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ON THE FIRST MAIN THEOREM OF HOLOMORPHIC MAPPINGS FROM C^2 INTO $Q_{n-1}(C)$

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0. Introduction

Let f be a holomorphic mapping of a complex line $\boldsymbol{C}$ into a complex projective space $P_n(\boldsymbol{C})$ and suppose that the image $f(\boldsymbol{C})$ is not contained in any hyperplane of $P_n(\boldsymbol{C})$. Put $V[t]=\{z\in\boldsymbol{C}: \log|z|<t\}$, and for a hyperplane ξ in $P_n(\boldsymbol{C})$ let $n(t, \xi)$ be the number of points in $V[t]\cap f^{-1}(\xi)$. Let Ω be the colsed form of degree 2 associated with the Fubini-Study metric on $P_n(\boldsymbol{C})$ and normalized as $\int_{P_n}\Omega^n=1$. The counting function $N(r, \xi)$ and the order function $T(r)$ being defined by

$$N(r, \xi) = \int_0^r n(t, \xi)dt\,, \tag{0.1}$$

$$T(r) = \int_0^r dt\int_{V[t]} f^*\Omega \tag{0.2}$$

respectively, the following equation is known as the First Main Theorem:

$$N(r, \xi)+(m(r, \xi)-m(0, \xi)) = T(r)\,, \tag{0.3}$$

where $m(r, \xi)$ is a non-negative function defined for $r\in\boldsymbol{R}^+$ and hyperplanes ξ in $P_n(\boldsymbol{C})$. The term $(m(r, \xi)-m(0, \xi))$ is called the compensating term. It follows from the equation (0.3) that the image $f(\boldsymbol{C})$ intersects with almost all hyperplanes in $P_n(\boldsymbol{C})$. Furthermore it is known that the number of hyperplanes in general position not intersecting with $f(\boldsymbol{C})$ is at most $n+1$. These results are originally due to Ahlfors, and treated also by H. Wu [6] and S. S. Chern [1] in a modernized form.

Let f be a holomorphic mapping of $\boldsymbol{C}^2$ into a complex quadratic $Q_{n-1}(\boldsymbol{C})$ ($n\geqq 3$) satisfying certain non-degenerate conditions [§2]. We consider $Q_{n-1}(\boldsymbol{C})$ as a fixed hypersurface in $P_n(\boldsymbol{C})$. We consider a special family of $(n-2)$-dimensional projective spaces $P_{n-2}(\boldsymbol{C})$ in $P_n(\boldsymbol{C})$ parametrized by a Grassmann manifold $G(\boldsymbol{R})$ of 2-dimensional linear spaces in $\boldsymbol{R}^{n+1}$ [§1]. This family determines a family of $(n-3)$-dimensional complex quadratic $\xi_\alpha(\alpha\in G(\boldsymbol{R}))$ in $Q_{n-1}(\boldsymbol{C})$, each of whose elements is a Poincaré dual of the form Ω^2 in $Q_{n-1}(\boldsymbol{C})$.

In this paper, we shall consider a value distribution problem in two complex variables with respect to the holomorphic mapping f and the family $\{\xi_\alpha\}$. The complex quadratic $Q_{n-1}(\boldsymbol{C})$ being a double covering space of $G(\boldsymbol{R})$, we may take $Q_{n-1}(\boldsymbol{C})$ as a parametrizing space of the family $\{\xi_\alpha\}$ in place of $G(\boldsymbol{R})$. Thus we have a setting similar to the case of holomorphic curves (holomorphic mappings of $\boldsymbol{C}$ into $P_n(\boldsymbol{C})$). Furthermore Ω is an invariant form on $Q_{n-1}(\boldsymbol{C})$ by a certain transformation group [§5]. This fact also plays an important role as in the case of holomorphic curves [§6].

Our main results are as follows: (1) First Main Theorem [§4], (2) the Crofton formula [§6] and (3) the Distribution theorem [§7]. In more detail, put

$$\Delta(r)=\{(z_1, z_2)\in\boldsymbol{C}^2\colon \log|z_i|<r(i=1, 2)\}$$

and define

$$n(\Delta(r), \alpha)=\sum_{p_i\in\Delta(r), f(p_i)\in\xi_\alpha} n(p_i, \alpha)\,,$$

where $n(p_i, \alpha)$ is a certain real number [§3] such that $n(p_i, \alpha)=1$ if $f(\boldsymbol{C}^2)$ intersects transversely with ξ_α at $f(p_i)$. We also define the following functions:

$$N(r, \alpha)=\int_0^r n(\Delta(t), \alpha)dt \text{ (counting function)} \tag{0.4}$$

$$T(r)=\int_0^r dt\int_{\Delta(t)} f^*\Omega^2 \quad \text{(order function)}\,. \tag{0.5}$$

Then our First Main Theorem states:

$$N(r, \alpha)+m(r, \alpha)-m(0, \alpha)=T(r)\,, \tag{0.6}$$

where $m(r, \alpha)$ is a non-negative function defined for $r\in\boldsymbol{R}^+$ and submainifold ξ_α $(\alpha\in G(\boldsymbol{R}))$ [§4]. The Crofton formula is as follows:

$$\int_{Q_{n-1}} n(\Delta(t), \alpha)\Omega^{n-1}(\alpha)=2\int_{\Delta(t)} f^*\Omega^2\,. \tag{0.7}$$

Finally the distribution theorem says: The image $f(\boldsymbol{C}^2)$ intersects with almost all submanifolds in $\{\xi_\alpha\}$ $(\alpha\in G(\boldsymbol{R}))$ i.e., we have $\int_W \Omega^{n-1}=0$ for $W=\{\alpha\in Q_{n-1}(\boldsymbol{C})\colon f(\boldsymbol{C}^2)\cap\xi_\alpha=\phi\}$.

We note that W. Stoll [4], P. Griffths and J. King [2] also developed the First Main Theorem in several complex variables. But our setting is different from theirs.

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1. Preliminaries

We shall recall several basic facts about the complex projective space $P_n(\boldsymbol{C})$

and the complex quadratic $Q_{n-1}(\boldsymbol{C})$ (c.f. [3]), and moreover we shall define a special family of submanifolds in $Q_{n-1}(\boldsymbol{C})$. Let $\boldsymbol{C}^{n+1}$(resp. $\boldsymbol{R}^{n+1}$) be the complex (resp. real) vector space of $(n+1)$ tuples of complex numbers $(z^0, \cdots, z^n)$ (resp. real numbers $(x^0, \cdots, x^n)$). We define a symmetric bilinear form $(\ ,\ )$ on $\boldsymbol{C}^{n+1}$ by

$$(1.1)\qquad (Z, W) = z^0w^0 + \cdots + z^nw^n$$

for $Z=(z^0, \cdots, z^n)$ and $W=(w^0, \cdots, w^n)$. For $Z=(z^0, \cdots, z^n)$ we put $\bar{Z}=(\bar{z}^0, \cdots, \bar{z}^n)$, where the bar denotes the complex conjugation. A vector $Z\in\boldsymbol{C}^{n+1}-\{0\}$ is called real if $\bar{Z}=Z$. We define a hermitian inner product $\langle\ ,\ \rangle$ on $\boldsymbol{C}^{n+1}$ by

$$(1.2)\qquad \langle Z, W\rangle = (Z, \bar{W})$$

for $Z, W\in\boldsymbol{C}^{n+1}$. We put $||Z||=\langle Z, Z\rangle^{1/2}$. For the complex projective space $P_n(\boldsymbol{C})$ of dimension n, we have the natural holomorphic fibring (called the Hopf fibring)

$$(1.3)\qquad \Pi : \boldsymbol{C}^{n+1}-\{0\} \to P_n(\boldsymbol{C}),$$

where $\Pi(Z)$ is the line passing through the origin and Z. We remark that the natural conjugation $Z\mapsto\bar{Z}$ in $\boldsymbol{C}^{n+1}-\{0\}$ induces a diffeomorphism $z\in P_n(\boldsymbol{C})\to \bar{z}\in P_n(\boldsymbol{C})$. Let $\tilde{\Omega}$ be the 2-form of type (1, 1) on $\boldsymbol{C}^{n+1}-\{0\}$ given by

$$(1.4)\qquad \tilde{\Omega} = \frac{i}{2\pi}\frac{1}{||Z||^4}\{(\textstyle\sum_j |z^j|^2)(\sum_j dz^j \wedge d\bar{z}^j)-(\sum_j \bar{z}^j dz^j)\wedge(\sum_j z^j d\bar{z}^j)\}.$$

It is well-known that there exists a unique 2-form Ω of type (1,1) on $P_n(\boldsymbol{C})$ such that $\Pi^*\Omega=\tilde{\Omega}$. Then Ω is the Kähler form associated with the Fubini-Study metric on $P_n(\boldsymbol{C})$ and we have

$$(1.5)\qquad \int_{P_n(C)} \Omega^n = 1.$$

We consider a family of subspaces H of $\boldsymbol{C}^{n+1}$ such that H is of $(n-1)$-dimension and $\bar{Z}\in H$ whenever $Z\in H$. With such an H, we can associate uniquely a real subspace of $\boldsymbol{R}^{n+1}$ of dimension 2 by

$$(1.6)\qquad \{X\in\boldsymbol{R}^{n+1}: \langle X, H\rangle = 0\}.$$

We see that this gives a one to one correspondence, and hence the above family of H's is parametrized by the Grassmann manifold $G(\boldsymbol{R})$ of 2 planes in $\boldsymbol{R}^{n+1}$. Especially we note that $[H]=\Pi(H-\{0\})$ is an $(n-2)$-dimensional projective space in $P_n(\boldsymbol{C})$.

On $P_n(\boldsymbol{C})$ with homogeneous coordinate $z^0, \cdots, z^n$ the complex quadratic $Q_{n-1}(\boldsymbol{C})$ is a complex hypersurface defined by the equation

$$(1.7)\qquad (z^0)^2 + \cdots + (z^n)^2 = 0.$$

Now the unit sphere $S^{2n+1}=\{Z\in\boldsymbol{C}^{n+1}: ||Z||=1\}$ is a principal fibre bundle over

$P_n(\boldsymbol{C})$ with structure group S^1. For a point $q \in Q_{n-1}(\boldsymbol{C})$, take a point $Z \in S^{2n+1}$ such that $\Pi(Z)=q$. We can write Z uniquely in the form $Z=(X+iY)/\sqrt{2}$, where X and Y are orthonormal real vectors in $\boldsymbol{C}^{n+1}$. Conversely if $Z=(X+iY)/\sqrt{2} \in S^{2n+1}$ for orthonormal real vectors X and Y, then we have $\Pi(Z) \in Q_{n-1}(\boldsymbol{C})$. Therefore we have

$$S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\boldsymbol{C})) = \{Z = (X+iY)/\sqrt{2} : X \text{ and } Y \text{ are orthonormal real vectors}\}. \tag{1.8}$$

The group $SO(n+1)$, considered as a subgroup of $U(n+1)$, acts on S^{2n+1} and leaves the submanifold $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\boldsymbol{C}))$ invariant. Moreover $SO(n+1)$ acts transitively on $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\boldsymbol{C}))$. The isotropy subgroup of $SO(n+1)$ at $Z_0=(1/\sqrt{2}, i/\sqrt{2}, 0, \cdots, 0)$ coincides with the subgroup $SO(n-1)$ of $SO(n+1)$. We denote an element g of $SO(n+1)$ by

$$g = (X_0, X_1, \cdots, X_n),$$

where each X_i is a column vector. Then, in the space $SO(n+1)/SO(n-1)$, the coset including $g=(X_0, X_1, \cdots, X_n)$ can be represented by the first two vectors (X_0, X_1). Under this identification, we have a diffeomorphism i: $SO(n+1)/SO(n-1) \to S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\boldsymbol{C}))$ defined by

$$i((X_0, X_1)) = \frac{1}{\sqrt{2}}(X_0+iX_1). \tag{1.9}$$

From now on we also identify $SO(n+1)/SO(n-1)$ with $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\boldsymbol{C}))$ by the above diffeomorphism. We denote by Π_1 the projection: $SO(n+1)/SO(n-1) \to Q_{n-1}(\boldsymbol{C})$ defined by

$$\Pi_1((X_0, X_1)) = \Pi((X_0+iX_1)/\sqrt{2}) \tag{1.10}$$

for $(X_0, X_1) \in SO(n+1)/SO(n-1)$. Note that the space $Q_{n-1}(\boldsymbol{C})$ also can be identified canonically with $SO(n+1)/SO(2) \times SO(n-1)$.

