Taniguchi, S.
Osaka J. Math.
22 (1985), 805–810

FINITE DIRECT SUM OF UNIFORM MODULES

SHUUHEI TANIGUCHI

(Received October 8, 1984)

In a paper of M. Harada [3], a right Artinian serial (resp. coserial) ring is characterized as a right QF-2 (resp. QF-2*) ring satisfying that the class of all finite direct sums of hollow (resp. uniform) modules is closed under submodules (resp. factor modules). In his another paper [1], a new class of right Artinian rings satisfying the above condition and that any hollow module is quasi-projective is determined as a generalization of right serial rings. The main purpose of this paper is to give a generalization of right coserial rings in dual manner.

In this paper, R denotes a right Artinian ring with identity element and every module is a unitary right R-module, unless otherwise stated. For a module M, we denote its socle and injective hull as Soc(M) and $E(M)$, respectively, and put $S_0(M)=0$ and $S_n(M)/S_{n-1}(M)=\mathrm{Soc}(M/S_{n-1}M))$, inductively. We denote a direct sum of k-copies of M as $M^{(k)}$.

Let U and V be uniform modules of finite length with $\mathrm{Soc}(U)\simeq\mathrm{Soc}(V)$, and set $S=\mathrm{Soc}(U)$ and $E=E(U)$, then we may assume that V is a submodule of E. We shall write Δ for $\mathrm{End}_R(S)$. We can obtain the mapping φ from $\mathrm{End}_R(E)$ to Δ by the restriction to S. Since E is injective, φ is an epimorphism. While we shall denote the image of the restriction mapping from $\mathrm{Hom}_R(U, V)$ to Δ as $\Delta(U, V)$ and $\Delta(U)$ instead of $\Delta(U, U)$. It is known that $\Delta(U)$ is a subdivision ring of Δ, so we shall denote the left dimension of Δ over $\Delta(U)$ as dim U, if it is finite.

A right coserial ring R satisfies the following conditions:

d–I: Every factor module of any direct sum of uniform modules of finite length is also a direct sum of uniform modules.

d-II: Every uniform module is quasi-injective.

Our purpose is to determine rings which satisfy the above both conditions, that is, we shall give the following theorem:

Theorem 1 [cf. 1: Theorem 2]. *For a right Artinian ring R, the following statements are equivalent*:

(1) *R satisfies the conditions* d-I *and* d-II.

(2) *R satisfies the condition* d-I *for direct sum of three uniform modules, and the condition* d-II.

(3) *For every indecomposable injective module E with $S=Soc(E)$, there are two uniserial modules A and B such that $E/S=A/S\oplus B/S$, and no factor of composition series of A/S is isomorphic to any one of B/S.*

The conditions d-I and d-II are inherited by the factor ring, namely:

Lemma 2 [cf. 1: Lemma 1]. *Let P be an ideal of R. If R satisfies the condition* d-I (*or* d-II), *then so does R/P.*

Proof. Let M and N be R/P-modules. M is R-uniform if and only if M is R/P-uniform, since any R-submodule of M is R/P-module. And if M is R-quasi-injective, then M is also R/P-quasi-injective, since $\mathrm{Hom}_R(N, M) = \mathrm{Hom}_{R/P}(N, M)$.

Regarding the condition d-I, we will consider the following conditions for a direct sum D of uniform modules of finite length:

d-(*): Every factor module, with respect to any simple submodule, of D is also a direct sum of uniform modules.

d-(**): Every simple submodule of D is contained in a non-trivial direct summand of D.

The conditions d-I and d-(*) are equivalent from next lemma:

Lemma 3. *A ring R satisfies the condition* d-I *if and only if every direct sum of uniform modules of finite length satisfies* d-(*).

Proof. To see the last of proof of Theorem 5 in [4].

Lemma 4 [4: Lemma 1]. *Let $\{U_i\}_{i=1}^{n+1}$ be a set of uniform modules of finite length. If $D'=\sum_{i=1}^{n}\oplus U_i$ satisfies* d-(**), *then $D=\sum_{i=1}^{n+1}\oplus U_i$ satisfies* d-(**).

