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APPROXIMATION OF JUMP PROCESSES ON FRACTALS

MICHAEL HINZ

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Abstract

The article studies approximations for stable like jump processes on fractal sets $F \subset \mathbb{R}^n$. Processes on d-sets are approximated by jump processes on the ε-parallel sets. For the special case of self-similar sets with equal contraction ratios, approximations in terms of finite Markov chains are provided. In either case, the convergence of Dirichlet forms, semigroups and resolvents are established as well as the convergence of the finite-dimensional distributions under canonical initial distributions. In the self-similar case also the weak convergence of the laws under these initial distributions in $D_F([0, t_0])$ is proved.

1. Introduction and setup

The question how to approximate a certain stochastic process is of particular interest for theory, physical models and numerical simulations. The present article considers approximations of jump processes on fractal sets. On d-sets and some generalizations such processes have been studied intensely, see e.g. [27], [21], [8] or [9]. On the other hand, fractional diffusions on self-similar sets and their generalizations have been considered by many authors, see e.g. [2] or [19] and the references therein. The idea to consider the energy forms of these processes as limits of discrete Dirichlet forms is well known to be a convenient way to construct them. See also [22]. There are recent results concerning the approximation of jump processes on $\mathbb{R}^n$, see [16], by methods similar to former works on continuous processes on $\mathbb{R}^n$, cf. [4], [28]. In [20] jump processes on increasing domains were considered, whereas in [15] processes on d-sets were approximated via processes on parallel sets decreasing to the d-set.

The aim of this work is to provide certain approximations for jump processes on d-sets as well as on self-similar sets. In the latter case we restrict ourselves to the case of equal contraction ratios. Our approach is somewhat different from the works mentioned above, except the last one, which made already use of such methods.

We describe the settings and main ideas. For subsets A of $\mathbb{R}^n$, we use the short notation $|A|$ to denote the n-dimensional Lebesgue measure, for finite words of form $w = (w_1, \dots, w_m)$, $|w|$ denotes their length m and for finite sets $\{x_1, \dots, x_m\}$ the cardinality $|\{x_1, \dots, x_m\}| = m$. For $x \in \mathbb{R}^n$, the Euclidean norm of x is also denoted by $|x|$. The respective meaning will be clear or pointed out. First consider the case of an

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arbitrary *d-set*, that is a compact subset $F \subset \mathbb{R}^n$ which carries a finite Radon measure μ such that $F = \operatorname{supp}\mu$ and

$$C_1 r^d \le \mu(B(x,r)) \le C_2 r^d \tag{1}$$

with $C_1, C_2 > 0$ holds for all $r < r_0$ and $x \in F$ with $0 < d \le n$. See e.g. [12], [18], or [29]. We will assume $\mu(\mathbb{R}^n) = 1$. We approximate processes on d-sets by processes whose state spaces are the ε-parallel sets of F, of course each of positive n-dimensional Lebesgue measure. A generalized type of Mosco convergence for the associated Dirichlet forms will be established, the convergence of the associated spectral structures in the sense of Kuwae and Shioya follows, see [23]. To do so, we make use of a special construction which is well adapted to our purposes. Some spatial averaging is combined with the mentioned concept of convergence, a related method has been described in [15].

Given a d-set $F \subset \mathbb{R}^n$, consider the closed ε-parallel sets

$$F_\varepsilon = \{x \in \mathbb{R}^n : \operatorname{dist}(x,F) \le \varepsilon\}, \tag{2}$$

where $\operatorname{dist}(x,F) = \inf_{y\in F}|x-y|$. Sometimes we will use $\varepsilon = 1$ to have F embedded into some compact set F_1. On F_ε we introduce probability measures μ_ε such that for each $\varepsilon > 0$, μ_ε is an n-measure on $\mathbb{R}^n$. In particular it is then equivalent to the restriction of n-dimensional Lebesgue measure to F_ε. These measures enjoy the following averaging identity: For a function $f \in L_1(F_\varepsilon)$ we have

$$\int f(x)\,\mu_\varepsilon(dx) = \int (f)_\varepsilon(x)\,\mu(dx), \tag{3}$$

where

$$(f)_\varepsilon(x) := \frac{1}{|B(x,2\varepsilon)\cap F_\varepsilon|}\int_{B(x,2\varepsilon)\cap F_\varepsilon} f(y)\,dy. \tag{4}$$

Here $|\cdot|$ denotes n-dimensional Lebesgue measure. For any function $f \in C(F_1)$, $\lim_{\varepsilon\to 0}(f)_\varepsilon(x)$ exists at all $x \in F$ and equals $f(x)$. In particular, the measures μ_ε converge weakly to μ on $\mathbb{R}^n$.

Consider the quadratic form given by

$$\mathcal{E}(u,u) = \iint_{(F\times F)\setminus D} (u(x)-u(y))^2 J(x,y)\,\mu(dx)\,\mu(dy), \quad u \in L_2(\mu), \tag{5}$$

where for $x, y \in F$,

$$J(x,y) = \frac{1}{|x-y|^\alpha \mu(B(x,|x-y|))}, \quad \alpha \in (0,2) \tag{6}$$

and $D = \{(x, x) \colon x \in F\}$ denotes the diagonal. Set

$$\mathcal{F} := \{u \in L_2(\mu) \colon \mathcal{E}(u, u) < \infty\}.$$

If we equip $\mathcal{F}$ with the usual norm given by

$$\left(\int_F u(x)^2 \, \mu(dx)\right)^{1/2} + \mathcal{E}(u, u)^{1/2},$$

$u \in \mathcal{F}$, then $\mathcal{F}$ coincides with $H^{\alpha/2}(F)$, which is the trace on F of the space $H^{\alpha/2+(n-d)/2}(\mathbb{R}^n)$ of Bessel potentials $f = G_{(\alpha+n-d)/2} * g$, $g \in L_2(\mathbb{R}^n)$, where $G_{(\alpha+n-d)/2}$ is the Bessel kernel of order $(\alpha + n - d)/2$ and $*$ denotes the convolution. More precisely, there are a bounded linear restriction operator $\mathcal{R} \colon H^{\alpha/2+(n-d)/2}(\mathbb{R}^n) \mapsto H^{\alpha/2}(F)$ and a bounded linear extension operator $E_0 \colon H^{\alpha/2}(F) \mapsto H^{\alpha/2+(n-d)/2}(\mathbb{R}^n)$ such that $\mathcal{R} \circ E_0$ is the identity mapping on $H^{\alpha/2}(F)$. For continuous functions, the restriction coincides with the pointwise restriction. See [18], Chapter V, Theorem 1 p. 103, Chapter VI, Theorem 1, p. 141 and Theorem 3, p. 155. Notice that $\Lambda^{2,2}_{\beta}(\mathbb{R}^n)$ in the notation there coincides with $H^{\beta}(\mathbb{R}^n)$ in the sense of equivalent norms. See also Theorem 1 p. 182 for the case $d < n$ and [17] and [29] for further methods. By this procedure and since the continuous functions with compact support are dense in $H^{\alpha/2+(n-d)/2}(\mathbb{R}^n)$, $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L_2(\mu)$. Further, define the approximating forms by

$$\mathcal{E}^{\varepsilon}(w, w) = \iint_{(F_\varepsilon \times F_\varepsilon)\setminus D} (w(x) - w(y))^2 J_\varepsilon(x, y) \, \mu_\varepsilon(dx) \, \mu_\varepsilon(dy), \quad w \in L_2(\mu_\varepsilon). \tag{7}$$

Here $D = \{(x, x) \colon x \in F_\varepsilon\}$, for brevity we use the same symbol. For $x, y \in F$,

$$J_\varepsilon(x, y) = \frac{1}{|x - y|^\alpha \mu_\varepsilon(B(x, |x - y|))}, \quad \alpha \in (0, 2). \tag{8}$$

Set $\mathcal{F}_\varepsilon = \{w \in L_2(\mu_\varepsilon)\colon \mathcal{E}^\varepsilon(w, w) < \infty\}$. Again $\mathcal{F} = H^{\alpha/2}(F_\varepsilon)$ and each $(\mathcal{E}^\varepsilon, \mathcal{F}_\varepsilon)$ is a regular Dirichlet form on $L_2(\mu_\varepsilon)$ as a consequence of the tracing procedure with respect to the n-set F_ε.

By the general theory, cf. [13], there exists a μ_ε-symmetric Hunt processes X^ε on each F_ε and a μ-symmetric Hunt process X on F, uniquely determined by $\mathcal{E}^\varepsilon$ resp. $\mathcal{E}$.

Now suppose X is given and the objective is to approximate it. We prove that the spectral structures of X^ε converge to those of X in the sense of [23] as ε tends to zero, as a consequence the finite dimensional distributions of X^ε with initial distributions μ_ε weakly converge to those of X with initial distribution μ.

The second situation we study is that of a self-similar set. We will introduce a familiy of discrete probability measures $\{\mu_m \colon m \in \mathbb{N}\}$ on F and then follow the same path as in the first case. This allows to approximate a given jump process X on F by finite Markov chains with the associated Dirichlet forms admitting simple discrete representations.

Let $F \subset \mathbb{R}^n$ be a the unique compact set satisfying

$$F = \Psi(F) = \bigcup_{i=1}^{N} \psi_i(F) \tag{9}$$

with contractive similarities ψ_i, $i = 1, \ldots, N$ all having the contraction ratio $r_1 = \cdots = r_N = s$. F carries the self-similar probability measure μ uniquely determined by

$$\mu = s^d \sum_{i=1}^{N} \mu \circ \psi_i^{-1}. \tag{10}$$

Under the open set condition μ does also satisfy the volume growth property (1) with d according to (10) and $d = -\log N/\log s$ is just the Hausdorff dimension of F. References on self-similar sets may be found e.g. in [12].

As will be shown below, the measures μ_m we introduce are just weighted sums of point mass measures supported on discrete sets V_m. For $f \in L_1(\mu_m)$, again the relation

$$\int f(x)\,\mu_m(dx) = \int (f)_m(x)\,\mu(dx) \tag{11}$$

holds, where $(f)_m$ is an average of f such that for any continuous function $f \in C(F)$,

$$\lim_{m\to\infty} (f)_m(x) = f(x), \quad x \in F. \tag{12}$$

In particular, the measures μ_m converge weakly to μ on F as m tends to infinity.

Again, consider the Dirichlet form given by (5) and $\mathcal{F}$ defined as above. Alternatively, we can use $j = |x-y|^{-\alpha-d}$ in place of J in (5). For $w \in L_2(\mu_m)$, set

$$\mathcal{E}^m(w, w) = \iint_{(F\times F)\setminus D} (w(x) - w(y))^2 J_m(x, y)\,\mu_m(dx)\,\mu_m(dy), \tag{13}$$

with

$$J_m(x, y) = \frac{1}{|x-y|^{\alpha}\mu_m(B(x, |x-y|))}, \quad x, y \in F. \tag{14}$$

If j was used instead of J for $\mathcal{E}$, use j in place of J_m. For $m \in \mathbb{N}$, $(\mathcal{E}^m, L_2(\mu_m))$ is a regular Dirichlet form. It is associated to a continuous time Markov chain Y^m with finite state space $V_m = \operatorname{supp} \mu_m$.

Again we prove the convergence of the spectral structures via generalized Mosco convergence, similarly as above the convergence of the finite dimensional distributions as above follows. Now we additionally obtain the weak convergence of the laws of the approximating Markov chains Y^m to the law of X in the Skorohod space $D_F([0, t_0])$ of right-continuous functions on $[0, t_0]$ with left limits and values in F, considered under initial distributions μ_m and μ, respectively.

