  $\sigma$  and  $\phi\text{-}m(X([0, 1]^N))$  is the  $\phi$ -Hausdorff measure of  $X([0, 1]^N)$ .

If  $N > \alpha d$ , then by a result of Pitt [17],  $X([0, 1]^N)$  a.s. has interior points and hence has positive  $d$ -dimensional Lebesgue measure. In the case of  $N = \alpha d$ , the problem of finding  $\phi\text{-}m(X([0, 1]^N))$  is still open even in the fractional Brownian motion case.

The paper is organized as follows. In Section 2 we recall the definition and some basic facts of Hausdorff measure, Gaussian processes and regularly varying functions. In Section 3 we prove the upper bound and in Section 4, we prove the lower bound for  $\phi\text{-}m(X([0, 1]^N))$ . We also give some examples showing that the hypotheses in Theorem 1.1 are satisfied by a large class of Gaussian random fields including fractional Brownian motion.

Another important example of Gaussian random fields is the Brownian sheet or  $N$ -parameter Wiener process  $W(t)$  ( $t \in \mathbf{R}_+^N$ ), see Orey and Pruitt [16]. Since  $W(t)$  ( $t \in \mathbf{R}_+^N$ ) is not locally nondeterministic, Theorem 1.1 does not apply. The problem of finding exact Hausdorff measure of  $W([0, 1]^N)$  was solved by Ehm [7].

We will use  $K$  to denote an unspecified positive constant which may be different in each appearance.

**2. Preliminaries**

Let  $\Phi$  be the class of functions  $\phi:(0,\delta)\rightarrow(0,1)$  which are right continuous, monotone increasing with  $\phi(0+)=0$  and such that there exists a finite constant  $K>0$  for which

$$\frac{\phi(2s)}{\phi(s)} \leq K, \text{ for } 0 < s < \frac{1}{2}\delta.$$

For  $\phi \in \Phi$ , the  $\phi$ -Hausdorff measure of  $E \subseteq \mathbf{R}^N$  is defined by

$$\phi\text{-}m(E) = \liminf_{\varepsilon \rightarrow 0} \left\{ \sum_i \phi(2r_i) : E \subseteq \cup_{i=1}^{\infty} B(x_i, r_i), r_i < \varepsilon \right\},$$

where  $B(x,r)$  denotes the open ball of radius  $r$  centered at  $x$ . It is known that  $\phi\text{-}m$  is a metric outer measure and every Borel set in  $\mathbf{R}^N$  is  $\phi\text{-}m$  measurable. The Hausdorff dimension of  $E$  is defined by

$$\begin{aligned} \dim E &= \inf \{ \alpha > 0 : s^\alpha\text{-}m(E) = 0 \} \\ &= \sup \{ \alpha > 0 : s^\alpha\text{-}m(E) = \infty \}. \end{aligned}$$

We refer to [F] for more properties of Hausdorff measure and Hausdorff dimension.

The following lemma can be easily derived from the results in [18] (see [23]), which gives a way to get a lower bound for  $\phi\text{-}m(E)$ . For any Borel measure  $\mu$  on  $\mathbf{R}^N$  and  $\phi \in \Phi$ , the upper  $\phi$ -density of  $\mu$  at  $x \in \mathbf{R}^N$  is defined by

$$\bar{D}_\mu^\phi(x) = \limsup_{r \rightarrow 0} \frac{\mu(B(x,r))}{\phi(2r)}.$$

**Lemma 2.1.** *For a given  $\phi \in \Phi$  there exists a positive constant  $K$  such that for any Borel measure  $\mu$  on  $\mathbf{R}^N$  and every Borel set  $E \subseteq \mathbf{R}^N$ , we have*

$$\phi\text{-}m(E) \geq K \mu(E) \inf_{x \in E} \{ \bar{D}_\mu^\phi(x) \}^{-1}.$$

Now we summarize some basic facts about Gaussian processes. Let  $Z(t)$  ( $t \in S$ ) be a Gaussian process. We provide  $S$  with the following metric

$$d(s,t) = \|Z(s) - Z(t)\|_2,$$

where  $\|Z\|_2 = (E(Z^2))^{\frac{1}{2}}$ . We denote by  $N_d(S,\varepsilon)$  the smallest number of open  $d$ -balls

of radius  $\varepsilon$  needed to cover  $S$  and write  $D = \sup\{d(s,t) : s, t \in S\}$ .

The following lemma is well known. It is a consequence of the Gaussian isopermetric inequality and Dudley's entropy bound ([11], see also [22]).

**Lemma 2.2.** *There exists an absolute constant  $K > 0$  such that for any  $u > 0$ , we have*

$$P\left\{\sup_{s, t \in S} |Z(s) - Z(t)| \geq K(u + \int_0^D \sqrt{\log N_d(S, \varepsilon) d\varepsilon}\right\} \leq \exp\left(-\frac{u^2}{D^2}\right).$$

**Lemma 2.3.** *Consider a function  $\Psi$  such that  $N_d(S, \varepsilon) \leq \Psi(\varepsilon)$  for all  $\varepsilon > 0$ . Assume that for some constant  $C > 0$  and all  $\varepsilon > 0$  we have*

$$\Psi(\varepsilon) / C \leq \Psi\left(\frac{\varepsilon}{2}\right) \leq C\Psi(\varepsilon).$$

Then

$$P\left\{\sup_{s, t \in S} |Z(s) - Z(t)| \leq u\right\} \geq \exp(-K\Psi(u)),$$

where  $K > 0$  is a constant depending only on  $C$ .

