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Osaka University
HAUSDORFF MEASURE OF THE SAMPLE PATHS
OF GAUSSIAN RANDOM FIELDS

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(Received November 17, 1995)

1. Introduction

Let \( Y(t) \) \((t \in \mathbb{R}^N)\) be a real-valued, centered Gaussian random field with \( Y(0) = 0 \). We assume that \( Y(t) \) \((t \in \mathbb{R}^N)\) has stationary increments and continuous covariance function \( R(t,s) = EY(t)Y(s) \) given by

\[
R(t,s) = \int_{\mathbb{R}^N} (e^{i\langle t,\lambda \rangle} - 1)(e^{-i\langle s,\lambda \rangle} - 1)\Delta(d\lambda),
\]

where \( \langle x,y \rangle \) is the ordinary scalar product in \( \mathbb{R}^N \) and \( \Delta(d\lambda) \) is a nonnegative symmetric measure on \( \mathbb{R}^N \setminus \{0\} \) satisfying

\[
\int_{\mathbb{R}^N} \frac{|\lambda|^2}{1 + |\lambda|^2}\Delta(d\lambda) < \infty.
\]

Then there exists a centered complex-valued Gaussian random measure \( W(d\lambda) \) such that

\[
Y(t) = \int_{\mathbb{R}^N} (e^{i\langle t,\lambda \rangle} - 1)W(d\lambda)
\]

and for any Borel sets \( A, B \subseteq \mathbb{R}^N \)

\[
E(W(A)W(B)) = \Delta(A \cap B) \quad \text{and} \quad W(-A) = W(A).
\]

It follows from (1.3) that

\[
E[(Y(t + h) - Y(t))^2] = 2\int_{\mathbb{R}^N} (1 - \cos \langle h, \lambda \rangle)\Delta(d\lambda).
\]

We assume that there exist constants \( \delta_0 > 0, 0 < c_1 \leq c_2 < \infty \) and a non-decreasing, continuous function \( \sigma : [0, \delta_0) \to [0, \infty) \) which is regularly varying at the origin with index \( \alpha \) \((0 < \alpha < 1)\) such that for any \( t \in \mathbb{R}^N \) and \( h \in \mathbb{R}^N \) with \( |h| \leq \delta_0 \)

\[
E[(Y(t + h) - Y(t))^2] \leq c_1 \sigma^2(|h|).
\]
and for all \( t \in \mathbb{R}^N \) and any \( 0 < r \leq \min\{|t|, \delta_0\} \)

\[
(1.6) \quad \text{Var}(Y(t)|Y(s); r \leq |s-t| \leq \delta_0) \geq c_2 \sigma^2(r).
\]

If (1.5) and (1.6) hold, we shall say that \( Y(t) (t \in \mathbb{R}^N) \) is strongly locally \( \sigma \)-nondeterministic. We refer to Monrad and Pitt [14], Berman [4] [5] and Cuzick and Du Peez [6] for more information on (strongly) locally nondeterminism.

We associate with \( Y(t) (t \in \mathbb{R}^N) \) a Gaussian random field \( X(t) (t \in \mathbb{R}^d) \) in \( \mathbb{R}^d \) by

\[
(1.7) \quad X(t) = (X_1(t), \ldots, X_d(t)),
\]

where \( X_1, \ldots, X_d \) are independent copies of \( Y \). The most important example of such Gaussian random fields is the fractional Brownian motion of index \( \alpha \) (see Example 4.1 below).

It is well known (see [1], Chapter 8) that with probability 1

\[
\text{dim } X([0,1]^N) = \min \left( d, \frac{N}{\alpha} \right).
\]

The objective of this paper is to consider the exact Hausdorff measure of the image set \( x([0,1]^N) \). The main result is the following theorem, which generalizes a theorem of Talagrand [22].

**Theorem 1.1.** If \( N < ad \), then with probability 1

\[
(1.8) \quad 0 < \phi-m(X([0,1]^N)) < \infty,
\]

where \( \phi(s) = \psi(s)^N \log \log \frac{1}{s} \), \( \psi \) is the inverse function of \( \sigma \) and \( \phi-m(X([0,1]^N)) \) is the \( \phi \)-Hausdorff measure of \( X([0,1]^N) \).

If \( N > ad \), then by a result of Pitt [17], \( X([0,1]^N) \) a.s. has interior points and hence has positive \( d \)-dimensional Lebesgue measure. In the case of \( N = ad \), the problem of finding \( \phi-m(X([0,1]^N)) \) is still open even in the fractional Brownian motion case.

The paper is organized as follows. In Section 2 we recall the definition and some basic facts of Hausdorff measure, Gaussian processes and regularly varying functions. In Section 3 we prove the upper bound and in Section 4, we prove the lower bound for \( \phi-m(X([0,1]^N)) \). We also give some examples showing that the hypotheses in Theorem 1.1 are satisfied by a large class of Gaussian random fields including fractional Brownian motion.

Another important example of Gaussian random fields is the Brownian sheet or \( N \)-parameter Wiener process \( W(t) (t \in \mathbb{R}_+^N) \), see Orey and Pruitt [16]. Since \( W(t) (t \in \mathbb{R}_+^N) \) is not locally nondeterministic, Theorem 1.1 does not apply. The problem of finding exact Hausdorff measure of \( W([0,1]^N) \) was solved by Ehren [7].
We will use \( K \) to denote an unspecified positive constant which may be different in each appearance.