To each point $\alpha=\Pi_1((X_0, X_1))$ in $Q_{n-1}(\boldsymbol{C})$, we assign the 2-dimensional linear space spanned by $\{X_0, X_1\}$ in $\boldsymbol{R}^{n+1}$. Through this assignment, $Q_{n-1}(\boldsymbol{C})$ is a double covering space of $G(\boldsymbol{R})$. We see that the function $|\langle Z, W\rangle|^2$ on $S^{2n+1} \times S^{2n+1}$ induces a function $|\Pi(Z), \Pi(W)|^2$ on $P_n(\boldsymbol{C}) \times P_n(\boldsymbol{C})$. For each $\alpha \in Q_{n-1}(\boldsymbol{C})$, we consider a complex submainifold ξ_α of $Q_{n-1}(\boldsymbol{C})$, defined by

$$\xi_\alpha = \{\beta \in Q_{n-1}(\boldsymbol{C}) : |\beta, \alpha|^2 + |\beta, \bar{\alpha}|^2 = 0\}. \tag{1.11}$$

Let $(X_0, X_1) \in SO(n+1)/SO(n-1)$ and set $\Pi_1((X_0, X_1))=\alpha$. Consider the complex subspace H of $\boldsymbol{C}^{n+1}$ orthogonal to the vectors X_0, X_1. We have $\xi_\alpha = Q_{n-1}(\boldsymbol{C}) \cap [H]$. $[H]$ is a Poincaré dual of the form Ω^2 in $P_n(\boldsymbol{C})$, and hence ξ_α is also, in $Q_{n-1}(\boldsymbol{C})$, a Poincaré dual of the form Ω^2 restricted to $Q_{n-1}(\boldsymbol{C})$. Finally we remark that each ξ_α is a complex quadratic $Q_{n-3}(\boldsymbol{C})$ and $\xi_\alpha = \xi_{\bar{\alpha}}$.

2. Holomorphic mapping

Let f be a holomorphic mapping of $\boldsymbol{C}^2$ into $Q_{n-1}(\boldsymbol{C})$ $(n \geqq 3)$. We consider the following two conditions on f.

Condition (A): *f is an immersion.*

Condition (B): *For each $\alpha \in Q_{n-1}(\boldsymbol{C})$, the set $\{p \in \boldsymbol{C}^2 : f(p) \in \xi_\alpha\}$ is discrete.*

For each point $p \in \boldsymbol{C}^2$, we can take a small neighborhood $U(p)$ of p such that there exists a holomorphic lift $F=(f^0, \cdots, f^n)$ of f on $U(p)$ into $\boldsymbol{C}^{n+1}-\{0\}$ i.e., $\Pi F=f$.

Proposition 2.1. *Condition (A) is equivalent to the following: for each point p of $\boldsymbol{C}^2$, choose a holomorphic lift $F=(f^0, \cdots, f^n)$ of f on a neighborhood U of p, then we have*

$$\text{(2.1)} \qquad \operatorname{rank}\begin{pmatrix} f^0, & \cdots, & f^n \\ \dfrac{\partial f^0}{\partial w_1}, & \cdots, & \dfrac{\partial f^n}{\partial w_1} \\ \dfrac{\partial f^0}{\partial w_2}, & \cdots, & \dfrac{\partial f^n}{\partial w_2} \end{pmatrix}(p) = 3\,,$$

where (w_1, w_2) is a coordinate system on the neighborhood U.

Proof. We identify the real tangent space $T_Z(\boldsymbol{C}^{n+1})$ at a point Z in $\boldsymbol{C}^{n+1}$ with $\boldsymbol{C}^{n+1}$ in the ususal way. For p, we take $(X_0, X_1, \cdots, X_n) \in SO(n+1)$ such that $(X_0+iX_1)/\sqrt{2}=(F/\|F\|)(p)$. Then the tangent space $T_{(X_0+iX_1)/\sqrt{2}}(S^{2n+1})$ has a basis $i(X_0+iX_1)$, X_0-iX_1, $i(X_0-iX_1)$, $X_2, \cdots, X_n, iX_2, \cdots, iX_n$. Let $T_{f(p)}$ be the subspace spanned by $X_2, \cdots, X_n, iX_2, \cdots, iX_n$. The projection $\tilde{\Pi}=\Pi_{|S^{2n+1}\cap \Pi^{-1}(Q_{n-1}(C))}$ induces a linear isomorphism $\tilde{\Pi}_*\colon T_{f(p)} \to T_{f(p)}(Q_{n-1}(\boldsymbol{C}))$ (c.f. [3] p.p. 279). Hence, $T_{f(p)}(Q_{n-1}(\boldsymbol{C}))$ is identified with the subspace of $\boldsymbol{C}^{n+1}$ orthogonal to the vectors $(F/\|F\|)(p)$ and $(\bar{F}/\|F\|)(p)$ with respect to $\langle\ ,\ \rangle$. Since we have $\langle F, \bar{F}\rangle=0$ on U, we see $\langle dF, \bar{F}\rangle=0$. We have

$$\text{(2.2)} \qquad d\Big(\frac{F}{\|F\|}\Big)=\frac{1}{\|F\|}\sum_{j=1}^{2}\Big(\frac{\partial F}{\partial w_j}-\Big\langle\frac{\partial F}{\partial w_j}, \frac{F}{\|F\|}\Big\rangle\frac{F}{\|F\|}\Big)dw_j + \sum_{j=1}^{2} iF\frac{\partial}{\partial y^j}\Big(\frac{1}{\|F\|}\Big)dx^j-\sum_{j=1}^{2} iF\frac{\partial}{\partial x^j}\Big(\frac{1}{\|F\|}\Big)dy^j\,,$$

where $w_j=x^j+iy^j$. Therefore we get

$$\text{(2.3)} \qquad df=\sum_{j=1}^{2}\tilde{\Pi}_*\Big[\frac{1}{\|F\|}\Big(\frac{\partial F}{\partial w_j}-\Big\langle\frac{\partial F}{\partial w_j}, \frac{F}{\|F\|}\Big\rangle\frac{F}{\|F\|}\Big)\Big]dw_j\,.$$

This shows Proposition 2.1. Q.E.D.

We define

$$(2.4)\qquad Q_{n-3}(f(p)^{\perp})=\{\alpha\in Q_{n-1}(\boldsymbol{C}):\ |f(p),\alpha|^2+|f(p),\bar{\alpha}|^2=0\}\,,$$

that is,

$$Q_{n-3}(f(p)^{\perp})=\{\alpha\in Q_{n-1}(\boldsymbol{C}): f(p)\in\xi_{\alpha}\}\,.$$

Then $Q_{n-3}(f(p)^{\perp})$ can be identified with $SO(n-1)/SO(2)\times SO(n-3)$ as follows: Choose an element $(X_0, X_1, \cdots, X_n)\in SO(n+1)$ such that $(X_0+iX_1)/\sqrt{2}=(F/||F||)(p)$. Let $(A_2, A_3)\in SO(n-1)/SO(n-3)$ where $A_i=(a_{i2},\cdots,a_{in})^t$ ($i=2, 3$). Consider the mapping

$$(2.5)\qquad (A_2, A_3)\to(\textstyle\sum_{i=2}^{n}a_{2i}X_i,\ \sum_{i=2}^{n}a_{3i}X_i)\,.$$

We see easily that this gives an identification of $SO(n-1)/SO(2)\times SO(n-3)$ with $Q_{n-3}(f(p)^{\perp})$, which is independent of the choice of lift F.

For $\alpha\in Q_{n-3}(f(p)^{\perp})$ we take $(X_0, X_1)\in SO(n+1)/SO(n-1)$ such that $\Pi_1((X_0, X_1))=\alpha$. Then the following condition is independent of the choice of (X_0, X_1),

$$(2.6)\qquad \begin{vmatrix}\langle(\partial F/\partial w_1)(p),(X_0+iX_1)/\sqrt{2}\rangle, & \langle(\partial F/\partial w_2)(p),(X_0+iX_1)/\sqrt{2}\rangle\\ \langle(\partial F/\partial w_1)(p),(X_0-iX_1)/\sqrt{2}\rangle, & \langle(\partial F/\partial w_2)(p),(X_0-iX_1)/\sqrt{2}\rangle\end{vmatrix}\neq 0\,.$$

Proposition 2.2. *The condition* (2.6) *holds if and only if f intersects transversely with ξ_{α} at $f(p)$.*

Proof. Put $(F/||F||)(p)=(X_2+iX_3)/\sqrt{2}$. Then we take an element $(X_0, X_1, X_2, X_3, \cdots, X_n)\in SO(n+1)$. As in the proof of Proposition 2.1, we see that the tangent space $T_{f(p)}(Q_{n-1}(\boldsymbol{C}))$ is spanned by the vectors $X_0, iX_0, X_1, iX_1, X_4, iX_4, \cdots, X_n, iX_n$ and the tangent space $T_{f(p)}(\xi_{\alpha})$ is spanned by $X_4, iX_4, \cdots, X_n, iX_n$ through the identification by $\tilde{\Pi}_*\colon T_{(X_2+iX_3)/\sqrt{2}}(S^{2n+1}\cap\Pi^{-1}(Q_{n-1}(\boldsymbol{C})))\to T_{f(p)}(Q_{n-1}(\boldsymbol{C}))$. Therefore by (2.3) (or (2.2)) it is sufficient to show that the condition (2.6) is equivalent to $\mathrm{rank}_{\boldsymbol{R}}\ ((\partial F/\partial w_1)(p), i(\partial F/\partial w_1)(p), (\partial F/\partial w_2)(p), i(\partial F/\partial w_2)(p), X_2, iX_2, \cdots, X_n, iX_n)=2(n+1)$. Now this can be seen easily.

Q.E.D.

Now we consider the following condition for $\alpha=\Pi_1((X_0, X_1))\in Q_{n-3}(f(p)^{\perp})$

$$(2.7)\qquad \begin{vmatrix}\langle(\partial F/\partial w_1)(p),(X_0+iX_1)/\sqrt{2}\rangle, & \langle(\partial F/\partial w_2)(p),(X_0+iX_1)/\sqrt{2}\rangle\\ \langle(\partial F/\partial w_1)(p),(X_0-iX_1)/\sqrt{2}\rangle, & \langle(\partial F/\partial w_2)(p),(X_0-iX_1)/\sqrt{2}\rangle\end{vmatrix}=0$$

Since the vectors $(\partial F/\partial w_1)(p)$ and $(\partial F/\partial w_2)(p)$ are linearly independent, the set of elements $\alpha\in Q_{n-3}(f(p)^{\perp})$ satisfying the condition (2.7) has measure zero in $Q_{n-3}(f(p)^{\perp})$.

REMARK 1. We shall remark here a certain sufficient condition for Condition (B). For $w\in\boldsymbol{C}$ we put $\boldsymbol{C}_w^1=\{(z, w): z\in\boldsymbol{C}\}$ and $\boldsymbol{C}_w^2=\{(w, z): z\in\boldsymbol{C}\}$.

Assume the following condition (C): *none of* $f(C_w^i)$ $(i=1, 2, w\in C)$ *is contained in a hyperplane in* $P_n(C)$. Let $f(p)\in\xi_\alpha$ and set $\prod_1((X_0, X_1))=\alpha$. We put $g_1(w_1,w_2)=\langle F, (X_0+iX_1)/\sqrt{2}\rangle(w_1, w_2)$ and $g_2(w_1, w_2)=\langle F, (X_0-iX_1)/\sqrt{2}\rangle(w_1, w_2)$ on $U(p)$, where (w_1, w_2) is a coordinate system on $U(p)$ such that $w_i(p)=0$ $(i=1, 2)$. Using the Weierstrass' preparation theorem we have the following representations

$$\begin{aligned} g_1(w_1, w_2) &= (a_0(w_1)+a_1(w_1)w_2+\cdots+a_{l_1}(w_1)w_2^{l_1})h_1(w_1, w_2) \\ g_2(w_1, w_2) &= (b_0(w_1)+b_1(w_1)w_2+\cdots+b_{l_2}(w_1)w_2^{l_2})h_2(w_1, w_2)\,, \end{aligned} \tag{2.8}$$

where $a_i(w_1)$, $b_i(w_1)$ and $h_i(w_1, w_2)$ are holomorphic such that $a_i(0)=0$ for $0\leqslant i<l_1$, $a_{l_1}(0)\neq 0$, $b_i(0)=0$ for $0\leqslant i<l_2$, $b_{l_2}(0)\neq 0$ and $h_i(w_1, w_2)\neq 0$ $(i=1, 2)$. We denote by $R(w_1)$ the resultant of $(a_0(w_1)+\cdots+a_{l_1}(w_1)w_2^{l_1})$ and $(b_0(w_1)+\cdots+b_{l_2}(w_1)w_2^{l_2})$. Since the function $R(w_1)$ is holomorphic, we have that $R(w_1)\equiv 0$ or the following (D): *the set* $\{w_1: R(w_1)=0\}$ *is discrete*. If, under the assumption of (C), f satisfies (D) for each $p\in C^2$ and $\alpha\in Q_{n-1}(C)$ such that $f(p)\in\xi_\alpha$, then Condition (B) holds.

