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ON THE FIRST MAIN THEOREM OF HOLOMORPHIC MAPPINGS FROM \mathbf{C}^2 INTO $Q_{n-1}(\mathbf{C})$

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0. Introduction

Let f be a holomorphic mapping of a complex line \mathbf{C} into a complex projective space $P_n(\mathbf{C})$ and suppose that the image $f(\mathbf{C})$ is not contained in any hyperplane of $P_n(\mathbf{C})$. Put $V[t]=\{z \in \mathbf{C}: \log|z| < t\}$, and for a hyperplane ξ in $P_n(\mathbf{C})$ let $n(t, \xi)$ be the number of points in $V[t] \cap f^{-1}(\xi)$. Let Ω be the colsed form of degree 2 associated with the Fubini-Study metric on $P_n(\mathbf{C})$ and normalized as $\int_{P_n} \Omega^n = 1$. The counting function $N(r, \xi)$ and the order function $T(r)$ being defined by

$$(0.1) \quad N(r, \xi) = \int_0^r n(t, \xi) dt,$$

$$(0.2) \quad T(r) = \int_0^r dt \int_{V[t]} f^* \Omega$$

respectively, the following equation is known as the First Main Theorem:

$$(0.3) \quad N(r, \xi) + (m(r, \xi) - m(0, \xi)) = T(r),$$

where $m(r, \xi)$ is a non-negative function defined for $r \in \mathbf{R}^+$ and hyperplanes ξ in $P_n(\mathbf{C})$. The term $(m(r, \xi) - m(0, \xi))$ is called the compensating term. It follows from the equation (0.3) that the image $f(\mathbf{C})$ intersects with almost all hyperplanes in $P_n(\mathbf{C})$. Furthermore it is known that the number of hyperplanes in general position not intersecting with $f(\mathbf{C})$ is at most $n+1$. These results are originally due to Ahlfors, and treated also by H. Wu [6] and S. S. Chern [1] in a modernized form.

Let f be a holomorphic mapping of \mathbf{C}^2 into a complex quadratic $Q_{n-1}(\mathbf{C})$ ($n \geq 3$) satisfying certain non-degenerate conditions [§2]. We consider $Q_{n-1}(\mathbf{C})$ as a fixed hypersurface in $P_n(\mathbf{C})$. We consider a special family of $(n-2)$ -dimensional projective spaces $P_{n-2}(\mathbf{C})$ in $P_n(\mathbf{C})$ parametrized by a Grassmann manifold $G(\mathbf{R})$ of 2-dimensional linear spaces in \mathbf{R}^{n+1} [§1]. This family determines a family of $(n-3)$ -dimensional complex quadratic $\xi_\alpha (\alpha \in G(\mathbf{R}))$ in $Q_{n-1}(\mathbf{C})$, each of whose elements is a Poincaré dual of the form Ω^2 in $Q_{n-1}(\mathbf{C})$.

In this paper, we shall consider a value distribution problem in two complex variables with respect to the holomorphic mapping f and the family $\{\xi_\alpha\}$. The complex quadratic $Q_{n-1}(\mathbf{C})$ being a double covering space of $G(\mathbf{R})$, we may take $Q_{n-1}(\mathbf{C})$ as a parametrizing space of the family $\{\xi_\alpha\}$ in place of $G(\mathbf{R})$. Thus we have a setting similar to the case of holomorphic curves (holomorphic mappings of \mathbf{C} into $P_n(\mathbf{C})$). Furthermore Ω is an invariant form on $Q_{n-1}(\mathbf{C})$ by a certain transformation group [§5]. This fact also plays an important role as in the case of holomorphic curves [§6].

Our main results are as follows: (1) First Main Theorem [§4], (2) the Crofton formula [§6] and (3) the Distribution theorem [§7]. In more detail, put

$$\Delta(r) = \{(z_1, z_2) \in \mathbf{C}^2 : \log|z_i| < r (i = 1, 2)\}$$

and define

$$n(\Delta(r), \alpha) = \sum_{p_i \in \Delta(r), f(p_i) \in \xi_\alpha} n(p_i, \alpha),$$

where $n(p_i, \alpha)$ is a certain real number [§3] such that $n(p_i, \alpha) = 1$ if $f(\mathbf{C}^2)$ intersects transversely with ξ_α at $f(p_i)$. We also define the following functions:

$$(0.4) \quad N(r, \alpha) = \int_0^r n(\Delta(t), \alpha) dt \text{ (counting function)}$$

$$(0.5) \quad T(r) = \int_0^r dt \int_{\Delta(t)} f^* \Omega^2 \text{ (order function).}$$

Then our First Main Theorem states:

$$(0.6) \quad N(r, \alpha) + m(r, \alpha) - m(0, \alpha) = T(r),$$

where $m(r, \alpha)$ is a non-negative function defined for $r \in \mathbf{R}^+$ and submanifold ξ_α ($\alpha \in G(\mathbf{R})$) [§4]. The Crofton formula is as follows:

$$(0.7) \quad \int_{Q_{n-1}} n(\Delta(t), \alpha) \Omega^{n-1}(\alpha) = 2 \int_{\Delta(t)} f^* \Omega^2.$$

Finally the distribution theorem says: The image $f(\mathbf{C}^2)$ intersects with almost all submanifolds in $\{\xi_\alpha\}$ ($\alpha \in G(\mathbf{R})$) i.e., we have $\int_W \Omega^{n-1} = 0$ for $W = \{\alpha \in Q_{n-1}(\mathbf{C}) : f(\mathbf{C}^2) \cap \xi_\alpha = \emptyset\}$.

We note that W. Stoll [4], P. Griffiths and J. King [2] also developed the First Main Theorem in several complex variables. But our setting is different from theirs.

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1. Preliminaries

We shall recall several basic facts about the complex projective space $P_n(\mathbf{C})$

and the complex quadratic $Q_{n-1}(\mathbf{C})$ (c.f. [3]), and moreover we shall define a special family of submanifolds in $Q_{n-1}(\mathbf{C})$. Let \mathbf{C}^{n+1} (resp. \mathbf{R}^{n+1}) be the complex (resp. real) vector space of $(n+1)$ tuples of complex numbers (z^0, \dots, z^n) (resp. real numbers (x^0, \dots, x^n)). We define a symmetric bilinear form $(,)$ on \mathbf{C}^{n+1} by

$$(1.1) \quad (Z, W) = z^0 w^0 + \dots + z^n w^n$$

for $Z=(z^0, \dots, z^n)$ and $W=(w^0, \dots, w^n)$. For $Z=(z^0, \dots, z^n)$ we put $\bar{Z}=(\bar{z}^0, \dots, \bar{z}^n)$, where the bar denotes the complex conjugation. A vector $Z \in \mathbf{C}^{n+1} - \{0\}$ is called real if $\bar{Z}=Z$. We define a hermitian inner product \langle , \rangle on \mathbf{C}^{n+1} by

$$(1.2) \quad \langle Z, W \rangle = (Z, \bar{W})$$

for $Z, W \in \mathbf{C}^{n+1}$. We put $\|Z\| = \langle Z, Z \rangle^{1/2}$. For the complex projective space $P_n(\mathbf{C})$ of dimension n , we have the natural holomorphic fibring (called the Hopf fibring)

$$(1.3) \quad \Pi: \mathbf{C}^{n+1} - \{0\} \rightarrow P_n(\mathbf{C}),$$

where $\Pi(Z)$ is the line passing through the origin and Z . We remark that the natural conjugation $Z \mapsto \bar{Z}$ in $\mathbf{C}^{n+1} - \{0\}$ induces a diffeomorphism $z \in P_n(\mathbf{C}) \rightarrow \bar{z} \in P_n(\mathbf{C})$. Let $\tilde{\Omega}$ be the 2-form of type $(1, 1)$ on $\mathbf{C}^{n+1} - \{0\}$ given by

$$(1.4) \quad \tilde{\Omega} = \frac{i}{2\pi} \frac{1}{\|Z\|^4} \{(\sum_j |z^j|^2)(\sum_j dz^j \wedge d\bar{z}^j) - (\sum_j z^j dz^j) \wedge (\sum_j \bar{z}^j d\bar{z}^j)\}.$$

It is well-known that there exists a unique 2-form Ω of type $(1,1)$ on $P_n(\mathbf{C})$ such that $\Pi^* \Omega = \tilde{\Omega}$. Then Ω is the Kähler form associated with the Fubini-Study metric on $P_n(\mathbf{C})$ and we have

$$(1.5) \quad \int_{P_n(\mathbf{C})} \Omega^n = 1.$$

We consider a family of subspaces H of \mathbf{C}^{n+1} such that H is of $(n-1)$ -dimension and $\bar{Z} \in H$ whenever $Z \in H$. With such an H , we can associate uniquely a real subspace of \mathbf{R}^{n+1} of dimension 2 by

$$(1.6) \quad \{X \in \mathbf{R}^{n+1}: \langle X, H \rangle = 0\}.$$

We see that this gives a one to one correspondence, and hence the above family of H 's is parametrized by the Grassmann manifold $G(\mathbf{R})$ of 2 planes in \mathbf{R}^{n+1} . Especially we note that $[H] = \Pi(H - \{0\})$ is an $(n-2)$ -dimensional projective space in $P_n(\mathbf{C})$.

On $P_n(\mathbf{C})$ with homogeneous coordinate z^0, \dots, z^n the complex quadratic $Q_{n-1}(\mathbf{C})$ is a complex hypersurface defined by the equation

$$(1.7) \quad (z^0)^2 + \dots + (z^n)^2 = 0.$$

Now the unit sphere $S^{2n+1} = \{Z \in \mathbf{C}^{n+1}: \|Z\|=1\}$ is a principal fibre bundle over

$P_n(\mathbf{C})$ with structure group S^1 . For a point $q \in Q_{n-1}(\mathbf{C})$, take a point $Z \in S^{2n+1}$ such that $\Pi(Z) = q$. We can write Z uniquely in the form $Z = (X + iY)/\sqrt{2}$, where X and Y are orthonormal real vectors in \mathbf{C}^{n+1} . Conversely if $Z = (X + iY)/\sqrt{2} \in S^{2n+1}$ for orthonormal real vectors X and Y , then we have $\Pi(Z) \in Q_{n-1}(\mathbf{C})$. Therefore we have

$$(1.8) \quad S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C})) = \{Z = (X + iY)/\sqrt{2} : X \text{ and } Y \text{ are orthonormal real vectors}\}.$$

The group $SO(n+1)$, considered as a subgroup of $U(n+1)$, acts on S^{2n+1} and leaves the submanifold $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))$ invariant. Moreover $SO(n+1)$ acts transitively on $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))$. The isotropy subgroup of $SO(n+1)$ at $Z_0 = (1/\sqrt{2}, i/\sqrt{2}, 0, \dots, 0)$ coincides with the subgroup $SO(n-1)$ of $SO(n+1)$. We denote an element g of $SO(n+1)$ by

$$g = (X_0, X_1, \dots, X_n),$$

where each X_i is a column vector. Then, in the space $SO(n+1)/SO(n-1)$, the coset including $g = (X_0, X_1, \dots, X_n)$ can be represented by the first two vectors (X_0, X_1) . Under this identification, we have a diffeomorphism $i: SO(n+1)/SO(n-1) \rightarrow S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))$ defined by

$$(1.9) \quad i((X_0, X_1)) = \frac{1}{\sqrt{2}}(X_0 + iX_1).$$

From now on we also identify $SO(n+1)/SO(n-1)$ with $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))$ by the above diffeomorphism. We denote by Π_1 the projection: $SO(n+1)/SO(n-1) \rightarrow Q_{n-1}(\mathbf{C})$ defined by

$$(1.10) \quad \Pi_1((X_0, X_1)) = \Pi((X_0 + iX_1)/\sqrt{2})$$

for $(X_0, X_1) \in SO(n+1)/SO(n-1)$. Note that the space $Q_{n-1}(\mathbf{C})$ also can be identified canonically with $SO(n+1)/SO(2) \times SO(n-1)$.