2. Preliminaries

Let \( \Phi \) be the class of functions \( \phi : (0, \delta) \rightarrow (0, 1) \) which are right continuous, monotone increasing with \( \phi(0^+) = 0 \) and such that there exists a finite constant \( K > 0 \) for which

\[
\frac{\phi(2s)}{\phi(s)} \leq K, \quad \text{for} \quad 0 < s < \frac{1}{2} \delta.
\]

For \( \phi \in \Phi \), the \( \phi \)-Hausdorff measure of \( E \subseteq \mathbb{R}^N \) is defined by

\[
\phi-m(E) = \liminf_{\epsilon \to 0} \left\{ \sum_i \phi(2r_i) : E \subseteq \bigcup_{i=1}^\infty B(x_i, r_i), \, r_i < \epsilon \right\},
\]

where \( B(x, r) \) denotes the open ball of radius \( r \) centered at \( x \). It is known that \( \phi-m \) is a metric outer measure and every Borel set in \( \mathbb{R}^N \) is \( \phi-m \) measurable. The Hausdorff dimension of \( E \) is defined by

\[
\dim E = \inf \{ \alpha > 0 : s^\alpha \cdot m(E) = 0 \} = \sup \{ \alpha > 0 : s^\alpha \cdot m(E) = \infty \}.
\]

We refer to [F] for more properties of Hausdorff measure and Hausdorff dimension.

The following lemma can be easily derived from the results in [18] (see [23]), which gives a way to get a lower bound for \( \phi-m(E) \). For any Borel measure \( \mu \) on \( \mathbb{R}^N \) and \( \phi \in \Phi \), the upper \( \phi \)-density of \( \mu \) at \( x \in \mathbb{R}^N \) is defined by

\[
\limsup_{r \to 0} \frac{\mu(B(x, r))}{\phi(2r)}.
\]

**Lemma 2.1.** For a given \( \phi \in \Phi \) there exists a positive constant \( K \) such that for any Borel measure \( \mu \) on \( \mathbb{R}^N \) and every Borel set \( E \subseteq \mathbb{R}^N \), we have

\[
\phi-m(E) \geq K \mu(E) \inf_{x \in E} \left\{ \frac{\mu(B(x, r))}{\phi(2r)} \right\}^{-1}.
\]

Now we summarize some basic facts about Gaussian processes. Let \( Z(t) (t \in S) \) be a Gaussian process. We provide \( S \) with the following metric

\[
d(s, t) = \| Z(s) - Z(t) \|_2,
\]

where \( \| Z \|_2 = (E(Z^2))^{1/2} \). We denote by \( N_{d}(S, \epsilon) \) the smallest number of open \( d \)-balls
of radius $\varepsilon$ needed to cover $S$ and write $D = \sup \{d(s, t): s, t \in S\}$.

The following lemma is well known. It is a consequence of the Gaussian isoperimetric inequality and Dudley's entropy bound ([11], see also [22]).

**Lemma 2.2.** There exists an absolute constant $K > 0$ such that for any $u > 0$, we have

$$P\left(\sup_{s, t \in S} |Z(s) - Z(t)| \geq K(u + \int_0^D \sqrt{\log N_d(S, \varepsilon, d\varepsilon)} \, d\varepsilon) \right) \leq \exp\left(-\frac{u^2}{D^2}\right).$$

**Lemma 2.3.** Consider a function $\Psi$ such that $N_d(S, \varepsilon) \leq \Psi(\varepsilon)$ for all $\varepsilon > 0$. Assume that for some constant $C > 0$ and all $\varepsilon > 0$ we have

$$\frac{\Psi(\varepsilon)}{C} \leq \Psi\left(\frac{\varepsilon}{2}\right) \leq C \Psi(\varepsilon).$$

Then

$$P\left\{\sup_{s, t \in S} |Z(s) - Z(t)| \leq u\right\} \geq \exp(-K\Psi(u)),$$

where $K > 0$ is a constant depending only on $C$.

This is proved in [21]. It gives an estimate for the lower bound of the small ball probability of Gaussian processes. Similar problems have also been considered be Monrad and Rootzén [15] and by Shao [20].

We end this section with some lemmas about regularly varying functions. Let $\sigma(s)$ be a regularly varying function with index $\alpha (0 < \alpha < 1)$. Then $\sigma$ can be written as

$$\sigma(s) = s^\alpha L(s),$$

where $L(s): [0, \delta_0) \to [0, \infty)$ is slowly varying at the origin in the sense of Karamata and hence can be represented by

$$L(s) = \exp\left(\eta(s) + \int_s^A \frac{\varepsilon(t)}{t} \, dt\right),$$

where $\eta(s): [0, \delta_0) \to R$, $\varepsilon(s): (0, A] \to R$ are bounded measurable functions and

$$\lim_{s \to 0} \eta(s) = c, \quad |c| < \infty; \quad \lim_{s \to 0} \varepsilon(s) = 0.$$

In the following, Lemma 2.4 is an easy consequence of (2.1) and Lemma 2.5 can be deduced from Theorem 2.6 and 2.7 in Seneta [19] directly.
Lemma 2.4. Let $L(s)$ be a slowly varying function at the origin and let $U = U(s) : [0, \infty) \to [0, \infty)$ satisfying

$$\lim_{s \to 0} U(s) = \infty \quad \text{and} \quad \lim_{s \to 0} s U(s) = 0.$$ 