Though of different type, the approximations in either setting follow by the same method. Therefore the article is organized as follows: First, in Section 2, we define both geometric settings, describe the concept of convergence we employ and state the main results for either case. In Section 3 we state simple properties of the measures μ_ε and μ_m. Rewriting the approximating Dirichlet forms $\mathcal{E}^m$, we obtain the conductivities for the approximating Markov chains. The following two sections then contain the proof of the generalized Mosco-convergence which implies the main Theorems. Section 4 establishes the pointwise convergence of the Dirichlet forms on the space of Hölder continuous functions. Then (3) resp. (11) allow to carry over some standard type arguments known for Mosco-convergence in the case of a single Hilbert space to our setting. This is done in Section 5. For the case of equal contraction ratios, Nash inequalities w.r.t. the μ_m are proved in the last section which lead to tightness bounds. We obtain the weak convergence of the laws of the processes in $D_F(0, t_0)$.

$B(z, r)$ denotes the open ball with center x and radius $r > 0$. For $A \subset \mathbb{R}^n$, $|A|$ denotes the n-dimensional Lebesgue measure of A, for a finite set B, $|B|$ stands for the number of elements. For a finite word $w = w_1 w_2 \cdots w_k$, $|w|$ is its length k, see below.

2. Definitions and main results

We define the notions of convergence we make use of. Then the two settings we investigate are described precisely and the main results are stated in either case.

The following definitions are formulated more generally situation to cover both cases. Let I be any directed index set and $\{\mu_i : i \in I\}$ a family of measures on some compact separable metric space (M, ϱ). Consider $L_2(\mu_i)$ normed by $\|\cdot\|_i$ and with scalar product $\langle \cdot, \cdot \rangle_i$. Let μ be another measure on M, and let $L_2(\mu)$, $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ be defined similarly.

Let $\mathcal{C}$ be a dense subspace in $L_2(\mu)$. Suppose for any $i \in I$ there is a bounded linear operator $\Phi_i : \mathcal{C} \to L_2(\mu_i)$ such that for any $u \in \mathcal{C}$ we have

$$\lim_i \|\Phi_i u\|_i = \|u\|.$$

Then the spaces $L_2(\mu_i)$ converge to $L_2(\mu)$ in the sense of Kuwae and Shioya, see [23]. This will be assumed throughout the following. In Section 3 below we will see that the families of measures $\{\mu_m : m \in \mathbb{N}\}$ and $\{\mu_\varepsilon : \varepsilon > 0\}$ on F resp. $F_1 \supset F$ satisfy these hypotheses.

We recall the notions of convergence of functions, operators and quadratic forms as introduced in [23].

DEFINITION 2.1. (i) A sequence of functions $\{u_i\}_{i\in I}$, $u_i \in L_2(\mu_i)$, *KS-strongly converges* to a function $u \in L_2(\mu)$ if there exists a sequence $\{\varphi_j\}_{j=1}^{\infty} \subset \mathcal{C}$ such that

$$\lim_{j\to\infty} \limsup_i \|\Phi_i\varphi_j - u_i\|_i = 0 \quad \text{and} \quad \lim_{j\to\infty} \|\varphi_j - u\| = 0.$$

(ii) A sequence of functions $\{u_i\}_{i\in I}$, $u_i \in L_2(\mu_i)$, *KS-weakly converges* to a function $u \in L_2(\mu)$ if for any sequence $\{v_i\}_{i\in I}$, $v_i \in L_2(\mu_i)$, that strongly converges to some $v \in L_2(\mu)$,

$$\lim_i \langle u_i, v_i\rangle_i = \langle u, v\rangle.$$

(iii) A sequence of bounded linear operators $\{B_i\}_{i\in I}$, $B_i\colon L_2(\mu_i) \to L_2(\mu_i)$ *KS-strongly converges* to a bounded linear operator $B\colon L_2(\mu) \to L_2(\mu)$ if for any sequence $\{u_i\}_{i\in I}$, $u_i \in L_2(\mu_i)$, that strongly converges to some $u \in L_2(\mu)$, the $B_i u_i$ KS-strongly converge to Bu.
(iv) A sequence $\{\mathcal{E}^i\}_{i\in I}$ of quadratic forms $\mathcal{E}^i\colon L_2(\mu_i) \times L_2(\mu_i) \to \mathbb{R} \cup \{-\infty, \infty\}$ *generalized Mosco-converges* to a quadratic form $\mathcal{E}\colon L_2(\mu) \times L_2(\mu) \to \mathbb{R} \cup \{-\infty, \infty\}$ if the following conditions are fulfilled:

(a) For any sequence $\{u_i\}_{i\in I}$ KS-weakly converging to $u \in L_2(\mu)$,

$$\liminf_i \mathcal{E}^i(u_i, u_i) \geq \mathcal{E}(u, u).$$

(b) For any $u \in L_2(\mu)$ there exists a sequence $\{u_i\}_{i\in I}$ KS-strongly converging to u such that

$$\limsup_i \mathcal{E}^i(u_i, u_i) \leq \mathcal{E}(u, u).$$

In (iv), the definition of a quadratic form $\mathcal{E}^i$ is extended to the whole of $L_2(\mu_i)$, setting $\mathcal{E}^i(u,u) = +\infty$ for a function u which is not in its domain. Similarly for a form $\mathcal{E}$ on $L_2(\mu)$, cf. [23] or [25].

Depending on the geometric settings we obtain similar results for different types of approximating processes.

The first situation is that of an arbitrary d-*set* $F \subset \mathbb{R}^n$, by definition there exists a normed Radon measure μ on $\mathbb{R}^n$ with $F = \operatorname{supp}\mu$ and such that (1) holds. Let the closed ε-parallel sets F_ε be defined by (2). Now consider the measures μ_ε on $\mathbb{R}^n$ given by

$$\mu_\varepsilon(A) = \int \frac{1}{|B(x, 2\varepsilon) \cap F_\varepsilon|} \int_{B(x,2\varepsilon)\cap F_\varepsilon} \mathbf{1}_A(y)\, dy\, \mu(dx), \quad A \subset \mathbb{R}^n. \tag{15}$$

Consider the spaces $L_2(\mu_\varepsilon)$ resp. $L_2(\mu)$ normed by $\|\cdot\|_\varepsilon$ resp. $\|\cdot\|$ and with scalar products $\langle\cdot,\cdot\rangle_\varepsilon$ resp. $\langle\cdot,\cdot\rangle$. Later it will be shown the μ_ε are normed n-measures with $\operatorname{supp}\mu_\varepsilon = F_\varepsilon$. Below we will prove that the $L_2(\mu_\varepsilon)$ converge to $L_2(\mu)$ in the sense of [23] and therefore the following statements make sense. Denote the operator semi-groups and resolvents associated to the processes X^ε and X by $(P_t^\varepsilon)_{t\geq 0}$ and $(G_\lambda^\varepsilon)_{\lambda>0}$ respectively $(P_t)_{t\geq 0}$ and $(G_\lambda)_{\lambda>0}$.

Theorem 2.1. (i) *The Dirichlet forms $\mathcal{E}^\varepsilon$ generalized Mosco-converge to $\mathcal{E}$ as ε tends to zero.*
(ii) *For any $\lambda > 0$, G_λ^ε KS-strongly converges to G_λ as ε tends to zero.*
(iii) *For any $t \geq 0$, P_t^ε KS-strongly converges to P_t as ε tends to zero.*

In particular, we observe

Corollary 2.1. *The finite dimensional distributions of the X^ε with initial distributions μ_ε weakly converge to those of X under initial distribution μ, i.e. for any $0 < t_1 < t_2 < \cdots < t_k < \infty$ and any $u \in C(F_1^{k+1})$ we have*

$$\lim_{\varepsilon\to 0} \mathbb{E}^{\mu_\varepsilon} u(X_0^\varepsilon, X_{t_1}^\varepsilon, \ldots, X_{t_k}^\varepsilon) = \mathbb{E}^{\mu} u(X_0, X_{t_1}, \ldots, X_{t_k}).$$

The second setting we study is that of a self-similar set.

DEFINITION 2.2. Let $F \subset \mathbb{R}^n$ be compact and $S = \{1, \ldots, N\}$. For any $i \in S$ let $\psi_i : F \to F$ be a contractive similarity and suppose there is a continuous injection $\pi : S^{\mathbb{N}} \to F$ such that $\psi_i \circ \pi = \pi \circ \sigma_i$, where $\sigma_i(w_1 w_2 \cdots) = i w_1 w_2 \cdots$. Then F is called a *self-similar set*. Let $W_m := S^m$ denote the words $w = (w_1, \ldots, w_m)$ of length $|w| = m$ and set $W_\infty = \bigcup_{m\geq 0} W_m$. For any finite word $w \in W_m$ of length m define $\psi_w := \psi_{w_1} \circ \cdots \circ \psi_{w_m}$. $F_w := \psi_w(F)$ is called an m-cell.

Then F satisfies (9), that is

$$F = \Psi(F) = \bigcup_{i=1}^{N} \psi_i(F).$$

Suppose the *open set condition* (*OSC*) holds, i.e. there exist $O \subset \mathbb{R}^n$ open sucht that $\Psi(O) \subset O$ and $\psi_i(O) \cap \psi_j(O) = \emptyset$ whenever $i \neq j$. Then the uniquely associated self-similar probability measure μ is equivalent to the d-dimensional Hausdorff measure $\mathcal{H}^d$ with d from (10) and satifies (1). We further assume that the *contraction ratios* r_i of the contractions ψ_i, $i = 1, \ldots, N$, *all equal* a given number $s \in (0, 1)$.

EXAMPLES 2.1. Among the sets fitting these assumptions are e.g. Cantor sets, Sierpinski gaskets, the Sierpinski carpet and the Koch curve. But also any closed interval in $\mathbb{R}$ and any closed cube or block in $\mathbb{R}^n$.

Let $V_0 = \{x_1, \dots, x_N\}$ denote the N distinct fixed points x_i of the contractions ψ_i. We assume the maximum distance between two points in V_0 is one. Let Ψ be given by (9) and $\Psi^m = \Psi \circ \Psi^{m-1}$. Set $V_m := \Psi^m(V_0)$. Further, for $x \in F$, let $F_m(x)$ denote the union of all 'step-m-copies' which contain x,

$$F_m(x) := \bigcup_{w \in W_m :\, x \in F_w} F_w. \tag{16}$$

Let $V_m(x) := F_m(x) \cap V_m$ and put

$$\mu_m(A) := \int \frac{1}{|V_m(x)|} \sum_{y \in V_m(x)} \delta_y(A)\, \mu(dx), \quad A \subset F, \tag{17}$$

where δ_y assigns mass one to y and is zero anywhere else. Here $|\cdot|$ denotes the cardinality of a finite set. By dominated convergence, the μ_m define normed measures on F. Obviously for $f \in L_1(\mu_m)$ the averaging relation (11) holds if we set

$$(f)_m(x) := \frac{1}{|V_m(x)|} \sum_{y \in V_m(x)} f(y). \tag{18}$$

For continuous functions $f \in C(F)$, (12) holds and therefore the measures μ_m converge weakly to μ on F as m tends to infinity. Now consider the spaces $L_2(\mu_m)$ resp. $L_2(\mu)$ with norms $\|\cdot\|_m$ resp. $\|\cdot\|$ and scalar products $\langle \cdot, \cdot \rangle_m$ resp. $\langle \cdot, \cdot \rangle$. The $L_2(\mu_m)$ also converge to $L_2(\mu)$ in the appropriate sense, see below.

Let $\mathcal{E}$ and $\mathcal{E}^m$ be given according to (5) and (13). Denote the operator semigroups and resolvents associated to the Dirichlet forms $\mathcal{E}^m$ and $\mathcal{E}$ by $(P_t^m)_{t \ge 0}$ and $(G_\lambda^m)_{\lambda > 0}$ respectively $(P_t)_{t \ge 0}$ and $(G_\lambda)_{\lambda > 0}$.