This is proved in [21]. It gives an estimate for the lower bound of the small ball probability of Gaussian processes. Similar problems have also been considered by Monrad and Rootzén [15] and by Shao [20].

We end this section with some lemmas about regularly varying functions. Let  $\sigma(s)$  be a regularly varying function with index  $\alpha$  ( $0 < \alpha < 1$ ). Then  $\sigma$  can be written as

$$\sigma(s) = s^\alpha L(s),$$

where  $L(s) : [0, \delta_0) \rightarrow [0, \infty)$  is slowly varying at the origin in the sense of Karamata and hence can be represented by

$$(2.1) \quad L(s) = \exp\left(\eta(s) + \int_s^A \frac{\varepsilon(t)}{t} dt\right),$$

where  $\eta(s) : [0, \delta_0) \rightarrow \mathbf{R}$ ,  $\varepsilon(s) : (0, A] \rightarrow \mathbf{R}$  are bounded measurable functions and

$$\lim_{s \rightarrow 0} \eta(s) = c, \quad |c| < \infty; \quad \lim_{s \rightarrow 0} \varepsilon(s) = 0.$$

In the following, Lemma 2.4 is an easy consequence of (2.1) and Lemma 2.5 can be deduced from Theorem 2.6 and 2.7 in Seneta [19] directly.

**Lemma 2.4.** *Let  $L(s)$  be a slowly varying function at the origin and let  $U = U(s) : [0, \infty) \rightarrow [0, \infty)$  satisfying*

$$\lim_{s \rightarrow 0} U(s) = \infty \quad \text{and} \quad \lim_{s \rightarrow 0} sU(s) = 0.$$

*Then for any  $\varepsilon > 0$ , as  $s$  small enough we have*

$$U(s)^{-\varepsilon} L(s) \leq L(sU(s)) \leq U(s)^\varepsilon L(s)$$

*and*

$$U(s)^{-\varepsilon} L(s) \leq L(sU(s)^{-1}) \leq U(s)^\varepsilon L(s).$$

**Lemma 2.5.** *Let  $\sigma$  be a regularly varying function at the origin with index  $\alpha > 0$ . Then there is a constant  $K > 0$  such that for  $r > 0$  small enough, we have*

$$(2.2) \quad \int_1^\infty \sigma(re^{-u^2}) du \leq K\sigma(r),$$

$$(2.3) \quad \int_0^1 \sigma(rs) ds \leq K\sigma(r),$$

$$(2.4) \quad \int_0^1 \sigma(rs)s^{N-1} ds \leq K\sigma(r).$$

Let  $\sigma : [0, \delta_0) \rightarrow [0, \infty)$  be non-decreasing and let  $\psi$  be the inverse function of  $\sigma$ , that is

$$\psi(s) = \inf\{t \geq 0 : \sigma(t) \geq s\}.$$

then  $\psi(s) = s^{1/\alpha} f(s)$ , where  $f(s)$  is also a slowly varying function and

$$(2.5) \quad \sigma(\psi(s)) \sim s \quad \text{and} \quad \psi(\sigma(s)) \sim s \quad \text{as} \quad s \rightarrow 0.$$

### 3. Upper bound for $\phi$ - $m(X([0, 1]^N))$

Let  $Y(t) (t \in \mathbf{R}^N)$  be a real-valued, centered Gaussian random field with stationary increments and a continuous covariance function  $R(t, s)$  given by (1.1). We assume that  $Y(0) = 0$  and (1.5) holds. Let  $X(t) (t \in \mathbf{R}^N)$  be the  $(N, d)$  Gaussian random field defined by (1.7).

We start with the following lemma.

**Lemma 3.1.** *Let  $Y(t) (t \in \mathbf{R}^N)$  be a Gaussian process with  $Y(0) = 0$  satisfying (1.5). Then*

(i) For any  $r > 0$  small enough and  $u \geq K\sigma(r)$ , we have

$$(3.1) \quad P \left\{ \sup_{|t| \leq r} |Y(t)| \geq u \right\} \leq \exp \left( - \frac{u^2}{K\sigma^2(r)} \right).$$

(ii) Let

$$\omega_Y(h) = \sup_{t, t+s \in [0,1]^N, |s| \leq h} |Y(t+s) - Y(t)|$$

be the uniform modulus of continuity of  $Y(t)$  on  $[0,1]^N$ . Then

$$(3.2) \quad \limsup_{h \rightarrow 0} \frac{\omega_Y(h)}{\sigma(h)\sqrt{2c_1 \log \frac{1}{h}}} \leq 1, \quad a.s.$$