Then for any $\varepsilon > 0$, as $s$ small enough we have

$$U(s)^{-1} L(s) \leq L(s U(s)) \leq U(s)^{1} L(s)$$

and

$$U(s)^{-1} L(s) \leq L(s U(s)^{-1}) \leq U(s)^{1} L(s).$$

Lemma 2.5. Let $\sigma$ be a regularly varying function at the origin with index $\alpha > 0$. Then there is a constant $K > 0$ such that for $r > 0$ small enough, we have

$$\int_{1}^{\infty} \sigma(re^{-u^2})du \leq K\sigma(r),$$

(2.2)

$$\int_{0}^{1} \sigma(rs)ds \leq K\sigma(r),$$

(2.3)

$$\int_{0}^{1} \sigma(rs)s^{-1}ds \leq K\sigma(r).$$

(2.4)

Let $\sigma : [0, \delta_0) \to [0, \infty)$ be non-decreasing and let $\psi$ be the inverse function of $\sigma$, that is

$$\psi(s) = \inf\{t \geq 0 : \sigma(t) \geq s\},$$

then $\psi(s) = s^{1/\alpha} f(s)$, where $f(s)$ is also a slowly varying function and

$$\sigma(\psi(s)) \sim s \quad \text{and} \quad \psi(\sigma(s)) \sim s \quad \text{as} \quad s \to 0.$$  

(2.5)

3. Upper bound for $\phi-m(X([0,1]^N))$

Let $Y(t) (t \in \mathbb{R}^N)$ be a real-valued, centered Gaussian random field with stationary increments and a continuous covariance function $R(t,s)$ given by (1.1). We assume that $Y(0) = 0$ and (1.5) holds. Let $X(t) (t \in \mathbb{R}^N)$ be the $(N,d)$ Gaussian random field defined by (1.7).

We start with the following lemma.

Lemma 3.1. Let $Y(t) (t \in \mathbb{R}^N)$ be a Gaussian process with $Y(0) = 0$ satisfying (1.5). Then
(i) For any $r > 0$ small enough and $u \geq K\sigma(r)$, we have

$$P \left\{ \sup_{|t| \leq r} |Y(t)| \geq u \right\} \leq \exp \left( -\frac{u^2}{K\sigma^2(r)} \right).$$

(ii) Let

$$\omega_Y(h) = \sup_{t, t + s \in [0,1]^n, |s| \leq h} |Y(t+s) - Y(t)|$$

be the uniform modulus of continuity of $Y(t)$ on $[0,1]^n$. Then

$$\limsup_{h \to 0} \frac{\omega_Y(h)}{\sigma(h)\sqrt{2c_1 \log \frac{1}{h}}} \leq 1, \quad \text{a.s.}$$

Proof. Let $r < \delta_0$ and $S = \{ t : |t| \leq r \}$. Since $d(s,t) \leq c_1 \sigma(|t-s|)$, we have

$$N_d(S,\varepsilon) \leq K \left( \frac{r}{\psi(\varepsilon)} \right)^N$$

and

$$D = \sup \{ d(s,t) ; s, t \in S \} \leq K\sigma(r).$$

By simple calculations

$$\int_0^r \sqrt{\log N_d(S,\varepsilon)} \, d\varepsilon \leq K \int_0^{K\sigma(r)} \sqrt{\log (Kr) / \psi(\varepsilon)} \, d\varepsilon \leq K \int_0^{Kr} \sqrt{\log (Kr) / \varepsilon} \, d\varepsilon \leq K \left( \sigma(r) + \int_0^K \frac{1}{u \sqrt{\log K / u}} \sigma(ur) \, du \right) \leq K \left( \sigma(r) + \int_K^{\infty} \sigma(re^{-u^2}) \, du \right) \leq K\sigma(r),$$

where the last inequality follows from (2.2). If $u \geq K\sigma(r)$, then by Lemma 2.2 we have

$$P \left\{ \sup_{|t| \leq r} |Y(t)| \geq 2Ku \right\} \leq P \left\{ \sup_{|t| \leq r} |Y(t)| \geq Ku + \int_0^D \sqrt{\log N_d(S,\varepsilon)} \, d\varepsilon \right\}.$$
This proves (3.1). The inequality (3.2) can be derived from Lemma 2.2 directly in a standard way (see also [13]).

In order to get the necessary independence, we will make use of the spectral representation (1.3). Given \(0 < a < b < \infty\), we consider the process

\[
Y(a, b, t) = \int_{\lambda \leq |t| \leq b} (e^{i \langle t, \lambda \rangle} - 1) W(d\lambda).
\]

Then for any \(0 < a < b < a' < b' < \infty\), the processes \(Y(a, b, t)\) and \(Y(a', b', t)\) are independent. The next lemma expresses how well \(Y(a, b, t)\) approximates \(Y(t)\).

**Lemma 3.2.** Let \(Y(t) (t \in \mathbb{R}^N)\) be defined by (1.3). If (1.5) holds, then there exists a constant \(B > 0\) such that for any \(B < a < b\) we have

\[
\|Y(a, b, t) - Y(t)\|_2 \leq K [t^2 a^2 \sigma^2 (a^{-1}) + \sigma^2 (b^{-1})].
\]

**Proof.** First we claim that for any \(u > 0\) and any \(h \in \mathbb{R}^N\) with \(|h| = 1 / u\) we have

\[
\int_{|\lambda| < u} \langle h, \lambda \rangle^2 \Delta(d\lambda) \leq K \int_{\mathbb{R}^N} (1 - \cos \langle h, \lambda \rangle) \Delta(d\lambda)
\]

\[
\int_{|\lambda| \geq u} \Delta(d\lambda) \leq K \left( \frac{u}{2} \right)^N \int_{[-1/u, 1/u]^N} dv \int_{\mathbb{R}^N} (1 - \cos \langle v, \lambda \rangle) \Delta(d\lambda).
\]

For \(N = 1\), (3.4) and (3.5) are the truncation inequalities in [12] p209. For \(N > 1\) a similar proof yields (3.4) and (3.5).