Theorem 5 [cf. 2: Theorem 2]. *Let $\{U_i\}_{i=1}^{n}$ be a set of uniform modules such that $\mathrm{Soc}(U_i)\cong\mathrm{Soc}(U_1)$ and $|U_1|\leqq|U_2|\leqq\cdots\leqq|U_n|$. Then $D=\sum_{i=1}^{n}\oplus U_i$ satisfies* d-(**) *if and only if for any sequence $\delta_2, \delta_3, \cdots, \delta_n$ of $n-1$ elements in Δ, there are an integer t with $2\leqq t\leqq n$, and $\bar{y}_i\in\Delta(U_i, U_t)$ for $2\leqq i\leqq t$ such that $\sum_{i=2}^{t}\bar{y}_i\delta_i\in\Delta(U_1, U_t)$ and $\bar{y}_t\neq 0$.*

Proof. Let π_i: $D\rightarrow U_i$ and σ_i: $U_i\rightarrow D$ be the projection and the injection, respectively. Put $S=\mathrm{Soc}(U_1)$ and $E=E(U_1)$, then we can assume that all U_i are submodules of E. Assume that D satisfies d-(**). Let $\delta_1=1_S$ and $\delta_2, \delta_3, \cdots, \delta_n$ be elements in Δ, and $S^*=\{\sum_{i=1}^{n}\delta_i(s)\,|\,s\in S\}$ a simple submodule of D. Then there is a direct decomposition $D=D_1\oplus D_2$ such that $D_1\supset S^*$ and D_2 is uniform. Let p: $D\rightarrow D_2$ be the projection. Since $1_{D_2}=p|_{D_2}=\sum_{i=1}^{n}p\sigma_i\pi_i|_{D_2}$ and $\mathrm{End}_R(D_2)$ is a local ring, there is an integer j such that $p\sigma_j\pi_j|_{D_2}$ is a unit in $\mathrm{End}_R(D_2)$, hence $\pi_j|_{D_2}$ and $p\sigma_j$ are isomorphisms. Let $z_i\in\mathrm{End}_R(E)$ be an extension of $\pi_j p\sigma_i\in\mathrm{Hom}_R(U_i, U_j)$ and $\bar{z}_i=\varphi(z_i)$, for $1\leqq i\leqq n$. Then there is an integer t such that $\bar{z}_t \neq 0$ but $\bar{z}_k = 0$ for any k with $k > t$, and t is not less than j, for $\bar{z}_j \neq 0$. Put $y_i = z_i$, if $t = j$, otherwise $y_i = z_t^{-1} z_i$. Then we get that $\varphi(y_i) = \bar{y}_i \in \Delta(U_i, U_t)$ and $\sum_{i=1}^n \bar{y}_i \bar{\delta}_i = 0$, since $\sum_{i=1}^n z_i \bar{\delta}_i(s) = \pi_j p(\sum_{i=1}^n \delta_i(s)) = 0$ for any $s \in S$. Now we can get from $\bar{z}_k = 0$ that $\bar{y}_k = 0$ for any k with $k > t$. Therefore we get that $\sum_{i=2}^t \bar{y}_i \bar{\delta}_i = -\bar{y}_1 \in \Delta(U_1, U_t)$. Conversely, let S^* be a simple submodule of D, $\delta_1 = 1_E$ and δ_i elements in $\mathrm{End}_R(R)$ such that $S^* = \{\sum_{i=1}^n \delta_i(s) \mid s \in S\}$, where $\bar{\delta}_i = \varphi(\delta_i)$. By our assumption, there are an integer t and $\bar{y}_i \in \Delta(U_i, U_t)$ such that $\sum_{i=2}^t \bar{y}_i \bar{\delta}_i \in \Delta(U_1, U_t)$ and $\bar{y}_t \neq 0$. Let $y_i \in \mathrm{End}_R(E)$ such that $y_i(U_i) \subset U_t$ and $\bar{y}_i = \varphi(y_i)$ for $2 \leqq i \leqq t$, and $\varphi(y_1) = -\sum_{i=2}^t \bar{y}_i \bar{\delta}_i$. Let $D' = \{u - \theta(u) \mid u \in \sum_{i \neq t} \oplus U_i\} \subset D$, where $\theta\colon \sum_{i \neq t} \oplus U_i \to U_t$ is a homomorphism given by setting $\theta(\sum_{i \neq t} x_i) = \sum_{i=1}^{t-1} y_t^{-1} y_i(x_i)$. Then $D = D' \oplus U_t$ and $D' \supset S^*$, since for any $s^* = \sum_{i=1}^n \bar{\delta}_i(s) \in S^*$, set $u = \sum_{i \neq t} \bar{\delta}_i(s)$, then $\theta(u) = \sum_{i=1}^{t-1} y_t^{-1} y_i \bar{\delta}_i(s)$ $= y_t^{-1}(-y_t \bar{\delta}_t(s)) = -\bar{\delta}_t(s)$, therefore $s^* = u + \bar{\delta}_t(s) = u - \theta(u) \in D'$.