Theorem 2.2. (i) *The Dirichlet forms $\mathcal{E}^m$ generalized Mosco-converge to $\mathcal{E}$ as m tends to infinity.*
(ii) *For any $\lambda > 0$, G_λ^m KS-strongly converges to G_λ as m tends to infinity.*
(iii) *For any $t \ge 0$, P_t^m KS-strongly converges to P_t as m tends to infinity.*

Recall Y^m and X denote the Markov processes corresponding to $\mathcal{E}^m$ resp. $\mathcal{E}$.

Corollary 2.2. *The finite dimensional distributions of the Y^m with initial distributions μ_m weakly converge to those of X with initial distribution μ, that is for any $0 < t_1 < t_2 < \cdots < t_k < \infty$ and any $u \in C(F^{k+1})$ we have*

$$\lim_{m \to \infty} \mathbb{E}^{\mu_m} u(X_0^m, X_{t_1}^m, \dots, X_{t_k}^m) = \mathbb{E}^{\mu} u(X_0, X_{t_1}, \dots, X_{t_k}).$$

Employing Corollary 2.2, we can establish one more convergence result. For $t_0 > 0$, let $D_F([0, t_0])$ denote the space of F-valued right continuous functions on $[0, t_0]$ with left limits.

Theorem 2.3. *The the laws of the processes Y^m under $\mathbb{P}^{\mu_m}$ weakly converge to the law of X under $\mathbb{P}^{\mu}$ in $D_F([0, t_0])$.*

3. Approximation measures

This section investigates simple properties of the measures μ_ε resp. μ_m and shows the above notions of convergence are well defined in our settings.

Choose some $\gamma \in (\alpha, 2)$ and let $C^{\gamma/2}(F)$ denote the space of all $\gamma/2$-Hölder continuous functions on F, endowed with the norm

$$\|u\|_{\gamma/2} = \|u\|_\infty + \sup_{x,y\in F} \frac{|u(x) - u(y)|}{|x - y|^{\gamma/2}},$$

where $\|u\|_\infty = \sup_{x\in F}|u(x)|$. Similarly, $C^{\gamma/2}(F_\varepsilon)$ denotes the space of all bounded $\gamma/2$-Hölder continuous functions on F_ε, $\|\cdot\|_{\gamma/2,\varepsilon}$ defined as above but with the above suprema taken over the whole of F_ε. The space $C^{\gamma/2}(F)$ is dense in $L_2(\mu)$, this follows from $C^{\gamma/2}(\mathbb{R}^n)$ being dense in $H^{\alpha/2+(n-d)/2}(\mathbb{R}^n)$ and the mentioned tracing procedure, recall the detailed references given in the introduction. In particular, the restriction of a function from $C^{\gamma/2}(\mathbb{R}^n)$ to F resp. F_ε, then in the pointwise sense, is a function in $C^{\gamma/2}(F)$ resp. $C^{\gamma/2}(F_\varepsilon)$. Further, let E denote the Whitney extension operator associated to the set F, cf. [26] Chapter VI or [18], Chapter I.2, p. 21. E is a bounded linear operator from $C^{\gamma/2}(F)$ into $C^{\gamma/2}(\mathbb{R}^n)$. This follows from [26], Chapter VI, Section 2.2, the Proposition on p. 172 and Theorem 3 on p. 174. Since the measures μ_ε are normed, E is equibounded from $C^{\gamma/2}(F)$ into the spaces $L_2(\mu_\varepsilon)$ with operator norm bounded independently of ε. Now Recall (15).

Lemma 3.1. (i) *For any $\varepsilon > 0$, $\operatorname{supp} \mu_\varepsilon = F_\varepsilon$.*
(ii) *There is a constant $a_1 > 0$ such that for arbitrary $\varepsilon > 0$, $z \in F_\varepsilon$ and $0 < r < \varepsilon$ we have*

$$a_1\varepsilon^{d-n}r^n \le \mu_\varepsilon(B(z, r)).$$

(iii) *There is a constant $a_2 > 0$ such that for arbitrary $\varepsilon > 0$, $z \in F_\varepsilon$ and $r > 0$ we have*

$$\mu_\varepsilon(B(z, r)) \le a_2\varepsilon^{d-n}r^n.$$

(iv) *For any $u \in C^{\gamma/2}(F)$,*

$$\lim_{\varepsilon\to 0}\|Eu\|_\varepsilon = \|u\|.$$

By (i), (ii) and (iii), the μ_ε are n-measures. (iv) shows the spaces $L_2(\mu_\varepsilon)$ converge to $L_2(\mu)$ in the sense of Kuwae and Shioya.

Proof. (i) is obvious. (iv) directly follows from the weak convergence $\mu_\varepsilon \Rightarrow \mu$. For (iii), notice that by Fubini

$$\begin{aligned}\mu_\varepsilon(B(z,r)) &= \int \frac{1}{|B(x,2\varepsilon)\cap F_\varepsilon|}\int \mathbf{1}_{B(z,r)}(y)\mathbf{1}_{B(x,2\varepsilon)\cap F_\varepsilon}(y)\,dy\,\mu(dx)\\ &= \int \mathbf{1}_{B(z,r)\cap F_\varepsilon}(y)\int \frac{1}{|B(x,2\varepsilon)\cap F_\varepsilon|}\mathbf{1}_{B(y,2\varepsilon)}(x)\,\mu(dx)\,dy.\end{aligned}$$

For any $y \in B(z,r)\cap F_\varepsilon$ there is some $y' \in F$ such that $B(y,2\varepsilon) \subset B(y',3\varepsilon)$. Then the inner integral in the last line is bounded above by

$$\int_{F\cap B(y',3\varepsilon)} \frac{\mu(dx)}{|B(x,2\varepsilon)\cap F_\varepsilon|} \le \frac{\mu(B(y',3\varepsilon))}{|B(0,\varepsilon)|}$$

since for any $x \in F = \operatorname{supp}\mu$ we have $B(x,\varepsilon) \subset B(x,2\varepsilon)\cap F_\varepsilon$. Now (iii) follows.

Assertion (ii) holds, since $r < \varepsilon$ implies the existence of some $z' \in B(z,r)$ such that $B(z',r/2) \subset B(z,r)\cap F_\varepsilon$. For any $y \in B(z',r/2)$, there is some $y' \in F$ such that $B(y',\varepsilon) \subset B(y,2\varepsilon)$. We obtain

$$\int_{F\cap B(y,2\varepsilon)} \frac{\mu(dx)}{|B(x,2\varepsilon)\cap F_\varepsilon|} \ge \frac{\mu(B(y',\varepsilon))}{|B(0,2\varepsilon)|}$$

and

$$\mu_\varepsilon(B(z,r)) \ge \int \mathbf{1}_{B(z',r/2)}(y)\int \frac{1}{|B(x,2\varepsilon)|}\mathbf{1}_{B(y,2\varepsilon)}(x)\,\mu(dx)\,dy. \qquad \square$$

Turn to the self-similar case. Recall (17). For some $z \in V_m$ let $\omega_m(z)$ denote the total number of different words w of length m such that $z \in F_w$, i.e.

$$\omega_m(z) := \left|\{w_1w_2\cdots w_m : \text{there exist } w_{m+1}w_{m+2}\cdots \text{ with } \pi(w_1w_2\cdots) = z\}\right|. \tag{19}$$

Lemma 3.2. (i) *Let $m \in \mathbb{N}$. Then* $\operatorname{supp}\mu_m = V_m$ *and for any $A \subset F$,*

$$\mu_m(A) = \frac{1}{N^m|V_0|}\sum_{z\in V_m}\omega_m(z)\delta_z(A).$$

In particular for any $z \in V_m$, $\mu_m(\{z\}) = \omega_m(z)N^{-1}s^{md}$.
(ii) *There are constants $b_1, b_2 > 0$ such that for all $m \in \mathbb{N}$ and all $z \in V_m$, $r < r_0$,*

$$b_1(r^d + s^{md}) \le \mu_m(B(z,r)) \le b_2(r^d + s^{md}).$$

(iii) *The spaces $L_2(\mu_m)$ converge to $L_2(\mu)$ in the sense of Kuwae and Shioya.*

Proof. To see (i), let $A \subset F$ be such that $A \cap V_m = \emptyset$, then

$$\mu_m(A) = \int \frac{1}{|V_m(x)|} \sum_{y \in V_m(x)} \mathbf{1}_A(y)\, \mu(dx) = 0.$$

Let $z \in V_m$ be arbitrary, then

$$\begin{aligned}\mu_m(\{z\}) &= \int_F \frac{1}{|V_m(x)|} \sum_{y \in V_m(x)} \mathbf{1}_{\{z\}}(y)\, \mu(dx)\\ &= \int_{F \setminus E_m} \frac{1}{|V_m(x)|} \mathbf{1}_{V_m(x)}(z)\, \mu(dx),\end{aligned}$$

where

$$E_m = \bigcup_{w,v \in W_m} (F_w \cap F_v),$$

which by the (OSC) is of zero measure μ. For $x \in F \setminus E_m$, there is exactly one m-cell $F_{w_1 \cdots w_m}$ that contains x, i.e. $F_m(x) = F_{w_1 \cdots w_m}$. In particular $V_m(x) = \pi_{w_1 \cdots w_m}(V_0)$ and $|V_m(x)| = |V_0|$, recall $V_m(x) = F_m(x) \cap V_m$. This implies

$$\mu_m(\{z\}) = \frac{1}{|V_0|} \int_{F \setminus E_m} \mathbf{1}_{F_m(x)}(z)\, \mu(dx) = \frac{\omega_m(z)}{|V_0| N^m}, \tag{20}$$

showing the first assertion in (i). The second follows using $|V_0| = N$ and $s^d = N^{-1}$. For (ii), first notice we may assume $r_0 < s$. If $r \le s^m$, then $B(z,r) \cap V_m = \{z\}$. If $s^m \le r < s^{m-1}$, then $B(z,r)$ contains $F_m(z)$ and intersects at most $F_{m-1}(z)$ and all adjacent $(m-1)$-cells, there are no more than $\omega_{m-1}(z)N$. Since $\omega_m(z) \le N$ and an $(m-1)$-cell contains $|V_1| \le N|V_0|$ points of V_m, therefore

$$|V_0| \le |B(z,r) \cap V_m| \le N^3 |V_0|.$$

If $s^{m-1} \le r < s^{m-2}$, then $B(z,r)$ contains $F_{m-1}(z)$ and intersects $F_{m-2}(z)$ and maybe adjacent $(m-2)$-cells of which there are at most $\omega_{m-2}(z)N \le N^2$. Each $(m-2)$-cell contains $|V_2| \le N|V_1|$ points of V_m, hence

$$|V_1| \le |B(z,r) \cap V_m| \le N^3 |V_1|.$$

Continuing, we observe that for $k = 1, \dots, m-1$ and $s^{m-k+1} < r \le s^{m-k}$,

$$|V_{k-1}| \le |B(z,r) \cap V_m| \le N^3 |V_{k-1}|. \tag{21}$$

(ii) holds since $N = s^{-d}$ and for $s^{m-k+1} < r \leq s^{m-k}$ we have by (21) and (20),

$$\frac{a_1}{|V_0|} N^{k-m-1} \leq \frac{|B(x,r) \cap V_m|}{|V_0| N^m} \leq \mu_m(B(x,r) \cap V_m)$$
$$\leq N \frac{|B(x,r) \cap V_m|}{|V_0| N^m} \leq \frac{a_2}{|V_0|} N^{k-m+1}.$$

