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# A THEOREM OF PAINLEVÉ ON PARAMETRIC SINGULARITIES OF ALGEBRAIC DIFFERENTIAL EQUATIONS OF THE FIRST ORDER 

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## 0. Introduction

Consider an algebraic differential equation of the first order $F\left(y, y^{\prime}\right)=0$ over an algebraically closed ordinary differential field $k$ of characteristic 0 , where $F$ is an irreducible polynomial over $k$. Recently Matsuda [3] presented a dif-ferential-algebraic definition for $F=0$ to be free from parametric singularities and gave a purely algebraic proof of the following theorem essentially due to Fuchs [2] and Poincaré [8]: Suppose that $F=0$ is free from parametric singularities. Then it is reduced to a Riccati equation or a defining equation of elliptic function by a birational transformation over $k$ if the genus of $F=0$ is 0 or 1 respectively. The author [4] proved that under the above assumption it is reduced to an equation of Clairaut type by a birational transformation over $k$ if the genus is greater than 1. This theorem is essentially due to Poincaré [8], Painlevé [5] and Picard [6].

Here a differential-algebraic formulation and its proof of the following theorem which is essentially due to Painleve [5], [6] will be given: The general solution $\eta$ of $F=0$ depends algebraically upon an arbitrary constant over some differential extension field of $k$ if and only if there exists an algebraic differential equation of the first order $G=0$ over $k$ such that it is free from parametric singularities and the general solution of $G=0$ is a rational function of $\eta$ and $\eta^{\prime}$ over $k$. Here, we assume that $k$ contains non-constants.

Let $k$ be an algebraically closed ordinary differential field of characteristic 0 , and $\Omega$ be a universal differential extension field of $k$. Suppose that $K$ is a differential subfield of $\Omega$ and it is an algebraic function field of one variable over $k$. Let $P$ be a prime divisor of $K$ and $K_{P}$ be the completion of $K$ with respect to $P$. Then $K_{P}$ is a differential extension field of $K$ and the differentiation is continuous in the metric of $K_{P}$ (cf. [1, p. 114]). Let $\nu_{P}$ and $t_{P}$ denote respectively the normalized valuation belonging to $P$ and a prime element in $P$. The following definition is due to Matsuda [3]: $K$ is said to be free from parametric singularities over $k$ if we have $\nu_{P}\left(t_{P}^{\prime}\right) \geqq 0$ for each prime divisor $P$ of
$K$. Here, we do not set the assumption that $K$ takes the form $k\left(y, y^{\prime}\right)$ with some element $y$ of $K$, which is done in [3]. In this general situation is the author's paper [4] which will be quoted later.

Let $k^{*}$ be a differential extension field of $k$ in $\Omega$; we take for granted that the field of constants of $k^{*}$ is the same as that of $k$ and $K, k^{*}$ are independent over $k$. Since $k$ is algebraically closed, $K$ and $k^{*}$ are linearly disjoint over $k$ (cf. [11, p. 19]). $K^{*}, k_{0}$ and $K_{0}^{*}$ will indicate $k^{*}(K)$, the fields of constants of $k$ and $K^{*}$ respectively; $k_{0}$ is algebraically closed.

Definition. $K$ will be said to be of Painlevé type over $k$ if there exists such $k^{*}$ that $K_{0}^{*} \neq k_{0}$.

If $K_{1}$ is of Painlevé type over $k$ and $K_{2}$ is an algebraic extension field of $K_{1}$ of finite degree, then $K_{2}$ is so over $k$ : For $K_{2}$ and $k^{*}$ are independent over $k$.

Theorem. $K$ is of Painlevé type over $k$ if and only if there exists a differential subfield of $K$ which is free from parametric singularities over $k$.