We can get all theorems and corollaries in Section 2 of [4] as a corollary of Theorem 5. Among there, the next corollaries are principal.

Corollary 6 [4: Corollary 1 of Theorem 3]. *Let U be a uniform module of finite length. Then $D = U^{(k+1)}$ satisfies* d-(**) *if and only if* dim $U \leqq k$.

Corollary 7 [4: Theorem 4 and its Corollary 2]. *Let $\{U_i\}_{i=1}^n$ be a set of uniform modules of finite length with* $\mathrm{Soc}(U_i) \cong \mathrm{Soc}(U_1)$ *and $k_i =$* dim U_i *for all i. Then $D = \sum_{i=1}^n \oplus U_i^{(k_i)}$ satisfies* d-(**) *if and only if there is a monomorphism from some U_i to another U_t.*

Proof. We may assume that $|U_1| \leqq |U_2| \leqq \cdots \leqq |U_n|$. We can take a set of linearly independent elements $\{\delta_{ij}\}_{j=1}^{k_i}$ in Δ over $\Delta(U_i)$. Applying Theorem for the set $\{\delta_{ij}\}_{i,j}$, there is a non-zero element $\bar{y}_{it} \in \Delta(U_i, U_t)$ for some i and t with $i < t$, which induces a monomorphism from U_i to U_t. Conversely if U_i is a submodule of U_t, then $U_i \oplus U_t^{(k_t)}$ satisfies d-(**), since $\Delta(U_t) \subset \Delta(U_i, U_t)$.

Let U be a quasi-injective uniform module, then it is clear that dim $U = 1$. Therefore assuming the condition d-II, Corollary above is gotten more simple. We shall use only such a case.

Lemma 8 [4: Lemma 3]. *Let $\{U_i\}_{i=1}^{n+1}$ be a set of uniform modules with $|U_i| = n$ for all i. If $D = \sum_{i=1}^{n+1} \oplus U_i$ satisfies* d-(*), *then D does* d-(**).

Proposition 9. *When any indecomposable injective module E has $S_2(E) = E$, Theorem* 1 *holds.*

Proof. (2) implies (3): Let E be an indecomposable injective module with $S = \mathrm{Soc}(E)$. Since the condition d-(**) holds for any direct sum of three submodules of E, which are of length two, by Lemma 8, there are at most two such submodules by Corollary 7 and the condition d-II. If $|E| \leqq 2$, E must be uniserial, hence the conclusion is clear. If $|E| \geqq 3$, we can write $E/S = U/S$