For (iii) recall again $C^{\gamma/2}(F)$ is dense in $L_2(\mu)$. Given $m \in \mathbb{N}$, let $\mathcal{R}_m \colon C^{\gamma/2}(F) \to L_2(\mu_m)$ denote the pointwise restriction $\mathcal{R}_m u := u|_{V_m}$, $u \in C^{\gamma/2}(F)$. Obviously each $\mathcal{R}_m$ is a bounded linear operator and the weak convergence implies

$$\lim_{m \to \infty} \|\mathcal{R}_m u\|_m = \|u\|$$

for any $u \in C^{\gamma/2}(F)$. $\square$

For $f \in L_1(\mu_m)$,

$$\int f(x)\, \mu_m(dx) = \sum_{z \in V_m} \mu_m(\{z\}) f(z). \tag{22}$$

Plugging this into (13), we observe

$$\mathcal{E}^m(u,u) = \sum_{w,z \in V_m} (u(w) - u(z))^2\, C^m_{w,z}, \tag{23}$$

where for $w, z \in V_m$

$$C^m_{w,z} = J_m(w,z)\mu_m(\{w\})\mu_m(\{z\}) = \frac{\omega_m(w)\omega_m(z)\mathbf{1}_{\{w \neq z\}} N^{-2} s^{2md}}{|w-z|^\alpha \mu_m(B(w, |w-z|))}. \tag{24}$$

Similarly in the case of j. Obviously $C^m_{w,z} = C^m_{w,z}$, $C^m_{z,w} \geq 0$ and $C^m_{w,z} > 0$ for $w \neq z$. Now let X^m denote the finite symmetric Markov chain on V_m defined by

$$\mathbb{P}(X_1^m = z \mid X_0^m = w) = \frac{C^m_{w,z}}{C^m_w} \quad \text{with} \quad C^m_w = \sum_{r \in V_m} C^m_{wr}.$$

Let $T_0 := 0$ and $T_k := \sum_{i=1}^k U_{X_i,i}$ where $U_{w,1}, U_{w,2}, \ldots$ are i.i.d. exponential random variables with parameter C^m_w for $w \in V_m$. The continuous time process Y^m constructed from X^m by letting $Y^m_t = X^m_n$ if $T_n \leq t < T_{n+1}$ then corresponds to the Dirichlet form $\mathcal{E}^m$. Y^m is right continuous and has left limits.

4. Pointwise convergence on the core

First consider the parallel set approximation. Recall that E denotes the Whitney extension operator associated to F.

Proposition 4.1. *For any* $u \in C^{\gamma/2}(F)$,

$$\lim_{\varepsilon \to 0} \mathcal{E}^{\varepsilon}(Eu, Eu) = \mathcal{E}(u, u).$$

Before heading into the proof, we collect some elementary prerequisites.

Lemma 4.1. *Let* $R > 0$. *Given* $\varepsilon_0 > 0$, *there is some* $\varepsilon' > 0$ *such that for all* $(x, r) \in F_1 \times [0, R]$ *with* $|(x, r) - (x', r')| < \varepsilon'$,

$$|\mu(B(x, r)) - \mu(B(x', r'))| < \varepsilon_0.$$

Proof. The function $(x, r) \mapsto \mu(B(x, r))$ is continuous at all$(x, r) \in F_1 \times [0, R]$ since from

$$|(x, r) - (x', r')| < \varepsilon'$$

follows that $B(x, r - 2\varepsilon') \subset B(x', r')$ as well as $B(x, r) \subset B(x', r' + 2\varepsilon')$ and therefore

$$\mu(B(x, r)) - \mu(B(x', r')) \leq \mu(B(x, r)) - \mu(B(x, r - 2\varepsilon'))$$

and

$$\mu(B(x', r')) - \mu(B(x, r)) \leq \mu(B(x, r + 2\varepsilon')) - \mu(B(x, r)),$$

which implies the continuity. $F_1 \times [0, R]$ being compact, we have uniform continuity and the assertion holds. $\square$

We use the short notation $b_\varepsilon(x) := B(x, 2\varepsilon) \cap F_\varepsilon$. Notice that for $(x, y) \in (F \times F) \setminus D$,

$$\big||x - y| - |w - z|\big| \leq |(x - y) - (w - z)| \leq |x - w| + |w - z| < 4\varepsilon \tag{25}$$

and in particular

$$\left|\frac{|w - z|}{|x - y|} - 1\right| < \frac{4\varepsilon}{|x - y|}, \tag{26}$$

whenever $w \in b_\varepsilon(x)$, $z \in b_\varepsilon(y)$.

Lemma 4.2. *For all* $\varepsilon > 0$, $x, y \in F$ *such that* $|x - y| > 16\varepsilon$ *and all* $w \in b_\varepsilon(x)$, $z \in b_\varepsilon(y)$,

$$\begin{aligned} &\big|\mu_\varepsilon(B(w, |w - z|)) - \mu(B(x, |x - y|))\big| \\ &\leq \mu(B(x, |x - y| + 8\varepsilon)) - \mu(B(x, |x - y| - 8\varepsilon)). \end{aligned}$$

Proof. We have

$$\left|\int \mathbf{1}_{B(w,|w-z|))}(\xi)\,\mu_\varepsilon(d\xi) - \int \mathbf{1}_{B(x,|x-y|))}(\xi)\,\mu(d\xi)\right|$$
$$\leq \int \frac{1}{|b_\varepsilon(\xi)|}\int_{b_\varepsilon(\xi)} |\mathbf{1}_{B(w,|w-z|))}(\eta) - \mathbf{1}_{B(x,|x-y|))}(\xi)|\,d\eta\,\mu(d\xi)$$

and

$$\begin{aligned}
&|\mathbf{1}_{B(\eta,|w-z|))}(w) - \mathbf{1}_{B(x,|x-y|))}(\xi)|\\
&= \mathbf{1}_{B(\eta,|w-z|))}(w)\mathbf{1}_{B(x,|x-y|))^c}(\xi) + \mathbf{1}_{B(\eta,|w-z|))^c}(w)\mathbf{1}_{B(x,|x-y|))}(\xi)\\
&\leq \mathbf{1}_{B(\xi,|x-y|+6\varepsilon))}(w)\mathbf{1}_{B(x,|x-y|))^c}(\xi) + \mathbf{1}_{B(\xi,|x-y|-6\varepsilon))^c}(w)\mathbf{1}_{B(x,|x-y|))}(\xi)\\
&\leq \mathbf{1}_{B(x,|x-y|+8\varepsilon))\setminus B(x,|x-y|))}(\xi) + \mathbf{1}_{B(x,|x-y|))\setminus B(x,|x-y|-8\varepsilon))}(\xi).
\end{aligned}$$

by (25). □

Lemma 4.3. *There is a constant $c_\mu > 0$ such that for any $\varepsilon > 0$ and $x, y \in F$ such that $|x-y| > 16\varepsilon$, we have*

$$\frac{\mu(B(x,\,|x-y|))}{\mu_\varepsilon(B(w,\,|w-z|))} \leq c_\mu,$$

whenever $w \in b_\varepsilon(x)$ and $z \in b_\varepsilon(y)$.

Proof.

$$\begin{aligned}
\int \mathbf{1}_{B(w,|w-z|))}(\xi)\,\mu_\varepsilon(d\xi) &= \int \frac{1}{|b_\varepsilon(\xi)|}\int_{b_\varepsilon(\xi)} \mathbf{1}_{B(\eta,|w-z|))}(w)\,d\eta\,\mu(d\xi)\\
&\geq \int \frac{1}{|b_\varepsilon(\xi)|}\int_{b_\varepsilon(\xi)} \mathbf{1}_{B(\xi,|w-z|-2\varepsilon))}(w)\,d\eta\,\mu(d\xi)\\
&= \int \mathbf{1}_{B(w,|w-z|-2\varepsilon))}(\xi)\,\mu(d\xi)\\
&\geq \mu(B(x,\,|w-z|-4\varepsilon))\\
&\geq \mu(B(x,\,|x-y|-8\varepsilon))\\
&\geq \mu(B(x,\,|x-y|/2)),
\end{aligned}$$

where we used (25) and $|x-y| - 8\varepsilon > |x-y| - |x-y|/2 = |x-y|/2$. Now the assertion follows since μ possesses a doubling property. □

We establish Proposition 4.1.

Proof. Throughout the following we will also write u to denote the Whitney extension Eu, recall $Eu \in C^{\gamma/2}(F_1)$. By the identity (3),

$$\begin{aligned}&\mathcal{E}^\varepsilon(u,u)\\ &= \iint \frac{1}{|b_\varepsilon(x)|\,|b_\varepsilon(y)|}\int_{b_\varepsilon(x)}\int_{b_\varepsilon(y)} \frac{(u(w)-u(z))^2}{|w-z|^\alpha \mu_\varepsilon(B(w,|w-z|))}\,dw\,dz\,\mu(dx)\,\mu(dy).\end{aligned}$$

STEP 1. We consider the part of $\mathcal{E}^\varepsilon(u,u)$ related to the relatively larger jumps,

$$\iint_{|x-y|>16\varepsilon} \Phi_\varepsilon(x,y)\frac{1}{\mu(B(x,|x-y|))}\,\mu(dx)\,\mu(dy),$$

where for each (x,y) with $|x-y|>16\varepsilon$,

$$\Phi_\varepsilon(x,y) = \frac{1}{|b_\varepsilon(x)|\,|b_\varepsilon(y)|}\int_{b_\varepsilon(x)}\int_{b_\varepsilon(y)} \frac{(u(w)-u(z))^2}{|w-z|^\alpha}\,\frac{\mu(B(x,|x-y|))}{\mu_\varepsilon(B(w,|w-z|))}\,dw\,dz.$$

Set

$$\Psi_\varepsilon(x,y) = \frac{1}{|b_\varepsilon(x)|\,|b_\varepsilon(y)|}\int_{b_\varepsilon(x)}\int_{b_\varepsilon(y)} \frac{(u(w)-u(z))^2}{|w-z|^\alpha}\,dw\,dz$$

and

$$\Phi(x,y) = \frac{(u(x)-u(y))^2}{|x-y|^\alpha}.$$

For any $(x,y)\in(F\times F)\setminus D$ then

$$\lim_{\varepsilon\to 0}\Psi_\varepsilon(x,y) = \Phi(x,y) \tag{27}$$

since $u\in C^{\gamma/2}(F_1)$. We will prove

$$\lim_{\varepsilon\to 0}\Phi_\varepsilon(x,y)\mathbf{1}_{\{|x-y|>16\varepsilon\}}(x,y) = \Phi(x,y). \tag{28}$$

Consider the left member in (28), for small ε it does not vanish and $|x-y|/2 > 4\varepsilon$. Clipping with (26),

$$\frac{|w-z|}{|x-y|} > \frac{|x-y|-4\varepsilon}{|x-y|} > \frac{1}{2}, \tag{29}$$

provided $w\in b_\varepsilon(x)$, $z\in b_\varepsilon(y)$.