The "if" part is known: For, a differential subfield $M$ is of Painlevé type over $k$ in our sense if $M$ is free from parametric singularities over $k$ (cf. [4]). Suppose that $K$ is of Painlevé type over $k$. Let $\Gamma$ be the totality of those prime divisors $P$ of $K$ satisfying $\nu_{P}\left(t_{P}^{\prime}\right)<0$. Assume that $K$ is not free from parametric singularities over $k$. Then $\Gamma$ is not empty. Let $P$ be an element of $\Gamma$. Then the number $n_{P}$ defined by $n_{P}=1-\nu_{P}\left(t_{P}^{\prime}\right)$ does not depend on the choice of a prime element $t_{P}$ in $P$. It is an integer greater than 1 . We define $G_{P}$ as the group of all differential $k$-automorphisms of $K_{P}$ that are continuous in the metric of $K_{P}$. By a theorem of Rosenlicht [9, Th. 3] we have the following:

Lemma. $\quad G_{P}$ is a cyclic group of order $n_{P}$.
Let $L$ denote the totality of those element of $K$ each of which is left invariant under $G_{P}$ for any $P$ in $\Gamma$. Then $L$ is a differential extension field of $k$. It is proved to be free from parametric singularities over $k$. Thus Theorem is obtained. If $k$ contains non-constants then $L$ takes the form $k\left(y, y^{\prime}\right)$ with some element $y$ of $L$.

In case $\Gamma=\phi$, we set $L=K$.
Proposition. Suppose that $K$ is of Painleve type over $k$. Then, every differential subfield of $K$ which is free from parametric singularities over $k$ is contained in $L$.

Lemma, Theorem and Proposition will be proved in $\S 1, \S 2$ and $\S 3$ respectively. In the last $\S 4$ some examples will be given.

Remark 1. Suppose that $K$ is of Painlevé type over $k$. Then, there exists such $k_{1}^{*}$ that $\left(K_{1}^{*}\right)_{0} \neq k_{0}$ and $\left[K^{*}: k^{*}\left(K_{0}^{*}\right)\right]=\left[K_{1}^{*}: k_{1}^{*}\left(\left(K_{1}^{*}\right)_{0}\right)\right]$ if $k^{*} \supset k_{1}^{*}$,
where $K_{1}^{*}=k_{1}^{*}(K)$ and $\left(K_{1}^{*}\right)_{0}$ denotes the field of constants of $K_{1}^{*}$.
Remark 2. By a result (3) in §2 we have the following: If $K^{*}=k^{*}\left(K_{0}^{*}\right)$ for some $k^{*}$, then $K$ is free from parametric singularities over $k$. Hence, $K$ is free from parametric singularities over $k$ if $K^{*}$ is so over some algebraically closed $k^{*}$.

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## 1. Proof of Lemma

Let $P$ be an element of $\Gamma$ and $n$ be $n_{P}$. By a theorem of Rosenlicht [9, Th. 3] there exists a prime element $t$ in $P$ such that $t^{\prime}=c t^{1-n}$ with a nonzero element $c$ of $k$. Suppose that $\sigma$ is an element of $G_{P}$. Then, we have $\nu_{P}(\sigma x)=\nu_{P}(x)$ for each $x$ in $K_{P}$, since $\sigma$ is continuous in the metric of $K_{P}$. We shall prove that $\sigma t=\varepsilon t$ with $\varepsilon^{n}=1, \varepsilon \in k_{0}$. In $K_{P}, \sigma t=\sum_{k=1}^{\infty} a_{i} i^{i+1} ; a_{i} \in k, a_{0} \neq 0$. Differentiation of this expression of $\sigma t$ gives us

$$
\begin{aligned}
(\sigma t)^{\prime} & =\sum_{i=0}^{\infty}(i+1) a_{i} t^{i} t^{\prime}+\sum_{i=0}^{\infty} a_{i}^{\prime} t^{i+1} \\
& =t^{1-n} \sum_{i=0}^{\infty}\left[(i+1) c a_{i}+a_{j-n}^{\prime}\right] t^{i} ;
\end{aligned}
$$

here we assume that $a_{i}=0$ if $i<0$. On the other hand

$$
\sigma\left(t^{\prime}\right)=c \sigma\left(t^{1-n}\right)=c t^{1-n}\left(\sum_{i=0}^{\infty} a_{i} t^{i}\right)^{1-n} .
$$