3. Certain forms on $Q_{n-1}(C)-\xi_\alpha$

We define one 2-form Ω_α on $Q_{n-1}(C)-\xi_\alpha$ by

$$\Omega_\alpha(\beta) = dd^c \log\{|\beta, \alpha|^2+|\beta, \bar\alpha|^2\}\,, \tag{3.1}$$

where $d^c = \dfrac{1}{4\pi i}(\partial-\bar\partial)$. We choose a unit vector Z_α such that $\prod(Z_\alpha)=\alpha$, and define a mapping P_α of $Q_{n-1}(C)-\xi_\alpha$ into $P_1(C)$ by

$$P_\alpha(\beta) = \hat{\prod}\left[\frac{1}{(|\beta, \alpha|^2+|\beta, \bar\alpha|^2)^{1/2}}(\langle Z_\beta, Z_\alpha\rangle, \langle Z_\beta, \bar Z_\alpha\rangle)\right], \tag{3.2}$$

where $Z_\beta\in S^{2n+1}$ such that $\prod(Z_\beta)=\beta$, and $\hat{\prod}$ is the Hopf fibring $S^3\to P_1(C)$. P_α is well-defined and holomorphic. Let ω be the Kähler 2-form associated with the Fubini-Study metric on $P_1(C)$ and normalized as $\int_{P_1(C)}\omega=1$. Then $P_\alpha^*\omega$ is independent of the choice of Z_α. From now on we also denote by Ω the restriction of the form Ω to $Q_{n-1}(C)$.

Lemma 3.1. *We have*

$$\Omega_\alpha = P_\alpha^*\omega-\Omega \qquad on\ \ Q_{n-1}(C)-\xi_\alpha\,. \tag{3.3}$$

Proof. Let σ be a local holomorphic cross-section of the Hopf fibring $\prod$: $C^{n+1}-\{0\}\to P_n(C)$ defined on an open set U in $Q_{n-1}(C)-\xi_\alpha$. Then we have

$$\begin{aligned} \Omega_\alpha &= dd^c \log\left\{\left|\left\langle\frac{\sigma}{\|\sigma\|}, Z_\alpha\right\rangle\right|^2+\left|\left\langle\frac{\sigma}{\|\sigma\|}, \bar Z_\alpha\right\rangle\right|^2\right\} \\ &= dd^c \log\{|\langle\sigma, Z_\alpha\rangle|^2+|\langle\sigma, \bar Z_\alpha\rangle|^2\}-dd^c\log\|\sigma\|^2 \\ &= P_\alpha^*\omega-\Omega\,. \end{aligned}$$

Q.E.D.

We define another 2-form Ω'_α on $Q_{n-1}(\boldsymbol{C})-\xi_\alpha$ by

$$\Omega'_\alpha=\Omega+P_\alpha^*\omega \qquad \text{on } Q_{n-1}(\boldsymbol{C})-\xi_\alpha . \tag{3.4}$$

Put

$$\Omega''_\alpha=-\Omega_\alpha\wedge\Omega'_\alpha \qquad \text{on } Q_{n-1}(\boldsymbol{C})-\xi_\alpha . \tag{3.5}$$

By (3.3) and (3.4), we have

$$\begin{aligned}\Omega''_\alpha&=(\Omega-P_\alpha^*\omega)\wedge(\Omega+P_\alpha^*\omega)\\&=\Omega^2-P_\alpha^*(\omega\wedge\omega)=\Omega^2 \qquad \text{on } Q_{n-1}(\boldsymbol{C})-\xi_\alpha .\end{aligned} \tag{3.5$'$}$$

Let $f\colon \boldsymbol{C}^2\to Q_{n-1}(\boldsymbol{C})$ $(n\geqq 3)$ be a holomorphic mapping satisfying Conditions (A) and (B) in §2. For a point p in $\boldsymbol{C}^2$, we take a small neighborhood $U(p)$ of p and a coordinate system (w_1, w_2) on it satisfying $w_i(p)=0$ $(i=1, 2)$. Let F be a holomorphic lift of f on $U(p)$ into $\boldsymbol{C}^{n+1}-\{0\}$. Set $f(p)\in\xi_\alpha$. Then we define a real number $n(p, \alpha)$ by

$$n(p,\alpha)=\lim_{\varepsilon\downarrow 0}\int_{\partial U_\varepsilon(p)} d^c\cdot\log\{|\langle F, Z_\alpha\rangle|^2+|\langle F, \bar{Z}_\alpha\rangle|^2\}\wedge f^*P_\alpha^*\omega , \tag{3.6}$$

where $U_\varepsilon(p)=\{(w_1, w_2)\in U(p)\colon |w_1|^2+|w_2|^2<\varepsilon^2\}$ and $\Pi(Z_\alpha)=\alpha$.

Lemma 3.2. *$n(p, \alpha)$ is well-defined and finite. Especially if f intersects transversely with ξ_α at $f(p)$, then we have $n(p, \alpha)=1$.*

Proof. First we choose a local lift F and a local coordinate system (w_1, w_2) such that $w_i(p)=0$. Take two positive real numbers ε_1 and ε_2 such that $U(p)\supset U_{\varepsilon_1}(p)\supset U_{\varepsilon_2}(p)$. Then we have

$$\begin{aligned}0&=\int_{U_{\varepsilon_1}(p)-U_{\varepsilon_2}(p)} f^*P_\alpha^*(\omega\wedge\omega)\\&=\int_{\partial U_{\varepsilon_1}(p)-\partial U_{\varepsilon_2}(p)} d^c\log\{|\langle F, Z_\alpha\rangle|^2+|\langle F, \bar{Z}_\alpha\rangle|^2\}\wedge f^*P_\alpha^*\omega .\end{aligned} \tag{3.7}$$

Therefore we obtain

$$\begin{aligned}&\int_{\partial U_{\varepsilon_1}(p)} d^c\log\{|\langle F, Z_\alpha\rangle|^2+|\langle F, \bar{Z}_\alpha\rangle|^2\}\wedge f^*P_\alpha^*\omega\\&\quad=\lim_{\varepsilon\downarrow 0}\int_{\partial U_\varepsilon(p)} d^c\log\{|\langle F, Z_\alpha\rangle|^2+|\langle F, \bar{Z}_\alpha\rangle|^2\}\wedge f^*P_\alpha^*\omega .\end{aligned} \tag{3.8}$$

The left hand-side of the equation (3.8) is finite and hence so is the right side. In the same way, we see that $n(p, \alpha)$ is independent of the choice of a local coordinate system. Now we shall show that $n(p, \alpha)$ is independent of the choice of F. Take two holomorphic lift F_1 and F_2 of f. Then there exists a holomorphic function g such that $F_1=gF_2$ and $g(q)\neq 0$ at any $q\in U(p)$. We have

$$\begin{aligned}(3.9)\qquad & d^c\log\{|\langle F_1, Z_\alpha\rangle|^2+|\langle F_1, \bar{Z}_\alpha\rangle|^2\} \\ &= d^c\log|g|^2+d^c\log\{|\langle F_2, Z_\alpha\rangle|^2+|\langle F_2, \bar{Z}_\alpha\rangle|^2\} \\ &= \frac{1}{4\pi i}[d\log g-d\log \bar{g}]+d^c\log\{|\langle F_2, Z_\alpha\rangle|^2+|\langle F_2, \bar{Z}_\alpha\rangle|^2\}\,.\end{aligned}$$

Since the form $f^*P_\alpha^*\omega$ is closed on $\partial U_\varepsilon(p)$, $n(p, \alpha)$ is independent of the choice of F.

Next suppose that f intersects transversely with ξ_α at $f(p)$. Then

$$\begin{vmatrix}\langle\partial F/\partial w_1, Z_\alpha\rangle, & \langle\partial F/\partial w_2, Z_\alpha\rangle \\ \langle\partial F/\partial w_1, \bar{Z}_\alpha\rangle, & \langle\partial F/\partial w_2, \bar{Z}_\alpha\rangle\end{vmatrix}(p)\neq 0\,,$$

and hence we can choose $(w_1, w_2)=(\langle F, Z_\alpha\rangle, \langle F, \bar{Z}_\alpha\rangle)$ as a coordinate system on $U(p)$. We have

$$n(p, \alpha) = \lim_{\varepsilon\downarrow 0}\int_{|w_1|^2+|w_2|^2=\varepsilon^2} d^c\log(|w_1|^2+|w_2|^2)\wedge f^*P_\alpha^*\omega\,.$$

Putting $w_1=r_1e^{i\theta_1}$, $w_2=r_2e^{i\theta_2}$, $r_1=r\cos t$ and $r_2=r\sin t$ $(0\leqslant\theta_i\leqslant 2\pi,\ 0\leqslant t\leqslant\pi/2)$, we have

$$d^c\log(r_1^2+r_2^2) = \frac{1}{2\pi}\frac{1}{r_1^2+r_2^2}(r_1^2d\theta_1+r_2^2d\theta_2)\,,$$

and

$$\begin{aligned}f^*P_\alpha^*\omega = \frac{1}{\pi}\frac{1}{(r_1^2+r_2^2)}&(r_1r_2^2dr_1\wedge d\theta_1+r_1^2r_2dr_2\wedge d\theta_2 \\ &-r_1r_2^2dr_1\wedge d\theta_2-r_1^2r_2dr_2\wedge d\theta_1)\,.\end{aligned}$$

Thus we see

$$d^c\log(r_1^2+r_2^2)\wedge f^*P_\alpha^*\omega = \frac{1}{2\pi^2}\sin t\cos t\, d\theta_1\wedge dt\wedge d\theta_2$$

$$\text{on } r = \text{constant.}$$

On the sphere $\{(w_1, w_2)\in U(p)\colon |w_1|^2+|w_2|^2=r^2\}$, $d\theta_1\wedge dt\wedge d\theta_2$ is a positive form. Therefore we have $n(p, \alpha)=1$. Q.E.D.

We denote by (z_1, z_2) the standard coordinate system on C^2. Put $\Delta(r)=\{(z_1, z_2)\in C^2\colon \log|z_i|<r(i=1, 2)\}$.

Theorem 1. *Let $f\colon C^2\to Q_{n-1}(C)$ $(n\geqq 3)$ be a holomorphic mapping satisfying (A) and (B). Suppose $f(\partial\Delta(r))\cap\xi_\alpha=\phi$. Then we have*

$$(3.10)\quad \int_{\Delta(r)} f^*\Omega^2 = n(\Delta(r), \alpha)+\int_{\partial\Delta(r)} d^c[-\log(|f, \alpha|^2+|f, \bar{\alpha}|^2)f^*(\Omega+P_\alpha^*\omega)]\,,$$

where $n(\Delta(r), \alpha)=\sum_{f(p_i)\in\xi_\alpha,\, p_i\in\Delta(r)} n(p_i, \alpha)$.

Proof. By (3.1), Lemma 3.1, (3.5) and (3.5)′, we have

$$(3.11)\qquad \int_{\Delta(r)} f^*\Omega^2 = \lim_{\varepsilon\downarrow 0}\int_{\Delta(r)-\sum_i U_\varepsilon(p_i)} f^*\Omega^2$$
$$= \lim_{\varepsilon\downarrow 0}\int_{\Delta(r)-\sum_i U_\varepsilon(p_i)} -dd^c\cdot\log(|f,\alpha|^2+|f,\bar\alpha|^2)\wedge f^*(\Omega+P_\alpha^*\omega)$$
$$= \lim_{\varepsilon\downarrow 0}\int_{\Delta(r)-\sum_i U_\varepsilon(p_i)} dd^c[-\log(|f,\alpha|^2+|f,\bar\alpha|^2)f^*(\Omega+P_\alpha^*\omega)]\,,$$

where $U_\varepsilon(p_i)$ is such a neighborhood of p_i as given in the definition $n(p_i, \alpha)$. Applying Stokes Theorem to the equation (3.11), we have

$$(3.12)\qquad \int_{\Delta(r)} f^*\Omega^2 = \int_{\partial\Delta(r)} d^c[-\log(|f,\alpha|^2+|f,\bar\alpha|^2)f^*(\Omega+P_\alpha^*\omega)]$$
$$-\lim_{\varepsilon\downarrow 0}\sum_i\int_{\partial U_\varepsilon(p_i)} d^c[\log\|F_i\|^2 f^*(\Omega+P_\alpha^*\omega)]$$
$$+\lim_{\varepsilon\downarrow 0}\sum_i\int_{\partial U_\varepsilon(p_i)} d^c[\log\{|\langle F_i, Z_\alpha\rangle|^2+|\langle F_i, \bar Z_\alpha\rangle|^2\}f^*\Omega]$$
$$+\sum_i n(p_i, \alpha)\,,$$

where F_i is a holomorphic lift of f on $U(p_i)$. We have

$$(3.13)\qquad \lim_{\varepsilon\downarrow 0}\int_{\partial U_\varepsilon(p_i)} d^c[\log\|F_i\|^2\cdot f^*\Omega] = \lim_{\varepsilon\downarrow 0}\int_{U_\varepsilon(p_i)} f^*\Omega^2 = 0\,.$$

Set $r^2=|w_i^1|^2+|w_i^2|^2$, where (w_i^1, w_i^2) denotes a coordinate system on $U(p_i)$, we see

$$(3.14)\qquad d^c\log\{|\langle F_i, Z_\alpha\rangle|^2+|\langle F_i, \bar Z_\alpha\rangle|^2\} = 0\left(\frac{1}{r}\right)(dw_i^1+d\bar w_i^1+dw_i^2+d\bar w_i^2)$$

and

$$(3.15)\qquad dd^c\log\{|\langle F_i, Z_\alpha\rangle|^2+|\langle F_i, \bar Z_\alpha\rangle|^2\} = 0\left(\frac{1}{r^2}\right)(dw_i^1\wedge d\bar w_i^1+dw_i^1\wedge d\bar w_i^2$$
$$+dw_i^2\wedge d\bar w_i^2+dw_i^2\wedge d\bar w_i^1)\,.$$

Since $\|F_i\|$ is positive on $U(p_i)$, we have

$$(3.16)\qquad d^c\log\|F_i\|^2 = 0(1)(dw_i^1+d\bar w_i^1+dw_i^2+d\bar w_i^2)$$

and

$$(3.17)\qquad f^*\Omega = 0(1)(dw_i^1\wedge d\bar w_i^1+dw_i^1\wedge d\bar w_i^2+dw_i^2\wedge d\bar w_i^2+dw_i^2\wedge d\bar w_i^1)\,.$$

Since the both sides of the equation (3.8) are finite, comparing (3.14) and (3.15) with (3.16) and (3.17), we have

$$(3.18)\qquad \lim_{\varepsilon\downarrow 0}\int_{\partial U_\varepsilon(p_i)} d^c[\log\|F_i\|^2\cdot f^*P_\alpha^*\omega] = 0$$

$$(3.19)\qquad \lim_{\varepsilon\downarrow 0}\int_{\partial U_\varepsilon(p_i)} d^c[\log\{|\langle F_i, Z_\alpha\rangle|^2+|\langle F_i, \bar{Z}_\alpha\rangle|^2\}f^*\Omega] = 0\,.$$

Q.E.D.