Now fix $(x,y)\in(F\times F)\setminus D$ and let $\varepsilon_0>0$. By (25) and (29) we have

$$\left|\frac{1}{|x-y|^\alpha} - \frac{1}{|w-z|^\alpha}\right| \le \frac{2\varepsilon}{|x-y|^{2\alpha}}$$

whenever ε is small enough. By Lemmata 4.1, 4.2 and 4.3 then also

$$\left|\frac{\mu(B(x,|x-y|))}{\mu_\varepsilon(B(w,|w-z|))}-1\right|\le\frac{c_\mu\varepsilon_0}{\mu(B(x,|x-y|))}.$$

Consequently,

$$\left|\frac{1}{|b_\varepsilon(x)|\,|b_\varepsilon(y)|}\int_{b_\varepsilon(x)}\int_{b_\varepsilon(y)}\frac{(u(w)-u(z))^2}{|x-y|^\alpha}\,\frac{\mu(B(x,|x-y|))}{\mu_\varepsilon(B(w,|w-z|))}\,dw\,dz-\Psi_\varepsilon(x,y)\right|$$
$$\le\frac{c_\mu\varepsilon_0}{|x-y|^\alpha\mu(B(x,|x-y|))}F_\varepsilon(x,y),$$

where

$$F_\varepsilon(x,y)=\frac{1}{|b_\varepsilon(x)|\,|b_\varepsilon(y)|}\int_{b_\varepsilon(x)}\int_{b_\varepsilon(y)}(u(w)-u(z))^2\,dw\,dz\le 4\|u\|_\infty^2.$$

Similarly,

$$\left|\Phi_\varepsilon(x,y)-\frac{1}{|b_\varepsilon(x)|\,|b_\varepsilon(y)|}\int_{b_\varepsilon(x)}\int_{b_\varepsilon(y)}\frac{(u(w)-u(z))^2\,dw\,dz}{|x-y|^\alpha}\,\frac{\mu(B(x,|x-y|))}{\mu_\varepsilon(B(w,|w-z|))}\,dw\,dz\right|$$
$$\le\frac{2\varepsilon_0}{|x-y|^{2\alpha}}F_\varepsilon(x,y).$$

Combining,

$$\lim_{\varepsilon\to0}\Phi_\varepsilon(x,y)\mathbf{1}_{\{|x-y|>16\varepsilon\}}=\lim_{\varepsilon\to0}\Psi_\varepsilon(x,y)=\Phi(x,y)$$

for any $(x,y)\in(F\times F)\setminus D$.

STEP 2. By Lebesgue's theorem we have

$$\begin{aligned}&\lim_{\varepsilon\to0}\iint\Phi_\varepsilon(x,y)\mathbf{1}_{\{|x-y|>16\varepsilon\}}(x,y)\frac{1}{\mu(B(x,|x-y|))}\,\mu(dx)\,\mu(dy)\\ &=\iint\Phi(x,y)\frac{1}{\mu(B(x,|x-y|))}\,\mu(dx)\,\mu(dy)\\ &=\mathcal{E}(u,u),\end{aligned}\tag{30}$$

provided we can show that the $\Phi_\varepsilon(x,y)\mathbf{1}_{\{|x-y|>16\varepsilon\}}$ are dominated by an $(\mu(B(x,|x-y|)))^{-1}\,\mu(dx)\,\mu(dy)$-integrable function. To see this, note that for $|x-y|>16\varepsilon$,

$$\Psi_\varepsilon(x,y)\le\frac{\|u\|_{\gamma/2}^2}{|b_\varepsilon(x)|\,|b_\varepsilon(y)|}\int_{b_\varepsilon(x)}\int_{b_\varepsilon(y)}|w-z|^{\gamma-\alpha}\,dw\,dz\le c'\|u\|_{\gamma/2}^2|x-y|^{\gamma-\alpha}$$

since $u\in C^{\gamma/2}(F_1)$, and using Lemma 4.3,

$$\Phi_\varepsilon(x,y)\le c_\mu\Psi_\varepsilon(x,y)\le c\|u\|_{\gamma/2}^2|x-y|^{\gamma-\alpha},$$

$c > 0$ independent of ε. Now

$$\iint \frac{|x-y|^{\gamma-\alpha}}{\mu(B(x,|x-y|))}\,\mu(dx)\,\mu(dy) < \infty$$

justifies the above.

STEP 3. Turn to the small jump part of $\mathcal{E}^\varepsilon(u,u)$,

$$\begin{aligned}
&\iint_{|x-y|\le 16\varepsilon} \Phi_\varepsilon(x,y)\frac{1}{\mu(B(x,|x-y|))}\,\mu(dx)\,\mu(dy)\\
&\le \iint \frac{\mathbf{1}_{\{|w-z|\le 20\varepsilon\}}}{|b_\varepsilon(x)|\,|b_\varepsilon(y)|}\int_{b_\varepsilon(x)}\int_{b_\varepsilon(y)} \frac{(u(w)-u(z))^2}{|w-z|^\alpha \mu_\varepsilon(B(w,|w-z|))}\,dw\,dz\,\mu(dx)\,\mu(dy)\\
&= \iint_{|x-y|\le 20\varepsilon} \frac{(u(x)-u(y))^2}{|x-y|^\alpha \mu_\varepsilon(B(x,|x-y|))}\,\mu_\varepsilon(dx)\,\mu_\varepsilon(dy).
\end{aligned}$$

Apart from the factor $\|u\|_{\gamma/2}$, for arbitrary $x \in F$ the inner integral in the last line is bounded above by

$$\begin{aligned}
\int_0^{20\varepsilon} \frac{r^{\gamma-\alpha}}{\mu_\varepsilon(B(x,r))}\,\mu_\varepsilon(B(x,dr)) &= \sum_{i=j_0}^{\infty}\int_{2^{-i-1}}^{2^{-i}} \frac{r^{\gamma-\alpha}}{\mu_\varepsilon(B(x,r))}\mu_\varepsilon(B(x,dr))\\
&\le \sum_{i=j_0}^{\infty} 2^{-i(\gamma-\alpha)}\frac{\mu_\varepsilon(B(x,2^{-i}))}{\mu_\varepsilon(B(x,2^{-i-1}))}\\
&\le K\frac{2^{-j_0(\gamma-\alpha)}}{1-2^{-2(\gamma-\alpha)}}\\
&\le c\varepsilon^{\gamma-\alpha},
\end{aligned}$$

where $j_0 = j_0(\varepsilon)$ is the largest integer such that 2^{-j_0} is greater than 20ε. Note that by Lemma 3.1 there is a uniform doubling constant $K \ge 1$ such that

$$\mu_\varepsilon(B(x,r)) \le K\mu_\varepsilon(B(x,r/2))$$

for all $x \in F$ and $r \le 20\varepsilon$. Together with (30) the above shows the assertion. □

For the self-similar case the method simplifies since the properties of the measures μ_m are better, cf. Lemma 3.2.

Proposition 4.2. *For any $u \in C^{\gamma/2}(F)$,*

$$\lim_{m\to\infty} \mathcal{E}^m(u,u) = \mathcal{E}(u,u).$$

Notice that for $m \in \mathbb{N}$ and $x \in F$, $\operatorname{diam} F_m(x) \leq N s^m$. Notice that for $x, y \in F$ and $w \in V_m(x)$, $z \in V_m(y)$, similar to the other setting,

$$\big||w-z| - |x-y|\big| \leq 4Ns^m.$$

Lemma 4.4. *Let $\delta > 0$. For m large enough, all $x, y \in F$ such that $|x-y| > \delta$ and all $w \in V_m(x)$, $z \in V_m(y)$, we have*

$$\begin{aligned}
&\big|\mu_m(B(w, |w-z|)) - \mu(B(x, |x-y|))\big| \\
&\leq \mu(B(x, |x-y| + 4Ns^m)) - \mu(B(x, |x-y| - 4Ns^m)).
\end{aligned}$$

The proof is the same as for Lemma 4.2. We briefly sketch the proof of Proposition 4.2.

Proof. For $|x-y| > \delta$ and $w \in V_m(x)$, $z \in V_m(y)$,

$$|w-z| \geq |x-y| - 4Ns^m \geq \frac{\delta}{2}$$

for given $\varepsilon_0 > 0$ and large m then by Lemma 4.4,

$$\left|\frac{\mu(B(x, |x-y|))}{\mu_m(B(w, |w-z|))} - 1\right| \leq \frac{\varepsilon_0}{a_1|w-z|^d} \leq \frac{2^d \varepsilon_0}{\delta^d}.$$

We claim

$$\begin{aligned}
&\lim_{m\to\infty} \iint_{|x-y|>\delta} \Phi_m(x, y) \frac{1}{\mu(B(x, |x-y|))}\, \mu(dx)\, \mu(dy) \\
&= \iint_{|x-y|>\delta} \frac{(u(x)-u(y))^2}{|x-y|^\alpha \mu(B(x, |x-y|))}\, \mu(dx)\, \mu(dy),
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
&\Phi_m(x, y) \\
&= \frac{1}{|V_m(x)|\,|V_m(y)|} \sum_{w\in V_m(x)} \sum_{z \in V_m(y)} \frac{(u(w)-u(z))^2}{|x-y|^\alpha} \frac{\mu(B(x, |x-y|))}{\mu_m(B(w, |w-z|))}\, dw\, dz.
\end{aligned}$$

As before we conclude

$$\lim_m \Phi_m(x, y) = \frac{(u(x)-u(y))^2}{|x-y|^\alpha},$$

now for $(x, y) \in (F \times F) \setminus \{x, y \in F : |x-y| < \delta\}$. Since for such x, y,

$$\Phi_m(x, y) \leq (2^d \varepsilon_0 \delta^{-d} + 1)\|u\|_\infty^2,$$

Lebesgue's theorem yields (31). For the small part, note

$$\begin{aligned}&\iint_{|x-y|\le\delta} \Phi_m(x, y)\frac{1}{\mu(B(x, |x-y|))}\,\mu(dx)\,\mu(dy)\\&\le \iint_{|x-y|\le\delta+4Ns^m} \frac{(u(x)-u(y))^2}{|x-y|^\alpha\mu_m(B(x, |x-y|))}\,\mu(dx)\,\mu(dy)\end{aligned}$$

and

$$\begin{aligned}&\int_0^{\delta+4Ns^m} \frac{r^{\gamma-\alpha}}{\mu_m(B(x, r))}\,\mu_m(B(x, dr))\\&\le \sum_{i=j_m}^{\infty} 2^{-i(\gamma-\alpha)}\frac{\mu_m(B(x, 2^{-i}))}{\mu_m(B(x, 2^{-i-1}))} \le c(\delta+4Ns^m)^{\gamma-\alpha}\end{aligned}$$

by Lemma 3.2. j_m denotes the largest integer such that $2^{-j_m} \ge \delta + 4Ns^m$. A similar bound holds for the double integral w.r.t. μ, taken over $|x-y| \le \delta$. As $\delta > 0$ was arbitrary small, this completes the proof. □

For j in place of J and J_m the proof is similar but simpler.

5. Generalized Mosco-convergence

In order to prove the generalized Mosco-convergence of $\mathcal{E}^\varepsilon$ resp. $\mathcal{E}^m$ to $\mathcal{E}$ conditions (a) and (b) of Definition 2.1 (iv) will be verified. Basically we use a Banach-Saks type argument similar to [3].

We formulate the proof for the measures μ_ε, for the measures μ_m it is analogous. Choose some $\delta > 0$, let

$$\mathcal{E}^{\varepsilon,(\delta)}(u, u) = \iint_{(F_\varepsilon\times F_\varepsilon)\setminus D} (u(x)-u(y))^2 J_\varepsilon(x, y)\mathbf{1}_{\{|x-y|\ge\delta\}}\,\mu_\varepsilon(dy)\,\mu_\varepsilon(dx)$$

and

$$\mathcal{E}^{(\delta)}(u, u) = \iint_{(F\times F)\setminus D} (u(x)-u(y))^2 J(x, y)\mathbf{1}_{\{|x-y|\ge\delta\}}\,\mu(dy)\,\mu(dx).$$

Assume $\{u_\varepsilon\}$ KS-weakly converges to $u \in L_2(\mu)$. Without loss of generality, we may assume $\lim_{\varepsilon\to 0}\mathcal{E}^\varepsilon(u_\varepsilon, u_\varepsilon)$ exists and is finite.