To each point $\alpha = \Pi_1((X_0, X_1))$ in $Q_{n-1}(\mathbf{C})$, we assign the 2-dimensional linear space spanned by $\{X_0, X_1\}$ in \mathbf{R}^{n+1} . Through this assignment, $Q_{n-1}(\mathbf{C})$ is a double covering space of $G(\mathbf{R})$. We see that the function $|\langle Z, W \rangle|^2$ on $S^{2n+1} \times S^{2n+1}$ induces a function $|\Pi(Z), \Pi(W)|^2$ on $P_n(\mathbf{C}) \times P_n(\mathbf{C})$. For each $\alpha \in Q_{n-1}(\mathbf{C})$, we consider a complex submanifold ξ_α of $Q_{n-1}(\mathbf{C})$, defined by

$$(1.11) \quad \xi_\alpha = \{\beta \in Q_{n-1}(\mathbf{C}) : |\beta, \alpha|^2 + |\beta, \bar{\alpha}|^2 = 0\}.$$

Let $(X_0, X_1) \in SO(n+1)/SO(n-1)$ and set $\Pi_1((X_0, X_1)) = \alpha$. Consider the complex subspace H of \mathbf{C}^{n+1} orthogonal to the vectors X_0, X_1 . We have $\xi_\alpha = Q_{n-1}(\mathbf{C}) \cap [H]$. $[H]$ is a Poincaré dual of the form Ω^2 in $P_n(\mathbf{C})$, and hence ξ_α is also, in $Q_{n-1}(\mathbf{C})$, a Poincaré dual of the form Ω^2 restricted to $Q_{n-1}(\mathbf{C})$. Finally we remark that each ξ_α is a complex quadratic $Q_{n-3}(\mathbf{C})$ and $\xi_\alpha = \xi_{\bar{\alpha}}$.

2. Holomorphic mapping

Let f be a holomorphic mapping of \mathbb{C}^2 into $Q_{n-1}(\mathbb{C})$ ($n \geq 3$). We consider the following two conditions on f .

Condition (A): f is an immersion.

Condition (B): For each $\alpha \in Q_{n-1}(\mathbb{C})$, the set $\{p \in \mathbb{C}^2: f(p) \in \xi_\alpha\}$ is discrete.

For each point $p \in \mathbb{C}^2$, we can take a small neighborhood $U(p)$ of p such that there exists a holomorphic lift $F = (f^0, \dots, f^n)$ of f on $U(p)$ into $\mathbb{C}^{n+1} - \{0\}$ i.e., $\Pi F = f$.

Proposition 2.1. *Condition (A) is equivalent to the following: for each point p of \mathbb{C}^2 , choose a holomorphic lift $F = (f^0, \dots, f^n)$ of f on a neighborhood U of p , then we have*

$$(2.1) \quad \text{rank} \begin{pmatrix} f^0, \dots, f^n \\ \frac{\partial f^0}{\partial w_1}, \dots, \frac{\partial f^n}{\partial w_1} \\ \frac{\partial f^0}{\partial w_2}, \dots, \frac{\partial f^n}{\partial w_2} \end{pmatrix} (p) = 3,$$

where (w_1, w_2) is a coordinate system on the neighborhood U .

Proof. We identify the real tangent space $T_Z(\mathbb{C}^{n+1})$ at a point Z in \mathbb{C}^{n+1} with \mathbb{C}^{n+1} in the usual way. For p , we take $(X_0, X_1, \dots, X_n) \in SO(n+1)$ such that $(X_0 + iX_1)/\sqrt{2} = (F/\|F\|)(p)$. Then the tangent space $T_{(X_0 + iX_1)/\sqrt{2}}(S^{2n+1})$ has a basis $i(X_0 + iX_1), X_0 - iX_1, i(X_0 - iX_1), X_2, \dots, X_n, iX_2, \dots, iX_n$. Let $T_{f(p)}$ be the subspace spanned by $X_2, \dots, X_n, iX_2, \dots, iX_n$. The projection $\tilde{\Pi} = \Pi|_{S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbb{C}))}$ induces a linear isomorphism $\tilde{\Pi}_*: T_{f(p)} \rightarrow T_{f(p)}(Q_{n-1}(\mathbb{C}))$ (c.f. [3] p.p. 279). Hence, $T_{f(p)}(Q_{n-1}(\mathbb{C}))$ is identified with the subspace of \mathbb{C}^{n+1} orthogonal to the vectors $(F/\|F\|)(p)$ and $(\bar{F}/\|F\|)(p)$ with respect to \langle, \rangle . Since we have $\langle F, \bar{F} \rangle = 0$ on U , we see $\langle dF, \bar{F} \rangle = 0$. We have

$$(2.2) \quad d\left(\frac{F}{\|F\|}\right) = \frac{1}{\|F\|} \sum_{j=1}^2 \left(\frac{\partial F}{\partial w_j} - \left\langle \frac{\partial F}{\partial w_j}, \frac{F}{\|F\|} \right\rangle \frac{F}{\|F\|} \right) dw_j + \sum_{j=1}^2 iF \frac{\partial}{\partial x^j} \left(\frac{1}{\|F\|} \right) dx^j - \sum_{j=1}^2 iF \frac{\partial}{\partial x^j} \left(\frac{1}{\|F\|} \right) dy^j,$$

where $w_j = x^j + iy^j$. Therefore we get

$$(2.3) \quad df = \sum_{j=1}^2 \tilde{\Pi}_* \left[\frac{1}{\|F\|} \left(\frac{\partial F}{\partial w_j} - \left\langle \frac{\partial F}{\partial w_j}, \frac{F}{\|F\|} \right\rangle \frac{F}{\|F\|} \right) \right] dw_j.$$

This shows Proposition 2.1.

Q.E.D.

We define

$$(2.4) \quad Q_{n-3}(f(p)^\perp) = \{\alpha \in Q_{n-1}(\mathbf{C}) : |f(p), \alpha|^2 + |f(p), \bar{\alpha}|^2 = 0\},$$

that is,

$$Q_{n-3}(f(p)^\perp) = \{\alpha \in Q_{n-1}(\mathbf{C}) : f(p) \in \xi_\alpha\}.$$

Then $Q_{n-3}(f(p)^\perp)$ can be identified with $SO(n-1)/SO(2) \times SO(n-3)$ as follows: Choose an element $(X_0, X_1, \dots, X_n) \in SO(n+1)$ such that $(X_0 + iX_1)/\sqrt{2} = (F/\|F\|)(p)$. Let $(A_2, A_3) \in SO(n-1)/SO(n-3)$ where $A_i = (a_{i2}, \dots, a_{in})^t$ ($i=2, 3$). Consider the mapping

$$(2.5) \quad (A_2, A_3) \rightarrow (\sum_{i=2}^n a_{2i} X_i, \sum_{i=2}^n a_{3i} X_i).$$

We see easily that this gives an identification of $SO(n-1)/SO(2) \times SO(n-3)$ with $Q_{n-3}(f(p)^\perp)$, which is independent of the choice of lift F .

For $\alpha \in Q_{n-3}(f(p)^\perp)$ we take $(X_0, X_1) \in SO(n+1)/SO(n-1)$ such that $\Pi_1((X_0, X_1)) = \alpha$. Then the following condition is independent of the choice of (X_0, X_1) ,

$$(2.6) \quad \left| \begin{array}{l} \langle (\partial F/\partial w_1)(p), (X_0 + iX_1)/\sqrt{2} \rangle, \langle (\partial F/\partial w_2)(p), (X_0 + iX_1)/\sqrt{2} \rangle \\ \langle (\partial F/\partial w_1)(p), (X_0 - iX_1)/\sqrt{2} \rangle, \langle (\partial F/\partial w_2)(p), (X_0 - iX_1)/\sqrt{2} \rangle \end{array} \right| \neq 0.$$

Proposition 2.2. *The condition (2.6) holds if and only if f intersects transversely with ξ_α at $f(p)$.*

Proof. Put $(F/\|F\|)(p) = (X_2 + iX_3)/\sqrt{2}$. Then we take an element $(X_0, X_1, X_2, X_3, \dots, X_n) \in SO(n+1)$. As in the proof of Proposition 2.1, we see that the tangent space $T_{f(p)}(Q_{n-1}(\mathbf{C}))$ is spanned by the vectors $X_0, iX_0, X_1, iX_1, X_2, iX_2, \dots, X_n, iX_n$ and the tangent space $T_{f(p)}(\xi_\alpha)$ is spanned by $X_2, iX_2, \dots, X_n, iX_n$ through the identification by $\tilde{\Pi}_*: T_{(X_2+iX_3)/\sqrt{2}}(S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))) \rightarrow T_{f(p)}(Q_{n-1}(\mathbf{C}))$. Therefore by (2.3) (or (2.2)) it is sufficient to show that the condition (2.6) is equivalent to $\text{rank}_{\mathbf{R}}((\partial F/\partial w_1)(p), i(\partial F/\partial w_1)(p), (\partial F/\partial w_2)(p), i(\partial F/\partial w_2)(p), X_2, iX_2, \dots, X_n, iX_n) = 2(n+1)$. Now this can be seen easily.

Q.E.D.

Now we consider the following condition for $\alpha = \Pi_1((X_0, X_1)) \in Q_{n-3}(f(p)^\perp)$

$$(2.7) \quad \left| \begin{array}{l} \langle (\partial F/\partial w_1)(p), (X_0 + iX_1)/\sqrt{2} \rangle, \langle (\partial F/\partial w_2)(p), (X_0 + iX_1)/\sqrt{2} \rangle \\ \langle (\partial F/\partial w_1)(p), (X_0 - iX_1)/\sqrt{2} \rangle, \langle (\partial F/\partial w_2)(p), (X_0 - iX_1)/\sqrt{2} \rangle \end{array} \right| = 0$$

Since the vectors $(\partial F/\partial w_1)(p)$ and $(\partial F/\partial w_2)(p)$ are linearly independent, the set of elements $\alpha \in Q_{n-3}(f(p)^\perp)$ satisfying the condition (2.7) has measure zero in $Q_{n-3}(f(p)^\perp)$.

REMARK 1. We shall remark here a certain sufficient condition for Condition (B). For $w \in \mathbf{C}$ we put $\mathbf{C}_w^1 = \{(z, w) : z \in \mathbf{C}\}$ and $\mathbf{C}_w^2 = \{(w, z) : z \in \mathbf{C}\}$.

Assume the following condition (C): *none of $f(C_w^i)(i=1, 2, w \in C)$ is contained in a hyperplane in $P_n(C)$.* Let $f(p) \in \xi_\alpha$ and set $\prod_1((X_0, X_1)) = \alpha$. We put $g_1(w_1, w_2) = \langle F, (X_0 + iX_1)/\sqrt{2} \rangle(w_1, w_2)$ and $g_2(w_1, w_2) = \langle F, (X_0 - iX_1)/\sqrt{2} \rangle(w_1, w_2)$ on $U(p)$, where (w_1, w_2) is a coordinate system on $U(p)$ such that $w_i(p) = 0$ ($i=1, 2$). Using the Weierstrass' preparation theorem we have the following representations

$$(2.8) \quad \begin{aligned} g_1(w_1, w_2) &= (a_0(w_1) + a_1(w_1)w_2 + \dots + a_{l_1}(w_1)w_2^{l_1})h_1(w_1, w_2) \\ g_2(w_1, w_2) &= (b_0(w_1) + b_1(w_1)w_2 + \dots + b_{l_2}(w_1)w_2^{l_2})h_2(w_1, w_2), \end{aligned}$$

where $a_i(w_1), b_i(w_1)$ and $h_i(w_1, w_2)$ are holomorphic such that $a_i(0) = 0$ for $0 \leq i < l_1$, $a_{l_1}(0) \neq 0$, $b_i(0) = 0$ for $0 \leq i < l_2$, $b_{l_2}(0) \neq 0$ and $h_i(w_1, w_2) \neq 0$ ($i=1, 2$). We denote by $R(w_1)$ the resultant of $(a_0(w_1) + \dots + a_{l_1}(w_1)w_2^{l_1})$ and $(b_0(w_1) + \dots + b_{l_2}(w_1)w_2^{l_2})$. Since the function $R(w_1)$ is holomorphic, we have that $R(w_1) \equiv 0$ or the following (D): *the set $\{w_1 : R(w_1) = 0\}$ is discrete.* If, under the assumption of (C), f satisfies (D) for each $p \in C^2$ and $\alpha \in Q_{n-1}(C)$ such that $f(p) \in \xi_\alpha$, then Condition (B) holds.

3. Certain forms on $Q_{n-1}(C) - \xi_\alpha$

We define one 2-form Ω_α on $Q_{n-1}(C) - \xi_\alpha$ by

$$(3.1) \quad \Omega_\alpha(\beta) = dd^c \log \{ |\beta, \alpha|^2 + |\beta, \bar{\alpha}|^2 \},$$

where $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$. We choose a unit vector Z_α such that $\prod(Z_\alpha) = \alpha$, and define a mapping P_α of $Q_{n-1}(C) - \xi_\alpha$ into $P_1(C)$ by

$$(3.2) \quad P_\alpha(\beta) = \hat{\Pi} \left[\frac{1}{(|\beta, \alpha|^2 + |\beta, \bar{\alpha}|^2)^{1/2}} (\langle Z_\beta, Z_\alpha \rangle, \langle Z_\beta, \bar{Z}_\alpha \rangle) \right],$$

where $Z_\beta \in S^{2n+1}$ such that $\prod(Z_\beta) = \beta$, and $\hat{\Pi}$ is the Hopf fibring $S^3 \rightarrow P_1(C)$. P_α is well-defined and holomorphic. Let ω be the Kähler 2-form associated with the Fubini-Study metric on $P_1(C)$ and normalized as $\int_{P_1(C)} \omega = 1$. Then $P_\alpha^* \omega$ is independent of the choice of Z_α . From now on we also denote by Ω the restriction of the form Ω to $Q_{n-1}(C)$.