Hence,

$$
\begin{equation*}
\left(\sum_{i=0}^{\infty} a_{i} i^{i}\right)^{n-1} \sum_{i=0}^{\infty}\left[(i+1) c a_{i}+a_{i-n}^{\prime}\right] t^{i}=c . \tag{1}
\end{equation*}
$$

Comparing the constant terms of both sides, we have $c a_{0}^{n}=c$ and $a_{0}^{n}=1 ; a_{0}$ is a constant. Let us show that $a_{i}=0$ for any $i \geqq 1$. To the contrary assume that there exists an index $i \geqq 1$ with $a_{i} \neq 0$. Let $j$ be the minimum of those indices. The coefficient of $t^{j}$ on the left hand side of (1) is $(n+j) c a_{0}^{n-1} a_{j}$. Hence $a_{j}=0$. This is a contradiction. Thus $a_{i}=0$ for any $i \geqq 1$. Therefore $\sigma t=\varepsilon t$ with $\varepsilon=a_{0}$. Conversely let $\varepsilon$ be an $n$-th root in $k_{0}$ of 1 and $\theta$ be a mapping of $K_{P}$ to itself defined by

$$
\theta(x)=\sum b_{i}(\varepsilon t)^{i}, x=\sum b_{i} t^{i}, b_{i} \in k
$$

Then $\theta$ is a continuous $k$-automorphism of $K_{P}$. It is a differential one: For,

$$
\begin{aligned}
\theta\left(x^{\prime}\right) & \left.=\theta\left(\sum i c^{\prime}\right)_{i} t^{i-n}+\sum b_{i}^{\prime} t^{i}\right) \\
& =\sum i c b_{i}(\varepsilon t)^{i-n}+\sum b_{i}^{\prime}(\varepsilon t)^{i} \\
& =\sum i c b_{i} \varepsilon^{i} t^{i-n}+\sum b_{i}^{\prime} \varepsilon^{i} t^{i} \\
& =\left(\sum b_{i} \varepsilon^{i} t^{i}\right)^{\prime}=(\theta x)^{\prime} .
\end{aligned}
$$

## 2. Proof of Theorem

Suppose that $K$ is of Painlevé type over $k$. We may assume that $k^{*}$ is algebraically closed, since $K$ and the algebraic closure of $k^{*}$ are independent over $k$. If $\Gamma$ is empty, then $K$ is free from parametric singularities over $k$. We assume that $\Gamma$ is not empty. Let $P$ be an element of $\Gamma$. We have a prime element $t$ in $P$ such that $t^{\prime}=c t^{1-n}$ (cf. §1). There exists uniquely a prime divisor $P^{*}$ of $K^{*}$ such that the restriction of $\nu_{P}^{*}$ to $K$ is $\nu_{P}$, where $\nu_{P *}^{*}$ is the normalized valuation belonging to $P^{*}$. The completion $K_{P}$ of $K$ with respect to $P$ is a subfield of the completion $K_{P *}^{*}$ of $K^{*}$ with respect to $P^{*}$.

We shall show that $K^{*}$ and $K_{P}$ are linearly disjoint over $K$. They are so if and only if $k^{*}$ and $K_{P}$ are linearly disjoint over $k$, since $K^{*}=k^{*}(K)$ and $K, k^{*}$ are linearly disjoint over $k$. Assume that $m$ elements $a_{1}, \cdots, a_{m}$ of $k^{*}$ are linearly dependent over $K_{P}: \sum_{i=1}^{m} a_{i} u_{i}=0$ with $u_{i} \in K_{P}$ and $u_{i} \neq 0$ for some $i$. We may suppose that $\nu_{P}\left(u_{1}\right) \leqq \nu_{P}\left(u_{i}\right)$ for any $i$. For each $i$ let $b_{i}$ be an element of $k$ such that $\nu_{P}\left(u_{i} / u_{1}-b_{i}\right)>0$. Then $b_{1}=1$ and $\sum_{i=1}^{m} a_{i} b_{i}=0$. Hence $a_{1}, \cdots, a_{m}$ are linearly dependent over $k$. Thus $k^{*}, K_{P}$ are linearly disjoint over $k$ and $K^{*}, K_{P}$ are linearly disjoint over $K$.