$\oplus V/S$ where U and V are only submodules of length two. Further if U/S is isomorphic to V/S via f, then a submodule W with $W/S=\{\bar{u}+f(\bar{u})\,|\,\bar{u}\in U/S\}\subset E/S$ is equal to either U or V. Therefore it must be $U/S\not\cong V/S$.
(3) implies (1): The condition d-II holds clearly. Let E be an indecomposable injective module with $S=\mathrm{Soc}(E)$ and $\{U_i\}_{i=1}^n$ a set of submodules of E. We shall show that $D=\sum_{i=1}^n\oplus U_i$ satisfies d-(*). Since this is satisfied clearly when E is uniserial, we may assume that $E/S=U/S\oplus V/S$ where U and V are of length two. If some U_i is either injective or simple, then D satisfies d-(**) by Corollary 7, hence the conclusion is true by induction on n. Further if $n\geqq 3$, D satisfies d-(**), since some U_i is equal to another U_j. Therefore it is enough to show the implication in a case of $D=U\oplus V$. Let $S^*=\{s+\bar{f}(s)\,|\,s\in S\}$ be a simple submodule of D where $\bar{f}=\varphi(f)$ with some monomorphism $f\in\mathrm{End}_R(E)$. If $f(U)\subset V$, D satisfies d-(**), since $D\oplus\!> \{u+f(u)\,|\,u\in U\}\supset S^*$. Otherwise, $E=f(U)+V$, and we can define an epimorphism ψ: $D\rightarrow E$, given by setting $\psi(u+v)=f(u)-v$. Then Ker $\psi=S^*$ and $D/S^*\cong E$.

In order to prove that (2) implies (3) of Theorem 1, it is enough to show the following lemmata 10, 11 and 12. Hence we shall assume that the statement (2) in Theorem 2 holds, in these lemmata.

Lemma 10. *Let U be a uniform module of finite length with $S=\mathrm{Soc}(U)$. Then there are two submodules B and C of U such that $U/S=B/S\oplus C/S$, $S_2(B)$ and $S_2(C)$ are uniserial and $S_2(B)/S\not\cong S_2(C)/S$, if $U\neq S$.*

Proof. Put $E=E(U)$ and $E'=S_2(E)$, then it is known that $E'=1_E(J^2)$ and E' is R/J^2-injective, where J denotes the Jacobson radical of R and $1_E(J^2)$ the left annihilator of J^2 on E, that is, $1_E(J^2)=\{e\in E\,|\,eJ^2=0\}$. Hence there are two submodules B' and C' such that $E'/S=B'/S\oplus C'/S$, $|B'|$, $|C'|\leqq 2$ and $B'/S\not\cong C'/S$, by Proposition 9. Now there are submodules $\{V_i\}_{i=1}^n$ such that $U/S=\sum_{i=1}^n\oplus V_i/S$ and V_i/S are uniform, by the condition d-I. But $n=2$ and $S_2(V_1)/S\not\cong S_2(V_2)/S$, since $B'/S\oplus C'/S=S_2(E)/S=\mathrm{Soc}(E/S)\oplus\!>\mathrm{Soc}(U/S)=\sum_{i=1}^n\oplus\mathrm{Soc}(V_i/S)=\sum_{i=1}^n\oplus S_2(V_i)/S$.

Lemma 11. *Let U be a module of finite length. If $S_2(U)$ is uniserial, then U is also uniserial.*

Proof. Assume that U is not uniserial. Then there is an integer $m(\geqq 2)$ such that $S_m(U)$ is uniserial but $S_{m+1}(U)$ is not uniserial. Since $U/S_{m+1}(U)$ is uniform, there are two submodules B and C of U such that $U/S_m(U)=B/S_m(U)\oplus C/S_m(U)$, $S_{m+1}(B)$ and $S_{m+1}(C)$ are uniserial and $S_{m+1}(B)/S_m(U)\not\cong S_{m+1}(C)/S_m(U)$. Put $A_1=S_{m+1}(B)/S_{m-2}(B)$ and $A_2=S_{m+1}(C)/S_{m-2}(C)$, then A_1 are uniserial modules of length three such that $S_2(A_1)=S_2(A_2)$ $(=A$, say) and $A_1/A\not\cong A_2/A$. Put $S=\mathrm{Soc}(A)$ and $D=A_1\oplus A_2$, and let $S^*=\{s+s\,|\,s\in S\}$ be a simple submodule of D. Then there are uniform modules $\{U_j\}_{j=1}^n$ such that D/S^*