4. First Main Theorem

Let $f: C^2\to Q_{n-1}(C)$ ($n\geqq 3$) be a holomorphic mapping satisfying (A) and (B). For a point α in $Q_{n-1}(\boldsymbol{C})$, we choose two real numbers r_1 and r_2 such that $r_1>r_2$ and the image $f(\overline{(r(\Delta_1)\backslash\Delta(r_2)})$ does not intersect with ξ_α.

We see easily $|\beta, \alpha|^2+|\beta, \bar{\alpha}|^2\leqq 1$ for $\beta\in Q_{n-1}(\boldsymbol{C})$. Hence $\psi_\alpha=-\log(|f, \alpha|^2+|f, \bar{\alpha}|^2)f^*(\Omega+P_\alpha^*\omega)$ is a positive form (non-negative form, precisely) on $\Delta(r_1)\backslash\Delta(r_2)$. Putting $z_j=e^{s_j+i\theta_j}(j=1, 2)$, we can write ψ_α on $\Delta(r_1)\backslash(\Delta(r_2)\cup\{(z, 0)\in\boldsymbol{C}^2\}\cup\{0, z)\in\boldsymbol{C}^2\})$ as follows:

$$(4.1)\qquad \begin{aligned}\psi_\alpha &= -\log(|f, \alpha|^2+|f, \bar{\alpha}|^2)f^*(\Omega+P_\alpha^*\omega)\\ &= \psi_1 ds_1\wedge d\theta_1+\psi_2 ds_1\wedge d\theta_2+\psi_3 ds_2\wedge d\theta_1\\ &\quad +\psi_4 ds_2\wedge d\theta_2+\psi_5 d\theta_1\wedge d\theta_2+\psi_6 ds_1\wedge ds_2\,.\end{aligned}$$

REMARK 2. If we write ψ_α with the standard coordinate system (z_1, z_2) on $\boldsymbol{C}^2$, we see $\psi_1(z_1, z_2)=\tilde{\psi}_1(z_1, z_2)e^{2s_1}$, $\psi_4(z_1, z_2)=\tilde{\psi}_4(z_1, z_2)e^{2s_2}$ and $\psi_j(z_1, z_2)=e^{s_1}\cdot e^{s_2}\tilde{\psi}_j(z_1, z_2)$ $(j=2, 3, 5, 6)$ for certain functions $\tilde{\psi}_i(i=1, 2, \cdots, 6)$.

Lemma 4.1. *We have*

$$(4.2)\qquad \psi_1\geqq 0,\ \psi_4\geqq 0 \ \textit{and}\ \psi_2=\psi_3\,.$$

Proof. Choosing a holomorphic lift F on a sufficiently small open set U in $\Delta(r_1)\backslash\Delta(r_2)$, we have

$$(4.3)\qquad f^*(\Omega+P_\alpha^*\omega) = dd^c[\log\|F\|^2+\log(|\langle F, Z_\alpha\rangle|^2+|\langle F, \bar{Z}_\alpha\rangle|^2)]\,,$$

where $\Pi(Z_\alpha)=\alpha$. Now we obtain

$$(4.4)\qquad \begin{aligned} d^c &= \frac{1}{4\pi}\sum_{j=1}^{2}\left[\frac{\partial}{\partial s_j}d\theta_j-\frac{\partial}{\partial\theta_j}ds_j\right]\\ d &= \sum_{j=1}^{2}\left[\frac{\partial}{\partial\theta_j}d\theta_j+\frac{\partial}{\partial s_j}ds_j\right]\end{aligned}\quad \text{on } U\backslash(\{(0, z)\in\boldsymbol{C}^2\}\cup\{(z, 0)\in\boldsymbol{C}^2\})\,,$$

where $(e^{s_1+i\theta_1}, e^{s_2+i\theta_2})$ is the restriction to U of the standard coordinate system in $\boldsymbol{C}^2$. Putting $g=\log(|\langle F, Z_\alpha\rangle|^2+|\langle F, \bar{Z}_\alpha\rangle|^2)+\log\|F\|^2$, we have

$$(4.5)\qquad \begin{aligned} dd^c g = \frac{1}{4\pi}\Big[&\Big(\frac{\partial^2 g}{(\partial\theta_1)^2}+\frac{\partial^2 g}{(\partial s_1)^2}\Big)ds_1\wedge d\theta_1+\Big(\frac{\partial^2 g}{\partial\theta_2\partial\theta_1}+\frac{\partial^2 g}{\partial s_1\partial s_2}\Big)ds_1\wedge d\theta_2\\ &+\Big(\frac{\partial^2 g}{\partial\theta_1\partial\theta_2}+\frac{\partial^2 g}{\partial s_2\partial s_1}\Big)ds_2\wedge d\theta_1+\Big(\frac{\partial^2 g}{(\partial\theta_2)^2}+\frac{\partial^2 g}{(\partial s_2)^2}\Big)ds_2\wedge d\theta_2+\cdots\Big]\,.\end{aligned}$$

Comparing (4.1) with (4.5), we have $\psi_2=\psi_3$.

We shall show $\psi_1 \geqq 0$ and $\psi_4 \geqq 0$.

$$(4.6)\qquad dd^c \log(\textstyle\sum_j f^j \bar{f}^j) = \dfrac{i}{2\pi}\partial\bar{\partial}\cdot\log(\textstyle\sum_j f^j \bar{f}^j)$$

$$= \frac{i}{2\pi}\frac{1}{\|F\|^4}[\|F\|^2(\textstyle\sum_j df^j \wedge d\bar{f}^j) - (\sum_k df^k \bar{f}^k) \wedge (\sum_j f^j d\bar{f}^j)]$$

$$= \frac{i}{2\pi}\frac{1}{\|F\|^4}\Bigg[\left(\|F\|^2\left\|\frac{\partial F}{\partial z_1}\right\|^2 - \left|\left(\frac{\partial F}{\partial z_1}, F\right)\right|^2\right)dz_1 \wedge d\bar{z}_1$$

$$+\left(\|F\|^2\left\|\frac{\partial F}{\partial z_2}\right\|^2 - \left|\left(\frac{\partial F}{\partial z_2}, F\right)\right|^2\right)dz_2 \wedge d\bar{z}_2 + \cdots\Bigg],$$

where $F=(f^0, f^1, \cdots, f^n)$. By the Schwartz inequality and the linear independence of vectors F and $\partial F/\partial z_j$ ($j=1, 2$), we have

$$\|F\|^2\left\|\frac{\partial F}{\partial z_j}\right\|^2 > \left|\left(\frac{\partial F}{\partial z_j}, F\right)\right|^2, \text{ and } dz_j \wedge d\bar{z}_j = e^{2s_j}(-2i ds_j \wedge d\theta_j)$$

($j=1, 2$). Thus we have

$$\frac{1}{\pi}\frac{1}{\|F\|^4}\left[\|F\|^2\left\|\frac{\partial F}{\partial z_j}\right\|^2 - \left|\left\langle\frac{\partial F}{\partial z_j}, F\right\rangle\right|^2\right]e^{2s_j} > 0 \ (j=1, 2)$$

or

$$(4.7)\qquad \frac{1}{\pi}\frac{1}{(\sum_k f^k \bar{f}^k)^2}\left[(\sum_k f^k \bar{f}^k)\left(\sum_k \frac{\partial f^k}{\partial z_j}\overline{\frac{\partial f^k}{\partial z_j}}\right) - \left|\left(\sum_k \frac{\partial f^k}{\partial z_j}\bar{f}^k\right)\right|^2\right]e^{2s_j} > 0 \ (j=1,2).$$

As for $dd^c[\log(|\langle F, Z_\alpha\rangle|^2 + |\langle F, \bar{Z}_\alpha\rangle|^2)]$, putting $f^0 = \langle F, Z_\alpha\rangle$, $f^1 = \langle F, \bar{F}_\alpha\rangle$ and $f^j = 0$ ($j=2, \cdots, n$) in the equation (4.6), we have also the inequality (4.7) (in this case we replace $>$ by $\geqq 0$) with respect to the coefficient of $ds_j \wedge d\theta_j$ ($j=1, 2$).

Q.E.D.

Let r be in $[r_2, r_1]$. We devide $\partial\Delta(r)$ into $\partial\Delta_1(r)$ and $\partial\Delta_2(r)$, where

$$(4.8)\qquad \partial\Delta_i(r) = \{(z_1, z_2) \in \partial\Delta(r) : \log|z_i| = r\} \ (i=1, 2).$$

Lemma 4.2. *We have*

$$(4.9)\qquad \int_{\partial\Delta(r)} d^c\psi_\alpha = \frac{1}{4\pi}\Bigg[-\int_{S^1\times S^1}\psi_4(e^{r+i\theta_1}, e^{r+i\theta_2})d\theta_2 \wedge d\theta_1$$

$$-\int_{S^1\times S^1}\psi_1(e^{r+i\theta_1}, e^{r+i\theta_2})d\theta_1 \wedge d\theta_2\Bigg]$$

$$+\frac{1}{4\pi}\frac{\partial}{\partial r}\left[\int_{\partial\Delta_1(r)}\psi_\alpha \wedge d\theta_1 + \int_{\partial\Delta_2(r)}\psi_\alpha \wedge d\theta_2\right].$$

Proof. First we remark that $d\theta_1 \wedge ds_2 \wedge d\theta_2$ and $d\theta_2 \wedge ds_1 \wedge d\theta_1$ are positive forms on $\partial\Delta_1(r)$ and $\partial\Delta_2(r)$ respectively.

By (4.1) and the preceeding remark 2, we have

$$\int_{\partial\Delta_1(r)} d^c\psi_\alpha = \int_{\partial\Delta_1(r)\setminus\{(e^{r+i\theta_1},0)\}} d^c\psi_\alpha$$
$$= \frac{1}{4\pi}\int_{\partial\Delta_1(r)\setminus\{(e^{r+i\theta_1},0)\}}\left[-\frac{\partial\psi_3}{\partial s_2}+\frac{\partial\psi_4}{\partial s_1}+\frac{\partial\psi_5}{\partial\theta_2}\right]d\theta_1\wedge ds_2\wedge d\theta_2$$
$$= \frac{1}{4\pi}\int_{\partial\Delta_1(r)}\left[-\frac{\partial\psi_3}{\partial s_2}+\frac{\partial\psi_4}{\partial s_1}+\frac{\partial\psi_5}{\partial\theta_2}\right]d\theta_1\wedge ds_2\wedge d\theta_2\,.$$

Clearly we have

$$\int_{\partial\Delta_1(r)}\frac{\partial\psi_5}{\partial\theta_2}d\theta_1\wedge ds_2\wedge d\theta_2 = 0\,.$$

Therefore we obtain

$$\int_{\partial\Delta_1(r)} d^c\psi_\alpha = \frac{1}{4\pi}\int_{\partial\Delta_1(r)}\left[-\frac{\partial\psi_3}{\partial s_2}+\frac{\partial\psi_4}{\partial s_1}\right]d\theta_1\wedge ds_2\wedge d\theta_2\,. \tag{4.10}$$

Similarly we obtain

$$\int_{\partial\Delta_2(r)} d^c\psi_\alpha = \frac{1}{4\pi}\int_{\partial\Delta_2(r)}\left[\frac{\partial\psi_1}{\partial s_2}-\frac{\partial\psi_2}{\partial s_1}\right]d\theta_2\wedge ds_1\wedge d\theta_1\,. \tag{4.11}$$