Proof. Let  $r < \delta_0$  and  $S = \{t : |t| \leq r\}$ . Since  $d(s,t) \leq c_1\sigma(|t-s|)$ , we have

$$N_d(S, \varepsilon) \leq K \left( \frac{r}{\psi(\varepsilon)} \right)^N$$

and

$$D = \sup \{d(s,t); s, t \in S\} \leq K\sigma(r).$$

By simple calculations

$$\begin{aligned} \int_0^D \sqrt{\log N_d(S, \varepsilon)} d\varepsilon &\leq K \int_0^{K\sigma(r)} \sqrt{\log(Kr) / \psi(\varepsilon)} d\varepsilon \\ &\leq K \int_0^{Kr} \sqrt{\log(Kr) / t} d\sigma(t) \\ &\leq K \left( \sigma(r) + \int_0^K \frac{1}{u\sqrt{\log K/u}} \sigma(ur) du \right) \\ &\leq K \left( \sigma(r) + \int_K^\infty \sigma(re^{-u^2}) du \right) \\ &\leq K\sigma(r), \end{aligned}$$

where the last inequality follows from (2.2). If  $u \geq K\sigma(r)$ , then by Lemma 2.2 we have

$$\begin{aligned} &P \left\{ \sup_{|t| \leq r} |Y(t)| \geq 2Ku \right\} \\ &\leq P \left\{ \sup_{|t| \leq r} |Y(t)| \geq K(u + \int_0^D \sqrt{\log N_d(S, \varepsilon)} d\varepsilon) \right\} \end{aligned}$$

$$\leq \exp\left(-\frac{u^2}{K\sigma^2(r)}\right).$$

This proves (3.1). The inequality (3.2) can be derived from Lemma 2.2 directly in a standard way (see also [13]).

In order to get the necessary independence, we will make use of the spectral representation (1.3). Given  $0 < a < b < \infty$ , we consider the process

$$Y(a, b, t) = \int_{a \leq |t| \leq b} (e^{i\langle t, \lambda \rangle} - 1) W(d\lambda).$$

Then for any  $0 < a < b < a' < b' < \infty$ , the processes  $Y(a, b, t)$  and  $Y(a', b', t)$  are independent. The next lemma expresses how well  $Y(a, b, t)$  approximates  $Y(t)$ .

**Lemma 3.2.** *Let  $Y(t)$  ( $t \in \mathbf{R}^N$ ) be defined by (1.3). If (1.5) holds, then there exists a constant  $B > 0$  such that for any  $B < a < b$  we have*

$$(3.3) \quad \|Y(a, b, t) - Y(t)\|_2 \leq K[|t|^2 a^2 \sigma^2(a^{-1}) + \sigma^2(b^{-1})]^{\frac{1}{2}}.$$

*Proof.* First we claim that for any  $u > 0$  and any  $h \in \mathbf{R}^N$  with  $|h| = 1/u$  we have

$$(3.4) \quad \int_{|\lambda| < u} \langle h, \lambda \rangle^2 \Delta(d\lambda) \leq K \int_{\mathbf{R}^N} (1 - \cos \langle h, \lambda \rangle) \Delta(d\lambda)$$

$$(3.5) \quad \int_{|\lambda| \geq u} \Delta(d\lambda) \leq K \left(\frac{u}{2}\right)^N \int_{[-1/u, 1/u]^N} dv \int_{\mathbf{R}^N} (1 - \cos \langle v, \lambda \rangle) \Delta(d\lambda).$$

For  $N = 1$ , (3.4) and (3.5) are the truncation inequalities in [12] p209. For  $N > 1$  a similar proof yields (3.4) and (3.5).

Now for any  $a > \delta_0^{-1}$  and any  $t \in \mathbf{R}^N \setminus \{0\}$ , by (1.4), (1.5) and (3.4) we have

$$(3.6) \quad \begin{aligned} \int_{|\lambda| < a} (1 - \cos \langle t, \lambda \rangle) \Delta(d\lambda) &\leq \int_{|\lambda| < a} \langle t, \lambda \rangle^2 \Delta(d\lambda) \\ &= |t|^2 a^2 \int_{|\lambda| < a} \langle t/(a|t), \lambda \rangle^2 \Delta(d\lambda) \leq K|t|^2 a^2 \sigma^2(a^{-1}). \end{aligned}$$

For  $b > 0$  large enough, by (3.5), (1.4), (1.5) and (2.4) we have

$$(3.7) \quad \begin{aligned} \int_{|\lambda| \geq b} \Delta(d\lambda) &\leq K \left(\frac{b}{2}\right)^N \int_{[-1/b, 1/b]^N} \sigma^2(|v|) dv \\ &\leq K b^N \int_0^{\sqrt{Nb}^{-1}} \sigma^2(\rho) \rho^{N-1} d\rho \leq K \sigma^2(b^{-1}). \end{aligned}$$

Combining (3.6) and (3.7), we see that there exists a constant  $B > 0$  such that  $B < a < b$  implies

$$\begin{aligned} E[(Y(a,b,t) - Y(t))^2] &= 2 \int_{\{|\lambda| < a\} \cup \{|\lambda| > b\}} (1 - \cos\langle t, \lambda \rangle) \Delta(d\lambda) \\ &\leq 2 \int_{|\lambda| < a} (1 - \cos\langle t, \lambda \rangle) \Delta(d\lambda) + 2 \int_{|\lambda| > b} \Delta(d\lambda) \\ &\leq K[|t|^2 a^2 \sigma^2(a^{-1}) + \sigma^2(b^{-1})]. \end{aligned}$$

This proves (3.3).