For each element $\sigma$ of $G_{P}$ there exists uniquely a continuous differential $k^{*}$ automorphism $\sigma^{*}$ of $K_{P}^{*}$ whose restriction to $K_{P}$ is $\sigma$. Set $G_{P}^{*}=\left\{\sigma^{*} ; \sigma \in G_{P}\right\}$. Let us define a subset $L^{*}(P)$ of $K^{*}$ as the totality of those elements of $K^{*}$ each of which is left invariant under $G_{P}^{*}$. Put $L^{*}=\cap L^{*}(P), P \in \Gamma$. Then, $L^{*}$ is a differential extension field of $k^{*}$.

We shall prove that $K_{0}^{*}$ is contained in $L^{*}$. Let $\gamma$ be a constant of $K^{*}$ that is transcendental over $k$. Take an element $P$ of $\Gamma$. In $K_{P *}^{*}$ we have

$$
\gamma=\sum_{i=p}^{\infty} a_{i} t^{i}, a_{i} \in k^{*}, a_{P} \neq 0 .
$$

Differentiation of this expression of $\gamma$ gives us

$$
0=\gamma^{\prime}=\sum_{i=p}^{\infty}\left[i c a_{i}+a_{i-n}^{\prime}\right] t^{i-n}
$$

here we assume that $a_{i}=0$ if $i<p$. This implies

$$
\begin{equation*}
i c a_{i}+a_{i-n}^{\prime}=0(p \leqq i) . \tag{2}
\end{equation*}
$$

In particular, $p c a_{p}+a_{p-n}^{\prime}=p c a_{p}=0$. Hence $p=0$. We shall show that $i \equiv 0(\bmod n)$ if $a_{i} \neq 0$. To the contrary assume that there exists an index $i$ such that $a_{i} \neq 0$ with $i \neq 0(\bmod n) . \quad$ Let $j$ be the minimum of those indices. Then we get $a_{j-n}$ $=0$, and $a_{j}=0$ by (2). Hence our assertion is true. Since $\sigma^{*} t^{n}=t^{n}, \gamma$ is contained in $L^{*}(P)$. Hence $\gamma \in L^{*}$.

Put $L(P)=L^{*}(P) \cap K$ and $L=L^{*} \cap K=\cap L(P), P \in \Gamma$. By the definition of $L^{*}$ and $L, L^{*} \supset k^{*}(L)$. We prove that

$$
\begin{equation*}
L^{*}=k^{*}(L) \tag{3}
\end{equation*}
$$

Let $x$ be an element of $L^{*}$. Since $L^{*} \subset K^{*}=k^{*}(K)$, we have

$$
\sum_{i=1}^{r} a_{i} u_{i}-x \sum_{j=1}^{s} b_{j} v_{j}=0 ;
$$

here $a_{i}, b_{j} \in k^{*}$ and $u_{i}, v_{j} \in K$. Among those expressions of $x$ pick one with a minimal $s$. We may assume that $a_{1}, \cdots, a_{r}$ are linearly independent over $K$ and $v_{s}=1$. Then

$$
\begin{equation*}
a_{1}, \cdots, a_{r}, x b_{1}, \cdots, x b_{s-1} \tag{4}
\end{equation*}
$$

are linearly independent over $K$ by the minimality of $s$. Let $P$ be an element of $\Gamma$. Then the members of (4) are linearly independent over $K_{P}$, since $K^{*}$ and $K_{P}$ are linearly disjoint over $K$. Take an element $\sigma^{*}$ of $G_{P}^{*}$. Then,