Lemma 3.1. *We have*

$$(3.3) \quad \Omega_\alpha = P_\alpha^* \omega - \Omega \quad \text{on } Q_{n-1}(C) - \xi_\alpha.$$

Proof. Let σ be a local holomorphic cross-section of the Hopf fibring $\Pi: C^{n+1} - \{0\} \rightarrow P_n(C)$ defined on an open set U in $Q_{n-1}(C) - \xi_\alpha$. Then we have

$$\begin{aligned} \Omega_\alpha &= dd^c \log \left\{ \left| \left\langle \frac{\sigma}{\|\sigma\|}, Z_\alpha \right\rangle \right|^2 + \left| \left\langle \frac{\sigma}{\|\sigma\|}, \bar{Z}_\alpha \right\rangle \right|^2 \right\} \\ &= dd^c \log \{ |\langle \sigma, Z_\alpha \rangle|^2 + |\langle \sigma, \bar{Z}_\alpha \rangle|^2 \} - dd^c \log \|\sigma\|^2 \\ &= P_\alpha^* \omega - \Omega. \end{aligned} \quad \text{Q.E.D.}$$

We define another 2-form Ω'_α on $Q_{n-1}(\mathbf{C})-\xi_\alpha$ by

$$(3.4) \quad \Omega'_\alpha = \Omega + P_\alpha^* \omega \quad \text{on } Q_{n-1}(\mathbf{C})-\xi_\alpha .$$

Put

$$(3.5) \quad \Omega''_\alpha = -\Omega_\alpha \wedge \Omega'_\alpha \quad \text{on } Q_{n-1}(\mathbf{C})-\xi_\alpha .$$

By (3.3) and (3.4), we have

$$(3.5)' \quad \begin{aligned} \Omega''_\alpha &= (\Omega - P_\alpha^* \omega) \wedge (\Omega + P_\alpha^* \omega) \\ &= \Omega^2 - P_\alpha^*(\omega \wedge \omega) = \Omega^2 \quad \text{on } Q_{n-1}(\mathbf{C})-\xi_\alpha . \end{aligned}$$

Let $f: \mathbf{C}^2 \rightarrow Q_{n-1}(\mathbf{C})$ ($n \geq 3$) be a holomorphic mapping satisfying Conditions (A) and (B) in §2. For a point p in \mathbf{C}^2 , we take a small neighborhood $U(p)$ of p and a coordinate system (w_1, w_2) on it satisfying $w_i(p) = 0$ ($i = 1, 2$). Let F be a holomorphic lift of f on $U(p)$ into $\mathbf{C}^{n+1} - \{0\}$. Set $f(p) \in \xi_\alpha$. Then we define a real number $n(p, \alpha)$ by

$$(3.6) \quad n(p, \alpha) = \lim_{\varepsilon \downarrow 0} \int_{\partial U_\varepsilon(p)} d^c \cdot \log \{ |\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2 \} \wedge f^* P_\alpha^* \omega ,$$

where $U_\varepsilon(p) = \{(w_1, w_2) \in U(p) : |w_1|^2 + |w_2|^2 < \varepsilon^2\}$ and $\Pi(Z_\alpha) = \alpha$.

Lemma 3.2. $n(p, \alpha)$ is well-defined and finite. Especially if f intersects transversely with ξ_α at $f(p)$, then we have $n(p, \alpha) = 1$.

Proof. First we choose a local lift F and a local coordinate system (w_1, w_2) such that $w_i(p) = 0$. Take two positive real numbers ε_1 and ε_2 such that $U(p) \supset U_{\varepsilon_1}(p) \supset U_{\varepsilon_2}(p)$. Then we have

$$(3.7) \quad \begin{aligned} 0 &= \int_{U_{\varepsilon_1}(p) - U_{\varepsilon_2}(p)} f^* P_\alpha^*(\omega \wedge \omega) \\ &= \int_{\partial U_{\varepsilon_1}(p) - \partial U_{\varepsilon_2}(p)} d^c \log \{ |\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2 \} \wedge f^* P_\alpha^* \omega . \end{aligned}$$

Therefore we obtain

$$(3.8) \quad \begin{aligned} &\int_{\partial U_{\varepsilon_1}(p)} d^c \log \{ |\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2 \} \wedge f^* P_\alpha^* \omega \\ &= \lim_{\varepsilon \downarrow 0} \int_{\partial U_\varepsilon(p)} d^c \log \{ |\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2 \} \wedge f^* P_\alpha^* \omega . \end{aligned}$$

The left hand-side of the equation (3.8) is finite and hence so is the right side. In the same way, we see that $n(p, \alpha)$ is independent of the choice of a local coordinate system. Now we shall show that $n(p, \alpha)$ is independent of the choice of F . Take two holomorphic lift F_1 and F_2 of f . Then there exists a holomorphic function g such that $F_1 = gF_2$ and $g(q) \neq 0$ at any $q \in U(p)$. We have

$$\begin{aligned}
 (3.9) \quad & d^c \log \{ |\langle F_1, Z_\alpha \rangle|^2 + |\langle F_1, \bar{Z}_\alpha \rangle|^2 \} \\
 &= d^c \log |g|^2 + d^c \log \{ |\langle F_2, Z_\alpha \rangle|^2 + |\langle F_2, \bar{Z}_\alpha \rangle|^2 \} \\
 &= \frac{1}{4\pi i} [d \log g - d \log \bar{g}] + d^c \log \{ |\langle F_2, Z_\alpha \rangle|^2 + |\langle F_2, \bar{Z}_\alpha \rangle|^2 \}.
 \end{aligned}$$

Since the form $f^*P_\alpha^*\omega$ is closed on $\partial U_\varepsilon(p)$, $n(p, \alpha)$ is independent of the choice of F .

Next suppose that f intersects transversely with ξ_α at $f(p)$. Then

$$\begin{vmatrix} \langle \partial F / \partial w_1, Z_\alpha \rangle, \langle \partial F / \partial w_2, Z_\alpha \rangle \\ \langle \partial F / \partial w_1, \bar{Z}_\alpha \rangle, \langle \partial F / \partial w_2, \bar{Z}_\alpha \rangle \end{vmatrix} (p) \neq 0,$$

and hence we can choose $(w_1, w_2) = (\langle F, Z_\alpha \rangle, \langle F, \bar{Z}_\alpha \rangle)$ as a coordinate system on $U(p)$. We have

$$n(p, \alpha) = \lim_{\varepsilon \downarrow 0} \int_{|w_1|^2 + |w_2|^2 = \varepsilon^2} d^c \log (|w_1|^2 + |w_2|^2) \wedge f^*P_\alpha^*\omega.$$

Putting $w_1 = r_1 e^{i\theta_1}$, $w_2 = r_2 e^{i\theta_2}$, $r_1 = r \cos t$ and $r_2 = r \sin t$ ($0 \leq \theta_i \leq 2\pi$, $0 \leq t \leq \pi/2$), we have

$$d^c \log (r_1^2 + r_2^2) = \frac{1}{2\pi} \frac{1}{r_1^2 + r_2^2} (r_1^2 d\theta_1 + r_2^2 d\theta_2),$$

and

$$\begin{aligned}
 f^*P_\alpha^*\omega &= \frac{1}{\pi} \frac{1}{(r_1^2 + r_2^2)} (r_1 r_2^2 dr_1 \wedge d\theta_1 + r_1^2 r_2 dr_2 \wedge d\theta_2 \\
 &\quad - r_1 r_2^2 dr_1 \wedge d\theta_2 - r_1^2 r_2 dr_2 \wedge d\theta_1).
 \end{aligned}$$

Thus we see

$$d^c \log (r_1^2 + r_2^2) \wedge f^*P_\alpha^*\omega = \frac{1}{2\pi^2} \sin t \cos t d\theta_1 \wedge dt \wedge d\theta_2$$

on $r = \text{constant}$.

On the sphere $\{(w_1, w_2) \in U(p) : |w_1|^2 + |w_2|^2 = r^2\}$, $d\theta_1 \wedge dt \wedge d\theta_2$ is a positive form. Therefore we have $n(p, \alpha) = 1$. Q.E.D.

We denote by (z_1, z_2) the standard coordinate system on \mathbb{C}^2 . Put $\Delta(r) = \{(z_1, z_2) \in \mathbb{C}^2 : \log |z_i| < r (i=1, 2)\}$.

Theorem 1. *Let $f: \mathbb{C}^2 \rightarrow Q_{n-1}(\mathbb{C})$ ($n \geq 3$) be a holomorphic mapping satisfying (A) and (B). Suppose $f(\partial\Delta(r)) \cap \xi_\alpha = \emptyset$. Then we have*

$$(3.10) \quad \int_{\Delta(r)} f^*\Omega^2 = n(\Delta(r), \alpha) + \int_{\partial\Delta(r)} d^c [-\log(|f, \alpha|^2 + |f, \bar{\alpha}|^2) f^*(\Omega + P_\alpha^*\omega)],$$

where $n(\Delta(r), \alpha) = \sum_{f(p_i) \in \xi_\alpha, p_i \in \Delta(r)} n(p_i, \alpha)$.

Proof. By (3.1), Lemma 3.1, (3.5) and (3.5)', we have

$$\begin{aligned}
 (3.11) \quad \int_{\Delta(r)} f^* \Omega^2 &= \lim_{\varepsilon \downarrow 0} \int_{\Delta(r) - \sum_i U_{\varepsilon}(p_i)} f^* \Omega^2 \\
 &= \lim_{\varepsilon \downarrow 0} \int_{\Delta(r) - \sum_i U_{\varepsilon}(p_i)} -dd^c \cdot \log(|f, \alpha|^2 + |f, \bar{\alpha}|^2) \wedge f^*(\Omega + P_{\alpha}^* \omega) \\
 &= \lim_{\varepsilon \downarrow 0} \int_{\Delta(r) - \sum_i U_{\varepsilon}(p_i)} dd^c[-\log(|f, \alpha|^2 + |f, \bar{\alpha}|^2) f^*(\Omega + P_{\alpha}^* \omega)],
 \end{aligned}$$

where $U_{\varepsilon}(p_i)$ is such a neighborhood of p_i as given in the definition $n(p_i, \alpha)$. Applying Stokes Theorem to the equation (3.11), we have

$$\begin{aligned}
 (3.12) \quad \int_{\Delta(r)} f^* \Omega^2 &= \int_{\partial \Delta(r)} d^c[-\log(|f, \alpha|^2 + |f, \bar{\alpha}|^2) f^*(\Omega + P_{\alpha}^* \omega)] \\
 &\quad - \lim_{\varepsilon \downarrow 0} \sum_i \int_{\partial U_{\varepsilon}(p_i)} d^c[\log \|F_i\|^2 f^*(\Omega + P_{\alpha}^* \omega)] \\
 &\quad + \lim_{\varepsilon \downarrow 0} \sum_i \int_{\partial U_{\varepsilon}(p_i)} d^c[\log\{|\langle F_i, Z_{\alpha} \rangle|^2 + |\langle F_i, \bar{Z}_{\alpha} \rangle|^2\} f^* \Omega] \\
 &\quad + \sum_i n(p_i, \alpha),
 \end{aligned}$$

where F_i is a holomorphic lift of f on $U(p_i)$. We have

$$(3.13) \quad \lim_{\varepsilon \downarrow 0} \int_{\partial U_{\varepsilon}(p_i)} d^c[\log \|F_i\|^2 \cdot f^* \Omega] = \lim_{\varepsilon \downarrow 0} \int_{U_{\varepsilon}(p_i)} f^* \Omega^2 = 0.$$