**Lemma 3.3.** *There exists a constant  $B > 0$  such that for any  $B < a < b$  and  $0 < r < B^{-1}$  the following holds: let  $A = r^2 a^2 \sigma^2(a^{-1}) + \sigma^2(b^{-1})$  such that  $\psi(\sqrt{A}) \leq \frac{1}{2}r$ , then for any*

$$u \geq K \left( A \log \frac{Kr}{\psi(\sqrt{A})} \right)^{\frac{1}{2}}$$

we have

$$(3.8) \quad P \left\{ \sup_{|t| \leq r} |Y(t) - Y(a,b,t)| \geq u \right\} \leq \exp \left( -\frac{u^2}{KA} \right).$$

*Proof.* Let  $S = \{t : |t| \leq r\}$  and  $Z(t) = Y(t) - Y(a,b,t)$ . Then

$$d(s,t) = \|Z(t) - Z(s)\|_2 \leq c_1 \sigma(|t - s|).$$

Hence

$$N_d(S, \varepsilon) \leq K \left( \frac{r}{\psi(\varepsilon)} \right)^N.$$

By Lemma 3.2 we have  $D \leq K\sqrt{A}$ . As in the proof of Lemma 3.1,

$$\begin{aligned} \int_0^D \sqrt{\log N_d(S, \varepsilon)} d\varepsilon &\leq K \int_0^{K\sqrt{A}} \sqrt{\log(Kr) / \psi(\varepsilon)} d\varepsilon \\ &\leq K \int_0^{K\psi(\sqrt{A})/r} \sqrt{\log K / t} d\sigma(rt) \\ &\leq K \left[ \sqrt{\log K / t} \sigma(rt) \Big|_0^{K\psi(\sqrt{A})/r} + \int_0^{K\psi(\sqrt{A})/r} \frac{1}{t\sqrt{\log K / t}} \sigma(rt) dt \right] \end{aligned}$$

$$\begin{aligned} &\leq K\sqrt{A \log Kr / \psi(\sqrt{A})} + K \int_{\sqrt{\log Kr / \psi(\sqrt{A})}}^{\infty} \sigma(Kre^{-u^2}) du \\ &\leq K\sqrt{A \log Kr / \psi(\sqrt{A})}, \end{aligned}$$

at least for  $r > 0$  small enough, where the last step follows from (2.2). Hence (3.8) follows immediately from Lemma 2.2.

Let  $X_1(a, b, t), \dots, X_d(a, b, t)$  be independent copies of  $Y(a, b, t)$  and let

$$X(a, b, t) = (X_1(a, b, t), \dots, X_d(a, b, t)) \quad (t \in \mathbb{R}^N).$$

Then we have the following corollary of Lemma 3.3.

**Corollary 3.1.** *Consider  $B < a < b$  and  $0 < r < B^{-1}$ . Let  $A = r^2 a^2 \sigma^2(a^{-1}) + \sigma^2(b^{-1})$  with  $\psi(\sqrt{A}) \leq \frac{1}{2}r$ . Then for any*

$$u \geq K \left( A \log \frac{Kr}{\psi(\sqrt{A})} \right)^{\frac{1}{2}}$$

we have

$$(3.9) \quad P \left\{ \sup_{|t| \leq r} |X(t) - X(a, b, t)| \geq u \right\} \leq \exp \left( -\frac{u^2}{KA} \right).$$

**Lemma 3.4.** *Given  $0 < r < \delta_0$  and  $\varepsilon < \sigma(r)$ . Then for any  $0 < a < b$  we have*

$$(3.10) \quad P \left\{ \sup_{|t| \leq r} |X(a, b, t)| \leq \varepsilon \right\} \geq \exp \left( -\frac{r^N}{K\psi(\varepsilon)^N} \right).$$

*Proof.* It is sufficient to prove (3.10) for  $Y(a, b, t)$ . Let  $S = \{t : |t| \leq r\}$  and define a distance  $d$  on  $S$  by

$$d(s, t) = \|Y(a, b, t) - Y(a, b, s)\|_2.$$

Then  $d(s, t) \leq c_1 \sigma(|t - s|)$  and

$$N_d(S, \varepsilon) \leq K \left( \frac{r}{\psi(\varepsilon)} \right)^N.$$

By Lemma 2.3 we have

$$P \left\{ \sup_{|t| \leq r} |Y(a, b, t)| \leq \varepsilon \right\} \geq \exp \left( -\frac{r^N}{K\psi(\varepsilon)^N} \right).$$

This proves lemma 3.4.

**Proposition 3.1.** *There exists a constant  $\delta_1 > 0$  such that for any  $0 < r_0 \leq \delta_1$ , we have*

$$(3.11) \quad P \left\{ \exists r \in [r_0^2, r_0] \text{ such that } \sup_{|t| \leq r} |X(t)| \leq K\sigma(r(\log \log \frac{1}{r})^{-\frac{1}{N}}) \right\} \geq 1 - \exp\left(-(\log \frac{1}{r_0})^{\frac{1}{2}}\right).$$

*Proof.* We follow the line of Talagrand [22]. Let  $U = U(r_0) \geq 1$ , where  $U(r)$  satisfying

$$(3.12) \quad U(r) \rightarrow \infty \text{ as } r \rightarrow 0$$

and for any  $\varepsilon > 0$

$$(3.13) \quad r^\varepsilon U(r) \rightarrow 0 \text{ as } r \rightarrow 0,$$

will be chosen later. For  $k \geq 0$ , let  $r_k = r_0 U^{-2k}$ . Let  $k_0$  be the largest integer such that

$$k_0 \leq \frac{\log \frac{1}{r_0}}{2 \log U},$$

then for any  $0 \leq k \leq k_0$  we have  $r_0^2 \leq r_k \leq r_0$ . In order to prove (3.11), it suffices to show that

$$(3.14) \quad P \left\{ \exists k \leq k_0 \text{ such that } \sup_{|t| \leq r_k} |X(t)| \leq K\sigma(r_k(\log \log \frac{1}{r_k})^{-\frac{1}{N}}) \right\} \geq 1 - \exp\left(-(\log \frac{1}{r_0})^{\frac{1}{2}}\right).$$