Now for any \(a > \delta_0^{-1}\) and any \(t \in \mathbb{R}^N \setminus \{0\}\), by (1.4), (1.5) and (3.4) we have

\[
\int_{|\lambda| < a} (1 - \cos \langle t, \lambda \rangle) \Delta(d\lambda) \leq \int_{|\lambda| < a} \langle t, \lambda \rangle^2 \Delta(d\lambda)
\]

\[
= |t|^2 a^2 \int_{|\lambda| < a} \langle t / |t|, \lambda \rangle^2 \Delta(d\lambda) \leq K |t|^2 a^2 \sigma^2 (a^{-1}).
\]

For \(b > 0\) large enough, by (3.5), (1.4), (1.5) and (2.4) we have

\[
\int_{|\lambda| \geq b} \Delta(d\lambda) \leq K \left( \frac{b}{2} \right)^N \int_{[-1/b, 1/b]^N} \sigma^2(|v|) dv
\]

\[
\leq K b^N \int_0^{N b^{-1}} \sigma^2(\rho) \rho^{N-1} d\rho \leq K \sigma^2 (b^{-1}).
\]
Combining (3.6) and (3.7), we see that there exists a constant $B>0$ such that $B<a<b$ implies

$$
E[(Y(a,b,t) - Y(t))^2] = 2 \int_{|\lambda| < a} (1 - \cos\langle t, \lambda \rangle) \Delta(d\lambda) + 2 \int_{|\lambda| > b} \Delta(d\lambda)
$$

This proves (3.3).

**Lemma 3.3.** There exists a constant $B>0$ such that for any $B<a<b$ and $0<r < B^{-1}$ the following holds: let $A = r^2 a^2 \sigma^2(a^{-1}) + \sigma^2(b^{-1})$ such that $\psi(\sqrt{A}) \leq \frac{1}{2} r$, then for any

$$
u \geq K \left( A \log \frac{Kr}{\psi(\sqrt{A})} \right)^{1/4}
$$

we have

$$
P\left\{ \sup_{|t| \leq r} |Y(t) - Y(a,b,t)| \geq u \right\} \leq \exp\left( - \frac{u^2}{KA} \right).
$$

**Proof.** Let $S = \{ t : |t| \leq r \}$ and $Z(t) = Y(t) - Y(a,b,t)$. Then

$$
d(s,t) = \| Z(t) - Z(s) \|_2 \leq c_1 \sigma(|t-s|).
$$

Hence

$$
N_d(S,\varepsilon) \leq K \left( \frac{r}{\psi(\varepsilon)} \right)^N.
$$

By Lemma 3.2 we have $D \leq K\sqrt{A}$. As in the proof of Lemma 3.1,

$$
\int_0^B \sqrt{\log N_d(S,\varepsilon)} d\varepsilon \leq K \int_0^{K\sqrt{A}} \sqrt{\log(Kr) / \psi(\varepsilon)} d\varepsilon
$$

$$
\leq K \int_0^{K\psi(\sqrt{A})/r} \sqrt{\log K / t} d\sigma(rt)
$$

$$
\leq K \left[ \sqrt{\log K / t} \sigma(rt) \right]_{0}^{r^{K\psi(\sqrt{A})/r}} + \int_0^{r^{K\psi(\sqrt{A})/r}} \frac{1}{t\sqrt{\log K / t}} \sigma(rt) dt
$$
at least for \( r > 0 \) small enough, where the last step follows from (2.2). Hence (3.8) follows immediately from Lemma 2.2.

Let \( X_1(a,b,t), \ldots, X_d(a,b,t) \) be independent copies of \( Y(a,b,t) \) and let

\[
X(a,b,t) = (X_1(a,b,t), \ldots, X_d(a,b,t)) \quad (t \in \mathbb{R}^N).
\]

Then we have the following corollary of Lemma 3.3.

**Corollary 3.1.** Consider \( B < a < b \) and \( 0 < r < B^{-1} \). Let \( A = r^2 a^2 \sigma^2(a^{-1}) + \sigma^2(b^{-1}) \) with \( \psi(\sqrt{A}) \leq \frac{1}{2} r \). Then for any

\[
u \geq K \left( A \log \frac{Kr}{\psi(\sqrt{A})} \right)^4
\]

we have

\[
P\left\{ \sup_{|t| \leq r} |X(t) - X(a,b,t)| \geq \nu \right\} \leq \exp\left( - \frac{\nu^2}{KA} \right).
\]

**Lemma 3.4.** Given \( 0 < r < \delta_0 \) and \( \varepsilon < \sigma(r) \). Then for any \( 0 < a < b \) we have

\[
P\left\{ \sup_{|t| \leq r} |X(a,b,t)| \leq \varepsilon \right\} \geq \exp\left( - \frac{r^N}{K\psi(\varepsilon)^N} \right).
\]

Proof. It is sufficient to prove (3.10) for \( Y(a,b,t) \). Let \( S = \{ t : |t| \leq r \} \) and define a distance \( d \) on \( S \) by

\[
d(s,t) = \| Y(a,b,t) - Y(a,b,s) \|_2.
\]

Then \( d(s,t) \leq c_1 \sigma(|t - s|) \) and

\[
N_d(S, \varepsilon) \leq K \left( \frac{r}{\psi(\varepsilon)} \right)^N.
\]

By Lemma 2.3 we have

\[
P\left\{ \sup_{|t| \leq r} |Y(a,b,t)| \leq \varepsilon \right\} \geq \exp\left( - \frac{r^N}{K\psi(\varepsilon)^N} \right).
\]
This proves lemma 3.4.