$$
\begin{aligned}
0 & =\sigma^{*}\left(\sum_{i=1}^{r} a_{i} u_{i}-x \sum_{j=1}^{s} b_{j} v_{j}\right) \\
& =\sum_{i=1}^{r} a_{i} \sigma u_{i}-x \sum_{j=1}^{s} b_{j} \sigma v_{j},
\end{aligned}
$$

since $\sigma^{*} x=x$. We have $\sigma v_{s}=v_{s}$ by $v_{s}=1$. Hence,

$$
\sum_{i=1}^{r} a_{i}\left(u_{i}-\sigma u_{i}\right)-x \sum_{j=1}^{s-1} b_{j}\left(v_{j}-\sigma v_{j}\right)=0 .
$$

Since each of $u_{i}-\sigma u_{i}$ and $v_{j}-\sigma v_{j}$ is in $K_{P}$, we obtain $u_{i}=\sigma u_{i}$ and $v_{j}=\sigma v_{j}(1 \leqq i$ $\leqq r, 1 \leqq j \leqq s)$. Hence, $u_{i}, v_{j} \in L(P)$. They are in $L$ and $x \in k^{*}(L)$. Therefore (3) holds.

By (3) we get $L \neq k$, since $L^{*} \supset K_{0}^{*} \supseteq k_{0}$. We shall prove that $L$ is free from parametric singularities over $k$. Let $Q, \bar{\nu}_{Q}$ and $\tau$ be a prime divisor of $L$, the normalized valuation belonging to $Q$ and a prime element in $Q$ respectively. Suppose that $P$ is an extension of $Q$ to $K$. Then, $L_{Q} \subset K_{P}$. Let $e$ be the ramification exponent of $P$ with respect to $L$. Take a prime element $t_{1}$ in $P$ such that $\tau=t_{1}^{e}$. Then,

$$
e \bar{e}_{Q}\left(\tau^{\prime} \mid \tau\right)=\nu_{P}\left(e t_{1}^{\prime} / t_{1}\right)=\nu_{P}\left(t_{1}^{\prime}\right)-1,
$$

and

$$
\begin{equation*}
e \bar{\nu}_{Q}\left(\tau^{\prime}\right)=\nu_{P}\left(t_{1}^{\prime}\right)+e-1 . \tag{5}
\end{equation*}
$$

If $P \notin \Gamma$, we have $\nu_{P}\left(t_{1}^{\prime}\right) \geqq 0$ and $\bar{\nu}_{Q}\left(\tau^{\prime}\right) \geqq 0$. Let us assume that $P \in \Gamma$. Then each element of $L_{Q}$ is left invariant under $G_{P}$. By (5), $e \bar{\nu}_{Q}\left(\tau^{\prime}\right)=e-n$. For each $\sigma$ of $G_{P}, \sigma t=\varepsilon t, \varepsilon^{n}=1$. There exists an element $\sigma$ of $G_{P}$ such that $\varepsilon$ is a primitive $n$-th root of 1 . Since $\nu_{P}\left(t^{e} / \tau\right)=0$, there exists a nonzero element $a$ of $k$ such that $\nu_{P}\left(t^{e} \mid \tau-a\right)>0$. We have

$$
\nu_{P}\left(\sigma\left(t^{e} / \tau-a\right)\right)=\nu_{P}\left(\varepsilon^{e} t^{e} / \tau-a\right)>0,
$$

since $\sigma t=\varepsilon t$ and $\sigma \tau=\tau$. Hence,

$$
\nu_{P}\left(\left(\varepsilon^{e}-1\right) t^{e} / \tau\right) \geqq \operatorname{Min}\left\{\nu_{\dot{P}}\left(\varepsilon^{e} t^{e} / \tau-a\right), \nu_{P}\left(t^{e} / \tau-a\right)\right\}>0 .
$$

We have $\varepsilon^{e}=1$, because $\nu_{P}\left(t^{e} / \tau\right)=0$. Therefore, $n$ divides $e$, since $\varepsilon$ is a primitive $n$-th root of $1: e \bar{\nu}_{Q}\left(\tau^{\prime}\right)=e-n \geqq 0$. Thus $L$ is free from parametric singularities over $k$.