Let  $a_k = r_0^{-1} U^{2k-1}$  and we define for  $k=0, 1, \dots$

$$X_k(t) = X(a_k, a_{k+1}, t),$$

then  $X_0, X_1, \dots$  are independent. By Lemma 3.4 we can take a constant  $K_1$  such that for  $r_0 > 0$  small enough

$$(3.15) \quad P \left\{ \sup_{|t| \leq r_k} |X_k(t)| \leq K_1 \sigma(r_k(\log \log \frac{1}{r_k})^{-\frac{1}{N}}) \right\} \geq \exp\left(-\frac{1}{4} \log \log \frac{1}{r_k}\right)$$

$$= \frac{1}{(\log \frac{1}{r_k})^{\frac{1}{2}}}$$

Thus, by independence we have

$$(3.16) \quad P \left\{ \exists k \leq k_0, \sup_{|t| \leq r_k} |X_k(t)| \leq K_1 \sigma(r_k) (\log \log \frac{1}{r_k})^{-1/N} \right\} \\ \geq 1 - \left( 1 - \frac{1}{(2 \log 1/r_0)^{1/4}} \right)^{k_0} \\ \geq 1 - \exp \left( - \frac{k_0}{(2 \log 1/r_0)^{1/4}} \right).$$

Let

$$A_k = r_k^2 a_k^2 \sigma^2(a_k^{-1}) + \sigma^2(a_{k+1}^{-1}) \\ = U^{-2+2\alpha} r_k^{2\alpha} L^2(r_k U) + U^{-2\alpha} r_k^{2\alpha} L^2(r_k / U).$$

Let  $\beta = 2 \min\{1 - \alpha, \alpha\}$  and fix an  $\varepsilon < \frac{1}{2}\beta$ . Then by Lemma 2.4, we see that as  $r_0$  small enough

$$U^{-\beta - \varepsilon} \sigma^2(r_k) \leq A_k \leq U^{-\beta + \varepsilon} \sigma^2(r_k).$$

Notice that  $r_0$  for small enough we have

$$\psi(\sqrt{A_k}) \geq \psi(U^{-(\beta + \varepsilon)/2} \sigma(r_k)) \\ = (U^{-\beta/2} \sigma(r_k))^{1/\alpha} f(U^{-\beta/2} \sigma(r_k)) \\ = U^{-\beta/(2\alpha)} r_k L(r_k)^{1/\alpha} f(U^{-\beta/2} \sigma(r_k)) \\ \geq K U^{-(\beta + \varepsilon)/2/(2\alpha)} r_k,$$

the last inequality follows from (2.5). It follows from Corollary 3.1 that for

$$u \geq K \sigma(r_k) U^{-\frac{\beta - \varepsilon}{2}} (\log U)^{1/2},$$

we have

$$(3.17) \quad P \left\{ \sup_{|t| \leq r_k} |X(t) - X_k(t)| \geq u \right\} \leq \exp \left( - \frac{u^2 U^{\beta - \varepsilon}}{K \sigma^2(r_k)} \right).$$

Hence, if we take

$$U = (\log 1/r_0)^{\frac{1}{\beta - \varepsilon}},$$

then as  $r_0$  small enough

$$\sigma(r_k)U^{-\beta-\frac{\varepsilon}{2}}(\log U)^{1/2} \leq \sigma(r_k(\log \log \frac{1}{r_0})^{-\frac{1}{N}}).$$

Hence by taking

$$u = \frac{K_1}{2} \sigma(r_k(\log \log \frac{1}{r_0})^{-\frac{1}{N}})$$

in (3.17), we obtain

$$(3.18) \quad P \left\{ \sup_{|t| \leq r_k} |X(t) - X_k(t)| \geq \frac{K_1}{2} \sigma(r_k(\log \log \frac{1}{r_0})^{-\frac{1}{N}}) \right\} \leq \exp \left( -\frac{u^2 U^{\beta-\varepsilon}}{K\sigma^2(r_k)} \right).$$

Combining (3.16) and (3.18) we have

$$(3.19) \quad P \left\{ \exists k \leq k_0 \text{ such that } \sup_{|t| \leq r_k} |X(t)| \leq 2K_1 \sigma(r_k(\log \log \frac{1}{r_0})^{-1/N}) \right\} \\ \geq 1 - \exp \left( -\frac{k_0}{2(\log 1/r_0)^{1/4}} \right) - k_0 \exp \left( -\frac{U^{\beta-\varepsilon}}{K(\log \log 1/r_0)^{(2\alpha)/N + \varepsilon}} \right).$$

We recall that

$$\frac{\log \frac{1}{r_0}}{4 \log U} \leq k_0 \leq \log \frac{1}{r_0}.$$

and hence for  $r_0$  small enough, (3.11) follows from (3.19).

Now we are in a position to prove the upper bound for  $\phi$ - $m(X([0,1]^N))$ .