**Proposition 3.1.** There exists a constant $\delta_1 > 0$ such that for any $0 < r_0 \leq \delta_1$, we have

\[
P\left\{ \exists r \in [r_0^2, r_0] \text{ such that } \sup_{|t| \leq r} |X(t)| \leq K_0 \sigma(r (\log \log \frac{1}{r})^{-\frac{1}{N}}) \right\} 
\geq 1 - \exp\left(-\frac{1}{r_0}\right).
\]

Proof. We follow the line of Talagrand [22]. Let $U = U(r_0) \geq 1$, where $U(r)$ satisfying

\[
U(r) \to \infty \quad \text{as} \quad r \to 0
\]

and for any $\epsilon > 0$

\[
r^\epsilon U(r) \to 0 \quad \text{as} \quad r \to 0,
\]

will be chosen later. For $k \geq 0$, let $r_k = r_0 U^{-2k}$. Let $k_0$ be the largest integer such that

\[
k_0 \leq \frac{\log \frac{1}{r_0}}{2 \log U},
\]

then for any $0 \leq k \leq k_0$ we have $r_0^2 \leq r_k \leq r_0$. In order to prove (3.11), it suffices to show that

\[
P\left\{ \exists k \leq k_0 \text{ such that } \sup_{|t| \leq r_k} |X(t)| \leq K_0 \sigma(r_k (\log \log \frac{1}{r_k})^{-\frac{1}{N}}) \right\} 
\geq 1 - \exp\left(-\frac{1}{r_0}\right).
\]

Let $a_k = r_0^{-1} U^{2k-1}$ and we define for $k = 0, 1, \ldots$

\[X_k(t) = X(a_k, a_{k+1}, t),\]

then $X_0, X_1, \ldots$ are independent. By Lemma 3.4 we can take a constant $K_1$ such that for $r_0 > 0$ small enough

\[
P\left\{ \sup_{|t| \leq r_k} |X_k(t)| \leq K_1 \sigma(r_k (\log \log \frac{1}{r_k})^{-\frac{1}{N}}) \right\} 
\geq \exp\left(-\frac{1}{4 \log \log \frac{1}{r_k}}\right).
\]
Thus, by independence we have

\begin{equation}
P \left\{ \exists k \leq k_0, \sup_{|t| \leq r_k} |X_k(t)| \leq K_1 \sigma(r_k (\log \log \frac{1}{r_k})^{-1/N}) \right\} \\
\geq 1 - \left( 1 - \frac{1}{(2 \log 1/ r_0)^{1/4}} \right)^{k_0} \\
\geq 1 - \exp \left( - \frac{k_0}{(2 \log 1/ r_0)^{1/4}} \right).
\end{equation}

Let

\[ A_k = r_k^2 a_k^2 \sigma^2 (a_k^{-1} + \sigma^2 (a_k^{-1} + 1) \\
= U^{-2 + 2 \alpha} L^2 (r_k U) + U^{-2 \alpha} L^2 (r_k / U). \]

Let \( \beta = 2 \min \{1 - \alpha, \alpha\} \) and fix an \( \varepsilon < \frac{1}{2} \beta \). Then by Lemma 2.4, we see that as \( r_0 \) small enough

\[ U^{-\beta - \varepsilon} \sigma^2 (r_k) \leq A_k \leq U^{-\beta + \varepsilon} \sigma^2 (r_k). \]

Notice that \( r_0 \) for small enough we have

\[ \psi(\sqrt{A_k}) \geq \psi(U^{-\beta + \varepsilon} \sigma(r_k)) \\
= (U^{-\beta + \varepsilon} \sigma(r_k))^{1/2} f(U^{-\beta} \sigma(r_k)) \\
= U^{-\beta / (2 \alpha)} L(r_k) \psi(U^{-\beta/2} \sigma(r_k)) \\
\geq KU^{-\beta / (2 / \alpha) + \varepsilon} r_k \]

the last inequality follows from (2.5). It follows from Corollary 3.1 that for

\[ u \geq K \sigma(r_k) U^{-\beta/2} (\log U)^{1/2}, \]

we have

\begin{equation}
P \left\{ \sup_{|t| \leq r_k} |X(t) - X_k(t)| \geq u \right\} \leq \exp \left( \frac{-u^2 U^{\beta - \varepsilon}}{K \sigma^2 (r_k)} \right).
\end{equation}

Hence, if we take

\[ U = (\log 1 / r_0)^{\beta - \varepsilon}, \]
then as \( r_0 \) small enough

\[
\sigma(r_k) U^{-\frac{\theta}{2} \left( \log U \right)^{1/2}} \leq \sigma(r_k (\log \log \frac{1}{r_0})^{-\frac{1}{N}}).
\]

