Nishioka, K.
Osaka J. Math.
18 (1981), 249–255

A THEOREM OF PAINLEVÉ ON PARAMETRIC SINGULARITIES OF ALGEBRAIC DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Keiji NISHIOKA

(Received September 7, 1979)

0. Introduction

Consider an algebraic differential equation of the first order $F(y,y')=0$ over an algebraically closed ordinary differential field k of characteristic 0, where F is an irreducible polynomial over k. Recently Matsuda [3] presented a differential-algebraic definition for $F=0$ to be free from parametric singularities and gave a purely algebraic proof of the following theorem essentially due to Fuchs [2] and Poincaré [8]: Suppose that $F=0$ is free from parametric singularities. Then it is reduced to a Riccati equation or a defining equation of elliptic function by a birational transformation over k if the genus of $F=0$ is 0 or 1 respectively. The authoı [4] proved that under the above assumption it is reduced to an equation of Clairaut type by a birational transformation over k if the genus is greater than 1. This theorem is essentially due to Poincaré [8], Painlevé [5] and Picard [6].

Here a differential-algebraic formulation and its proof of the following theorem which is essentially due to Painlevé [5], [6] will be given: The general solution η of $F=0$ depends algebraically upon an arbitrary constant over some differential extension field of k if and only if there exists an algebraic differential equation of the first order $G=0$ over k such that it is free from parametric singularities and the general solution of $G=0$ is a rational function of η and η' over k. Here, we assume that k contains non-constants.

Let k be an algebraically closed ordinary differential field of characteristic 0, and Ω be a universal differential extension field of k. Suppose that K is a differential subfield of Ω and it is an algebraic function field of one variable over k. Let P be a prime divisor of K and K_P be the completion of K with respect to P. Then K_P is a differential extension field of K and the differentiation is continuous in the metric of K_P (cf. [1, p. 114]). Let ν_P and t_P denote respectively the normalized valuation belonging to P and a prime element in P. The following definition is due to Matsuda [3]: K is said to be *free from parametric singularities* over k if we have $\nu_P(t'_P)\geqq 0$ for each prime divisor P of

K. Here, we do not set the assumption that K takes the form $k(y,y')$ with some element y of K, which is done in [3]. In this general situation is the author's paper [4] which will be quoted later.

Let k^* be a differential extension field of k in Ω; we take for granted that the field of constants of k^* is the same as that of k and K, k^* are independent over k. Since k is algebraically closed, K and k^* are linearly disjoint over k (cf. [11, p. 19]). K^*, k_0 and K_0^* will indicate $k^*(K)$, the fields of constants of k and K^* respectively; k_0 is algebraically closed.

DEFINITION. K will be said to be of *Painlevé type* over k if there exists such k^* that $K_0^* \neq k_0$.

If K_1 is of Painlevé type over k and K_2 is an algebraic extension field of K_1 of finite degree, then K_2 is so over k: For K_2 and k^* are independent over k.

Theorem. *K is of Painlevé type over k if and only if there exists a differential subfield of K which is free from parametric singularities over k.*

The "if" part is known: For, a differential subfield M is of Painlevé type over k in our sense if M is free from parametric singularities over k (cf. [4]). Suppose that K is of Painlevé type over k. Let Γ be the totality of those prime divisors P of K satisfying $\nu_P(t_P') < 0$. Assume that K is not free from parametric singularities over k. Then Γ is not empty. Let P be an element of Γ. Then the number n_P defined by $n_P = 1 - \nu_P(t_P')$ does not depend on the choice of a prime element t_P in P. It is an integer greater than 1. We define G_P as the group of all differential k-automorphisms of K_P that are continuous in the metric of K_P. By a theorem of Rosenlicht [9, Th. 3] we have the following:

Lemma. *G_P is a cyclic group of order n_P.*

Let L denote the totality of those element of K each of which is left invariant under G_P for any P in Γ. Then L is a differential extension field of k. It is proved to be free from parametric singularities over k. Thus Theorem is obtained. If k contains non-constants then L takes the form $k(y,y')$ with some element y of L.

In case $\Gamma = \phi$, we set $L = K$.