Now we shall consider the equation (4.10). We have

$$\begin{aligned}(4.12)\qquad &\frac{1}{4\pi}\int_{\partial\Delta_1(r)}\frac{\partial\psi_3}{\partial s_2}d\theta_1\wedge ds_2\wedge d\theta_2\\ &= \frac{1}{4\pi}\int_{\partial\Delta_1(r)} d(\psi_3\, d\theta_2\wedge d\theta_1)\\ &= \frac{1}{4\pi}\int_{\partial\Delta_1(r)\cap\partial\Delta_2(r)}\psi_3 d\theta_2\wedge d\theta_1\\ &= \frac{1}{4\pi}\int_{S^1\times S^1}\psi_3(e^{r+i\theta_1}, e^{r+i\theta_2})d\theta_2\wedge d\theta_1\,.\end{aligned}$$

Since we have

$$\int_{\partial\Delta_1(r)}\psi_4 d\theta_1\wedge ds_2\wedge d\theta_2$$
$$= \int_{\partial\Delta_1(r)} d\left\{\left(\int_{-\infty}^{s_2}\psi_4(e^{r+i\theta_1}, e^{t+i\theta_2})dt\right)d\theta_2\wedge d\theta_1\right\}$$
$$= \int_{S^1\times S^1}\left(\int_{-\infty}^{r}\psi_4(e^{r+i\theta_1}, e^{t+i\theta_2})\,dt\right)d\theta_2\wedge d\theta_1\,,$$

we obtain

$$(4.13)\qquad \frac{\partial}{\partial r}\int_{\partial\Delta_1(r)}\psi_4 d\theta_1\wedge ds_2\wedge d\theta_2$$
$$=\int_{S^1\times S^1}\psi_4(e^{r+i\theta_1}, e^{r+i\theta_2})d\theta_2\wedge d\theta_1$$
$$+\int_{S^1\times S^1}\left(\int_{-\infty}^{r}\frac{\partial\psi_4}{\partial r}(e^{r+i\theta_1}, e^{t+i\theta_2})dt\right)d\theta_2\wedge d\theta_1\,.$$

By (4.10), (4.12) and (4.13), we obtain

$$(4.14)\qquad \int_{\partial\Delta_1(r)}d^c\psi_\alpha=\frac{1}{4\pi}\int_{S^1\times S^1}[-\psi_3-\psi_4](e^{r+i\theta_1}, e^{r+i\theta_2})d\theta_2\wedge d\theta_1$$
$$+\frac{1}{4\pi}\frac{\partial}{\partial r}\int_{\partial\Delta_1(r)}\psi_4 d\theta_1\wedge ds_2\wedge d\theta_2\,.$$

By the similar argument as we derived (4.14) from (4.10), we derive the following from (4.11)

$$(4.15)\qquad \frac{1}{4\pi}\int_{\partial\Delta_2(r)}d^c\psi_\alpha=\frac{1}{4\pi}\int_{S^1\times S^1}[-\psi_2-\psi_1](e^{r+i\theta_1}, e^{r+i\theta_2})d\theta_1\wedge d\theta_2$$
$$+\frac{1}{4\pi}\frac{\partial}{\partial r}\int_{\partial\Delta_2(r)}\psi_1 d\theta_2\wedge ds_1\wedge d\theta_1\,.$$

By (4.14), (4.15) and the definition of ψ_α we obtain (4.9). Q.E.D.

Lemma 4.3. *We have*

$$(4.16)\qquad \int_{\Delta(r)}f^*\Omega^2=\frac{1}{4\pi}\frac{\partial}{\partial r}\left[\int_{\partial\Delta_1(r)}\psi_\alpha\wedge d\theta_1+\int_{\partial\Delta_2(r)}\psi_\alpha\wedge d\theta_2\right]+n(\Delta(r), \alpha)\,.$$

Proof. By Theorem 1 and Lemma 4.2, we have only to prove that

$$\frac{1}{4\pi}\int_{S^1\times S^1}[\psi_4-\psi_1](e^{r+i\theta_1}, e^{r+i\theta_2})d\theta_2\wedge d\theta_1=0\,.$$

We define a mapping h: $\boldsymbol{C}^2\to\boldsymbol{C}^2$ by $h((z_1, z_2))=(z_2, z_1)$. Then $(f\circ h)$ satisfies Conditions (A) and (B), and we have

$$(|f\circ h, \alpha|^2+|f\circ h, \bar{\alpha}|^2)(z_1, z_2)=(|f, \alpha|^2+|f, \bar{\alpha}|^2)(z_2, z_1)$$

and

$$n_f((z_1, z_2), \alpha)=\lim_{\varepsilon\downarrow 0}\int_{\partial U_\varepsilon((z_1,z_2))}d^c\log[|\langle F, Z_\alpha\rangle|^2+|\langle F, \bar{Z}_\alpha\rangle|^2]\wedge f^*P_\alpha^*\omega$$
$$=\lim_{\varepsilon\downarrow 0}\int_{\partial U_\varepsilon((z_2,z_1))}d^c\log[|\langle F\circ h, Z_\alpha\rangle|^2+|\langle F\circ h, \bar{Z}_\alpha\rangle|^2]\wedge (fh)^*P_\alpha^*\omega$$
$$=n_{f\cdot h}((z_2, z_1), \alpha)\,.$$

On the other hand, we have from (4.1)

$$(4.17)\qquad \begin{aligned}(h^*\psi_\alpha) = \psi_1\circ h\, ds_2\wedge d\theta_2+\psi_2\circ h\, ds_2\wedge d\theta_1+\psi_3\circ h\, ds_1\wedge d\theta_2\\ +\psi_4\circ h\, ds_1\wedge d\theta_1+\psi_5\circ h\, d\theta_2\wedge d\theta_1+\psi_6\circ h\, ds_2\wedge ds_1\,.\end{aligned}$$

By Theorem 1, (4.14) and (4.15) in Lemma 4.2, comparing (4.1) with (4.17) we have

$$(4.18)\qquad \begin{aligned}&\int_{\Delta(r)} f^*\Omega^2 = \int_{\Delta(r)} h^*f^*\Omega^2 = n(\Delta(r), \alpha)\\ &+\frac{1}{4\pi}\left[-\int_{S^1\times S^1}\psi_1\circ h(e^{r+i\theta_1}, e^{r+i\theta_2})d\theta_2\wedge d\theta_1-\int_{S^1\times S^1}\psi_4\circ h(e^{r+i\theta_1}, e^{r+i\theta_2})d\theta_1\wedge d\theta_2\right]\\ &+\frac{1}{4\pi}\,\frac{\partial}{\partial r}\left[\int_{\partial\Delta_1(r)}\psi_1\circ h\, d\theta_1\wedge ds_2\wedge d\theta_2+\int_{\partial\Delta_2(r)}\psi_4\circ h\, d\theta_2\wedge ds_1\wedge d\theta_1\right].\end{aligned}$$

We see easily

$$\begin{aligned}\int_{\partial\Delta_1(r)}\psi_1\circ h\, d\theta_1\wedge ds_2\wedge d\theta_2 &= \int_{\partial\Delta_2(r)}\psi_1 d\theta_2\wedge ds_1\wedge d\theta_1\\ &= \int_{\partial\Delta_2(r)}\psi_\alpha\wedge d\theta_2\end{aligned}$$

and

$$\begin{aligned}\int_{\partial\Delta_2(r)}\psi_4\circ h\, d\theta_2\wedge ds_1\wedge d\theta_1 &= \int_{\partial\Delta_1(r)}\psi_4 d\theta_1\wedge ds_2\wedge d\theta_2\\ &= \int_{\partial\Delta_1(r)}\psi_\alpha\wedge d\theta_1\,.\end{aligned}$$

Therefore we have only to prove

$$\int_{S^1\times S^1}((\psi_i\circ h)-\psi_i)(e^{r+i\theta_1}, e^{r+i\theta_2})d\theta_1\wedge d\theta_2 = 0 \quad (i = 1, 4)\,.$$

For any α, $\beta\in[0, 2\pi]$, we have

$$\begin{aligned}((\psi_i\circ h)-\psi_i)(e^{r+i\alpha}, e^{r+i\beta}) &= \psi_i(e^{r+i\beta}, e^{r+i\alpha})-\psi_i(e^{r+i\alpha}, e^{r+i\beta})\\ ((\psi_i\circ h)-\psi_i)(e^{r+i\beta}, e^{r+i\alpha}) &= \psi_i(e^{r+i\alpha}, e^{r+i\beta})-\psi_i(e^{r+i\beta}, e^{r+i\alpha})\end{aligned}$$

Thus we obtain

$$((\psi_i\circ h)-\psi_i)(e^{r+i\alpha}, e^{r+i\beta}) = -((\psi_i\circ h)-\psi_i)(e^{r+i\beta}, e^{r+i\alpha})\,.$$

Q.E.D.

For the holomorphic mapping f: $\boldsymbol{C}^2\to Q_{n-1}(\boldsymbol{C})$ $(n\geqq 3)$ satisfying Conditions (A) and (B), we put

$$T(r) = \int_0^r dt\int_{\Delta(t)} f^*\Omega^2 \qquad \text{(order function)}$$

(4.19) $$N(r, \alpha)=\int_0^r n(\Delta(t), \alpha)dt \text{ (counting function)}$$
$$m(r, \alpha)=\frac{1}{4\pi}\left[\int_{\partial\Delta_1(r)}\psi_\alpha\wedge d\theta_1+\int_{\partial\Delta_2(r)}\psi_\alpha\wedge d\theta_2\right].$$

We need the following lemma, which can be proved in a similar way as ([5] p.p. 502).

Lemma 4.4. *For any α, $m(r, \alpha)$ is continuous with respect to $r\in[0, \infty)$.*

Theorem 2. *We have*

(4.20) $$T(r)=m(r, \alpha)-m(0, \alpha)+N(r, \alpha) \quad \textit{for any } r\geq 0,$$

and $m(r, \alpha)$ is non-negative.

Proof. Integrating the equation in Lemma 4.3 with respect to $r\in[r_2, r_1]$, we have

$$\int_{r_2}^{r_1}dr\int_{\Delta(r)}f^*\Omega^2=\int_{r_2}^{r_1}n(\Delta(r), \alpha)dr+m(r_1, \alpha)-m(r_2, \alpha).$$

By Lemma 4.4 we obtain the equation (4.20). It follows from Lemma 4.1 and Lemma 4.4 that the function $m(r, \alpha)$ is non-negative. Q.E.D.

Lemma 4.5. *For any r, $m(r, \alpha)$ is continuous with respect to $\alpha\in Q_{n-1}(C)$.*

We also omit this proof by the same reason as in Lemma 4.4. (c.f. [5] p.p. 504).

Theorem 3. *There exists a positive constant C satisfying*

(4.21) $$T(r)+C>N(r, \alpha) \quad \textit{whenever} \quad r\geqslant 0 \text{ and } \alpha\in Q_{n-1}(C).$$

Proof. By Theorem 2 we have

$$T(r)+m(0, \alpha)\geqq N(r, \alpha) \quad \text{for any } r\geqslant 0.$$

Therefore by Lemma 4.5 we have the equation (4.21). Q.E.D.

5. Induced form by f

We denote by $(X_0, X_1, \cdots, X_n)$ an element of $SO(n+1)$, where X_i's$(0\leqslant i\leqslant n)$ are column vectors, and we put $X_i=(x_{i0}, \cdots, x_{in})^t$. The left invariant forms θ_{ij} $(0\leqslant i, j\leqslant n)$ on $SO(n+1)$ are defined by the following equation:

(5.1) $$-\begin{pmatrix} dX_0^t \\ dX_1^t \\ \vdots \\ dX_n^t \end{pmatrix}(X_0, \cdots, X_n)=\begin{pmatrix} X_0^t \\ X_1^t \\ \vdots \\ X_n^t \end{pmatrix}(dX_0, \cdots, dX_n)=\begin{pmatrix} 0, & \theta_{10}, & \cdots, & \theta_{n0} \\ \theta_{01} & 0, & \cdots, & \theta_{n1} \\ \vdots & \vdots & & \vdots \\ \theta_{0n}, & \theta_{1n}, & \cdots, & 0 \end{pmatrix},$$

where $\theta_{ij}=-\theta_{ji}$.