Hence by taking

\[
u = \frac{K_1}{2} \sigma(r_k (\log \log \frac{1}{r_0})^{-\frac{1}{N}})
\]

in (3.17), we obtain

\[
P\left\{ \sup_{|t| \leq r_k} |X(t) - X_k(t)| \geq \frac{K_1}{2} \sigma(r_k (\log \log \frac{1}{r_0})^{-\frac{1}{N}}) \right\} \leq \exp \left(- \frac{u^2 U^{\theta - \varepsilon}}{K \sigma^2(r_k)} \right).
\]

Combining (3.16) and (3.18) we have

\[
P\left\{ \exists k \leq k_0 \text{ such that } \sup_{|t| \leq r_k} |X(t)| \leq 2K_1 \sigma(r_k (\log \log \frac{1}{r_k})^{-1/N}) \right\} \geq 1 - \exp \left(- \frac{k_0}{2(\log 1/r_k)^{1/4}} \right) - k_0 \exp \left(- \frac{U^{\theta - \varepsilon}}{K (\log \log 1/r_0) (2a)^N + \varepsilon} \right).
\]

We recall that

\[
\frac{\log \frac{1}{r_0}}{4 \log U} \leq k_0 \leq \frac{1}{r_0}.
\]

and hence for \( r_0 \) small enough, (3.11) follows from (3.19).

Now we are in a position to prove the upper bound for \( \phi_m(\mathcal{X}([0,1]^N)) \).

**Theorem 3.1.** Let \( \phi(s) = \psi(s)^N \log \log \frac{1}{s} \). Then with probability 1

\[\phi_m(\mathcal{X}([0,1]^N)) < \infty.\]

**Proof.** For \( k \geq 1 \), consider the set

\[
R_k = \left\{ t \in [0,1]^N : \exists r \in [2^{-2k}, 2^{-k}] \text{ such that } \sup_{|s-t| \leq r} |X(s) - X(t)| \leq K \sigma(r (\log \log \frac{1}{r})^{-1/N}) \right\}.
\]

By Proposition 3.1 we have

\[
P\{ t \in R_k \} \geq 1 - \exp(-\sqrt{k}/2).
\]
Denote the Lebesgue measure in $\mathbb{R}^N$ by $L_N$. It follows from Fubini's theorem that $P(\Omega_o)=1$, where

$$\Omega_o = \{ \omega : L_N(R_k) \geq 1 - \exp(-\sqrt{k/4}) \text{ infinitely often} \}.$$

On the other hand, by Lemma 3.1 ii), there exists an event $\Omega_1$ such that $P(\Omega_1)=1$ and for all $\omega \in \Omega_1$, there exists $n_1 = n_1(\omega)$ large enough such that for all $n \geq n_1$ and any dyadic cube $C$ of order $n$ in $\mathbb{R}^N$, we have

$$$$(3.20) \quad \sup_{s,t \in C} |X(t) - X(s)| \leq K\sigma(2^{-n})\sqrt{n}.$$ $$

Now fix an $\omega \in \Omega_0 \cap \Omega_1$, we show that $\phi_m(X([0,1]^N)) < \infty$. Consider $k \geq 1$ such that

$$L_N(R_k) \geq 1 - \exp(-\sqrt{k/4}).$$

For any $x \in R_k$ we can find $n$ with $k \leq n \leq 2k + k_0$ (where $k_0$ depends on $N$ only) such that

$$$$(3.21) \quad \sup_{s,t \in C_n(x)} |X(t) - X(s)| \leq K\sigma(2^{-n}(\log \log 2^n)^{-1/N}),$$ $$

where $C_n(x)$ is the unique dyadic cube of order $n$ containing $x$. Thus we have

$$R_k \subseteq V = \bigcup_{n=k}^{2k+k_0} V_n$$

and each $V_n$ is a union of dyadic cubes $C_n$ of order $n$ for which (3.21) holds. Clearly $X(C_n)$ can be covered by a ball of radius

$$\rho_n = K\sigma(2^{-n}(\log \log 2^n)^{-1/N}).$$

Since $\phi(2\rho_n) \leq K2^{-nN} = KL_N(C_n)$, we have

$$$$(3.22) \quad \sum_n \sum_{C \in V_n} \phi(2\rho_n) \leq \sum_n \sum_{C \in V_n} KL_N(C_n)$$

$$= KL_N(V) < \infty.$$ $$

On the other hand, $[0,1]^N \setminus V$ is contained in a union of dyadic cubes of order $q = 2k + k_0$, none of which meets $R_k$. There can be at most

$$2^{Nq}L_N([0,1]^N \setminus V) \leq K2^{Nq}\exp(-\sqrt{k}/4)$$

of such cubes. For each of these cubes, $X(C)$ is contained in a ball of radius $\rho = K\sigma(2^{-q})\sqrt{q}$. Thus for any $\varepsilon > 0$

$$$$(3.23) \quad \sum \phi(2\rho) \leq K2^{Nq}\exp(-\sqrt{k}/4)2^{-Nq}\varepsilon^{N(2a)+\varepsilon} \leq 1.$$ $$
for \( k \) large enough. Since \( k \) can be arbitrarily large, Theorem 3.1 follows from (3.22) and (3.23).