Given $\varepsilon_0 > 0$, by Lemma 4.1, Lemma 4.2 and Lemma 4.3,

$$\begin{aligned}|J_\varepsilon(w, z) - J(x, y)| &\le \frac{1}{|x-y|^\alpha}\left|\frac{1}{\mu_\varepsilon(B(w, |w-z|))} - \frac{1}{\mu(B(x, |x-y|))}\right|\\&\le c_\mu\delta^\alpha\frac{\mu(B(x, |x-y|+8\varepsilon)) - \mu(B(x, |x-y|-8\varepsilon))}{\mu(B(x, \delta))}\\&< \varepsilon_0\end{aligned}$$

whenever ε is sufficiently small uniformly for all $x, y \in F$ such that $|x-y| \geq \delta$ and all $w \in b_\varepsilon(x)$, $z \in b_\varepsilon(y)$. Recall that by Lemma 4.1, $\mu(B(x,r))$ is uniformly continuous on the compact set $F_1 \times [\delta, 2\operatorname{diam} F]$. By (4) and Fubini therefore

$$\begin{aligned}
&\mathcal{E}^\varepsilon(u_\varepsilon, u_\varepsilon) \geq \mathcal{E}^{\varepsilon,(\delta)}(u_\varepsilon, u_\varepsilon)\\
&\geq \iint \frac{\mathbf{1}_{\{|x-y|\geq\delta\}}}{|b_\varepsilon(x)|\,|b_\varepsilon(y)|} \int_{b_\varepsilon(x)}\int_{b_\varepsilon(y)} (u_\varepsilon(w)-u_\varepsilon(z))^2 J_\varepsilon(w,z)\,dw\,dz\,\mu(dy)\,\mu(dx)\\
&\geq \iint \frac{\mathbf{1}_{\{|x-y|\geq\delta\}}}{|b_\varepsilon(x)|\,|b_\varepsilon(y)|} \int_{b_\varepsilon(x)}\int_{b_\varepsilon(y)} (u_\varepsilon(w)-u_\varepsilon(z))^2 (J(x,y)-\varepsilon_0)\,dw\,dz\,\mu(dy)\,\mu(dx)\\
&\geq \iint_{|x-y|\geq\delta} ((u_\varepsilon)_\varepsilon(x)-(u_\varepsilon)_\varepsilon(y))^2 J(x,y)\,\mu(dx)\,\mu(dy) - 2\varepsilon_0 \|(u_\varepsilon)_\varepsilon\|\\
&= \mathcal{E}^{(\delta)}((u_\varepsilon)_\varepsilon, (u_\varepsilon)_\varepsilon) - 2\varepsilon_0\|(u_\varepsilon)_\varepsilon\|
\end{aligned}$$

because

$$\begin{aligned}
(u_\varepsilon)_\varepsilon(x)-(u_\varepsilon)_\varepsilon(y) &= \frac{1}{|b_\varepsilon(x)|\,|b_\varepsilon(y)|}\int_{b_\varepsilon(x)}\int_{b_\varepsilon(y)} (u_\varepsilon(w)-u_\varepsilon(z))\,dw\,dz\\
&\leq \left(\frac{1}{|b_\varepsilon(x)|\,|b_\varepsilon(y)|}\int_{b_\varepsilon(x)}\int_{b_\varepsilon(y)} (u_\varepsilon(w)-u_\varepsilon(z))^2\,dw\,dz\right)^{1/2}
\end{aligned}$$

by Hölder's inequality.

Since $\{u_\varepsilon\}_\varepsilon$ KS-weakly converges, we have $\sup_\varepsilon \|u_\varepsilon\|_\varepsilon < \infty$, see [23], Lemma 2.3. By (3), $\|(u_\varepsilon)_\varepsilon\| \leq \|u_\varepsilon\|_\varepsilon$. By the above, $\{(u_\varepsilon)_\varepsilon\}_\varepsilon$ is therefore bounded in the Hilbert space formed by $L_2(\mu)$ with norm $(\mathcal{E}^{(\delta)}(\cdot,\cdot) + \|\cdot\|^2)^{1/2}$. Fix an arbitrary $\{\varepsilon_j\}_j$ with $\varepsilon_j \to 0$. By the Banach-Saks theorem there exists a subsequence $\{\varepsilon_{j_k}\}_{j_k}$, we write $u_k := u_{\varepsilon_{j_k}}$, $(u_k)_k := (u_{\varepsilon_{j_k}})_{\varepsilon_{j_k}}$, such that $(u_k)_k$ weakly converges to some v in $L_2(\mu)$,

$$\lim_{k\to\infty} \left\| \frac{1}{n}\sum_{k=1}^n (u_k)_k - v \right\| = 0$$

and

$$\lim_{k\to\infty} \mathcal{E}^{(\delta)}\left(\frac{1}{n}\sum_{k=1}^n (u_k)_k - v,\ \frac{1}{n}\sum_{k=1}^n (u_k)_k - v\right) = 0.$$

On the other hand, the KS-weak convergence implies that for $\varphi \in C(F)$

$$\begin{aligned}
\lim_{k\to\infty}\langle (u_k)_k, \varphi\rangle &= \lim_{k\to\infty}\int \frac{1}{|b_{\varepsilon_{j_k}}(x)|}\int_{b_{\varepsilon_{j_k}}(x)} u_k(y)\,dy\,\varphi(x)\,\mu(dx)\\
&= \lim_{k\to\infty}\int \frac{1}{|b_{\varepsilon_{j_k}}(x)|}\int_{b_{\varepsilon_{j_k}}(x)} u_k(y)E\varphi(y)\,dy\,\mu(dx)
\end{aligned}$$

$$= \lim_{k\to\infty} \langle u_k, E\varphi\rangle_{\varepsilon_{j_k}}$$
$$= \langle u, \varphi\rangle.$$

$C(F)$ being dense in $L_2(\mu)$, we see that $(u_k)_k$ weakly converges to u in $L_2(\mu)$ and therefore $v = u$. Clipping the statements, for any fixed sequence $\{\varepsilon_j\}_j$ with resulting subsequence as above,

$$\liminf_{j\to\infty} \mathcal{E}^{\varepsilon_j}(u_{\varepsilon_j}, u_{\varepsilon_j}) \geq \liminf_{j\to\infty} \mathcal{E}^{\varepsilon_j,(\delta)}(u_{\varepsilon_j}, u_{\varepsilon_j}) \geq \limsup_{j\to\infty} \mathcal{E}^{(\delta)}((u_{\varepsilon_j})_{\varepsilon_j}, (u_{\varepsilon_j})_{\varepsilon_j})$$
$$\geq \lim_{k\to\infty} \mathcal{E}^{(\delta)}\left(\frac{1}{n}\sum_{k=1}^{n}(u_k)_k, \frac{1}{n}\sum_{k=1}^{n}(u_k)_k\right) = \mathcal{E}^{(\delta)}(u, u),$$

notice

$$\mathcal{E}^{(\delta)}\left(\frac{1}{n}\sum_{k=1}^{n}(u_k)_k, \frac{1}{n}\sum_{k=1}^{n}(u_k)_k\right)^{1/2} \leq \frac{1}{n}\sum_{k=1}^{n}\mathcal{E}^{(\delta)}((u_k)_k, (u_k)_k)^{1/2}$$

by the triangle inequality. The above holds for any $\delta > 0$ and any chosen subsequence, therefore necessarily

$$\liminf_{\varepsilon\to 0} \mathcal{E}^{\varepsilon}(u_\varepsilon, u_\varepsilon) \geq \mathcal{E}(u, u),$$

which is condition (a).

To see condition (b), we make use of Proposition 4.1. For any $u \in \mathcal{F}$, there is a sequence $\{\varphi_j\}_{j=1}^{\infty} \subset C^{\gamma/2}(F)$ such that

(32) $$\lim_{j\to\infty} \mathcal{E}(\varphi_j, \varphi_j) = \mathcal{E}(u, u) \quad \text{and} \quad \lim_{j\to\infty} \|\varphi_j - u\| = 0.$$

For $j \in \mathbb{N}$, consider the Whitney extensions $E\varphi_j \subset C^{\gamma/2}(F_1)$ of φ_j, for brevity denote it again by φ_j. By μ_ε converging weakly to μ and by Proposition 4.1, there exists some $\varepsilon_1 > 0$ such that

$$\left|\|\varphi_1 - \varphi_0\|_\varepsilon - \|\varphi_1 - \varphi_0\|\right| < 2^{-1}$$

and

$$|\mathcal{E}^{\varepsilon}(\varphi_1, \varphi_1) - \mathcal{E}(\varphi_1, \varphi_1)| < 2^{-1}$$

for $\varepsilon < \varepsilon_1$. There also exists some $\varepsilon_2 < \varepsilon_1$ such that

$$\left| \|\varphi_2 - \varphi_0\|_\varepsilon - \|\varphi_2 - \varphi_0\| \right| < 2^{-2}$$

and

$$\left| \|\varphi_2 - \varphi_1\|_\varepsilon - \|\varphi_2 - \varphi_1\| \right| < 2^{-2}$$

and

$$|\mathcal{E}^\varepsilon(\varphi_2, \varphi_2) - \mathcal{E}(\varphi_2, \varphi_2)| < 2^{-2}$$

whenever $\varepsilon < \varepsilon_2$. Continuing this way, for any $j \in \mathbb{N}$ there is some ε_j such that

$$\left| \|\varphi_j - \varphi_i\|_\varepsilon - \|\varphi_j - \varphi_i\| \right| < 2^{-j} \quad \text{for} \quad i < j \quad \text{and} \quad |\mathcal{E}^\varepsilon(\varphi_j, \varphi_j) - \mathcal{E}(\varphi_j, \varphi_j)| < 2^{-j} \tag{33}$$

if $\varepsilon < \varepsilon_j$. For any $k \in \mathbb{N}$, there is some $i(k)$ such that $\|\varphi_j - \varphi_i\| < 2^{-k}$ if $i, j \geq i(k)$ due to (32). In particular

$$\|\varphi_j - \varphi_{i(k)}\| < 2^{-k} \quad \text{for} \quad j > i(k).$$

Clipping with (33),

$$\|\varphi_j - \varphi_{i(k)}\|_\varepsilon \leq \|\varphi_j - \varphi_{i(k)}\| + 2^{-j} \leq 2^{-k} + 2^{-j}$$

whenever $j > i(k)$ and $\varepsilon < \varepsilon_j$. Set $u_\varepsilon = 0$ for $\varepsilon \geq \varepsilon_1$, $u_\varepsilon = \varphi_1$ for $\varepsilon_2 \leq \varepsilon < \varepsilon_1$ and $u_\varepsilon = \varphi_j$ for $\varepsilon_{j+1} \leq \varepsilon < \varepsilon_j$. Put $\psi_k = \varphi_{i(k)}$, then for ε small enough

$$\|u_\varepsilon - \psi_k\|_\varepsilon \leq 2^{-j} + 2^{-k},$$

hence

$$\limsup_\varepsilon \|u_\varepsilon - \psi_k\|_\varepsilon \leq 2^{-k}.$$

Since $\lim_k \|u - \psi_k\| = 0$, the u_ε KS-strongly converge to u.

On the other hand, by (32) and (33),

$$\lim_\varepsilon \mathcal{E}^\varepsilon(u_\varepsilon, u_\varepsilon) = \lim_\varepsilon \mathcal{E}(u_\varepsilon, u_\varepsilon) = \mathcal{E}(u, u).$$

For $u \in L_2(\mu) \setminus \mathcal{F}$ we have $\mathcal{E}(u, u) = +\infty$, thus condition (b) is verified.

This proves assertion (i) in Theorem 2.1 and Theorem 2.2. Assertions (ii) and (iii) in either case directly follow from Theorem 2.4 in [23].

It remains to conclude the Corollaries 2.1 and 2.2. We prove Corollary 2.1, the other proof is similar.