Set $r^2 = |w_i^1|^2 + |w_i^2|^2$, where (w_i^1, w_i^2) denotes a coordinate system on $U(p_i)$, we see

$$(3.14) \quad d^c \log\{|\langle F_i, Z_{\alpha} \rangle|^2 + |\langle F_i, \bar{Z}_{\alpha} \rangle|^2\} = 0 \left(\frac{1}{r} \right) (dw_i^1 + d\bar{w}_i^1 + dw_i^2 + d\bar{w}_i^2)$$

and

$$\begin{aligned}
 (3.15) \quad dd^c \log\{|\langle F_i, Z_{\alpha} \rangle|^2 + |\langle F_i, \bar{Z}_{\alpha} \rangle|^2\} &= 0 \left(\frac{1}{r^2} \right) (dw_i^1 \wedge d\bar{w}_i^1 + dw_i^1 \wedge d\bar{w}_i^2 \\
 &\quad + dw_i^2 \wedge d\bar{w}_i^2 + dw_i^2 \wedge d\bar{w}_i^1).
 \end{aligned}$$

Since $\|F_i\|$ is positive on $U(p_i)$, we have

$$(3.16) \quad d^c \log \|F_i\|^2 = 0(1)(dw_i^1 + d\bar{w}_i^1 + dw_i^2 + d\bar{w}_i^2)$$

and

$$(3.17) \quad f^* \Omega = 0(1)(dw_i^1 \wedge d\bar{w}_i^1 + dw_i^1 \wedge d\bar{w}_i^2 + dw_i^2 \wedge d\bar{w}_i^2 + dw_i^2 \wedge d\bar{w}_i^1).$$

Since the both sides of the equation (3.8) are finite, comparing (3.14) and (3.15) with (3.16) and (3.17), we have

$$(3.18) \quad \lim_{\varepsilon \downarrow 0} \int_{\partial U_{\varepsilon}(p_i)} d^c[\log \|F_i\|^2 \cdot f^* P_{\alpha}^* \omega] = 0$$

$$(3.19) \quad \lim_{\varepsilon \downarrow 0} \int_{\partial U_\varepsilon(\rho_i)} d^c [\log \{ |\langle F_i, Z_\alpha \rangle|^2 + |\langle F_i, \bar{Z}_\alpha \rangle|^2 \} f^* \Omega] = 0.$$

Q.E.D.

4. First Main Theorem

Let $f : C^2 \rightarrow Q_{n-1}(C)$ ($n \geq 3$) be a holomorphic mapping satisfying (A) and (B). For a point α in $Q_{n-1}(C)$, we choose two real numbers r_1 and r_2 such that $r_1 > r_2$ and the image $f((r(\Delta_1) \setminus \Delta(r_2)))$ does not intersect with ξ_α .

We see easily $|\beta, \alpha|^2 + |\beta, \bar{\alpha}|^2 \leq 1$ for $\beta \in Q_{n-1}(C)$. Hence $\psi_\alpha = -\log(|f, \alpha|^2 + |f, \bar{\alpha}|^2) f^*(\Omega + P_\alpha^* \omega)$ is a positive form (non-negative form, precisely) on $\Delta(r_1) \setminus \Delta(r_2)$. Putting $z_j = e^{s_j + i\theta_j}$ ($j = 1, 2$), we can write ψ_α on $\Delta(r_1) \setminus (\Delta(r_2) \cup \{(z, 0) \in C^2\} \cup \{0, z\} \in C^2)$ as follows:

$$(4.1) \quad \begin{aligned} \psi_\alpha &= -\log(|f, \alpha|^2 + |f, \bar{\alpha}|^2) f^*(\Omega + P_\alpha^* \omega) \\ &= \psi_1 ds_1 \wedge d\theta_1 + \psi_2 ds_1 \wedge d\theta_2 + \psi_3 ds_2 \wedge d\theta_1 \\ &\quad + \psi_4 ds_2 \wedge d\theta_2 + \psi_5 d\theta_1 \wedge d\theta_2 + \psi_6 ds_1 \wedge ds_2. \end{aligned}$$

REMARK 2. If we write ψ_α with the standard coordinate system (z_1, z_2) on C^2 , we see $\psi_1(z_1, z_2) = \tilde{\psi}_1(z_1, z_2)e^{2s_1}$, $\psi_4(z_1, z_2) = \tilde{\psi}_4(z_1, z_2)e^{2s_2}$ and $\psi_j(z_1, z_2) = e^{s_1} \cdot e^{s_2} \tilde{\psi}_j(z_1, z_2)$ ($j = 2, 3, 5, 6$) for certain functions $\tilde{\psi}_i$ ($i = 1, 2, \dots, 6$).

Lemma 4.1. *We have*

$$(4.2) \quad \psi_1 \geq 0, \psi_4 \geq 0 \text{ and } \psi_2 = \psi_3.$$

Proof. Choosing a holomorphic lift F on a sufficiently small open set U in $\Delta(r_1) \setminus \Delta(r_2)$, we have

$$(4.3) \quad f^*(\Omega + P_\alpha^* \omega) = dd^c [\log \|F\|^2 + \log (|\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2)],$$

where $\Pi(Z_\alpha) = \alpha$. Now we obtain

$$(4.4) \quad \begin{aligned} d^c &= \frac{1}{4\pi} \sum_{j=1}^2 \left[\frac{\partial}{\partial s_j} d\theta_j - \frac{\partial}{\partial \theta_j} ds_j \right] \\ d &= \sum_{j=1}^2 \left[\frac{\partial}{\partial \theta_j} d\theta_j + \frac{\partial}{\partial s_j} ds_j \right] \end{aligned} \quad \text{on } U \setminus (\{(0, z) \in C^2\} \cup \{(z, 0) \in C^2\}),$$

where $(e^{s_1 + i\theta_1}, e^{s_2 + i\theta_2})$ is the restriction to U of the standard coordinate system in C^2 . Putting $g = \log (|\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2) + \log \|F\|^2$, we have

$$(4.5) \quad \begin{aligned} dd^c g &= \frac{1}{4\pi} \left[\left(\frac{\partial^2 g}{(\partial \theta_1)^2} + \frac{\partial^2 g}{(\partial s_1)^2} \right) ds_1 \wedge d\theta_1 + \left(\frac{\partial^2 g}{\partial \theta_2 \partial \theta_1} + \frac{\partial^2 g}{\partial s_1 \partial s_2} \right) ds_1 \wedge d\theta_2 \right. \\ &\quad \left. + \left(\frac{\partial^2 g}{\partial \theta_1 \partial \theta_2} + \frac{\partial^2 g}{\partial s_2 \partial s_1} \right) ds_2 \wedge d\theta_1 + \left(\frac{\partial^2 g}{(\partial \theta_2)^2} + \frac{\partial^2 g}{(\partial s_2)^2} \right) ds_2 \wedge d\theta_2 + \dots \right]. \end{aligned}$$

Comparing (4.1) with (4.5), we have $\psi_2 = \psi_3$.

We shall show $\psi_1 \geq 0$ and $\psi_4 \geq 0$.

$$\begin{aligned}
 (4.6) \quad dd^c \log(\sum_j f^j \bar{f}^j) &= \frac{i}{2\pi} \partial \bar{\partial} \cdot \log(\sum_j f^j \bar{f}^j) \\
 &= \frac{i}{2\pi} \frac{1}{\|F\|^4} [\|F\|^2 (\sum_j df^j \wedge d\bar{f}^j) - (\sum_k df^k \bar{f}^k) \wedge (\sum_j f^j d\bar{f}^j)] \\
 &= \frac{i}{2\pi} \frac{1}{\|F\|^4} \left[\left(\|F\|^2 \left\| \frac{\partial F}{\partial z_1} \right\|^2 - \left| \left(\frac{\partial F}{\partial z_1}, F \right) \right|^2 \right) dz_1 \wedge d\bar{z}_1 \right. \\
 &\quad \left. + \left(\|F\|^2 \left\| \frac{\partial F}{\partial z_2} \right\|^2 - \left| \left(\frac{\partial F}{\partial z_2}, F \right) \right|^2 \right) dz_2 \wedge d\bar{z}_2 + \dots \right],
 \end{aligned}$$

where $F = (f^0, f^1, \dots, f^n)$. By the Schwartz inequality and the linear independence of vectors F and $\partial F / \partial z_j$ ($j=1, 2$), we have

$$\|F\|^2 \left\| \frac{\partial F}{\partial z_j} \right\|^2 > \left| \left(\frac{\partial F}{\partial z_j}, F \right) \right|^2, \text{ and } dz_j \wedge d\bar{z}_j = e^{2s_j} (-2ids_j \wedge d\theta_j)$$

($j=1, 2$). Thus we have

$$\frac{1}{\pi} \frac{1}{\|F\|^4} \left[\|F\|^2 \left\| \frac{\partial F}{\partial z_j} \right\|^2 - \left| \left(\frac{\partial F}{\partial z_j}, F \right) \right|^2 \right] e^{2s_j} > 0 \quad (j=1, 2)$$

or

$$(4.7) \quad \frac{1}{\pi} \frac{1}{(\sum_k f^k \bar{f}^k)^2} \left[(\sum_k f^k \bar{f}^k) \left(\sum_k \frac{\partial f^k}{\partial z_j} \frac{\partial \bar{f}^k}{\partial z_j} \right) - \left| \left(\sum_k \frac{\partial f^k}{\partial z_j} \bar{f}^k \right) \right|^2 \right] e^{2s_j} > 0 \quad (j=1, 2).$$

As for $dd^c[\log(|\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2)]$, putting $f^0 = \langle F, Z_\alpha \rangle$, $f^1 = \langle F, \bar{Z}_\alpha \rangle$ and $f^j = 0$ ($j=2, \dots, n$) in the equation (4.6), we have also the inequality (4.7) (in this case we replace $>$ by ≥ 0) with respect to the coefficient of $ds_j \wedge d\theta_j$ ($j=1, 2$).

Q.E.D.

Let r be in $[r_2, r_1]$. We divide $\partial\Delta(r)$ into $\partial\Delta_1(r)$ and $\partial\Delta_2(r)$, where

$$(4.8) \quad \partial\Delta_i(r) = \{(z_1, z_2) \in \partial\Delta(r) : \log|z_i| = r\} \quad (i=1, 2).$$

Lemma 4.2. *We have*

$$\begin{aligned}
 (4.9) \quad \int_{\partial\Delta(r)} d^c \psi_\alpha &= \frac{1}{4\pi} \left[- \int_{S^1 \times S^1} \psi_4(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1 \right. \\
 &\quad \left. - \int_{S^1 \times S^1} \psi_1(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_1 \wedge d\theta_2 \right] \\
 &\quad + \frac{1}{4\pi} \frac{\partial}{\partial r} \left[\int_{\partial\Delta_1(r)} \psi_\alpha \wedge d\theta_1 + \int_{\partial\Delta_2(r)} \psi_\alpha \wedge d\theta_2 \right].
 \end{aligned}$$

Proof. First we remark that $d\theta_1 \wedge ds_2 \wedge d\theta_2$ and $d\theta_2 \wedge ds_1 \wedge d\theta_1$ are positive forms on $\partial\Delta_1(r)$ and $\partial\Delta_2(r)$ respectively.