4. Lower bound for \( \phi-m(X([0,1]^N)) \)

Let \( Y(t) (t \in \mathbb{R}^N) \) be a real-valued, centered Gaussian random field with stationary increments and a continuous covariance function \( R(t,s) \) given by (1.1). We assume that \( Y(0)=0 \) and (1.6) holds. Let \( X(t) (t \in \mathbb{R}^N) \) be the \((N,d)\) Gaussian random field defined by (1.7). In this section, we prove that if \( N<\alpha d \), then

\[
\phi-m(X([0,1]^N)) > 0 \quad \text{a.s.}
\]

For simplicity we assume \( \delta_0=1 \) and let \( I=[0,1]^N \cap B(0,1) \) (otherwise we consider a smaller cube). For any \( 0<r<1 \) and \( y \in \mathbb{R}^d \). Let

\[
T_y(r) = \int_{I} 1_{B(y,r)}(X(t))dt
\]

be the sojourn time of \( X(t) (t \in I) \) in the open ball \( B(y,r) \). If \( y=0 \), we write \( T(r) \) for \( T_0(r) \).

**Proposition 4.1.** There exist \( \delta_2 > 0 \) and \( b>0 \) such that for any \( 0<r<\delta_2 \)

\[
E(\exp(b\psi(r)^{-N}T(r))) \leq K < \infty.
\]

Proof. We first prove that there exists a constant \( 0< K < \infty \) such that for any \( n \geq 1 \)

\[
E(T(r))^n \leq K^n n! \psi(r)^{Nn}.
\]

For \( n=1 \), by (2.4) and (2.5) we have

\[
ET(r) = \int_I P(X(t) \in B(0,r))dt
\]

\[
\leq \int_I \min\{1,K\frac{r}{\sigma(|t|)}\}dt
\]

\[
\leq K \int_0^1 \min\{1,K\frac{r}{\sigma^d}\} \rho^{N-1} d\rho
\]

\[
\leq K \int_0^{K\psi(r)} \rho^{N-1} d\rho + K \int_{K\psi(r)}^1 \frac{r^d \rho^{N-1}}{\sigma^d} d\rho
\]

\[
\leq K \psi(r)^N + Kr^d \psi(r)^{N-\alpha d} \int_1^\infty \frac{1}{t^{1+\alpha d-N}L(\psi(r)t^d)} dt
\]
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\[ \leq K\psi(r)^N + Kr^d\psi(r)^{N-d}/L(\psi(r))^d \]
\[ \leq K\psi(r)^N. \]

For \( n \geq 2 \)

\( (4.4) \quad E(T(r)^n) = \int_{\mathbb{R}^n} P(|X(t_1)| < r, \ldots, |X(t_n)| < r) dt_1 \cdots dt_n. \)

Consider \( t_1, \ldots, t_n \in I \) satisfying

\[ t_j \neq 0 \quad \text{for} \quad j = 1, \ldots, n, \quad t_j \neq t_k \quad \text{for} \quad j \neq k. \]

Let \( \eta = \min\{|t_i|, |t_i - t_j|, i = 1, \ldots, n - 1\} \). Then by (1.6) we have

\[ (4.5) \quad \text{Var}(X(t_0)|X(t_1), \ldots, X(t_{n-1})) \geq c_2\sigma^2(\eta). \]

Since conditional distributions in Gaussian processes are still Gaussian, it follows from (4.5) that

\[ (4.6) \quad P(|X(t_0)| < r|X(t_1) = \ldots = X(t_n) = x_{n-1}) \]
\[ \leq K \int_{|u| < \sigma(\eta)} \frac{1}{\sigma(\eta)^d} \exp\left(-\frac{|u|^2}{K\sigma^2(\eta)}\right) du. \]

Similar to (4.3), we have

\[ (4.7) \quad \int_I dt_n \int_{|u| < \sigma(\eta)} \frac{1}{\sigma(\eta)^d} \exp\left(-\frac{|u|^2}{K\sigma^2(\eta)}\right) du \]
\[ \leq \int_I \min\{1, K\left(\frac{r}{\sigma(\eta)}\right)^d\} dt_n \]
\[ \leq K \int_I \sum_{i=0}^{n-1} \min\{1, K\left(\frac{r}{\sigma(|t_i - t_0|)}\right)^d\} dt_n \quad (t_0 = 0) \]
\[ \leq Kn \int_0^1 \min\{1, \frac{Kr^d}{\rho \sigma(\rho)}\} \rho^{n-1} d\rho \]
\[ \leq Kn\psi(r)^N. \]

By (4.4), (4.6) and (4.7), we obtain

\[ E(T(r))^n \leq K \int_{\mathbb{R}^n} P(|X_1(t_1)| < r, \ldots, |X(t_{n-1})| < r) dt_1 \cdots dt_{n-1} \]
\[ \cdot \int_I dt_n \int_{|u| < \sigma(\eta)} \frac{1}{\sigma(\eta)^d} \exp\left(-\frac{|u|^2}{K\sigma^2(\eta)}\right) du. \]
Hence, the inequality (4.2) follows from (4.3) and induction. Let $0 < b < 1 / K$, then by (4.2) we have

$$E \exp(b \psi(r)^N T(r)) = \sum_{n=0}^{\infty} (Kb)^n < \infty.$$ 

This proves (4.1)

**Proposition 4.2.** With probability 1

(4.8) \[ \limsup_{r \to 0} \frac{T(r)}{\phi(r)} \leq \frac{1}{b}. \]

where $\phi(r) = \psi(r)^N \log \log 1 / r$.