Write $\|\cdot\|_1$ and $\|\cdot\|_{\varepsilon,1}$ to denote the norms in $L_1(\mu)$ and $L_1(\mu_\varepsilon)$. For functions u_ε KS-strongly converging to $u \in L_2(\mu)$, by (3), Definition 2.1 and Theorem 2.1,

$$\limsup_\varepsilon \|(P_t^\varepsilon u_\varepsilon)_\varepsilon - P_t u\| \leq \lim_j \limsup_\varepsilon \|P_t^\varepsilon u_\varepsilon - \varphi_j\|_\varepsilon + \lim_j \|P_t u - \varphi_j\|$$

and thus by Hölder,

$$\lim_\varepsilon \|(P_t^\varepsilon u_\varepsilon)_\varepsilon - P_t u\|_1 \leq \lim_\varepsilon \|(P_t^\varepsilon u_\varepsilon)_\varepsilon - P_t u\| = 0.$$

Again by (3) then

$$\lim_\varepsilon \|P_t^\varepsilon u_\varepsilon\|_{\varepsilon,1} = \|P_t u\|_1.$$

The product of a function $v \in C(F_1)$ and a sequence of functions u_ε KS-strongly converging to $u \in L_2(\mu)$ KS-strongly converges to uv, notice that if $\{\varphi_j\} \subset C(F)$ is the sequence according to Definition 2.1, we have

$$\|u_\varepsilon v - \varphi_j v\|_\varepsilon \leq \|v\|_\infty \|u_\varepsilon - \varphi_j\|_\varepsilon$$

and

$$\|\varphi v - uv\| \to 0,$$

$\varphi_j v \in C(F)$. Iterating these arguments for $u_0, u_1, \ldots, u_k \in C(F_1)$ then yields

$$\begin{aligned}
&\lim_\varepsilon \mathbb{E}^{\mu_\varepsilon}[u_0(X_0^\varepsilon)u_1(X_{t_1}^\varepsilon)\cdots u_k(X_{t_k}^\varepsilon)] \\
&= \lim_\varepsilon \int u_0(x) P_{t_1}^\varepsilon(u_1 P_{t_2-t_1}^\varepsilon(u_2 \cdots P_{t_{k-1}-t_{k-2}}^\varepsilon(u_{k-1} P_{t_k-t_{k-1}}^\varepsilon u_k)\cdots))(x)\,\mu_\varepsilon(dx) \\
&= \int u_0(x) P_{t_1}(u_1 P_{t_2-t_1}(u_2 \cdots P_{t_{k-1}-t_{k-2}}(u_{k-1} P_{t_k-t_{k-1}} u_k)\cdots))(x)\,\mu(dx) \\
&= \mathbb{E}\mu[u_0(X_0)u_1(X_{t_1})\cdots u_k(X_{t_k})].
\end{aligned}$$

By the Stone-Weierstrass theorem we may pass from functions $u(x_0, \ldots, x_k) = u_0(x_0)\cdots u_k(x_k)$, $u_i \in C(F_1)$ to general $u \in C(F_1^{k+1})$. This proves the Corollary.

6. Nash inequalites, tightness and convergence in *D*

Adapting an idea used in [10], we will now obtain Nash inequalities for the approximating Dirichlet forms. We will use these inequalities to deduce a tightness bound on the Y^m and then verify the convergence of the processes in $D_F([0, t_0])$. The arguments we use are similar to those in [5], [8] and [16].

Let m be arbitrary but fixed. For $0 < r < 1$ and a function u on V_m set

$$u_r(x) = \frac{1}{\mu_m(B(x,r))} \int_{B(x,r)} u(y)\,\mu_m(dy), \quad x \in V_m.$$

The following local Poincaré inequality holds.

Lemma 6.1. *There is some $c_0 > 0$ such that for any function $u \in L_2(\mu_m)$, we have*

$$\|u - u_r\|_m^2 \leq c_0 r^\alpha \mathcal{E}^m(u, u).$$

Proof. For any $x \in V_m$,

$$|u(x) - u_r(x)| \leq \frac{1}{\mu_m(B(x, r))} \int_{B(x,r)} |u(x) - u(y)|^2 \, \mu_m(dy).$$

Integrating,

$$\begin{aligned}\|u - u_r\|_m^2 &= \iint_{B(x,r)\cap D^c} (u(x) - u(y))^2 J(x, y) \frac{\mu_m(dy)\, \mu_m(dx)}{J(x, y)\mu_m(B(x, r))} \\ &\leq b_1^{-1} r^\alpha \mathcal{E}^m(u, u)\end{aligned}$$

since for $|x - y| < r$,

$$\frac{1}{J(x, y)\mu_m(B(x, r))} \leq b_1^{-1} r^\alpha. \qquad \square$$

A Nash inequality follows. Let $\| \cdot \|_{m,1}$ denote the norm in $L_1(\mu_m)$.

Proposition 6.1. *There is a constant $c > 0$ such that for all $m \in \mathbb{N}$ and all $u \in L_2(\mu_m)$,*

$$\|u\|_m^{2+2\alpha/d} \leq c(\mathcal{E}^m(u, u) + r_0^{-1} \|u\|_m^2) \|u\|_{m,1}^{2\alpha/d}.$$

Proof. Let $0 < r < r_0$. We have $\|u\|_m^2 = \langle u - u_r, u\rangle_m + \langle u_r, u\rangle_m$. By the previous lemma,

$$\langle u - u_r, u\rangle_m \leq \|u - u_r\|_m \, \|u\|_m \leq c_0^{1/2} r^{\alpha/2} \mathcal{E}^m(u, u) \|u\|_m$$

and by Lemma 3.2,

$$\langle u_r, u\rangle_m \leq \|u_r\|_\infty \, \|u\|_{m,1} \leq b_1^{-1}(r + s^m)^{-d} \|u\|_{m,1}^2.$$

Then for all $r > 0$,

$$\|u\|_m^2 \leq c_0^{1/2} r^{\alpha/2} (\mathcal{E}^m(u, u) + c_0^{-1} r_0^{-\alpha} \|u\|_m^2)^{1/2} \|u\|_m + b_1^{-1} r^{-d} \|u\|_{m,1}^2.$$

Minimizing the right hand side yields

$$\|u\|_m^2 \le c(\alpha, d)[\mathcal{E}^m(u, u) + c_0^{-1} r_0^{-\alpha} \|u\|_m^2]^{d/(\alpha+2d)} \|u\|_m^{2d/(\alpha+2d)} \|u\|_{m,1}^{2\alpha/(\alpha+2d)},$$

where

$$c(\alpha, d) = \left(\left(\frac{2d}{\alpha} \right)^{\alpha/(\alpha+2d)} + \left(\frac{\alpha}{2d} \right)^{2d/(\alpha+2d)} \right) c_0^{d/(\alpha+2d)} b_1^{-\alpha/(\alpha+2d)}.$$

Simplifying, the result follows. □

As an additional result, we observe that Proposition 6.1 together with our notions of convergence allows to obtain a Nash inequality for $\mathcal{E}$, usually proved by other means, as a limit of the Nash inequalities for the forms $\mathcal{E}^m$.

Corollary 6.1. *For any $u \in \mathcal{F}$,*

$$\|u\|^{2+2\alpha/d} \le c(\mathcal{E}(u, u) + c_0^{-1} r_0^{-1} \|u\|^2) \|u\|_1^{4\alpha/d},$$

where c is the constant from Proposition 6.1 *and $\|\cdot\|_1$ denotes the norm in $L_1(\mu)$.*

Proof. For $u \in C(F)$ the result follows from the weak convergence together with Proposition 4.1. For general $u \in \mathcal{F}$, it holds since by definition $C(F)$ is dense in $\mathcal{F}$ w.r.t. $\mathcal{E}_1$. □

Proposition 6.1 allows to proceed to a uniform tightness bound by standard arguments. For $m \in \mathbb{N}$ and $A \subset F$, let

$$\tau(A; Y^m) = \inf\{t \ge 0 \colon Y_t^m \notin A\}$$

denote the first exit time.

Proposition 6.2. *Given $A > 0$ and $B \in (0, 1)$, there exists a constant $\gamma > 0$ such that for all $m \in \mathbb{N}$ and all $0 < r \le \operatorname{diam} F$,*

$$\mathbb{P}^{\mu_m}(\tau(B(z, Ar); Y^m) < \gamma r^\alpha) \le B.$$

The proof of this proposition is shifted to the appendix.

Together with Corollary 2.2, Proposition 6.2 now leads to Theorem 2.3. For $x \in F$ and $x_m \in V_m$ converging to x, let Q_m denote the law of Y^m in $D_F([0, t_0])$ under $\mathbb{P}^{\mu_m}$ and Q the law of X under $\mathbb{P}^\mu$.

We make use of Aldous' Theorem on tightness in $D_F([0, t_0])$, cf. [1], [11] or [6].

Theorem 6.1. *Suppose $t_0 > 0$ and (Y^m) is a sequence of processes in $D_F([0, t_0])$. Assume that for all sequences $\{\tau_n\}$ of random variables with values in $[0, t_0]$ such that τ_n is a stopping time w.r.t. the filtration $\sigma(Y_s^m\colon s \leq t)$ and for all sequences $\delta_m \geq 0$ with $\lim_{m\to\infty} \delta_m = 0$,*

$$\left|Y^m_{\sigma_m+\delta_m} - Y^m_{\sigma_m}\right| \to 0 \quad \textit{in probability as} \quad m \to \infty.$$

Assume either (Y_0^m) and $\max_{t\in[0,t_0]}|Y_t^m - Y_{t-}^m|$ are tight or Y_t^m is tight for every $t \in [0, t_0]$. Then the laws of $(Y^m)_m$ are tight in $D_F([0, t_0])$.

We prove Theorem 2.3. Let $t_0 > 0$, $x \in F$, let $(\tau_m)_m$ be a sequence of $[0, t_0]$-valued stopping times and $(\delta_m)_m$ a sequence tending to zero. Given $\eta > 0$ and $B \in (0, 1)$, Proposition 6.2 provides a constant $\gamma = \gamma(\eta, B) > 0$ such that

$$\mathbb{P}^{\mu_m}(\tau(B(x_m, \eta)); Y^m) \leq \gamma) \leq B \tag{34}$$

for all m. Whenever m is large enough, so that $\delta_m \leq \gamma$, the strong Markov property together with (34) imply

$$\begin{aligned}\mathbb{P}^{\mu_m}\left(\varrho\left(Y^m_{\tau_m+\delta_m}, Y^m_{\tau_m}\right) > \eta\right) &= \mathbb{P}^{\mu_m}\left(\varrho\left(Y^m_{\delta_m}, Y^m_0\right) > \eta\right)\\ &\leq \mathbb{P}^{\mu_m}(\tau(B(x_m, \eta); Y^m) \leq \delta_m) \leq B.\end{aligned}$$

The tightness of the (Y_t^m) for any $t \in [0, t_0]$ follows since F is compact.

By Theorem 6.1 the sequence $(Q_m)_m$ is tight in $D_F([0, t_0])$. Together with the weak convergence of the finite-dimensional distributions, Corollary 2.2, this shows that the Q_m weakly converge to Q and therefore proves Theorem 2.3, see e.g. [6], Theorem 13.1 or [11], Theorem 7.8.