$=\sum_{j=1}^{n}\oplus U_j$, by the condition d-II. Let ν_{ij}: $A_i \rightarrow U_j$ be composition mappings of the injection $A_i \rightarrow D$, the natural epimorphism ν: $D \rightarrow D/S^*$ and the projection $D/S^* \rightarrow U_j$. For Ker $\nu \supset \{s+0 \mid s \in S\}$, if no ν_{1j} is a monomorphism, we can assume that some ν_{1j} is a monomorphism, and that some ν_{2k} is also a monomorphism. If $j \neq k$, we get that $|D| > |D/S^*| \geqq |U_j \oplus U_k| \geqq |A_1 \oplus A_2| = |D|$, which is a contradiction. Hence we may assume that $j=k$ $(=1$, say). Since A_i are quasi-injective, $U_1=\nu_{11}(A_1)+\nu_{21}(A_2)=A_1+A_2=S_{m+1}(U)/S_{m-2}(U)$, hence taking length, $D/S^*=U_1 \oplus U_2$ and U_2 is simple. And $|\text{Ker } \nu_{i2}| \geqq |A_i| - |U_2|=2$, so Ker $(\nu_{12}+\nu_{22}) \supset A \oplus A$. But set $A^*=\text{Ker }(\nu_{11}+\nu_{21})$, then $|A^*|=2$ and $A^* \subset A \oplus A$. Hence $S^*=\text{Ker } \nu=\text{Ker }(\nu_{11}+\nu_{21}) \cap \text{Ker }(\nu_{12}+\nu_{22})=A^*$, which is a contradiction.

Let E be an indecomposable injective module with $S=\text{Soc}(E)$. Then E is of finite length, since $S_k(E)=1_E(J^k)$ and $J^n=0$ for any k and some n. Hence there are two uniserial submodules A and B such that $E/S=A/S \oplus \mathbf{C}/S$ and $S_2(A)/S \not\cong S_2(B)/S$, if $E \neq S$, from above lemmata. We show the remainder of (3) in Theorem 1.

Lemma 12. *Let E, S, A and B be as above, and set $A_i=S_i(A)$ and $B_j=S_j(B)$. Then $A_{i+1}/A_i \not\cong B_{j+1}/B_j$ for any pair i, j.*

Proof. We proceed by induction on $i+j$. The case of $i=j=1$ is done. Assume that $i+j>2$ and that A_{i+1}/A_i is isomorphic to B_{j+1}/B_j via f. Then A_{i+1}/A_i is not isomorphic to any factor of composition series of $A_i/S \oplus B_j/S$, by induction hypothesis. Put $K=A_i+B_j$, $C_0=A_{i+1}+B_j$ and $C_2=A_i+B_{j+1}$, and let C_1 be a submodule of E such that $C_1/K=\{\bar{c}+\psi_B^{-1} f \psi_A(\bar{c}) \mid \bar{c} \in C_0/K\}$ where ψ_A: $C_0/K \rightarrow A_{i+1}/A_i$ and ψ_B: $C_2/K \rightarrow B_{j+1}/B_j$ are the natural isomorphisms. Then C_1 is a hollow module with a maximal submodule K. Put $D=C_1 \oplus C_2$ and let $S^*=\{s+s \mid s \in S\}$ be a simple submodule of D. Then there are uniform modules $\{U_t\}_{t=1}^{n}$ such that $D/S^*=\sum_{t=1}^{n}\oplus U_t$. Let ν_{st}: $C_s \rightarrow U_t$ be composition mapping of the injection $C_s \rightarrow D$, the natural epimorphism ν: $D \rightarrow D/S^*$ and the projection $D/S^* \rightarrow U_t$. Then we can assume that ν_{11} and ν_{21} are monomorphisms and $U_1=C_1+C_2=A_{i+1}+B_{j+1}$, as in proof of Lemma 11. If $\nu_1=0$, where $\nu_s=\sum_{t=2}^{n}\nu_{st}$: $C_s \rightarrow \sum_{t=2}^{n}\oplus U_t$, then ν_2 is an epimorphism and $|\text{Ker } \nu_2|=|C_2|-|\sum_{t=2}^{n}\oplus U_t|=2$, which implies that Ker $\nu_2=A_2$ or B_2. Hence in case of Ker $\nu_2=A_2$, we get that Ker $\nu=\text{Ker }(\nu_{11}+\nu_{21}) \cap \text{Ker }(\nu_1+\nu_2) \supset \{\nu_{11}^{-1}\nu_{21}(a_2) - a_2 \mid a_2 \in A_2\}$, and similarly in case of Ker $\nu_2=B_2$, which are contradictions. Hence there is an integer t $(\geqq 2)$ such that $\nu_{1t} \neq 0$, and it holds that $\nu_{1t}(C_1)/\nu_{1t}(K) \cong C_1/K \cong A_{i+1}/A_i$, which means that A_{i+1}/A_i is isomorphic to some factor of composition series of U_t, and to some one of $A_i/S \oplus B_j/S$, by comparing the factors of composition series. Therefore we get a contradiction.