## 3. Proof of Proposition

Suppose that $M$ is a differential subfield of $K$ which is free from parametric singularities over $k$. Then $M^{*}=k^{*}\left(M_{0}^{*}\right)$ for some $k^{*}$, where $M^{*}=k^{*}(M)$ and $M_{0}^{*}$ is the field of constants of $M^{*}$ (cf. [4]). Since $K_{0}^{*} \supset M_{0}^{*} \supsetneq k_{0}, K_{0}^{*}$ contains $k_{0}$ properly. We may suppose that $k^{*}$ is algebraically closed. Since $L^{*}$ includes $K_{0}^{*}, M_{0}^{*} \subset K_{0}^{*} \subset L^{*}$. Hence, $M^{*}=k^{*}\left(M_{0}^{*}\right) \subset L^{*}$. Thus,

$$
M \subset K \cap M^{*} \subset K \cap L^{*}=L
$$

## 4. Examples

Example 1. Suppose that $K=k(t)$ and $t^{\prime}=t^{2}-1$. Then a $k$-automorphism $\sigma$ of $K$ defined by $\sigma t=1 / t$ is a differential one. Let $P$ be a prime divisor of $K$ determined by $\nu_{P}(t)=1$. Then $\sigma$ is not continuous in the metric of $K_{P}$.

Example 2. Rosenlicht [10] proved the following theorem: Suppose that $k=k_{0}$ and $K=k(y)$ : If $K$ is of Painlevé type over $k$ in our sense, then either $y^{\prime}=a f / f_{y}$ or $y^{\prime}=1 / g_{y}$ with $f, g \in k(y)$ and $a \in k$.

Example 3. Suppose that $K=k(t)$ and $2 t t^{\prime}=1$. Let $P$ be the prime divisor of $K$ determined by $\nu_{P}(t)=1$. Then $n_{P}=2$, and $\Gamma=\{P\}$. The generator $\sigma$ of $G_{P}$ satisfies $\sigma t=-t$, and $I=k\left(t^{2}\right)$.

## References

[1] C. Chevalley: Introduction to the theory of algebraic functions of one variable, Math. Surveys VI, Amer. Math. Soc., 1951.
[2] L. Fuchs: Über Differentialgleichungen, deren Integrale feste Verzweibungspunkte besitzen, S.-B. der Königl. Preuss. Akad. Wiss. Berlin, 32 (1884), 699-710.
[3] M. Matsuda: Algebraic differential equations of the first order free from parametric singularities from the differential-algebraic standpoint, J. Math. Soc. Japan 30 (1978), 447-455.
[4] K. Nishioka: Algebraic differential equations of Clairaut type from the differentialalgebraic standpoint, J. Math. Soc. Japan 31 (1979), 553-559.
[5] P. Painlevé: Mémoire sur les équations différentielles du premier order, Ann. Sci. Ecole Norm. Sup. $3^{e}$ Série 8 (1891), 9-58, 103-140, 201-226, 267-284; 9 (1892), 9-30, 101-144, 283-308.
[6] -_: Leçons sur la théorie analytique des équations différentielles, professées à Stockholm (1895), Hermann, Paris, 1897; Oeuvres, Tome I, 205-798, Centre National de la Recherche Scientifique, Paris, 1972.
[7] E. Picard: Traité d'analyse, Tome III, 2e Édition, Gauthier-Villars, Paris, 1908.
[8] H. Poincaré: Sur un théorème de M. Fuchs, Acta Math. 7 (1885), 1-32.
[9] M. Rosenlicht: Canonical forms for local derivations, Pacific J. Math. 42 (1972), 721-732.
[10] -: Nonminimality of the differential closure, Pacific J. Math. 52 (1974), 529-537.
[11] A. Weil: Foundations of algebraic geometry, Amer. Math. Soc. Colloq. Publ. Vol. 29, Providence, 1946.

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