Proof. For any $\varepsilon > 0$, it follows from (4.1) that

(4.9) \[ P\{T(r) \geq (1 / b + \varepsilon) \psi(r)^N \log \log 1 / r\} \leq \frac{K}{(\log 1 / r)^{1 + 2\varepsilon}}. \]

Take $r_n = \exp(-n / \log n)$, then by (4.9) we have

$$P\{T(r_n) \geq (1 / b + \varepsilon) \psi(r_n)^N \log \log 1 / r_n\} \leq \frac{K}{(n / \log n)^{1 + 2\varepsilon}}.$$ 

Hence by Borel-Cantelli lemma we have

(4.10) \[ \limsup_{n \to \infty} \frac{T(r_n)}{\phi(r_n)} \leq \frac{1}{b} + \varepsilon. \]

It is easy to verify that

(4.11) \[ \lim_{n \to \infty} \frac{\phi(r_n)}{\phi(r_{n+1})} = 1. \]

Hence by (4.10) and (4.11) we have

$$\limsup_{r \to 0} \frac{T(r)}{\phi(r)} \leq \frac{1}{b} + \varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrary, we obtain (4.8).

Since $X(t)$ ($t \in \mathbb{R}^N$) has stationary increments, we derive the following
Corollary 4.1. Fix $t_0 \in I$, then with probability 1
\[
\limsup_{r \to 0} \frac{T_{X(t_0)}(r)}{\phi(r)} \leq \frac{1}{b}.
\)

Theorem 4.1. If $N < ad$, then with probability 1
\[
\phi-\text{m}(X([0,1]^N)) > 0,
\]
where $\phi(r) = \psi(r)^N \log \log 1/r$.

Proof. We define a random Borel measure $\mu$ on $X(I)$ as follows. For any Borel set $B \subseteq \mathbb{R}^d$, let
\[
\mu(B) = L_N \{ t \in I, X(t) \in B \}.
\]
Then $\mu(\mathbb{R}^d) = \mu(X(I)) = L_N(I)$. By Corollary 4.1, for each fixed $t_0 \in I$, with probability 1
\[
\limsup_{r \to 0} \frac{\mu(B(X(t_0),r))}{\phi(r)} \leq \limsup_{r \to 0} \frac{T_{X(t_0)}(r)}{\phi(r)} \leq \frac{1}{b}.
\]
Let $E(\omega) = \{ X(t_0): t_0 \in I \text{ and } (4.13 \text{ holds}) \}$. Then $E(\omega) \subseteq X(I)$. A Fubini argument shows $\mu(E(\omega)) = 1$, a.s.. Hence by Lemma 2.1, we have
\[
\phi-\text{m}(E(\omega)) \geq Kb.
\]
This proves (4.12).

Proof of Theorem 1.1. It follows from Theorems 3.1 and 4.1 immediately.

Example 4.1. Let $Y(t)$ ($t \in \mathbb{R}^N$) be a real-valued fractional Brownian motion of index $\alpha$ ($0 < \alpha < 1$) (see [10], Chapter 18). Its covariance function has the representation
\[
R(s,t) = \frac{1}{2}(|s|^{2\alpha} + |t|^{2\alpha} - |t-s|^{2\alpha})
\]
\[
= c(\alpha) \int_{\mathbb{R}^N} (e^{i<s,\lambda>} - 1)(e^{-i<s,\lambda>} - 1) \frac{d\lambda}{||\lambda||^{N+2\alpha}},
\]
where $c(\alpha)$ is a normalizing constant. Then (1.5) is verified and by a result of Pitt [17], (1.6) is also verified. In this case, Theorem 1.1 is proved by Goldman [9]
for $\alpha = 1/2$ and by Talagrand [22] for $0 < \alpha < 1$.

**Example 4.2** Let $Z(t)$ ($t \in R^d$) be a real-valued mean zero stationary random field with covariance function

$$R(s,t) = \exp(-c|s-t|^{2\alpha}) \quad \text{with} \quad c > 0 \quad \text{and} \quad 0 < \alpha < 1.$$  

Then $Y(t) = Z(t) - Z(0)$ verifies the conditions (1.5) and (1.6). We can apply Theorem 1.1 to obtain the Hausdorff measure of $X([0,1]^N)$, where

$$X(t) = (X_1(t), \ldots, X_d(t))$$

and $X_1, \ldots, X_d$ are independent copies of $Z$. Other examples with absolutely continuous spectral measure can be found in Berman [2] p289, and Berman [4].

**Example 4.3.** Now we give an example with discrete spectral measure. Let $X_n$ ($n \geq 0$) and $Y_n$ ($n \geq 0$) be independent standard normal random variables and $a_n$ ($n \geq 0$) real numbers such that $\sum a_n^2 < \infty$. Then for each $t$, the random series

$$(4.14) \quad Z(t) = \sum_{n=0}^{\infty} a_n (X_n \cos nt + Y_n \sin nt)$$

converges with probability 1 (see [10]), and $Z(t)$ ($t \in R$) represents a stationary Gaussian process with mean 0 and covariance function

$$R(s,t) = \sum_{n=0}^{\infty} a_n^2 \cos n(t-s).$$

By a result of Berman [4], there are many choices of $a_n$ ($n \geq 0$) such that the process $Y(t) = Z(t) - Z(0)$ satisfies the hypotheses of Theorem 1.1 with

$$\sigma^2(s) = 2 \sum_{n=0}^{\infty} a_n^2 (1 - \cos ns).$$

Let $X(t)$ ($t \in R$) be the Gaussian process in $R^d$ associated with $Z(t)$ or $Y(t)$ ($t \in R$) by (1.7). If $l < ad$, then

$$0 < \phi-m(X([0,1]^N)) < \infty,$$

where $\phi(s) = \psi(s) \log \log \frac{1}{s}$ and $\psi$ is the inverse function of $\sigma$. A special case of (4.14) is Example 3.5 in Monrad and Rootzén [15].
References


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