**Theorem 3.1.** *Let  $\phi(s) = \psi(s)^N \log \log \frac{1}{s}$ . Then with probability 1*

$$\phi\text{-}m(X([0,1]^N)) < \infty.$$

*Proof.* For  $k \geq 1$ , consider the set

$$R_k = \left\{ t \in [0,1]^N : \exists r \in [2^{-2k}, 2^{-k}] \text{ such that } \sup_{|s-t| \leq r} |X(s) - X(t)| \leq K\sigma(r(\log \log \frac{1}{r})^{-1/N}) \right\}.$$

By Proposition 3.1 we have

$$P\{t \in R_k\} \geq 1 - \exp(-\sqrt{k/2}).$$

Denote the Lebesgue measure in  $\mathbb{R}^N$  by  $L_N$ . It follows from Fubini's theorem that  $P(\Omega_0) = 1$ , where

$$\Omega_0 = \{\omega : L_N(R_k) \geq 1 - \exp(-\sqrt{k/4}) \text{ infinitely often}\}.$$

On the other hand, by Lemma 3.1 ii), there exists an event  $\Omega_1$  such that  $P(\Omega_1) = 1$  and for all  $\omega \in \Omega_1$ , there exists  $n_1 = n_1(\omega)$  large enough such that for all  $n \geq n_1$  and any dyadic cube  $C$  of order  $n$  in  $\mathbb{R}^N$ , we have

$$(3.20) \quad \sup_{s,t \in C} |X(t) - X(s)| \leq K\sigma(2^{-n})\sqrt{n}.$$

Now fix an  $\omega \in \Omega_0 \cap \Omega_1$ , we show that  $\phi\text{-}m(X([0, 1]^N)) < \infty$ . Consider  $k \geq 1$  such that

$$L_N(R_k) \geq 1 - \exp(-\sqrt{k/4}).$$

For any  $x \in R_k$  we can find  $n$  with  $k \leq n \leq 2k + k_0$  (where  $k_0$  depends on  $N$  only) such that

$$(3.21) \quad \sup_{s,t \in C_n(x)} |X(t) - X(s)| \leq K\sigma(2^{-n}(\log \log 2^n)^{-1/N}),$$

where  $C_n(x)$  is the unique dyadic cube of order  $n$  containing  $x$ . Thus we have

$$R_k \subseteq V = \cup_{n=k}^{2k+k_0} V_n$$

and each  $V_n$  is a union of dyadic cubes  $C_n$  of order  $n$  for which (3.21) holds. Clearly  $X(C_n)$  can be covered by a ball of radius

$$\rho_n = K\sigma(2^{-n}(\log \log 2^n)^{-1/N}).$$

Since  $\phi(2\rho_n) \leq K2^{-nN} = KL_N(C_n)$ , we have

$$(3.22) \quad \sum_n \sum_{C \in V_n} \phi(2\rho_n) \leq \sum_n \sum_{C \in V_n} KL_N(C_n) = KL_N(V) < \infty.$$

On the other hand,  $[0, 1]^N \setminus V$  is contained in a union of dyadic cubes of order  $q = 2k + k_0$ , none of which meets  $R_k$ . There can be at most

$$2^{Nq}L_N([0, 1]^N \setminus V) \leq K2^{Nq}\exp(-\sqrt{k/4})$$

of such cubes. For each of these cubes,  $X(C)$  is contained in a ball of radius  $\rho = K\sigma(2^{-q})\sqrt{q}$ . Thus for any  $\varepsilon > 0$

$$(3.23) \quad \sum \phi(2\rho) \leq K2^{Nq}\exp(-\sqrt{k/4})2^{-Nq}q^{N/(2\alpha) + \varepsilon} \leq 1$$

for  $k$  large enough. Since  $k$  can be arbitrarily large, Theorem 3.1 follows from (3.22) and (3.23).

**4. Lower bound for  $\phi$ - $m(X([0, 1]^N))$**

Let  $Y(t) (t \in \mathbf{R}^N)$  be a real-valued, centered Gaussian random field with stationary increments and a continuous covariance function  $R(t,s)$  given by (1.1). We assume that  $Y(0)=0$  and (1.6) holds. Let  $X(t) (t \in \mathbf{R}^N)$  be the  $(N,d)$  Gaussian random field defined by (1.7). In this section, we prove that if  $N < \alpha d$ , then

$$\phi\text{-}m(X([0, 1]^N)) > 0 \quad \text{a.s.}$$

For simplicity we assume  $\delta_0=1$  and let  $I=[0, 1]^N \cap B(0, 1)$  (otherwise we consider a smaller cube). For any  $0 < r < 1$  and  $y \in \mathbf{R}^d$ . Let

$$T_y(r) = \int_I 1_{B(y,r)}(X(t)) dt$$

be the sojourn time of  $X(t) (t \in I)$  in the open ball  $B(y,r)$ . If  $y=0$ , we write  $T(r)$  for  $T_0(r)$ .