Therefore we have $-\langle dX_i, X_j\rangle=\theta_{ji}$ i.e.,

$$dX_i = \sum_j \theta_{ij} X_j . \tag{5.2}$$

Taking its exterior derivative, we see

$$d\theta_{01} = \textstyle\sum_k \theta_{0k}\wedge\theta_{k1} = -\sum_k \theta_{0k}\wedge\theta_{1k} . \tag{5.3}$$

We remark that $d\theta_{01}$ is a 2-form on $SO(n+1)/SO(n-1)$. Furthermore it is a lift of a 2-form on $Q_{n-1}(\boldsymbol{C})$ by Π_1. In fact, let U be an open neighborhood of $Q_{n-1}(\boldsymbol{C})$, and (X_0, X_1) be a local cross-section of U into $SO(n+1)/SO(n-1)$: $\Pi_1((X_0, X_1))=$identity on U. We have

$$\Pi_1^{-1}(\Pi_1(X_0, X_1)) = \left\{(X_0, X_1)\begin{pmatrix}\cos\theta, & -\sin\theta \\ \sin\theta, & \cos\theta\end{pmatrix} : 0\leqslant\theta<2\pi\right\} . \tag{5.4}$$

Then we have on $\Pi_1^{-1}(U)$,

$$\begin{aligned} d\theta_{01} &= d\langle d(\cos\theta\cdot X_0+\sin\theta\cdot X_1), (-\sin\theta\cdot X_1+\cos\theta\cdot X_1)\rangle \\ &= d(d\theta+\langle dX_0, X_1\rangle) = d\langle dX_0, X_1\rangle . \end{aligned} \tag{5.5}$$

Let σ be a local holomorphic cross-section on U into $\boldsymbol{C}^{n+1}-\{0\}$ with respect to the Hopf fibring: $\Pi\sigma=$identity on U. We can write σ in the form $\sigma=X+iY$ for orthogonal real vectors X and Y at each point of U. Then we see

$$\Omega = dd^c\log\|\sigma\|^2 = -\frac{1}{2\pi} d\langle d(X/\|X\|), Y/\|Y\|\rangle . \tag{5.6}$$

Thus, $d\theta_{01}$ is the lift of $-2\pi\Omega$ by Π_1^* i.e.,

$$\Pi_1^*\Omega = -\frac{1}{2\pi} d\theta_{01} . \tag{5.7}$$

In the equation (5.1) we defined θ_{0j}'s and θ_{1j}'s $(0\leqq j\leqq n)$ as 1-forms on $SO(n+1)$. They are also regarded as 1-forms on $SO(n+1)/SO(n-1)$. To prove this fact we shall identify $SO(n+1)/SO(n-1)$ with $S^{2n+1}\cap\Pi^{-1}(Q_{n-1}(\boldsymbol{C}))$. We take a local coordinate $x=(x^1, \cdots, x^{2n-1})$ on a small open set U in $S^{2n+1}\cap\Pi^{-1}(Q_{n-1}(\boldsymbol{C}))$ and write a point $Z(x)$ of U in the form $(X_0(x)+iX_1(x))/\sqrt{2}$, where $\langle X_0, X_0\rangle(x)=\langle X_1, X_1\rangle(x)=1$ and $\langle X_0, X_1\rangle(x)=0$. For each x, extending $X_0(x)$ and $X_1(x)$, we take a real orthonormal basis $X_0(x), \cdots, X_n(x)$ in $\boldsymbol{C}^{n+1}$ such that $(X_0, \cdots, X_n)(x)\in SO(n+1)$. Then the tangent space $T_{Z(x)}(S^{2n+1}\cap\Pi^{-1}(Q_{n-1}(\boldsymbol{C})))$ has a basis $(iX_0-X_1)(x), X_2(x), \cdots, X_n(x), iX_2(x), \cdots, X_n(x)$ (c.f. [3] p.p. 279). In the equation $dZ=\sum_{i=1}^{2n-1}\frac{\partial Z}{\partial x^i}dx^i$, we see $\frac{\partial Z}{\partial x^i}=Z_*\left(\frac{\partial}{\partial x^i}\right)(1\leqslant i\leqslant 2n-1)$ and hence $\frac{\partial Z}{\partial x^i}$'s are tangent vectors of $T_{Z(x)}(S^{2n+1}\cap\Pi^{-1}(Q_{n-1}(\boldsymbol{C})))$. Thus there exists 1-

forms θ_j's $(1\leqslant j\leqslant n)$ and $\tilde{\theta}_j$'s $(2\leqslant j\leqslant n)$ on U such that $dZ=\theta_1(iX_0-X_1)+\sum_{j=2}^{n}(\theta_j+i\tilde{\theta}_j)X_j$. Comparing this form with (5.2), we have $\theta_1=\theta_{10}/\sqrt{2}$, $\theta_j=\theta_{0j}/\sqrt{2}$ $(2\leqslant j\leqslant n)$ and $\tilde{\theta}_j=\theta_{1j}/\sqrt{2}$ $(2\leqslant j\leqslant n)$. Thus we have from (5.2), (5.3) and (5.7)

$$(5.8)\qquad (\Pi_1^*\Omega)_{(X_0,X_1)}=\frac{1}{2\pi}\sum_{j=2}^{n}\langle dX_0, X_j\rangle\wedge\langle dX_1, X_j\rangle\,,$$

where $(X_0, X_1, \cdots, X_n)\in SO(n+1)$. For the volume form Ω^{n-1} on $Q_{n-1}(\boldsymbol{C})$, we have

$$(5.9)\qquad (\Pi_1^*\Omega^{n-1})_{(X_0,X_1)}=\left(\frac{1}{2\pi}\right)^{n-1}(n-1)!\,\langle dX_0, X_2\rangle\wedge\langle dX_1, X_2\rangle\wedge\cdots\wedge\langle dX_0, X_n\rangle\wedge\langle dX_1, X_n\rangle\,.$$

We shall obtain a formula for $f^*\Omega^2$ on $\boldsymbol{C}^2$. Let F be a holomorphic lift of f on a neighborhood U in $\boldsymbol{C}^2$ by Π. Set $(X_0+iX_1)/\sqrt{2}=F/||F||$, where X_i $(i=0, 1)$ are the orthonormal real vectors. With the coordinate system (x_1+iy_1, x_2+iy_2) on $\boldsymbol{C}^2$, we can write:

$$(5.10)\qquad \begin{aligned} dX_0&=\omega_1X_1+\lambda_2\tilde{B}_2dx_1-\lambda_3\tilde{B}_3dy_1+\lambda_4\tilde{B}_4dx_2-\lambda_5\tilde{B}_5dy_2\\ dX_1&=\omega_2X_0+\lambda_3\tilde{B}_3dx_1+\lambda_2\tilde{B}_2dy_1+\lambda_5\tilde{B}_5dx_2+\lambda_4\tilde{B}_4dy_2\,,\end{aligned}$$

where $\tilde{B}_i$'s $(2\leqslant i\leqslant 5)$ are differentiable vectors satisfying $\langle\tilde{B}_i, \tilde{B}_i\rangle=1$, λ_i's $(2\leqslant i\leqslant 5)$ are differentiable functions and ω_i's $(1\leqslant i\leqslant 2)$ are 1-forms on U. Then we take differentiable orthonormal vectors $B_i(2\leqslant i\leqslant 5)$ such that $\tilde{B}_2=B_2$, $\tilde{B}_3=\alpha_2B_2+\alpha_3B_3$, $\tilde{B}_4=\beta_2B_2+\beta_3B_3+\beta_4B_4$ and $\tilde{B}_5=\gamma_2B_2+\gamma_3B_3+\gamma_4B_4+\gamma_5B_5$, where α_i, β_i and γ_i are differentiable functions satisfying $\sum\alpha_i^2=1$, $\sum\beta_i^2=1$ and $\sum\gamma_i^2=1$. We choose differentiable vectors $B_6, \cdots, B_n$ on U such that $(X_0, X_1, B_2, \cdots, B_n)\in SO(n+1)$ at each point of U. By (5.8) we have

$$(5.11)\qquad \begin{aligned} f^*\Omega&=\frac{1}{2\pi}\textstyle\sum_{j=2}^{n}\langle dX_0, B_j\rangle\wedge\langle dX_1, B_j\rangle\\ &=\frac{1}{2\pi}\{[\lambda_2\lambda_5\gamma_2-\lambda_3\lambda_4\alpha_2\beta_2-\lambda_3\lambda_4\beta_3\alpha_3](dx_1\wedge dx_2+dy_1\wedge dy_2)\\ &\quad+[\lambda_2^2+\lambda_3^2]dx_1\wedge dy_1+[\lambda_4^2+\lambda_5^2]dx_2\wedge dy_2\\ &\quad+[\lambda_2\lambda_4\beta_2+\lambda_3\lambda_5\alpha_2\gamma_2+\lambda_3\lambda_5\alpha_3\gamma_3](dx_1\wedge dy_2-dy_1\wedge dx_2)\}\,.\end{aligned}$$

Furthermore we obtain

$$(5.12)\qquad \begin{aligned} f^*\Omega^2&=\left(\frac{1}{2\pi}\right)^2\times2\times\{[\lambda_2^2+\lambda_3^2][\lambda_4^2+\lambda_5^2]\\ &\quad-[\lambda_2\lambda_4\beta_2+\lambda_3\lambda_5\alpha_2\gamma_2+\lambda_3\lambda_5\alpha_3\gamma_3]^2\\ &\quad-[\lambda_2\lambda_5\gamma_2-\lambda_3\lambda_4\alpha_2\beta_2-\lambda_3\lambda_4\alpha_3\beta_3]^2\}dx_1\wedge dy_1\wedge dx_2\wedge dy_2\,.\end{aligned}$$

6. Crofton formula

In §3 we have defined $n(\Delta(r), \alpha)$ for a holomorphic mapping $f\colon \boldsymbol{C}^2 \to Q_{n-1}(\boldsymbol{C})$ $(n \geqq 3)$ satisfying Conditions (A) and (B). Then we have:

Theorem 4 (Crofton formula). *Let D be an open set in $\boldsymbol{C}^2$ with compact closure. Then we have*

$$\int_{Q_{n-1}(C)} n(D, \xi)\, d\xi = 2\int_D f^*\Omega^2, \tag{6.1}$$

where $d\xi = d\xi_\alpha = d\alpha = \Omega^{n-1}$.

Proof. First we assume that D is so small that there exists a differentiable lift $\sigma = (X_0, X_1)$ of f on D: $\prod_1 \sigma = f$. Let q be a point in D and set $f(q) \in \xi_\alpha$. For any real orthonormal vectors Y_0, Y_1 such that $\prod_1((Y_0, Y_1)) = \alpha$, we have

$$\langle X_0(q), Y_0\rangle = \langle X_0(q), Y_1\rangle = \langle X_1(q), Y_0\rangle = \langle X_1(q), Y_1\rangle = 0\,. \tag{6.2}$$

We set

$$\begin{aligned} &Q_{n-3}(f(q)^\perp) = \{\alpha \in Q_{n-1}(\boldsymbol{C})\colon f(q) \in \xi_\alpha\} \\ &f(D)^\perp = \{\alpha \in Q_{n-1}(\boldsymbol{C})\colon f(D) \cap \xi_\alpha \neq \phi\}\,. \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} &D' = \textstyle\prod_1^{-1}(f(D)^\perp) \\ &D'' = \{(q, a)\colon q \in D,\ a = (A_2, A_3, \cdots, A_n) \in SO(n-1)\}\,. \end{aligned} \tag{6.4}$$

For $a = (A_2, A_3, \cdots, A_n) \in SO(n-1)$ we write its column vector A_i as $A_i = (a_{i2}, \cdots, a_{in})^t$. Then we define a mapping $t\colon D'' \to SO(n+1)$ by

$$t((q, a)) = (B_2, B_3, X_0, X_1, B_4, \cdots, B_n)(q) \tag{6.5}$$
$$\times \begin{pmatrix} a_{22} & a_{32} & 0 & 0 & a_{42} & \cdots & a_{n2} \\ a_{23} & a_{33} & 0 & 0 & a_{43} & \cdots & a_{n3} \\ 0 & 0 & 1 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 1 & 0 & & 0 \\ a_{24} & a_{34} & 0 & 0 & a_{44} & \cdots & a_{n4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{2n} & a_{3n} & 0 & 0 & a_{4n} & \cdots & a_{nn} \end{pmatrix},$$

where $(X_0, X_1, B_2, \cdots, B_n)(q)$ is the one given in §5. Let $\prod'$ be the projection $D \times (SO(n-1)/SO(n-3)) \to D \times Q_{n-3}(\boldsymbol{C})$ defined by $\prod'((q, (A_2, A_3))) = (q, \prod''((A_2, A_3)))$, where $\prod''$ is the projection with respect to the Hopf fibring $SO(n-1)/SO(n-3) \to Q_{n-3}(\boldsymbol{C})$. We consider the following diagram;

$$
(6.6)\qquad \begin{array}{ccc}
D\times(SO(n-1)/SO(n-3)) & \xrightarrow{t'} & D'\subset SO(n+1)/SO(n-1) \\
\downarrow \Pi' & & \downarrow \Pi_1 \\
D\times Q_{n-3}(\boldsymbol{C}) & \xrightarrow{t''} & f(D)^{\perp}\subset Q_{n-1}(\boldsymbol{C}),
\end{array}
$$

where $t'((q, (A_2, A_3)))=(\sum_{i=2}^{n}a_{2i}B_i(q), \sum_{i=2}^{n}a_{3i}B_i(q))$ and t'' is defined by $\Pi_1\circ t' = t''\circ\Pi'$. Then, in the above diagram, we remark that $t''((q, Q_{n-3}(\boldsymbol{C})))=Q_{n-3}(f(q)^{\perp})$ for each $q\in D$. Putting $t((q, a))=(X_0', X_1', \cdots, X_n')$, we obtain

$$
\begin{aligned}
(6.7)\quad & (\Pi')^*(t'')^*\Omega^{n-1} = (t')^*(\Pi_1)^*\Omega^{n-1} \\
&= \left(\frac{1}{2\pi}\right)^{n-1}(n-1)!\langle dX_0', X_2'\rangle\wedge\langle dX_1', X_2'\rangle\wedge\cdots\wedge\langle dX_0', X_n'\rangle\wedge\langle dX_1', X_n'\rangle \\
&= \left(\frac{1}{2\pi}\right)^{n-1}(n-1)!\times\frac{1}{16}\times\langle d(X_0+iX_1), X_0'+iX_1'\rangle\wedge\langle d(X_0-iX_1), X_0'-iX_1'\rangle \\
&\quad \wedge\langle d(X_0+iX_1), X_0'-iX_1'\rangle\wedge\langle d(X_0-iX_1), X_0'+iX_1'\rangle\wedge\langle dA_2, A_4\rangle \\
&\quad \wedge\langle dA_3, A_4\rangle\wedge\cdots\wedge\langle dA_2, A_n\rangle\wedge\langle dA_3, A_n\rangle \\
&= -\frac{1}{4}\left(\frac{1}{2\pi}\right)^2(n-1)(n-2)\left\|\begin{matrix} \langle\lambda_2\tilde{B}_2+i\lambda_3\tilde{B}_3, X_0'+iX_1'\rangle, & \langle\lambda_4\tilde{B}_4+i\lambda_5\tilde{B}_5, X_0'+iX_1'\rangle \\ \langle\lambda_2\tilde{B}_2+i\lambda_3\tilde{B}_3, X_0'-iX_1'\rangle, & \langle\lambda_4\tilde{B}_4+i\lambda_5\tilde{B}_5, X_0'-iX_1'\rangle \end{matrix}\right\|^2 \\
&\quad \times dx_1\wedge dy_1\wedge dx_2\wedge dy_2\wedge\left(\frac{1}{2\pi}\right)^{n-3}(n-3)!\langle dA_2, A_4\rangle\wedge\langle dA_3, A_4\rangle\wedge\cdots \\
&\quad \wedge\langle dA_2, A_n\rangle\wedge\langle dA_3, A_n\rangle .
\end{aligned}
$$