7. Appendix

We consider Proposition 6.2. Since the method has become quite standard meanwhile, so we only give a brief exposition. For further details we refer the reader to [5], [8], [14] or [16]. Fix $m \in \mathbb{N}$. For $\delta \in (0, D]$, D a number to be chosen later, introduce the measures

$$\mu_m^\delta(A) = \frac{\mu_m(\delta A)}{\delta^d}, \quad A \subset \delta^{-1}F$$

on $\delta^{-1}F = \{x \in X\colon \delta x \in F\}$. Then $\operatorname{supp}\mu_m^\delta = \delta^{-1}V_m$, analogously defined and for $z \in \delta^{-1}V_m$, $0 < r < \delta^{-1}r_0$,

$$b_1'(r^d + s^{md}\delta^{-d}) \leq \mu_m^\delta(B(x, r)) \leq b_2'(r^d + s^{md}\delta^{-d}),$$

recall Lemma 3.2. Consider the process $Y_t^{m,\delta} := \delta^{-1} Y^m_{\delta^\alpha t}$ with values in $\delta^{-1} V_m$ associated to the Dirichlet form

$$\mathcal{E}^{m,\delta}(u, u) = \delta^{\alpha-d}\mathcal{E}^m(f, f)$$

for $f(x) = u(\delta^{-1}x)$, $u \in L_2(\mu_m^\delta)$. For the corresponding small jump part Dirichlet form

$$\mathcal{C}^{m,\delta}(u, u) = \iint_{|x-y|\leq 1} \frac{((u(x) - u(y))^2}{|x-y|^\alpha \mu_m(B(x, |x-y|))}\, \mu_m^\delta(dx)\, \mu_m^\delta(dy)$$

we have

$$\begin{aligned} 0 &\leq \mathcal{E}^{m,\delta}(u, u) - \mathcal{C}^{m,\delta}(u, u) \\ &\leq 2\int u(x)^2 \left(\int_{|x-y|\leq 1 \vee \delta^{-1}s^m} \frac{\mu_m^\delta(dy)}{|x-y|^\alpha \mu_m(B(x, |x-y|))} \right) \mu_m^\delta(dx) \\ &\leq c\|u\|_{\delta,m}, \end{aligned}$$

with $c > 0$ independent of δ and m and $\|\cdot\|_{\delta,m}$ denoting the norm in $L_2(\mu_m^\delta)$. The Nash inequality proved in Proposition 6.1 can now be shifted over to a Nash inequality for $\mathcal{C}^{m,\delta}$, cf. [8]. It follows that the process $Z^{m,\delta}$ belonging to $\mathcal{C}^{m,\delta}$ possesses transition densities $p_m^\delta(t, x, y)$ which admit the bound

$$p_m^\delta(t, x, y) \leq c' t^{-d/\alpha} e^{-E(2t,x,y)+(\delta^\alpha + c)t}, \quad t > 0,\ x, y \in \delta^{-1}F$$

with constants c and c' independent of δ and m. For details, see [7], Theorems 2.1 and 3.25. In the above,

$$\begin{aligned} E(t, x, y) &= \sup\{|\psi(x) - \psi(y)| - \Lambda(\psi)^2 \colon \Lambda(\psi) < \infty\}, \\ \Lambda(\psi)^2 &= \max\{\|e^{-2\psi}\Gamma(e^\psi, e^\psi)\|_\infty, \|e^{2\psi}\Gamma(e^{-\psi}, e^{-\psi})\|_\infty\} \end{aligned}$$

and

$$\Gamma(e^\psi, e^\psi)(\xi) = \int_{|x-y|\leq 1 \vee s^m} \frac{(e^{\psi(\xi)} - e^{\psi(\eta)})^2}{|\xi-\eta|^\alpha \mu_m(B(\xi, |\xi-\eta|))}\, \mu_m^\delta(d\eta).$$

Using the cut-off function $\psi(\xi) = |\xi, x| \vee |x-y|$, we obtain $|\psi(\eta) - \psi(\xi)| \leq |\xi - \eta|$ and

$$\begin{aligned} e^{-2\psi(\xi)}\Gamma(e^\psi, e^\psi)(\xi) &= \int_{|x-y|\leq 1} \frac{(1 - e^{\psi(\eta)-\psi(\xi)})^2}{|\xi-\eta|^\alpha \mu_m(B(\xi, |\xi-\eta|))}\, \mu_m^\delta(d\eta) \\ &\leq \int_{|x-y|\leq 1} \frac{(\psi(\eta) - \psi(\xi))^2 e^{2|\psi(\eta)-\psi(\xi)|}}{|\xi-\eta|^\alpha \mu_m(B(\xi, |\xi-\eta|))}\, \mu_m^\delta(d\eta) \\ &\leq C \end{aligned}$$

with $C > 0$ independent of δ and m. Hence

$$p_m^\delta(t, x, y) \le c' t^{-d/\alpha} e^{-\varrho(x,y)+(\delta^\alpha + c)t}, \quad t > 0,\ x, y \in \delta^{-1} F.$$

For $t \in [1/2, 1]$, $\delta \in (0, D]$ and $\lambda > 0$ we obtain for any $x \in \delta^{-1} V_m$,

$$\mathbb{P}^x(|Z_t^{m,\delta} - Z_0^{m,\delta}| > \lambda) = \int_{|x-y|>\lambda} p_m^\delta(t, x, y)\, \mu_m^\delta(dy) \le c e^{-\lambda/2}, \tag{35}$$

$c > 0$ independent of δ and m. If now $\mathcal{L}^{m,\delta}$ and $\mathcal{A}^{m,\delta}$ denote the generators of $Y^{m,\delta}$ and $Z^{m,\delta}$ respectively, we have

$$\mathcal{L}^{m,\delta} = \mathcal{A}^{m,\delta} + \mathcal{B}^{m,\delta}$$

with

$$\mathcal{B}^{m,\delta} u(x) = \int_{\varrho(x,y)>1} \frac{(u(x) - u(y))^2}{|x - y|^\alpha \mu_m(B(x, |x - y|))}\, \mu_m^\delta(dy).$$

It is easily verified that there are positive constants c and c' independent of m such that for any $u \in L_2(\mu_m^\delta)$ and $v \in L_\infty(\delta^{-1} V_m)$,

$$\|\mathcal{B}^{m,\delta} u\|_{L_2(\mu_m^\delta)} \le c \|u\|_{L_2(\mu_m^\delta)}$$

and

$$\|\mathcal{B}^{m,\delta} v\|_\infty \le c' \|v\|_\infty.$$

If $(Q^{m,\delta})$ denotes the transition semigroup of $Z^{m,\delta}$,

$$S_0(t) := Q_t^{m,\delta}$$

and

$$S_k(t) := \int_0^t S_{k-1}(s) \mathcal{B}^{m,\delta} Q_{t-s}^{m,\delta}\, ds, \quad k \ge 1,$$

we obtain bounded linear operators $S_k(t)$ on $L_2(\mu_m^\delta)$ with operator norm bounded above by $(ct)^k/k!$, $c > 0$ independent of m. By [24], see also [11], $(P_t^{m,\delta})$ with $P_t^{m,\delta} = \sum_{k=0}^\infty S_k(t)$ then is the semigroup associated to $\mathcal{L}^{m,\delta}$. Similarly, each $S_k(t)$ is abounded linear operator on $L_\infty(\delta^{-1} V_m)$ with operator norm bounded above by $(c't)^k/k!$, $c' > 0$ independent of m. Then also $P_t^{m,\delta} = \sum_{k=0}^\infty S_k(t)$, with convergence in the operator norm on $L_2(\mu_m^\delta)$. In particular, for $v \in L(\delta^{-1}V_m)$,

$$\| P_t^{m,\delta} v - Q_t^{m,\delta} v\|_\infty \le \sum_{k=1}^{\infty} \frac{(c't)^k}{k!} \|v\|_\infty \le C' t e^{C't} \|v\|_\infty$$

and for any $x \in \delta^{-1}V_m$,

$$\mathbb{P}^x(|Y_t^{m,\delta} - x| > \lambda) \le \mathbb{P}^x(|Z_t^{m,\delta} - x| > \lambda) + C'te^{C't} \le ce^{-\lambda/4} + ct$$

for $t \in [1/2, 1]$ with with $c > 0$ independent of m and δ by (35). Introducing the exit times $\sigma(\lambda) = \inf\{t \ge 0\colon |Y_t^{m,\delta} - Y_0^{m,\delta}| > \lambda\}$ and using the strong Markov property we have for all $t \le 1/2$ and $x \in \delta^{-1}V_m$

$$\begin{aligned}
&\mathbb{P}^x\left(\sup_{s\le t}|Y_s^{m,\delta} - Y_0^{m,\delta}| > \lambda\right)\\
&= \mathbb{P}^x\left(\sigma(\lambda) < t; |Y_1^{m,\delta} - Y_0^{m,\delta}| > \frac{\lambda}{2}\right) + \mathbb{P}^x\left(\sigma(\lambda) < t; |Y_1^{m,\delta} - Y_0^{m,\delta}| \le \frac{\lambda}{2}\right)\\
&\le \mathbb{P}^x\left(|Y_1^{m,\delta} - Y_0^{m,\delta}| > \frac{\lambda}{2}\right) + \mathbb{P}^x\left(\sigma(\lambda) < t; |Y_1^{m,\delta} - Y_{\sigma(\lambda)}^{m,\delta}| \ge \frac{\lambda}{2}\right)\\
&\le ce^{-\lambda/4} + ct + \mathbb{E}^x\left[\sigma(\lambda) < t; \mathbb{P}^{Y_{\sigma(\lambda)}^{m,\delta}}\left(|Y_{1-\sigma(\lambda)}^{m,\delta} - Y_0^{m,\delta}| > \frac{\lambda}{2}\right)\right]\\
&\le ce^{-\lambda/4} + ct + \mathbb{E}^x\left[\sigma(\lambda) < t; \sup_{u\le 1/2} \mathbb{P}^{Y_{\sigma(\lambda)}^{m,\delta}}\left(|Y_{1-u}^{m,\delta} - Y_0^{m,\delta}| > \frac{\lambda}{2}\right)\right]\\
&\le ce^{-\lambda/4} + ct + \max_{y\in\delta^{-1}V_m} \sup_{u\le 1/2} \mathbb{P}^y\left(|Y_{1-u}^{m,\delta} - Y_0^{m,\delta}| > \frac{\lambda}{2}\right)
\end{aligned}$$

which is bounded by $ce^{-\lambda/4} + ct$ with $c > 0$ independent of m and δ. Integrating,

$$\int_{\delta^{-1}V_m} \mathbb{P}^z\left(\sup_{s\le t}|Y_s^{m,\delta} - Y_0^{m,\delta}| > \lambda\right) \mu_m^\delta(dz) \le \delta^{-d}(ce^{-\lambda/4} + ct)$$

for $t \in [0, 1]$. Scaling back,

$$\begin{aligned}
&\mathbb{P}^{\mu_m}\left(\sup_{s\le\delta^\alpha t}|Y_s^m - Y_0^m| > \delta\lambda\right)\\
&= \int_{V_m} \mathbb{P}^x\left(\sup_{s\le\delta^\alpha t}|Y_s^m - Y_0^m| > \delta\lambda\right) \mu_m(dx)\\
&= \delta^d \int_{\delta^{-1}V_m} \mathbb{P}^z\left(\sup_{s\le t}|Y_s^{m,\delta} - Y_0^{m,\delta}| > \lambda\right) \mu_m^\delta(dz)\\
&\le ce^{-\lambda/4} + ct.
\end{aligned} \tag{36}$$

Choose $t_0 \le 1/2$ and λ large enough that the right hand side is smaller than B and $A/\lambda < \operatorname{diam} F$. Then put $\delta := Ar/\lambda$ which is in $(0, D]$ for $D = (\operatorname{diam} F)^2$. Finally, set $\gamma = A^\alpha t_0/\lambda^\alpha$, what yields the assertion.

REMARK 7.1. Establishing a parabolic Harnack inequality and proving the equicontinuity of the $p_m(t, x, y)$ one can also deduce the convergence in $D_F([0, t_0])$ with arbitrary starting distributions. This follows along the lines of [16].

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University of Jena
Mathematical Institute
Ernst-Abbe-Platz 2
07737 Jena
Germany
e-mail: mhinz@minet.uni-jena.de