By (4.1) and the preceding remark 2, we have

$$\begin{aligned} \int_{\partial\Delta_1(r)} d^c\psi_\alpha &= \int_{\partial\Delta_1(r)\setminus\{(e^r+i\theta_1,0)\}} d^c\psi_\alpha \\ &= \frac{1}{4\pi} \int_{\partial\Delta_1(r)\setminus\{(e^r+i\theta_1,0)\}} \left[-\frac{\partial\psi_3}{\partial s_2} + \frac{\partial\psi_4}{\partial s_1} + \frac{\partial\psi_5}{\partial\theta_2} \right] d\theta_1 \wedge ds_2 \wedge d\theta_2 \\ &= \frac{1}{4\pi} \int_{\partial\Delta_1(r)} \left[-\frac{\partial\psi_3}{\partial s_2} + \frac{\partial\psi_4}{\partial s_1} + \frac{\partial\psi_5}{\partial\theta_2} \right] d\theta_1 \wedge ds_2 \wedge d\theta_2. \end{aligned}$$

Clearly we have

$$\int_{\partial\Delta_1(r)} \frac{\partial\psi_5}{\partial\theta_2} d\theta_1 \wedge ds_2 \wedge d\theta_2 = 0.$$

Therefore we obtain

$$(4.10) \quad \int_{\partial\Delta_1(r)} d^c\psi_\alpha = \frac{1}{4\pi} \int_{\partial\Delta_1(r)} \left[-\frac{\partial\psi_3}{\partial s_2} + \frac{\partial\psi_4}{\partial s_1} \right] d\theta_1 \wedge ds_2 \wedge d\theta_2.$$

Similarly we obtain

$$(4.11) \quad \int_{\partial\Delta_2(r)} d^c\psi_\alpha = \frac{1}{4\pi} \int_{\partial\Delta_2(r)} \left[\frac{\partial\psi_1}{\partial s_2} - \frac{\partial\psi_2}{\partial s_1} \right] d\theta_2 \wedge ds_1 \wedge d\theta_1.$$

Now we shall consider the equation (4.10). We have

$$\begin{aligned} (4.12) \quad & \frac{1}{4\pi} \int_{\partial\Delta_1(r)} \frac{\partial\psi_3}{\partial s_2} d\theta_1 \wedge ds_2 \wedge d\theta_2 \\ &= \frac{1}{4\pi} \int_{\partial\Delta_1(r)} d(\psi_3 d\theta_2 \wedge d\theta_1) \\ &= \frac{1}{4\pi} \int_{\partial\Delta_1(r) \cap \partial\Delta_2(r)} \psi_3 d\theta_2 \wedge d\theta_1 \\ &= \frac{1}{4\pi} \int_{S^1 \times S^1} \psi_3(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1. \end{aligned}$$

Since we have

$$\begin{aligned} & \int_{\partial\Delta_1(r)} \psi_4 d\theta_1 \wedge ds_2 \wedge d\theta_2 \\ &= \int_{\partial\Delta_1(r)} d \left\{ \left(\int_{-\infty}^{s_2} \psi_4(e^{r+i\theta_1}, e^{t+i\theta_2}) dt \right) d\theta_2 \wedge d\theta_1 \right\} \\ &= \int_{S^1 \times S^1} \left(\int_{-\infty}^r \psi_4(e^{r+i\theta_1}, e^{t+i\theta_2}) dt \right) d\theta_2 \wedge d\theta_1, \end{aligned}$$

we obtain

$$\begin{aligned}
 (4.13) \quad & \frac{\partial}{\partial r} \int_{\partial \Delta_1(r)} \psi_4 d\theta_1 \wedge ds_2 \wedge d\theta_2 \\
 &= \int_{S^1 \times S^1} \psi_4(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1 \\
 &+ \int_{S^1 \times S^1} \left(\int_{-\infty}^r \frac{\partial \psi_4}{\partial r}(e^{r+i\theta_1}, e^{t+i\theta_2}) dt \right) d\theta_2 \wedge d\theta_1.
 \end{aligned}$$

By (4.10), (4.12) and (4.13), we obtain

$$\begin{aligned}
 (4.14) \quad & \int_{\partial \Delta_1(r)} d^c \psi_\omega = \frac{1}{4\pi} \int_{S^1 \times S^1} [-\psi_3 - \psi_4](e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1 \\
 &+ \frac{1}{4\pi} \frac{\partial}{\partial r} \int_{\partial \Delta_1(r)} \psi_4 d\theta_1 \wedge ds_2 \wedge d\theta_2.
 \end{aligned}$$

By the similar argument as we derived (4.14) from (4.10), we derive the following from (4.11)

$$\begin{aligned}
 (4.15) \quad & \frac{1}{4\pi} \int_{\partial \Delta_2(r)} d^c \psi_\omega = \frac{1}{4\pi} \int_{S^1 \times S^1} [-\psi_2 - \psi_1](e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_1 \wedge d\theta_2 \\
 &+ \frac{1}{4\pi} \frac{\partial}{\partial r} \int_{\partial \Delta_2(r)} \psi_1 d\theta_2 \wedge ds_1 \wedge d\theta_1.
 \end{aligned}$$

By (4.14), (4.15) and the definition of ψ_ω we obtain (4.9). Q.E.D.

Lemma 4.3. *We have*

$$(4.16) \quad \int_{\Delta(r)} f^* \Omega^2 = \frac{1}{4\pi} \frac{\partial}{\partial r} \left[\int_{\partial \Delta_1(r)} \psi_\omega \wedge d\theta_1 + \int_{\partial \Delta_2(r)} \psi_\omega \wedge d\theta_2 \right] + n(\Delta(r), \alpha).$$

Proof. By Theorem 1 and Lemma 4.2, we have only to prove that

$$\frac{1}{4\pi} \int_{S^1 \times S^1} [\psi_4 - \psi_1](e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1 = 0.$$

We define a mapping $h: \mathbf{C}^2 \rightarrow \mathbf{C}^2$ by $h((z_1, z_2)) = (z_2, z_1)$. Then $(f \circ h)$ satisfies Conditions (A) and (B), and we have

$$(|f \circ h, \alpha|^2 + |f \circ h, \bar{\alpha}|^2)(z_1, z_2) = (|f, \alpha|^2 + |f, \bar{\alpha}|^2)(z_2, z_1)$$

and

$$\begin{aligned}
 n_f((z_1, z_2), \alpha) &= \lim_{\varepsilon \downarrow 0} \int_{\partial U_\varepsilon((z_1, z_2))} d^c \log [|\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2] \wedge f^* P_\alpha^* \omega \\
 &= \lim_{\varepsilon \downarrow 0} \int_{\partial U_\varepsilon((z_2, z_1))} d^c \log [|\langle F \circ h, Z_\alpha \rangle|^2 + |\langle F \circ h, \bar{Z}_\alpha \rangle|^2] \wedge (fh)^* P_\alpha^* \omega \\
 &= n_{f \circ h}((z_2, z_1), \alpha).
 \end{aligned}$$

On the other hand, we have from (4.1)

$$(4.17) \quad (h^*\psi_\alpha) = \psi_1 \circ h \, ds_2 \wedge d\theta_2 + \psi_2 \circ h \, ds_2 \wedge d\theta_1 + \psi_3 \circ h \, ds_1 \wedge d\theta_2 \\ + \psi_4 \circ h \, ds_1 \wedge d\theta_1 + \psi_5 \circ h \, d\theta_2 \wedge d\theta_1 + \psi_6 \circ h \, ds_2 \wedge ds_1.$$

By Theorem 1, (4.14) and (4.15) in Lemma 4.2, comparing (4.1) with (4.17) we have

$$(4.18) \quad \int_{\Delta(r)} f^*\Omega^2 = \int_{\Delta(r)} h^* f^*\Omega^2 = n(\Delta(r), \alpha) \\ + \frac{1}{4\pi} \left[- \int_{S^1 \times S^1} \psi_1 \circ h (e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1 - \int_{S^1 \times S^1} \psi_4 \circ h (e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_1 \wedge d\theta_2 \right] \\ + \frac{1}{4\pi} \frac{\partial}{\partial r} \left[\int_{\partial\Delta_1(r)} \psi_1 \circ h \, d\theta_1 \wedge ds_2 \wedge d\theta_2 + \int_{\partial\Delta_2(r)} \psi_4 \circ h \, d\theta_2 \wedge ds_1 \wedge d\theta_1 \right].$$

We see easily

$$\int_{\partial\Delta_1(r)} \psi_1 \circ h \, d\theta_1 \wedge ds_2 \wedge d\theta_2 = \int_{\partial\Delta_2(r)} \psi_1 \, d\theta_2 \wedge ds_1 \wedge d\theta_1 \\ = \int_{\partial\Delta_2(r)} \psi_\alpha \wedge d\theta_2$$

and

$$\int_{\partial\Delta_2(r)} \psi_4 \circ h \, d\theta_2 \wedge ds_1 \wedge d\theta_1 = \int_{\partial\Delta_1(r)} \psi_4 \, d\theta_1 \wedge ds_2 \wedge d\theta_2 \\ = \int_{\partial\Delta_1(r)} \psi_\alpha \wedge d\theta_1.$$

Therefore we have only to prove

$$\int_{S^1 \times S^1} ((\psi_i \circ h) - \psi_i) (e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_1 \wedge d\theta_2 = 0 \quad (i = 1, 4).$$

For any $\alpha, \beta \in [0, 2\pi]$, we have

$$((\psi_i \circ h) - \psi_i) (e^{r+i\alpha}, e^{r+i\beta}) = \psi_i (e^{r+i\beta}, e^{r+i\alpha}) - \psi_i (e^{r+i\alpha}, e^{r+i\beta}) \\ ((\psi_i \circ h) - \psi_i) (e^{r+i\beta}, e^{r+i\alpha}) = \psi_i (e^{r+i\alpha}, e^{r+i\beta}) - \psi_i (e^{r+i\beta}, e^{r+i\alpha})$$

Thus we obtain

$$((\psi_i \circ h) - \psi_i) (e^{r+i\alpha}, e^{r+i\beta}) = -((\psi_i \circ h) - \psi_i) (e^{r+i\beta}, e^{r+i\alpha}).$$

Q.E.D.

For the holomorphic mapping $f: \mathbf{C}^2 \rightarrow Q_{n-1}(\mathbf{C}) (n \geq 3)$ satisfying Conditions (A) and (B), we put

$$T(r) = \int_0^r dt \int_{\Delta(t)} f^*\Omega^2 \quad (\text{order function})$$

$$(4.19) \quad N(r, \alpha) = \int_0^r n(\Delta(t), \alpha) dt \quad (\text{counting function})$$

$$m(r, \alpha) = \frac{1}{4\pi} \left[\int_{\partial\Delta_1(r)} \psi_\alpha \wedge d\theta_1 + \int_{\partial\Delta_2(r)} \psi_\alpha \wedge d\theta_2 \right].$$

We need the following lemma, which can be proved in a similar way as ([5] p.p. 502).

Lemma 4.4. *For any α , $m(r, \alpha)$ is continuous with respect to $r \in [0, \infty)$.*

Theorem 2. *We have*

$$(4.20) \quad T(r) = m(r, \alpha) - m(0, \alpha) + N(r, \alpha) \quad \text{for any } r \geq 0,$$

and $m(r, \alpha)$ is non-negative.

Proof. Integrating the equation in Lemma 4.3 with respect to $r \in [r_2, r_1]$, we have

$$\int_{r_2}^{r_1} dr \int_{\Delta(r)} f^* \Omega^2 = \int_{r_2}^{r_1} n(\Delta(r), \alpha) dr + m(r_1, \alpha) - m(r_2, \alpha).$$

By Lemma 4.4 we obtain the equation (4.20). It follows from Lemma 4.1 and Lemma 4.4 that the function $m(r, \alpha)$ is non-negative. Q.E.D.

Lemma 4.5. *For any r , $m(r, \alpha)$ is continuous with respect to $\alpha \in Q_{n-1}(C)$.*

We also omit this proof by the same reason as in Lemma 4.4. (c.f. [5] p.p. 504).

Theorem 3. *There exists a positive constant C satisfying*

$$(4.21) \quad T(r) + C > N(r, \alpha) \quad \text{whenever } r \geq 0 \text{ and } \alpha \in Q_{n-1}(C).$$

Proof. By Theorem 2 we have

$$T(r) + m(0, \alpha) \geq N(r, \alpha) \quad \text{for any } r \geq 0.$$

Therefore by Lemma 4.5 we have the equation (4.21). Q.E.D.

5. Induced form by f

We denote by (X_0, X_1, \dots, X_n) an element of $SO(n+1)$, where X_i 's ($0 \leq i \leq n$) are column vectors, and we put $X_i = (x_{i0}, \dots, x_{in})^t$. The left invariant forms θ_{ij} ($0 \leq i, j \leq n$) on $SO(n+1)$ are defined by the following equation:

$$(5.1) \quad - \begin{pmatrix} dX_0^t \\ dX_1^t \\ \vdots \\ dX_n^t \end{pmatrix} (X_0, \dots, X_n) = \begin{pmatrix} X_0^t \\ X_1^t \\ \vdots \\ X_n^t \end{pmatrix} (dX_0, \dots, dX_n) = \begin{pmatrix} 0, & \theta_{10}, & \dots, & \theta_{n0} \\ \theta_{01}, & 0, & \dots, & \theta_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{0n}, & \theta_{1n}, & \dots, & 0 \end{pmatrix},$$

where $\theta_{ij} = -\theta_{ji}$.