We put $C=\{\beta\in f(D)^{\perp}$: there exists $\beta'\in(t'')^{-1}(\beta)$ such that $(dt'')(\beta')$ is singular$\}$. From Sard's Theorem the set C has measure zero. If we take $\alpha\in(f(D)^{\perp}\backslash C)$, the set $(t'')^{-1}(\alpha)$ consists of finite elements because of the compactness of $\bar{D}$ and Condition (B). We denote by n_α the number of elements $(t'')^{-1}(\alpha)$. Then, for each $\alpha\in(f(D)^{\perp}\backslash C)$ there exists a connected neighborhood V of α in $(f(D)^{\perp}\backslash C)$ such that $(t'')^{-1}(V)$ has n_α connected components and t'' maps each component onto V diffeomorphically. Let $\{V_i\}$ be a locally finite covering of $f(D)^{\perp}\backslash C$ by such open sets and $\{\phi_i\}$ be a partition of unity subordinated to $\{V_i\}$. Now we have

$$
\begin{aligned}
(6.8)\qquad \int_{f(D)^{\perp}} n_\alpha d\alpha &= \int_{f(D)^{\perp}-C} n_\alpha d\alpha = \sum_i\int_{f(D)^{\perp}-C}\phi_i(\alpha)n_\alpha d\alpha \\
&= \sum_i\int_{V_i} n_\alpha(\phi_i(\alpha)d\alpha) = \sum_i\int_{(t'')^{-1}(V_i)} -(t'')^*(\phi_i(\alpha)d\alpha) \\
&= \sum_i\int_{(t'')^{-1}(V_i)} -((t'')^*\phi_i(\alpha))((t'')^*d\alpha) \\
&= \int_{D\times Q_{n-3}-C'} -(t'')^*d\alpha = \int_{D\times Q_{n-3}} -(t'')^*d\alpha ,
\end{aligned}
$$

where C' is the set of critical points of t''. If

$$t''((q, \alpha_j)) = \alpha \text{ and } \begin{vmatrix} \langle \partial F/\partial z_1, Z_\alpha\rangle, \langle \partial F/\partial z_2, Z_\alpha\rangle \\ \langle \partial F/\partial z_1, \bar{Z}_\alpha\rangle, \langle \partial F/\partial z_2, \bar{Z}_\alpha\rangle \end{vmatrix} (q)$$

$$\left(\text{which is equal to } \frac{\|F\|}{2} \begin{vmatrix} \langle \lambda_2\tilde{B}_2 + i\lambda_3\tilde{B}_3, Z_\alpha\rangle, \langle \lambda_4\tilde{B}_4 + i\lambda_5\tilde{B}_5, Z_\alpha\rangle \\ \langle \lambda_2\tilde{B}_2 + i\lambda_3\tilde{B}_3, \bar{Z}_\alpha\rangle, \langle \lambda_4\tilde{B}_4 + i\lambda_5\tilde{B}_5, \bar{Z}_\alpha\rangle \end{vmatrix} (q)\right) = 0$$

for $\Pi(Z_\alpha) = \alpha$, then $dt''((q, \alpha_j))$ is singular because of (6.7). By Lemma 3.2 we have $n(D, \alpha) = n_\alpha$ on $f(D)^\perp \backslash C$. Therefore we have

$$(6.9) \quad \int_{Q_{n-1}} n(D, \alpha) d\alpha = \frac{1}{4}\left(\frac{1}{2\pi}\right)^2 (n-1)(n-2) \int_D dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2$$
$$\times \int_{Q_{n-3}(f(q)^\perp)} \left\| \begin{matrix} \langle \lambda_2\tilde{B}_2 + i\lambda_3\tilde{B}_3, X_0' + iX_1'\rangle, \langle \lambda_4\tilde{B}_4 + i\lambda_5\tilde{B}_5, X_0' + iX_1'\rangle \\ \langle \lambda_2\tilde{B}_2 + i\lambda_3\tilde{B}_3, X_0' - iX_1'\rangle, \langle \lambda_4\tilde{B}_4 + i\lambda_5\tilde{B}_5, X_0' - iX_1'\rangle \end{matrix} \right\|^2 \Omega^{n-3}.$$

Next we have the following equation:

$$(6.10) \quad \int_{Q_{n-3}(f(q)^\perp)} \left\| \begin{matrix} \langle \lambda_2\tilde{B}_2 + i\lambda_3\tilde{B}_3, X_0' + iX_1'\rangle, \langle \lambda_4\tilde{B}_4 + i\lambda_5\tilde{B}_5, X_0' + iX_1'\rangle \\ \langle \lambda_2\tilde{B}_2 + i\lambda_3\tilde{B}_3, X_0' - iX_1'\rangle, \langle \lambda_4\tilde{B}_4 + i\lambda_5\tilde{B}_5, X_0' - iX_1'\rangle \end{matrix} \right\|^2 \Omega^{n-3}$$
$$= [(\lambda_2\lambda_4\beta_3 - \lambda_3\lambda_5\alpha_2\gamma_3 + \lambda_3\lambda_5\alpha_3\gamma_2)^2 + (\lambda_3\lambda_4\alpha_2\beta_3 + \lambda_2\lambda_5\gamma_3 - \lambda_3\lambda_4\alpha_3\beta_2)^2](q)$$
$$\times \int_{Q_{n-3}(f(q)^\perp)} \left\| \begin{matrix} \langle B_2, X_0' + iX_1'\rangle, \langle B_3, X_0' + iX_1'\rangle \\ \langle B_2, X_0' - iX_1'\rangle, \langle B_3, X_0' - iX_1'\rangle \end{matrix} \right\|^2 \Omega^{n-3}$$
$$+ (\lambda_2^2 + \lambda_3^2\alpha_2^2)(\lambda_4^2\beta_4^2 + \lambda_5^2\gamma_4^2)(q) \int_{Q_{n-3}(f(q)^\perp)} \left\| \begin{matrix} \langle B_2, X_0' + iX_1'\rangle, \langle B_4, X_0' + iX_1'\rangle \\ \langle B_2, X_0' - iX_1'\rangle, \langle B_4, X_0' - iX_1'\rangle \end{matrix} \right\|^2 \Omega^{n-3}$$
$$+ (\lambda_2^2 + \lambda_3^2\alpha_2^2)(\lambda_5^2\gamma_5^2)(q) \int_{Q_{n-3}(f(q)^\perp)} \left\| \begin{matrix} \langle B_2, X_0' + iX_1'\rangle, \langle B_5, X_0' + iX_1'\rangle \\ \langle B_2, X_0' - iX_1'\rangle, \langle B_5, X_0' - iX_1'\rangle \end{matrix} \right\|^2 \Omega^{n-3}$$
$$+ (\lambda_3^2\alpha_3^2)(\lambda_4^2\beta_4^2 + \lambda_5^2\gamma_4^2)(q) \int_{Q_{n-3}(f(q)^\perp)} \left\| \begin{matrix} \langle B_3, X_0' + iX_1'\rangle, \langle B_4, X_0' + iX_1'\rangle \\ \langle B_3, X_0' - iX_1'\rangle, \langle B_4, X_0' - iX_1'\rangle \end{matrix} \right\|^2 \Omega^{n-3}$$
$$+ (\lambda_3^2\alpha_3^2\lambda_5^2\gamma_5^2)(q) \int_{Q_{n-3}(f(q)^\perp)} \left\| \begin{matrix} \langle B_3, X_0' + iX_1'\rangle, \langle B_5, X_0' + iX_1'\rangle \\ \langle B_3, X_0' - iX_1'\rangle, \langle B_5, X_0' - iX_1'\rangle \end{matrix} \right\|^2 \Omega^{n-3}.$$

In fact, the integral of the other terms which appear at the right hand side of (6.10) turns out to be zero. For example we consider the following integral:

$$l=\int_{Q_{n-3}(f(q)^{\perp})}\begin{vmatrix}\langle B_2(q), X_0'+iX_1'\rangle, \langle B_3(q), X_0'+iX_1'\rangle\\ \langle B_2(q), X_0'-iX_1'\rangle, \langle B_3(q), X_0'-iX_1'\rangle\end{vmatrix}\overline{\begin{vmatrix}\langle B_2(q), X_0'+iX_1'\rangle, \langle B_4(q), X_0'+iX_1'\rangle\\ \langle B_2(q), X_0'-iX_1'\rangle, \langle B_4(q), X_0'-iX_1'\rangle\end{vmatrix}}\Omega^{n-3}.$$

We have

$$l=\int_{SO(n-1)/SO(n-3)}\begin{vmatrix}(a_{22}-ia_{32}), (a_{23}-ia_{33})\\ (a_{22}+ia_{32}), (a_{23}+ia_{33})\end{vmatrix}\overline{\begin{vmatrix}(a_{22}-ia_{32}), (a_{24}-ia_{34})\\ (a_{22}+ia_{32}), (a_{24}+ia_{34})\end{vmatrix}}$$
$$\times\left(\frac{1}{2\pi}\right)^{n-2}(n-3)!\, d\theta\wedge\langle dA_2, A_4\rangle\wedge\langle dA_3, A_4\rangle\wedge\cdots\wedge\langle dA_2, A_n\rangle\wedge\langle dA_3, A_n\rangle,$$

where $0\leqslant\theta\leqslant 2\pi$. For each vector $A_i=(a_{i2}, a_{i3}, a_{i4}, \cdots, a_{in})^t$ we set $\tilde{A}_i$ by $\tilde{A}_i=(a_{i2}, -a_{i3}, a_{i4}, \cdots, a_{in})^t$. This induces a diffeomorphism k; $SO(n-1)\to SO(n-1)$ by $k((A_2, A_3, A_4, A_5, \cdots, A_n))=(\tilde{A}_2, \tilde{A}_3, \tilde{A}_5, \tilde{A}_4, \cdots, \tilde{A}_n)$. Then we have

$$l=\int_{SO(n-1)/SO(n-3)}-\begin{vmatrix}(a_{22}-ia_{32}), (a_{23}-ia_{33})\\ (a_{22}+ia_{32}), (a_{23}+ia_{33})\end{vmatrix}\overline{\begin{vmatrix}(a_{22}-ia_{32}), (a_{24}-ia_{34})\\ (a_{22}+ia_{32}), (a_{24}+ia_{34})\end{vmatrix}}$$
$$\times\left(\frac{1}{2\pi}\right)^{n-2}(n-3)!\, d\theta\wedge\langle d\tilde{A}_2, d\tilde{A}_5\rangle\wedge\langle d\tilde{A}_3, \tilde{A}_5\rangle\wedge\langle d\tilde{A}_2, \tilde{A}_4\rangle\wedge\langle d\tilde{A}_3, \tilde{A}_4\rangle$$
$$\wedge\langle d\tilde{A}_2, \tilde{A}_6\rangle\wedge\langle d\tilde{A}_3\, \tilde{A}_6\rangle\wedge\cdots\wedge\langle d\tilde{A}_2, \tilde{A}_n\rangle\langle d\tilde{A}_3, \tilde{A}_n\rangle.$$