**Proposition 4.1.** *There exist  $\delta_2 > 0$  and  $b > 0$  such that for any  $0 < r < \delta_2$*

$$(4.1) \quad E(\exp(b\psi(r)^{-N}T(r))) \leq K < \infty.$$

*Proof.* We first prove that there exists a constant  $0 < K < \infty$  such that for any  $n \geq 1$

$$(4.2) \quad E(T(r))^n \leq K^n n! \psi(r)^{Nn}.$$

For  $n=1$ , by (2.4) and (2.5) we have

$$(4.3) \quad \begin{aligned} ET(r) &= \int_I P\{X(t) \in B(0,r)\} dt \\ &\leq \int_I \min\{1, K(\frac{r}{\sigma(|t|)})^d\} dt \\ &\leq K \int_0^1 \min\{1, \frac{Kr^d}{\sigma(\rho)^d}\} \rho^{N-1} d\rho \\ &\leq K \int_0^{K\psi(r)} \rho^{N-1} d\rho + K \int_{K\psi(r)}^1 \frac{r^d \rho^{N-1}}{\sigma(\rho)^d} d\rho \\ &\leq K\psi(r)^N + Kr^d \psi(r)^{N-\alpha d} \int_1^\infty \frac{1}{t^{1+\alpha d-N} L(\psi(r)t)^d} dt \end{aligned}$$

$$\begin{aligned} &\leq K\psi(r)^N + Kr^d\psi(r)^{N-ad} / L(\psi(r))^d \\ &\leq K\psi(r)^N. \end{aligned}$$

For  $n \geq 2$

$$(4.4) \quad E(T(r)^n) = \int_{I^n} P\{|X(t_1)| < r, \dots, |X(t_n)| < r\} dt_1 \cdots dt_n.$$

Consider  $t_1, \dots, t_n \in I$  satisfying

$$t_j \neq 0 \text{ for } j=1, \dots, n, \quad t_j \neq t_k \text{ for } j \neq k.$$

Let  $\eta = \min\{|t_n|, |t_n - t_i|, i=1, \dots, n-1\}$ . Then by (1.6) we have

$$(4.5) \quad \text{Var}(X(t_n)|X(t_1), \dots, X(t_{n-1})) \geq c_2\sigma^2(\eta).$$

Since conditional distributions in Gaussian processes are still Gaussian, it follows from (4.5) that

$$(4.6) \quad \begin{aligned} &P\{|X(t_n)| < r | X(t_1) = x_1, \dots, X(t_{n-1}) = x_{n-1}\} \\ &\leq K \int_{|u| < r} \frac{1}{\sigma(\eta)^d} \exp\left(-\frac{|u|^2}{K\sigma^2(\eta)}\right) du. \end{aligned}$$

Similar to (4.3), we have

$$(4.7) \quad \begin{aligned} &\int_I dt_n \int_{|u| < r, \sigma(\eta)^d} \frac{1}{\sigma(\eta)^d} \exp\left(-\frac{|u|^2}{K\sigma^2(\eta)}\right) du \\ &\leq \int_I \min\{1, K\left(\frac{r}{\sigma(\eta)}\right)^d\} dt_n \\ &\leq K \int_I \sum_{i=0}^{n-1} \min\{1, K\left(\frac{r}{\sigma(|t_n - t_i|)}\right)^d\} dt_n \quad (t_0 = 0) \\ &\leq Kn \int_0^1 \min\{1, \frac{Kr^d}{\sigma(\rho)^d}\} \rho^{N-1} d\rho \\ &\leq Kn\psi(r)^N. \end{aligned}$$

By (4.4), (4.6) and (4.7), we obtain

$$\begin{aligned} E(T(r))^n &\leq K \int_{I^{n-1}} P\{|X_1(t_1)| < r, \dots, |X(t_{n-1})| < r\} dt_1 \cdots dt_{n-1} \\ &\quad \cdot \int_I dt_n \int_{|u| < r, \sigma(\eta)^d} \frac{1}{\sigma(\eta)^d} \exp\left(-\frac{|u|^2}{K\sigma^2(\eta)}\right) du \end{aligned}$$

$$\leq Kn\psi(r)^N E(T(r))^{n-1}.$$

Hence, the inequality (4.2) follows from (4.3) and induction. Let  $0 < b < 1/K$ , then by (4.2) we have

$$E \exp(b\psi(r)^{-N} T(r)) = \sum_{n=0}^{\infty} (Kb)^n < \infty.$$

This proves (4.1)

**Proposition 4.2.** *With probability 1*

$$(4.8) \quad \limsup_{r \rightarrow 0} \frac{T(r)}{\phi(r)} \leq \frac{1}{b},$$

where  $\phi(r) = \psi(r)^N \log \log 1/r$ .

*Proof.* For any  $\varepsilon > 0$ , it follows from (4.1) that

$$(4.9) \quad P\{T(r) \geq (1/b + \varepsilon)\psi(r)^N \log \log 1/r\} \leq \frac{K}{(\log 1/r)^{1+b\varepsilon}}.$$

Take  $r_n = \exp(-n/\log n)$ , then by (4.9) we have

$$P\{T(r_n) \geq (1/b + \varepsilon)\psi(r_n)^N \log \log 1/r_n\} \leq \frac{K}{(n/\log n)^{1+b\varepsilon}}.$$

Hence by Borel-Cantelli lemma we have

$$(4.10) \quad \limsup_{n \rightarrow \infty} \frac{T(r_n)}{\phi(r_n)} \leq \frac{1}{b} + \varepsilon.$$

It is easy to verify that

$$(4.11) \quad \lim_{n \rightarrow \infty} \frac{\phi(r_n)}{\phi(r_{n+1})} = 1.$$

Hence by (4.10) and (4.11) we have

$$\limsup_{r \rightarrow 0} \frac{T(r)}{\phi(r)} \leq \frac{1}{b} + \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we obtain (4.8).