Therefore we have $-\langle dX_i, X_j \rangle = \theta_{ji}$ i.e.,

$$(5.2) \quad dX_i = \sum_j \theta_{ij} X_j.$$

Taking its exterior derivative, we see

$$(5.3) \quad d\theta_{01} = \sum_k \theta_{0k} \wedge \theta_{k1} = -\sum_k \theta_{0k} \wedge \theta_{1k}.$$

We remark that $d\theta_{01}$ is a 2-form on $SO(n+1)/SO(n-1)$. Furthermore it is a lift of a 2-form on $Q_{n-1}(\mathbb{C})$ by Π_1 . In fact, let U be an open neighborhood of $Q_{n-1}(\mathbb{C})$, and (X_0, X_1) be a local cross-section of U into $SO(n+1)/SO(n-1)$: $\Pi_1((X_0, X_1)) = \text{identity on } U$. We have

$$(5.4) \quad \Pi_1^{-1}(\Pi_1(X_0, X_1)) = \{(X_0, X_1) \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} : 0 \leq \theta < 2\pi\}.$$

Then we have on $\Pi_1^{-1}(U)$,

$$(5.5) \quad \begin{aligned} d\theta_{01} &= d\langle d(\cos \theta \cdot X_0 + \sin \theta \cdot X_1), (-\sin \theta \cdot X_0 + \cos \theta \cdot X_1) \rangle \\ &= d(d\theta + \langle dX_0, X_1 \rangle) = d\langle dX_0, X_1 \rangle. \end{aligned}$$

Let σ be a local holomorphic cross-section on U into $\mathbb{C}^{n+1} - \{0\}$ with respect to the Hopf fibring: $\Pi\sigma = \text{identity on } U$. We can write σ in the form $\sigma = X + iY$ for orthogonal real vectors X and Y at each point of U . Then we see

$$(5.6) \quad \Omega = dd^c \log \|\sigma\|^2 = -\frac{1}{2\pi} d\langle d(X/\|X\|), Y/\|Y\| \rangle.$$

Thus, $d\theta_{01}$ is the lift of $-2\pi\Omega$ by Π_1^* i.e.,

$$(5.7) \quad \Pi_1^* \Omega = -\frac{1}{2\pi} d\theta_{01}.$$

In the equation (5.1) we defined θ_{0j} 's and θ_{1j} 's ($0 \leq j \leq n$) as 1-forms on $SO(n+1)$. They are also regarded as 1-forms on $SO(n+1)/SO(n-1)$. To prove this fact we shall identify $SO(n+1)/SO(n-1)$ with $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbb{C}))$. We take a local coordinate $x = (x^1, \dots, x^{2n-1})$ on a small open set U in $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbb{C}))$ and write a point $Z(x)$ of U in the form $(X_0(x) + iX_1(x))/\sqrt{2}$, where $\langle X_0, X_0 \rangle(x) = \langle X_1, X_1 \rangle(x) = 1$ and $\langle X_0, X_1 \rangle(x) = 0$. For each x , extending $X_0(x)$ and $X_1(x)$, we take a real orthonormal basis $X_0(x), \dots, X_n(x)$ in \mathbb{C}^{n+1} such that $(X_0, \dots, X_n)(x) \in SO(n+1)$. Then the tangent space $T_{Z(x)}(S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbb{C})))$ has a basis $(iX_0 - X_1)(x), X_2(x), \dots, X_n(x), iX_2(x), \dots, X_n(x)$ (c.f. [3] p.p. 279).

In the equation $dZ = \sum_{i=1}^{2n-1} \frac{\partial Z}{\partial x^i} dx^i$, we see $\frac{\partial Z}{\partial x^i} = Z_* \left(\frac{\partial}{\partial x^i} \right)$ ($1 \leq i \leq 2n-1$) and hence $\frac{\partial Z}{\partial x^i}$'s are tangent vectors of $T_{Z(x)}(S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbb{C})))$. Thus there exists 1-

forms θ_j 's ($1 \leq j \leq n$) and $\tilde{\theta}_j$'s ($2 \leq j \leq n$) on U such that $dZ = \theta_1(iX_0 - X_1) + \sum_{j=2}^n (\theta_j + i\tilde{\theta}_j)X_j$. Comparing this form with (5.2), we have $\theta_1 = \theta_{10}/\sqrt{2}$, $\theta_j = \theta_{0j}/\sqrt{2}$ ($2 \leq j \leq n$) and $\tilde{\theta}_j = \theta_{1j}/\sqrt{2}$ ($2 \leq j \leq n$). Thus we have from (5.2), (5.3) and (5.7)

$$(5.8) \quad (\Pi_1^* \Omega)_{(X_0, X_1)} = \frac{1}{2\pi} \sum_{j=2}^n \langle dX_0, X_j \rangle \wedge \langle dX_1, X_j \rangle,$$

where $(X_0, X_1, \dots, X_n) \in SO(n+1)$. For the volume form Ω^{n-1} on $Q_{n-1}(\mathbf{C})$, we have

$$(5.9) \quad (\Pi_1^* \Omega^{n-1})_{(X_0, X_1)} = \left(\frac{1}{2\pi} \right)^{n-1} (n-1)! \langle dX_0, X_2 \rangle \wedge \langle dX_1, X_2 \rangle \wedge \dots \wedge \langle dX_0, X_n \rangle \wedge \langle dX_1, X_n \rangle.$$

We shall obtain a formula for $f^* \Omega^2$ on \mathbf{C}^2 . Let F be a holomorphic lift of f on a neighborhood U in \mathbf{C}^2 by Π . Set $(X_0 + iX_1)/\sqrt{2} = F/\|F\|$, where X_i ($i=0, 1$) are the orthonormal real vectors. With the coordinate system $(x_1 + iy_1, x_2 + iy_2)$ on \mathbf{C}^2 , we can write:

$$(5.10) \quad \begin{aligned} dX_0 &= \omega_1 X_1 + \lambda_2 \tilde{B}_2 dx_1 - \lambda_3 \tilde{B}_3 dy_1 + \lambda_4 \tilde{B}_4 dx_2 - \lambda_5 \tilde{B}_5 dy_2, \\ dX_1 &= \omega_2 X_0 + \lambda_3 \tilde{B}_3 dx_1 + \lambda_2 \tilde{B}_2 dy_1 + \lambda_5 \tilde{B}_5 dx_2 + \lambda_4 \tilde{B}_4 dy_2, \end{aligned}$$

where \tilde{B}_i 's ($2 \leq i \leq 5$) are differentiable vectors satisfying $\langle \tilde{B}_i, \tilde{B}_i \rangle = 1$, λ_i 's ($2 \leq i \leq 5$) are differentiable functions and ω_i 's ($1 \leq i \leq 2$) are 1-forms on U . Then we take differentiable orthonormal vectors B_i ($2 \leq i \leq 5$) such that $\tilde{B}_2 = B_2$, $\tilde{B}_3 = \alpha_2 B_2 + \alpha_3 B_3$, $\tilde{B}_4 = \beta_2 B_2 + \beta_3 B_3 + \beta_4 B_4$ and $\tilde{B}_5 = \gamma_2 B_2 + \gamma_3 B_3 + \gamma_4 B_4 + \gamma_5 B_5$, where α_i , β_i and γ_i are differentiable functions satisfying $\sum \alpha_i^2 = 1$, $\sum \beta_i^2 = 1$ and $\sum \gamma_i^2 = 1$. We choose differentiable vectors B_6, \dots, B_n on U such that $(X_0, X_1, B_2, \dots, B_n) \in SO(n+1)$ at each point of U . By (5.8) we have

$$(5.11) \quad \begin{aligned} f^* \Omega &= \frac{1}{2\pi} \sum_{j=2}^n \langle dX_0, B_j \rangle \wedge \langle dX_1, B_j \rangle \\ &= \frac{1}{2\pi} \{ [\lambda_2 \lambda_5 \gamma_2 - \lambda_3 \lambda_4 \alpha_2 \beta_2 - \lambda_3 \lambda_4 \beta_3 \alpha_3] (dx_1 \wedge dx_2 + dy_1 \wedge dy_2) \\ &\quad + [\lambda_2^2 + \lambda_3^2] dx_1 \wedge dy_1 + [\lambda_4^2 + \lambda_5^2] dx_2 \wedge dy_2 \\ &\quad + [\lambda_2 \lambda_4 \beta_2 + \lambda_3 \lambda_5 \alpha_2 \gamma_2 + \lambda_3 \lambda_5 \alpha_3 \gamma_3] (dx_1 \wedge dy_2 - dy_1 \wedge dx_2) \}. \end{aligned}$$

Furthermore we obtain

$$(5.12) \quad \begin{aligned} f^* \Omega^2 &= \left(\frac{1}{2\pi} \right)^2 \times 2 \times \{ [\lambda_2^2 + \lambda_3^2] [\lambda_4^2 + \lambda_5^2] \\ &\quad - [\lambda_2 \lambda_4 \beta_2 + \lambda_3 \lambda_5 \alpha_2 \gamma_2 + \lambda_3 \lambda_5 \alpha_3 \gamma_3]^2 \\ &\quad - [\lambda_2 \lambda_5 \gamma_2 - \lambda_3 \lambda_4 \alpha_2 \beta_2 - \lambda_3 \lambda_4 \alpha_3 \beta_3]^2 \} dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2. \end{aligned}$$

6. Crofton formula

In §3 we have defined $n(\Delta(r), \alpha)$ for a holomorphic mapping $f: \mathbb{C}^2 \rightarrow Q_{n-1}(\mathbb{C})$ ($n \geq 3$) satisfying Conditions (A) and (B). Then we have:

Theorem 4 (Crofton formula). *Let D be an open set in \mathbb{C}^2 with compact closure. Then we have*

$$(6.1) \quad \int_{Q_{n-1}(\mathbb{C})} n(D, \xi) d\xi = 2 \int_D f^* \Omega^2,$$

where $d\xi = d\xi_\alpha = d\alpha = \Omega^{n-1}$.

Proof. First we assume that D is so small that there exists a differentiable lift $\sigma = (X_0, X_1)$ of f on D : $\Pi_1 \sigma = f$. Let q be a point in D and set $f(q) \in \xi_\alpha$. For any real orthonormal vectors Y_0, Y_1 such that $\Pi_1((Y_0, Y_1)) = \alpha$, we have

$$(6.2) \quad \langle X_0(q), Y_0 \rangle = \langle X_0(q), Y_1 \rangle = \langle X_1(q), Y_0 \rangle = \langle X_1(q), Y_1 \rangle = 0.$$

We set

$$(6.3) \quad \begin{aligned} Q_{n-3}(f(q)^\perp) &= \{ \alpha \in Q_{n-1}(\mathbb{C}) : f(q) \in \xi_\alpha \} \\ f(D)^\perp &= \{ \alpha \in Q_{n-1}(\mathbb{C}) : f(D) \cap \xi_\alpha \neq \emptyset \}. \end{aligned}$$

and

$$(6.4) \quad \begin{aligned} D' &= \Pi_1^{-1}(f(D)^\perp) \\ D'' &= \{ (q, a) : q \in D, a = (A_2, A_3, \dots, A_n) \in SO(n-1) \}. \end{aligned}$$

For $a = (A_2, A_3, \dots, A_n) \in SO(n-1)$ we write its column vector A_i as $A_i = (a_{i2}, \dots, a_{in})^t$. Then we define a mapping $t: D'' \rightarrow SO(n+1)$ by

$$(6.5) \quad t((q, a)) = (B_2, B_3, X_0, X_1, B_4, \dots, B_n)(q) \times \begin{pmatrix} a_{22} & a_{32} & 0 & 0 & a_{42} & \dots & a_{n2} \\ a_{23} & a_{33} & 0 & 0 & a_{43} & \dots & a_{n3} \\ 0 & 0 & 1 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 1 & 0 & & 0 \\ a_{24} & a_{34} & 0 & 0 & a_{44} & \dots & a_{n4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{2n} & a_{3n} & 0 & 0 & a_{4n} & \dots & a_{nn} \end{pmatrix},$$

where $(X_0, X_1, B_2, \dots, B_n)(q)$ is the one given in §5. Let Π' be the projection $D \times (SO(n-1)/SO(n-3)) \rightarrow D \times Q_{n-3}(\mathbb{C})$ defined by $\Pi'((q, (A_2, A_3))) = (q, \Pi''((A_2, A_3)))$, where Π'' is the projection with respect to the Hopf fibring $SO(n-1)/SO(n-3) \rightarrow Q_{n-3}(\mathbb{C})$. We consider the following diagram;

$$(6.6) \quad \begin{array}{ccc} D \times (SO(n-1)/SO(n-3)) & \xrightarrow{t'} & D' \subset SO(n+1)/SO(n-1) \\ \downarrow \Pi' & & \downarrow \Pi_1 \\ D \times Q_{n-3}(C) & \xrightarrow{t''} & f(D)^+ \subset Q_{n-1}(C), \end{array}$$

where $t'((q, (A_2, A_3))) = (\sum_{i=2}^n a_{2i} B_i(q), \sum_{i=2}^n a_{3i} B_i(q))$ and t'' is defined by $\Pi_1 \circ t' = t'' \circ \Pi'$. Then, in the above diagram, we remark that $t''((q, Q_{n-3}(C))) = Q_{n-3}(f(q)^+)$ for each $q \in D$. Putting $t((q, a)) = (X_0', X_1', \dots, X_n')$, we obtain

$$(6.7) \quad \begin{aligned} & (\Pi')^*(t'')^*\Omega^{n-1} = (t')^*(\Pi_1)^*\Omega^{n-1} \\ &= \left(\frac{1}{2\pi}\right)^{n-1} (n-1)! \langle dX_0', X_2' \rangle \wedge \langle dX_1', X_2' \rangle \wedge \cdots \wedge \langle dX_0', X_n' \rangle \wedge \langle dX_1', X_n' \rangle \\ &= \left(\frac{1}{2\pi}\right)^{n-1} (n-1)! \times \frac{1}{16} \times \langle d(X_0+iX_1), X_0'+iX_1' \rangle \wedge \langle d(X_0-iX_1), X_0'-iX_1' \rangle \\ & \quad \wedge \langle d(X_0+iX_1), X_0'-iX_1' \rangle \wedge \langle d(X_0-iX_1), X_0'+iX_1' \rangle \wedge \langle dA_2, A_4 \rangle \\ & \quad \wedge \langle dA_3, A_4 \rangle \wedge \cdots \wedge \langle dA_2, A_n \rangle \wedge \langle dA_3, A_n \rangle \\ &= -\frac{1}{4} \left(\frac{1}{2\pi}\right)^2 (n-1)(n-2) \left| \begin{array}{c} \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X_0'+iX_1' \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, \\ X_0'+iX_1' \rangle \\ \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X_0'-iX_1' \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, \\ X_0'-iX_1' \rangle \end{array} \right|^2 \\ & \quad \times dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \left(\frac{1}{2\pi}\right)^{n-3} (n-3)! \langle dA_2, A_4 \rangle \wedge \langle dA_3, A_4 \rangle \wedge \cdots \\ & \quad \wedge \langle dA_2, A_n \rangle \wedge \langle dA_3, A_n \rangle. \end{aligned}$$