Since we have $\langle dA_i, A_j\rangle=\langle d\tilde{A}_i, \tilde{A}_j\rangle$ $(2\leqslant i\leqslant 3,\ 4\leqslant j\leqslant n)$, we obtain $l=0$. In the equation (6.10), the integrals

$$\int_{Q_{n-3}(f(q)^{\perp})}\left|\begin{vmatrix}\langle B_2, X_0'+iX_1'\rangle, \langle B_3, X_0'+iX_1'\rangle\\ \langle B_2, X_0'-iX_1'\rangle, \langle B_3, X_0'-iX_1'\rangle\end{vmatrix}\right|^2\Omega^{n-3},$$

$$\int_{Q_{n-3}(f(q)^{\perp})}\left|\begin{vmatrix}\langle B_2, X_0'+iX_1'\rangle, \langle B_4, X_0'+iX_1'\rangle\\ \langle B_2, X_0'-iX_1'\rangle, \langle B_4, X_0'-iX_1'\rangle\end{vmatrix}\right|^2\Omega^{n-3},$$

$$\int_{Q_{n-3}(f(q)^{\perp})}\left|\begin{vmatrix}\langle B_2, X_0'+iX_1'\rangle, \langle B_5, X_0'+iX_1'\rangle\\ \langle B_2, X_0'-iX_1'\rangle, \langle B_5, X_0'-iX_1'\rangle\end{vmatrix}\right|^2\Omega^{n-3},$$

$$\int_{Q_{n-3}(f(q)^{\perp})}\left|\begin{vmatrix}\langle B_3, X_0'+iX_1'\rangle, \langle B_4, X_0'+iX_1'\rangle\\ \langle B_3, X_0'-iX_1'\rangle, \langle B_4, X_0'-iX_1'\rangle\end{vmatrix}\right|^2\Omega^{n-3}$$

and

$$\int_{Q_{n-3}(f(q)^{\perp})}\left|\begin{vmatrix}\langle B_3, X_0'+iX_1'\rangle, \langle B_5, X_0'+iX_1'\rangle\\ \langle B_3, X_0'-iX_1'\rangle, \langle B_5, X_0'-iX_1'\rangle\end{vmatrix}\right|^2\Omega^{n-3}$$

are all equal and furthermore its value is independent of q. We denote by C_0 its common value. Then by (5.12), (6.9) and (6.10) we have

$$(6.11)\qquad \int_{Q_{n-1}(C)} n(D,\alpha)d\alpha = \frac{1}{8}(n-1)(n-2)C_0\int_D f^*\Omega^2 .$$

We shall calculate the value C_0. Let $SO(n-1)/SO(n-3)\to Q_{n-3}(C)$ be the Hopf fibring. For arbitrary fixed pair (C_2, C_3) of $SO(n-1)/SO(n-3)$ we have

$$(6.12)\qquad C_0 = \int_{Q_{n-3}(C)} \left|\begin{vmatrix} \langle C_2, A_2+iA_3\rangle, & \langle C_3, A_2+iA_3\rangle \\ \langle C_2, A_2-iA_3\rangle, & \langle C_3, A_2-iA_3\rangle \end{vmatrix}\right|^2 \Omega^{n-3} .$$

We take an orthonormal pair (D_4, D_5) of $SO(n-1)/SO(n-3)$ such that $\langle C_i, D_j\rangle =0$ $(2\leqslant i\leqslant 3,\ 4\leqslant j\leqslant 5)$ and set real orthonormal vectors A_2, A_3, A_4 and A_5 by

$$(6.13)\qquad \begin{aligned} A_2 &= \sin\varphi(\sin\theta\cdot C_2-\cos\theta\cdot C_3)+\cos\varphi(\sin\alpha\cdot D_4-\cos\alpha\cdot D_5) \\ A_3 &= \sin\eta(\cos\theta\cdot C_2+\sin\theta\cdot C_3)+\cos\eta(\cos\alpha\cdot D_4+\sin\alpha\cdot D_5) \\ A_4 &= -\cos\varphi(\sin\theta\cdot C_2-\cos\theta\cdot C_3)+\sin\varphi(\sin\alpha\cdot D_4-\cos\alpha\cdot D_5) \\ A_5 &= -\cos\eta(\cos\theta\cdot C_2+\sin\theta\cdot C_3)+\sin\eta(\cos\alpha\cdot D_4+\sin\alpha\cdot D_5) , \end{aligned}$$

where $0<\theta$, $\alpha<\pi$, $-\pi/2<\varphi$, $\eta<\pi/2$. By extending A_2, A_3, A_4 and A_5 to an ordered real orthonormal basis $A_2, A_3, \cdots, A_n$ in C^{n-1} we get $(A_2, A_3, \cdots, A_n)\in SO(n-1)$. Take an open set $U\subset Q_{n-5}(C)$, where $Q_{n-5}(C)$ is a set $\{\beta\in Q_{n-3}(C)$: $|\beta, \Pi''((C_2, C_3))|^2+|\beta, \Pi''((C_2, -C_3))|^2=0\}$ in $Q_{n-3}(C)$, and a local cross-section $\sigma=(D_4, D_5)$ of U into $SO(n-3)/SO(n-5)$ with respect to the Hopf fibring: $SO(n-3)/SO(n-5)\to Q_{n-5}(C)$. Then we see easily the set $\{(A_2, A_3)\in SO(n-1)/SO(n-3)$: (A_2, A_3) is defined at (6.13) for $\sigma=(D_4, D_5)\}$ is a double covering of an open set in $Q_{n-3}(C)$. We have

$$(6.14)\qquad \begin{aligned} &\langle dA_2, A_4\rangle = -d\varphi,\ \langle dA_3, A_5\rangle = -d\eta , \\ &\langle dA_2, A_5\rangle = -\sin\varphi\cos\eta d\theta+\sin\eta\cos\varphi d\alpha+\cos\varphi\sin\eta\langle dD_4, D_5\rangle , \\ &\langle dA_3, A_4\rangle = \sin\eta\cos\varphi d\theta-\sin\varphi\cos\eta d\alpha-\cos\eta\sin\varphi\langle dD_4, D_5\rangle , \\ &\left.\begin{aligned}\langle dA_2, A_i\rangle &= \cos\varphi(\sin\alpha\langle dD_4, A_i\rangle-\cos\alpha\langle dD_5, A_i\rangle) \\ \langle dA_3, A_i\rangle &= \cos\eta(\cos\alpha\langle dD_4, A_i\rangle+\sin\alpha\langle dD_5, A_i\rangle)\end{aligned}\right\}\quad (i\geq 6) . \end{aligned}$$

By (6.14) we get

$$(6.15)\qquad \begin{aligned} &\langle dA_2, A_4\rangle\wedge\langle dA_3, A_4\rangle\wedge\cdots\wedge\langle dA_2, A_n\rangle\wedge\langle dA_3, A_n\rangle \\ &= (\sin^2\eta\cos^2\varphi-\sin^2\varphi\cos^2\eta)(\cos\varphi\cos\eta)^{n-5} \\ &\times d\varphi\wedge d\theta\wedge d\alpha\wedge d\eta\wedge\prod_{i\geq 6}\langle dD_4, A_i\rangle\wedge\langle dD_5, A_i\rangle , \end{aligned}$$

and

$$(6.16)\qquad \left|\begin{vmatrix} \langle C_2, A_2+iA_3\rangle, & \langle C_3, A_2+iA_3\rangle \\ \langle C_2, A_2-iA_3\rangle, & \langle C_3, A_2-iA_3\rangle \end{vmatrix}\right|^2 = 4|\sin\varphi\sin\eta|^2$$

Thus we obtain

$$(6.12)' \qquad C_0 = (n-3)(n-4)\int |\sin\varphi \sin\eta|^2 |\sin^2\eta\cos^2\varphi - \sin^2\varphi\cos^2\eta|$$

$$|\cos\varphi\cos\eta|^{n-5} d\varphi d\eta \times \int_{Q_{n-5}(C)} \Omega^{n-5}$$

$$= 2(n-3)(n-4)\int |\sin\varphi \sin\eta|^2 |\sin^2\eta\cos^2\varphi - \sin^2\varphi\cos^2\eta|$$

$$\times |\cos\varphi\cos\eta|^{n-5} d\varphi d\eta$$

$$= \frac{16}{(n-1)(n-2)},$$

because of $\int_{Q_i(C)} \Omega^i = 2$ and $\int_E (\sin\varphi\sin\eta)^2(\sin^2\varphi\cos^2\eta - \sin^2\eta\cos^2\varphi)$

$$\times (\cos\varphi\cos\eta)^{n-5} d\varphi\, d\eta = \frac{2}{(n-1)(n-2)(n-3)(n-4)}, \quad \text{where}$$

$E=\{(\eta, \varphi)\colon 0\leqslant\varphi\leqslant\pi/2 \text{ and } 0\leqslant\eta\leqslant\varphi\}$. Thus we have proved the equation (6.1) for a sufficiently small D. Now let D be an arbitrary open set in $\boldsymbol{C}^2$ with compact closure. We take a finite covering $\{D_s\}_{s=1}^{l}$ of D such that each D_s has a differentiable local cross-section of f into $SO(n+1)/SO(n-1)$. Let $\{g_s\}$ be a partition of unity subordinated to $\{D_s\}$. Taking a mapping $P_s\colon D_s\times Q_{n-3}(\boldsymbol{C})\to D_s$ defined by $P_s((q, \alpha))=q$ for $(q, \alpha)\in D_s\times Q_{n-3}(\boldsymbol{C})$, we put $n'(D_s, \alpha)=\sum_k n(p_k, \alpha)g_s(p_k)$. Then we obtain

$$(6.17) \qquad \int_{Q_{n-1}} n(D, \alpha)d\alpha = \sum_{s=1}^{l} \int_{Q_{n-1}} n'(D_s, \alpha)d\alpha$$

$$= \sum_s \int_{D_s\times Q_{n-3}} -g_s(P_s(\alpha'))\,(t_s'')^* d\alpha$$

$$= 2\sum_s \int_{D_s} g_s f^*\Omega^2$$

$$= 2\int_D f^*\Omega^2,$$

where t_s'' is a mapping of $D_s\times Q_{n-3}(\boldsymbol{C})$ onto $f(D_s)^\perp$ defined by (6.6). Q.E.D.

7. Equidistribution theorem

We define the defect $\delta(\alpha)$ of ξ_α by

$$(7.1) \qquad \delta(\alpha) = \liminf_{r\to\infty} \frac{m(r, \alpha)}{T(r)}.$$

Since $m(r, \alpha)$ is non-negative, $\delta(\alpha)$ is non-negative for any $\alpha\in Q_{n-1}(\boldsymbol{C})$. We see clearly that $\delta(\alpha)=\delta(\bar{\alpha})$ for any $\alpha\in Q_{n-1}(\boldsymbol{C})$. By Theorem 2, Lemma 4.5 and the fact that $T(r)\to\infty$ if $r\to\infty$, we have

$$(7.2) \qquad \delta(\alpha) = \liminf_{r\to\infty}\left(1-\frac{N(r, \alpha)}{T(r).}\right).$$

Then we have the following equidistribution theorem.

Theorem 5. *$\delta(\alpha)$ is equal to zero for almost all $\alpha \in Q_{n-1}(C)$ with respect to the volume Ω^{n-1}.*

Proof. By the Fatou's preparation theorem we have

$$\begin{aligned}
0 &\leqslant \int_{Q_{n-1}} \delta(\alpha) d\alpha \leqslant \int_{Q_{n-1}} \left\{ \liminf_{r\to\infty} \left(1 - \frac{N(r,\alpha)}{T(r)}\right)\right\} d\alpha \\
&\leqslant \liminf_{r\to\infty} \int_{Q_{n-1}} \left(1 - \frac{N(r,\alpha)}{T(r)}\right) d\alpha = \liminf_{r\to\infty} \left(2 - \frac{1}{T(r)} \int_{Q_{n-1}} N(r,\alpha) d\alpha\right) \\
&= \liminf_{r\to\infty} \left(2 - \frac{1}{T(r)} \int_{Q_{n-1}} \left\{\int_0^r n(\Delta(t),\alpha) dt\right\} d\alpha\right) \\
&= \liminf_{r\to\infty} \left(2 - \frac{1}{T(r)} \int_0^r dt \int_{Q_{n-1}} n(\Delta(t),\alpha) d\alpha\right) \\
&= \liminf_{r\to\infty} (2-2) = 0 \quad \text{(by Theorem 4).}
\end{aligned}$$

Thus we obtain $\delta(\alpha)=0$ for almost all $\alpha \in Q_{n-1}(C)$. Q.E.D.

If the image $f(C^2)$ does not intersect with ξ_α, we have $\delta(\alpha)=1$. So we have

Corollary. *Let f be a holomorphic mapping of C^2 into $Q_{n-1}(C)$ $(n \geqq 3)$ satisfying Conditions (A) and (B). We put $W = \{\alpha \in Q_{n-1}(C) : f(C^2) \cap \xi_\alpha = \phi\}$. Then the set W has measure zero with respect to volume Ω^{n-1}.*

REMARK 3. In the case of holomorphic curves (f: $C \to P_n(C)$ holomorphic mapping), it is known that $0 \leqslant \delta(\xi) \leqslant 1$ for each hyperplane ξ (c.f. [1], [5] and [6]). But in our case we can not prove that $\delta(\alpha) \leqslant 1$.

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