In order to show that (3) implies (1) of Theorem 1, we shall prepare one Lemma.

Lemma 13. *Let E be an indecomposable injective module with* $S=Soc(E)$. *Assume that A and B are uniserial modules such that* $E/S=A/S\oplus B/S$, *and set* $A_i=S_i(A)$ *and* $B_j=S_j(B)$. *Let* $U_k=A_{i_k}+B_{j_k}$; $1<i_1<i_2<\cdots<i_n$, $j_1>j_2>\cdots>j_n>1$, *then* $D=\sum_{k=1}^n\oplus U_k$ *satisfies* d-(*).

Proof. Let $S^*=\{\sum_{k=1}^n f_k f_{k-1}\cdots f_1(s)\,|\,s\in S\}$ be a simple submodule of D, where $f_k\in \mathrm{End}_R(E)$ are isomorphisms and $f_1=1_E$. Set $V_k=A_{i_{k+1}}\oplus B_{j_k}$ for any k with $1\leqq k\leqq n-1$ and $D'=(A_{i_1}/S)\oplus(\sum_{k=1}^{n-1}\oplus V_k)\oplus(B_{j_n}/S)$. Then we can define an epimorphism ψ: $D\rightarrow D'$ setting by $\psi(\sum_{k=1}^n(a_k+b_k))=(a_1+S)+\sum_{k=1}^{n-1}(f_{k+1}(a_k+b_k)-(a_{k+1}+b_{k+1}))+(b_n+S)$. Since if $\psi(\sum_{k=1}^n(a_k+b_k))=0$ for $a_k\in A_{i_k}$ and $b_k\in B_{j_k}$, then $a_1\in S$, $b_n\in S$ and $f_{k+1}(a_k+b_k)=a_{k+1}+b_{k+1}$ for any k, hence $f_{k+1}(a_k)-a_{k+1}=b_{k+1}-f_{k+1}(b_k)\in A_{i_{k+1}}\cap B_{j_k}=S$, and a_k and $b_k\in S$, inductively. Therefore $\sum_{k=1}^n(a_k+b_k)\in S^*$. Now for any $\sum_{k=1}^n f_k f_{k-1}\cdots f_1(s)\in S^*$, it holds that $\psi(\sum_{k=1}^n(f_k f_{k-1}\cdots f_1(s)+0))=0$. Therefore Ker $\psi=S^*$, and $D/S^*=D'$.

Proof (of (3) implies (1) of Theorem 1). The condition d-II holds clearly. Let $\{U_k\}_{k=1}^n$ be a set of uniform modules of finite length, we shall show that $D=\sum_{k=1}^n\oplus U_k$ satisfies d-(*). However if there is a monomorphism from some U_i to another U_j, then D satisfies d-(**) by Corollary 7, hence D satisfies d-(*) by induction on n. So we can assume that there is no U_i containing another U_j, namely that $\{U_k\}_{k=1}^n$ is as in Lemma 13. Therefore D satisfies d-(*).

References

[1] M. Harada: *On maximal submodules of a finite direct sum of hollow modules* III, Osaka J. Math. **22** (1985), 81–98.

[2] M. Harada and Y. Yukimoto: *On maximal submodules of a finite direct sum of hollow modules* IV, Osaka, J. Math. **22** (1985), 321–326.

[3] M. Harada: *Serial rings and direct decompositions,* J. Pure Appl. Algebra **31** (1984), 55–61.

[4] M. Harada: *Simple submodules in a finite direct sum of uniform modules,* (to appear)

Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka 558,
Japan