Since  $X(t)$  ( $t \in \mathbf{R}^N$ ) has stationary increments, we derive the following

**Corollary 4.1.** Fix  $t_0 \in I$ , then with probability 1

$$\limsup_{r \rightarrow 0} \frac{T_{X(t_0)}(r)}{\phi(r)} \leq \frac{1}{b}.$$

**Theorem 4.1.** If  $N < \alpha d$ , then with probability 1

$$(4.12) \quad \phi\text{-}m(X([0, 1]^N)) > 0,$$

where  $\phi(r) = \psi(r)^N \log \log 1/r$ .

*Proof.* We define a random Borel measure  $\mu$  on  $X(I)$  as follows. For any Borel set  $B \subseteq \mathbb{R}^d$ , let

$$\mu(B) = L_N\{t \in I, X(t) \in B\}.$$

Then  $\mu(\mathbb{R}^d) = \mu(X(I)) = L_N(I)$ . By Corollary 4.1, for each fixed  $t_0 \in I$ , with probability 1

$$(4.13) \quad \begin{aligned} \limsup_{r \rightarrow 0} \frac{\mu(B(X(t_0), r))}{\phi(r)} \\ \leq \limsup_{r \rightarrow 0} \frac{T_{X(t_0)}(r)}{\phi(r)} \leq \frac{1}{b}. \end{aligned}$$

Let  $E(\omega) = \{X(t_0) : t_0 \in I \text{ and (4.13) holds}\}$ . Then  $E(\omega) \subseteq X(I)$ . A Fubini argument shows  $\mu(E(\omega)) = 1$ , a.s.. Hence by Lemma 2.1, we have

$$\phi\text{-}m(E(\omega)) \geq Kb.$$

This proves (4.12).

*Proof of Theorem 1.1.* It follows from Theorems 3.1 and 4.1 immediately.

**EXAMPLE 4.1.** Let  $Y(t)$  ( $t \in \mathbb{R}^N$ ) be a real-valued fractional Brownian motion of index  $\alpha$  ( $0 < \alpha < 1$ ) (see [10], Chapter 18). Its covariance function has the representation

$$\begin{aligned} R(s, t) &= \frac{1}{2}(|s|^{2\alpha} + |t|^{2\alpha} - |t-s|^{2\alpha}) \\ &= c(\alpha) \int_{\mathbb{R}^N} (e^{i\langle t, \lambda \rangle} - 1)(e^{-i\langle s, \lambda \rangle} - 1) \frac{d\lambda}{|\lambda|^{N+2\alpha}}, \end{aligned}$$

where  $c(\alpha)$  is a normalizing constant. Then (1.5) is verified and by a result of Pitt [17], (1.6) is also verified. In this case, Theorem 1.1 is proved by Goldman [9]

for  $\alpha = 1/2$  and by Talagrand [22] for  $0 < \alpha < 1$ .

EXAMPLE 4.2 Let  $Z(t)$  ( $t \in \mathbf{R}^N$ ) be a real-valued mean zero stationary random field with covariance function

$$R(s, t) = \exp(-c|s - t|^{2\alpha}) \quad \text{with } c > 0 \quad \text{and } 0 < \alpha < 1.$$

Then  $Y(t) = Z(t) - Z(0)$  verifies the conditions (1.5) and (1.6). We can apply Theorem 1.1 to obtain the Hausdorff measure of  $X([0, 1]^N)$ , where

$$X(t) = (X_1(t), \dots, X_d(t))$$

and  $X_1, \dots, X_d$  are independent copies of  $Z$ . Other examples with absolutely continuous spectral measure can be found in Berman [2] p289, and Berman [4].

EXAMPLE 4.3. Now we give an example with discrete spectral measure. Let  $X_n$  ( $n \geq 0$ ) and  $Y_n$  ( $n \geq 0$ ) be independent standard normal random variables and  $a_n$  ( $n \geq 0$ ) real numbers such that  $\sum_n a_n^2 < \infty$ . Then for each  $t$ , the random series

$$(4.14) \quad Z(t) = \sum_{n=0}^{\infty} a_n (X_n \cos nt + Y_n \sin nt)$$

converges with probability 1 (see [10]), and  $Z(t)$  ( $t \in \mathbf{R}$ ) represents a stationary Gaussian process with mean 0 and covariance function

$$R(s, t) = \sum_{n=0}^{\infty} a_n^2 \cos n(t - s).$$

By a result of Berman [4], there are many choices of  $a_n$  ( $n \geq 0$ ) such that the process  $Y(t) = Z(t) - Z(0)$  satisfies the hypotheses of Theorem 1.1 with

$$\sigma^2(s) = 2 \sum_{n=0}^{\infty} a_n^2 (1 - \cos ns).$$

Let  $X(t)$  ( $t \in \mathbf{R}$ ) be the Gaussian process in  $\mathbf{R}^d$  associated with  $Z(t)$  or  $Y(t)$  ( $t \in \mathbf{R}$ ) by (1.7). If  $1 < \alpha d$ , then

$$0 < \phi\text{-}m(X([0, 1]^N)) < \infty,$$

where  $\phi(s) = \psi(s) \log \log \frac{1}{s}$  and  $\psi$  is the inverse function of  $\sigma$ . A special case of (4.14) is Example 3.5 in Monrad and Rootzén [15].

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