We put $C = \{\beta \in f(D)^+ : \text{there exists } \beta' \in (t'')^{-1}(\beta) \text{ such that } (dt'')(\beta') \text{ is singular}\}$. From Sard's Theorem the set C has measure zero. If we take $\alpha \in (f(D)^+ \setminus C)$, the set $(t'')^{-1}(\alpha)$ consists of finite elements because of the compactness of \bar{D} and Condition (B). We denote by n_α the number of elements $(t'')^{-1}(\alpha)$. Then, for each $\alpha \in (f(D)^+ \setminus C)$ there exists a connected neighborhood V of α in $(f(D)^+ \setminus C)$ such that $(t'')^{-1}(V)$ has n_α connected components and t'' maps each component onto V diffeomorphically. Let $\{V_i\}$ be a locally finite covering of $f(D)^+ \setminus C$ by such open sets and $\{\phi_i\}$ be a partition of unity subordinated to $\{V_i\}$. Now we have

$$(6.8) \quad \begin{aligned} \int_{f(D)^+} n_\alpha d\alpha &= \int_{f(D)^+ \setminus C} n_\alpha d\alpha = \sum_i \int_{f(D)^+ \setminus C} \phi_i(\alpha) n_\alpha d\alpha \\ &= \sum_i \int_{V_i} n_\alpha (\phi_i(\alpha) d\alpha) = \sum_i \int_{(t'')^{-1}(V_i)} -(t'')^*(\phi_i(\alpha) d\alpha) \\ &= \sum_i \int_{(t'')^{-1}(V_i)} -((t'')^*\phi_i(\alpha)) ((t'')^*d\alpha) \\ &= \int_{D \times Q_{n-3} \setminus C'} -(t'')^*d\alpha = \int_{D \times Q_{n-3}} -(t'')^*d\alpha, \end{aligned}$$

where C' is the set of critical points of t'' . If

$$t''((q, \alpha_j)) = \alpha \text{ and } \left| \begin{array}{l} \langle \partial F / \partial z_1, Z_\alpha \rangle, \langle \partial F / \partial z_2, Z_\alpha \rangle \\ \langle \partial F / \partial z_1, \bar{Z}_\alpha \rangle, \langle \partial F / \partial z_2, \bar{Z}_\alpha \rangle \end{array} \right| (q)$$

$$\left(\text{which is equal to } \frac{\|F\|}{2} \left| \begin{array}{l} \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, Z_\alpha \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, Z_\alpha \rangle \\ \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, \bar{Z}_\alpha \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, \bar{Z}_\alpha \rangle \end{array} \right| (q) \right) = 0$$

for $\Pi(Z_\alpha) = \alpha$, then $dt''((q, \alpha_j))$ is singular because of (6.7). By Lemma 3.2 we have $n(D, \alpha) = n_\alpha$ on $f(D)^+ \setminus C$. Therefore we have

$$(6.9) \quad \int_{Q_{n-1}} n(D, \alpha) d\alpha = \frac{1}{4} \left(\frac{1}{2\pi} \right)^2 (n-1)(n-2) \int_D dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2$$

$$\times \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{array}{l} \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X'_0 + iX'_1 \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, X'_0 + iX'_1 \rangle \\ \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X'_0 - iX'_1 \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, X'_0 - iX'_1 \rangle \end{array} \right|^2 \Omega^{n-3}.$$

Next we have the following equation:

$$(6.10) \quad \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{array}{l} \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X'_0 + iX'_1 \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, X'_0 + iX'_1 \rangle \\ \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X'_0 - iX'_1 \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, X'_0 - iX'_1 \rangle \end{array} \right|^2 \Omega^{n-3}$$

$$= [(\lambda_2 \lambda_4 \beta_3 - \lambda_3 \lambda_5 \alpha_2 \gamma_3 + \lambda_3 \lambda_5 \alpha_3 \gamma_2)^2 + (\lambda_3 \lambda_4 \alpha_2 \beta_3 + \lambda_2 \lambda_5 \gamma_3 - \lambda_3 \lambda_4 \alpha_3 \beta_2)^2] (q)$$

$$\times \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{array}{l} \langle B_2, X'_0 + iX'_1 \rangle, \langle B_3, X'_0 + iX'_1 \rangle \\ \langle B_2, X'_0 - iX'_1 \rangle, \langle B_3, X'_0 - iX'_1 \rangle \end{array} \right|^2 \Omega^{n-3}$$

$$+ (\lambda_2^2 + \lambda_3^2 \alpha_2^2) (\lambda_4^2 \beta_4^2 + \lambda_5^2 \gamma_4^2) (q) \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{array}{l} \langle B_2, X'_0 + iX'_1 \rangle, \\ \langle B_2, X'_0 - iX'_1 \rangle, \\ \langle B_4, X'_0 + iX'_1 \rangle \\ \langle B_4, X'_0 - iX'_1 \rangle \end{array} \right|^2 \Omega^{n-3}$$

$$+ (\lambda_2^2 + \lambda_3^2 \alpha_2^2) (\lambda_5^2 \gamma_5^2) (q) \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{array}{l} \langle B_2, X'_0 + iX'_1 \rangle, \langle B_5, X'_0 + iX'_1 \rangle \\ \langle B_2, X'_0 - iX'_1 \rangle, \langle B_5, X'_0 - iX'_1 \rangle \end{array} \right|^2 \Omega^{n-3}$$

$$+ (\lambda_3^2 \alpha_3^2) (\lambda_4^2 \beta_4^2 + \lambda_5^2 \gamma_4^2) (q) \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{array}{l} \langle B_3, X'_0 + iX'_1 \rangle, \langle B_4, X'_0 + iX'_1 \rangle \\ \langle B_3, X'_0 - iX'_1 \rangle, \langle B_4, X'_0 - iX'_1 \rangle \end{array} \right|^2 \Omega^{n-3}$$

$$+ (\lambda_3^2 \alpha_3^2 \lambda_5^2 \gamma_5^2) (q) \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{array}{l} \langle B_3, X'_0 + iX'_1 \rangle, \langle B_5, X'_0 + iX'_1 \rangle \\ \langle B_3, X'_0 - iX'_1 \rangle, \langle B_5, X'_0 - iX'_1 \rangle \end{array} \right|^2 \Omega^{n-3}.$$

In fact, the integral of the other terms which appear at the right hand side of (6.10) turns out to be zero. For example we consider the following integral:

$$l = \int_{\mathcal{Q}_{n-3}(\mathcal{F}(\mathcal{Q})^\perp)} \left| \begin{array}{cc} \langle B_2(q), X'_0 + iX'_1 \rangle, \langle B_3(q), X'_0 + iX'_1 \rangle \\ \langle B_2(q), X'_0 - iX'_1 \rangle, \langle B_3(q), X'_0 - iX'_1 \rangle \end{array} \right| \frac{\overline{\langle B_2(q), X'_0 + iX'_1 \rangle}, \langle B_2(q), X'_0 - iX'_1 \rangle}}{\langle B_4(q), X'_0 + iX'_1 \rangle, \langle B_4(q), X'_0 - iX'_1 \rangle} \Omega^{n-3}.$$

We have

$$l = \int_{SO(n-1)/SO(n-3)} \left| \begin{array}{cc} (a_{22} - ia_{32}), (a_{23} - ia_{33}) \\ (a_{22} + ia_{32}), (a_{23} + ia_{33}) \end{array} \right| \frac{\overline{(a_{22} - ia_{32}), (a_{24} - ia_{34})}}{(a_{22} + ia_{32}), (a_{24} + ia_{34})} \times \left(\frac{1}{2\pi}\right)^{n-2} (n-3)! d\theta \wedge \langle dA_2, A_4 \rangle \wedge \langle dA_3, A_4 \rangle \wedge \cdots \wedge \langle dA_2, A_n \rangle \wedge \langle dA_3, A_n \rangle,$$

where $0 \leq \theta \leq 2\pi$. For each vector $A_i = (a_{i2}, a_{i3}, a_{i4}, \dots, a_{in})^t$ we set \tilde{A}_i by $\tilde{A}_i = (a_{i2}, -a_{i3}, a_{i4}, \dots, a_{in})^t$. This induces a diffeomorphism $k; SO(n-1) \rightarrow SO(n-1)$ by $k((A_2, A_3, A_4, A_5, \dots, A_n)) = (\tilde{A}_2, \tilde{A}_3, \tilde{A}_5, \tilde{A}_4, \dots, \tilde{A}_n)$. Then we have

$$l = \int_{SO(n-1)/SO(n-3)} \left| \begin{array}{cc} (a_{22} - ia_{32}), (a_{23} - ia_{33}) \\ (a_{22} + ia_{32}), (a_{23} + ia_{33}) \end{array} \right| \frac{\overline{(a_{22} - ia_{32}), (a_{24} - ia_{34})}}{(a_{22} + ia_{32}), (a_{24} + ia_{34})} \times \left(\frac{1}{2\pi}\right)^{n-2} (n-3)! d\theta \wedge \langle d\tilde{A}_2, d\tilde{A}_5 \rangle \wedge \langle d\tilde{A}_3, \tilde{A}_5 \rangle \wedge \langle d\tilde{A}_2, \tilde{A}_4 \rangle \wedge \langle d\tilde{A}_3, \tilde{A}_4 \rangle \wedge \langle d\tilde{A}_2, \tilde{A}_6 \rangle \wedge \langle d\tilde{A}_3, \tilde{A}_6 \rangle \wedge \cdots \wedge \langle d\tilde{A}_2, \tilde{A}_n \rangle \wedge \langle d\tilde{A}_3, \tilde{A}_n \rangle.$$

Since we have $\langle dA_i, A_j \rangle = \langle d\tilde{A}_i, \tilde{A}_j \rangle$ ($2 \leq i \leq 3, 4 \leq j \leq n$), we obtain $l=0$. In the equation (6.10), the integrals

$$\begin{aligned} & \int_{\mathcal{Q}_{n-3}(\mathcal{F}(\mathcal{Q})^\perp)} \left| \begin{array}{cc} \langle B_2, X'_0 + iX'_1 \rangle, \langle B_3, X'_0 + iX'_1 \rangle \\ \langle B_2, X'_0 - iX'_1 \rangle, \langle B_3, X'_0 - iX'_1 \rangle \end{array} \right|^2 \Omega^{n-3}, \\ & \int_{\mathcal{Q}_{n-3}(\mathcal{F}(\mathcal{Q})^\perp)} \left| \begin{array}{cc} \langle B_2, X'_0 + iX'_1 \rangle, \langle B_4, X'_0 + iX'_1 \rangle \\ \langle B_2, X'_0 - iX'_1 \rangle, \langle B_4, X'_0 - iX'_1 \rangle \end{array} \right|^2 \Omega^{n-3}, \\ & \int_{\mathcal{Q}_{n-3}(\mathcal{F}(\mathcal{Q})^\perp)} \left| \begin{array}{cc} \langle B_2, X'_0 + iX'_1 \rangle, \langle B_5, X'_0 + iX'_1 \rangle \\ \langle B_2, X'_0 - iX'_1 \rangle, \langle B_5, X'_0 - iX'_1 \rangle \end{array} \right|^2 \Omega^{n-3}, \\ & \int_{\mathcal{Q}_{n-3}(\mathcal{F}(\mathcal{Q})^\perp)} \left| \begin{array}{cc} \langle B_3, X'_0 + iX'_1 \rangle, \langle B_4, X'_0 + iX'_1 \rangle \\ \langle B_3, X'_0 - iX'_1 \rangle, \langle B_4, X'_0 - iX'_1 \rangle \end{array} \right|^2 \Omega^{n-3} \end{aligned}$$

and

$$\int_{\mathcal{Q}_{n-3}(\mathcal{F}(\mathcal{Q})^\perp)} \left| \begin{array}{cc} \langle B_3, X'_0 + iX'_1 \rangle, \langle B_5, X'_0 + iX'_1 \rangle \\ \langle B_3, X'_0 - iX'_1 \rangle, \langle B_5, X'_0 - iX'_1 \rangle \end{array} \right|^2 \Omega^{n-3}$$

are all equal and furthermore its value is independent of q . We denote by C_0 its common value. Then by (5.12), (6.9) and (6.10) we have

$$(6.11) \quad \int_{Q_{n-1}(C)} n(D, \alpha) d\alpha = \frac{1}{8} (n-1)(n-2) C_0 \int_D f^* \Omega^2.$$

We shall calculate the value C_0 . Let $SO(n-1)/SO(n-3) \rightarrow Q_{n-3}(C)$ be the Hopf fibring. For arbitrary fixed pair (C_2, C_3) of $SO(n-1)/SO(n-3)$ we have

$$(6.12) \quad C_0 = \int_{Q_{n-3}(C)} \left| \begin{matrix} \langle C_2, A_2 + iA_3 \rangle, \langle C_3, A_2 + iA_3 \rangle \\ \langle C_2, A_2 - iA_3 \rangle, \langle C_3, A_2 - iA_3 \rangle \end{matrix} \right|^2 \Omega^{n-3}.$$

We take an orthonormal pair (D_4, D_5) of $SO(n-1)/SO(n-3)$ such that $\langle C_i, D_j \rangle = 0$ ($2 \leq i \leq 3, 4 \leq j \leq 5$) and set real orthonormal vectors A_2, A_3, A_4 and A_5 by

$$(6.13) \quad \begin{aligned} A_2 &= \sin\varphi(\sin\theta \cdot C_2 - \cos\theta \cdot C_3) + \cos\varphi(\sin\alpha \cdot D_4 - \cos\alpha \cdot D_5) \\ A_3 &= \sin\eta(\cos\theta \cdot C_2 + \sin\theta \cdot C_3) + \cos\eta(\cos\alpha \cdot D_4 + \sin\alpha \cdot D_5) \\ A_4 &= -\cos\varphi(\sin\theta \cdot C_2 - \cos\theta \cdot C_3) + \sin\varphi(\sin\alpha \cdot D_4 - \cos\alpha \cdot D_5) \\ A_5 &= -\cos\eta(\cos\theta \cdot C_2 + \sin\theta \cdot C_3) + \sin\eta(\cos\alpha \cdot D_4 + \sin\alpha \cdot D_5), \end{aligned}$$

where $0 < \theta, \alpha < \pi, -\pi/2 < \varphi, \eta < \pi/2$. By extending A_2, A_3, A_4 and A_5 to an ordered real orthonormal basis A_2, A_3, \dots, A_n in C^{n-1} we get $(A_2, A_3, \dots, A_n) \in SO(n-1)$. Take an open set $U \subset Q_{n-5}(C)$, where $Q_{n-5}(C)$ is a set $\{\beta \in Q_{n-3}(C) : |\beta, \Pi''((C_2, C_3))|^2 + |\beta, \Pi''((C_2, -C_3))|^2 = 0\}$ in $Q_{n-3}(C)$, and a local cross-section $\sigma = (D_4, D_5)$ of U into $SO(n-3)/SO(n-5)$ with respect to the Hopf fibring: $SO(n-3)/SO(n-5) \rightarrow Q_{n-5}(C)$. Then we see easily the set $\{(A_2, A_3) \in SO(n-1)/SO(n-3) : (A_2, A_3) \text{ is defined at (6.13) for } \sigma = (D_4, D_5)\}$ is a double covering of an open set in $Q_{n-3}(C)$. We have

$$(6.14) \quad \begin{aligned} \langle dA_2, A_4 \rangle &= -d\varphi, \quad \langle dA_3, A_5 \rangle = -d\eta, \\ \langle dA_2, A_5 \rangle &= -\sin\varphi \cos\eta d\theta + \sin\eta \cos\varphi d\alpha + \cos\varphi \sin\eta \langle dD_4, D_5 \rangle, \\ \langle dA_3, A_4 \rangle &= \sin\eta \cos\varphi d\theta - \sin\varphi \cos\eta d\alpha - \cos\eta \sin\varphi \langle dD_4, D_5 \rangle, \\ \langle dA_2, A_i \rangle &= \cos\varphi(\sin\alpha \langle dD_4, A_i \rangle - \cos\alpha \langle dD_5, A_i \rangle) \\ \langle dA_3, A_i \rangle &= \cos\eta(\cos\alpha \langle dD_4, A_i \rangle + \sin\alpha \langle dD_5, A_i \rangle) \quad (i \geq 6). \end{aligned}$$

By (6.14) we get

$$(6.15) \quad \begin{aligned} &\langle dA_2, A_4 \rangle \wedge \langle dA_3, A_4 \rangle \wedge \dots \wedge \langle dA_2, A_n \rangle \wedge \langle dA_3, A_n \rangle \\ &= (\sin^2\eta \cos^2\varphi - \sin^2\varphi \cos^2\eta) (\cos\varphi \cos\eta)^{n-5} \\ &\quad \times d\varphi \wedge d\theta \wedge d\alpha \wedge d\eta \wedge \prod_{i \geq 6} \langle dD_4, A_i \rangle \wedge \langle dD_5, A_i \rangle, \end{aligned}$$

and

$$(6.16) \quad \left| \begin{matrix} \langle C_2, A_2 + iA_3 \rangle, \langle C_3, A_2 + iA_3 \rangle \\ \langle C_2, A_2 - iA_3 \rangle, \langle C_3, A_2 - iA_3 \rangle \end{matrix} \right|^2 = 4 |\sin\varphi \sin\eta|^2$$

Thus we obtain

$$\begin{aligned}
 (6.12)' \quad C_0 &= (n-3)(n-4) \int |\sin\varphi\sin\eta|^2 |\sin^2\eta\cos^2\varphi - \sin^2\varphi\cos^2\eta| \\
 &\quad |\cos\varphi\cos\eta|^{n-5} d\varphi d\eta \times \int_{Q_{n-5}(\mathbf{C})} \Omega^{n-5} \\
 &= 2(n-3)(n-4) \int |\sin\varphi\sin\eta|^2 |\sin^2\eta\cos^2\varphi - \sin^2\varphi\cos^2\eta| \\
 &\quad \times |\cos\varphi\cos\eta|^{n-5} d\varphi d\eta \\
 &= \frac{16}{(n-1)(n-2)},
 \end{aligned}$$

because of $\int_{Q_i(\mathbf{C})} \Omega^i = 2$ and $\int_E (\sin\varphi\sin\eta)^2 (\sin^2\varphi\cos^2\eta - \sin^2\eta\cos^2\varphi) \times (\cos\varphi\cos\eta)^{n-5} d\varphi d\eta = \frac{2}{(n-1)(n-2)(n-3)(n-4)}$, where

$E = \{(\eta, \varphi) : 0 \leq \varphi \leq \pi/2 \text{ and } 0 \leq \eta \leq \varphi\}$. Thus we have proved the equation (6.1) for a sufficiently small D . Now let D be an arbitrary open set in \mathbf{C}^2 with compact closure. We take a finite covering $\{D_s\}_{s=1}^l$ of D such that each D_s has a differentiable local cross-section of f into $SO(n+1)/SO(n-1)$. Let $\{g_s\}$ be a partition of unity subordinated to $\{D_s\}$. Taking a mapping $P_s : D_s \times Q_{n-3}(\mathbf{C}) \rightarrow D_s$ defined by $P_s((q, \alpha)) = q$ for $(q, \alpha) \in D_s \times Q_{n-3}(\mathbf{C})$, we put $n'(D_s, \alpha) = \sum_k n(p_k, \alpha) g_s(p_k)$. Then we obtain

$$\begin{aligned}
 (6.17) \quad \int_{Q_{n-1}} n(D, \alpha) d\alpha &= \sum_{s=1}^l \int_{Q_{n-1}} n'(D_s, \alpha) d\alpha \\
 &= \sum_s \int_{D_s \times Q_{n-3}} -g_s(P_s(\alpha')) (t_s'')^* d\alpha \\
 &= 2 \sum_s \int_{D_s} g_s f^* \Omega^2 \\
 &= 2 \int_D f^* \Omega^2,
 \end{aligned}$$

where t_s'' is a mapping of $D_s \times Q_{n-3}(\mathbf{C})$ onto $f(D_s)^+$ defined by (6.6). Q.E.D.

7. Equidistribution theorem

We define the defect $\delta(\alpha)$ of ξ_α by

$$(7.1) \quad \delta(\alpha) = \liminf_{r \rightarrow \infty} \frac{m(r, \alpha)}{T(r)}.$$

Since $m(r, \alpha)$ is non-negative, $\delta(\alpha)$ is non-negative for any $\alpha \in Q_{n-1}(\mathbf{C})$. We see clearly that $\delta(\alpha) = \delta(\bar{\alpha})$ for any $\alpha \in Q_{n-1}(\mathbf{C})$. By Theorem 2, Lemma 4.5 and the fact that $T(r) \rightarrow \infty$ if $r \rightarrow \infty$, we have

$$(7.2) \quad \delta(\alpha) = \liminf_{r \rightarrow \infty} \left(1 - \frac{N(r, \alpha)}{T(r)} \right).$$

Then we have the following equidistribution theorem.

Theorem 5. $\delta(\alpha)$ is equal to zero for almost all $\alpha \in Q_{n-1}(\mathbf{C})$ with respect to the volume Ω^{n-1} .

Proof. By the Fatou's preparation theorem we have

$$\begin{aligned} 0 &\leq \int_{Q_{n-1}} \delta(\alpha) d\alpha \leq \int_{Q_{n-1}} \left\{ \liminf_{r \rightarrow \infty} \left(1 - \frac{N(r, \alpha)}{T(r)} \right) \right\} d\alpha \\ &\leq \liminf_{r \rightarrow \infty} \int_{Q_{n-1}} \left(1 - \frac{N(r, \alpha)}{T(r)} \right) d\alpha = \liminf_{r \rightarrow \infty} \left(2 - \frac{1}{T(r)} \int_{Q_{n-1}} N(r, \alpha) d\alpha \right) \\ &= \liminf_{r \rightarrow \infty} \left(2 - \frac{1}{T(r)} \int_{Q_{n-1}} \left\{ \int_0^r n(\Delta(t), \alpha) dt \right\} d\alpha \right) \\ &= \liminf_{r \rightarrow \infty} \left(2 - \frac{1}{T(r)} \int_0^r dt \int_{Q_{n-1}} n(\Delta(t), \alpha) d\alpha \right) \\ &= \liminf_{r \rightarrow \infty} (2 - 2) = 0 \quad (\text{by Theorem 4}). \end{aligned}$$

Thus we obtain $\delta(\alpha) = 0$ for almost all $\alpha \in Q_{n-1}(\mathbf{C})$. Q.E.D.

If the image $f(\mathbf{C}^2)$ does not intersect with ξ_a , we have $\delta(\alpha) = 1$. So we have

Corollary. Let f be a holomorphic mapping of \mathbf{C}^2 into $Q_{n-1}(\mathbf{C})$ ($n \geq 3$) satisfying Conditions (A) and (B). We put $W = \{\alpha \in Q_{n-1}(\mathbf{C}) : f(\mathbf{C}^2) \cap \xi_a = \phi\}$. Then the set W has measure zero with respect to volume Ω^{n-1} .

REMARK 3. In the case of holomorphic curves ($f: \mathbf{C} \rightarrow P_n(\mathbf{C})$ holomorphic mapping), it is known that $0 \leq \delta(\xi) \leq 1$ for each hyperplane ξ (c.f. [1], [5] and [6]). But in our case we can not prove that $\delta(\alpha) \leq 1$.

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