Proposition. *Suppose that K is of Painlevé type over k. Then, every differential subfield of K which is free from parametric singularities over k is contained in L.*

Lemma, Theorem and Proposition will be proved in §1, §2 and §3 respectively. In the last §4 some examples will be given.

REMARK 1. Suppose that K is of Painlevé type over k. Then, there exists such k_1^* that $(K_1^*)_0 \neq k_0$ and $[K^*: k^*(K_0^*)] = [K_1^*: k_1^*((K_1^*)_0)]$ if $k^* \supset k_1^*$, where $K_1^*=k_1^*(K)$ and $(K_1^*)_0$ denotes the field of constants of K_1^*.

REMARK 2. By a result (3) in §2 we have the following: If $K^*=k^*(K_0^*)$ for some k^*, then K is free from parametric singularities over k. Hence, K is free from parametric singularities over k if K^* is so over some algebraically closed k^*.

The author wishes to express his sincere gratitude to Professor M. Matsuda for his kind advice.

1. Proof of Lemma

Let P be an element of Γ and n be n_P. By a theorem of Rosenlicht [9, Th. 3] there exists a prime element t in P such that $t'=ct^{1-n}$ with a nonzero element c of k. Suppose that σ is an element of G_P. Then, we have $\nu_P(\sigma x)=\nu_P(x)$ for each x in K_P, since σ is continuous in the metric of K_P. We shall prove that $\sigma t=\varepsilon t$ with $\varepsilon^n=1$, $\varepsilon\in k_0$. In K_P, $\sigma t=\sum_{k=1}^{\infty}a_i t^{i+1}$; $a_i\in k$, $a_0\neq 0$. Differentiation of this expression of σt gives us

$$(\sigma t)' = \textstyle\sum_{i=0}^{\infty}(i+1)a_i t^i t'+\sum_{i=0}^{\infty}a_i' t^{i+1}$$
$$= t^{1-n}\textstyle\sum_{i=0}^{\infty}[(i+1)ca_i+a_{j-n}']t^i;$$

here we assume that $a_i=0$ if $i<0$. On the other hand

$$\sigma(t') = c\sigma(t^{1-n}) = ct^{1-n}(\textstyle\sum_{i=0}^{\infty}a_i t^i)^{1-n} .$$

Hence,

$$(1) \qquad (\textstyle\sum_{i=0}^{\infty}a_i t^i)^{n-1}\sum_{i=0}^{\infty}[(i+1)ca_i+a_{i-n}']t^i = c .$$

Comparing the constant terms of both sides, we have $ca_0^n=c$ and $a_0^n=1$; a_0 is a constant. Let us show that $a_i=0$ for any $i\geqq 1$. To the contrary assume that there exists an index $i\geqq 1$ with $a_i\neq 0$. Let j be the minimum of those indices. The coefficient of t^j on the left hand side of (1) is $(n+j)ca_0^{n-1}a_j$. Hence $a_j=0$. This is a contradiction. Thus $a_i=0$ for any $i\geqq 1$. Therefore $\sigma t=\varepsilon t$ with $\varepsilon=a_0$. Conversely let ε be an n-th root in k_0 of 1 and θ be a mapping of K_P to itself defined by

$$\theta(x) = \textstyle\sum b_i(\varepsilon t)^i,\ x = \sum b_i t^i,\ b_i\in k .$$

Then θ is a continuous k-automorphism of K_P. It is a differential one: For,

$$\theta(x') = \theta(\textstyle\sum icb_i t^{i-n}+\sum b_i' t^i)$$
$$= \textstyle\sum icb_i(\varepsilon t)^{i-n}+\sum b_i'(\varepsilon t)^i$$
$$= \textstyle\sum icb_i\varepsilon^i t^{i-n}+\sum b_i'\varepsilon^i t^i$$
$$= (\textstyle\sum b_i\varepsilon^i t^i)' = (\theta x)' .$$

2. Proof of Theorem

Suppose that K is of Painlevé type over k. We may assume that k^* is algebraically closed, since K and the algebraic closure of k^* are independent over k. If Γ is empty, then K is free from parametric singularities over k. We assume that Γ is not empty. Let P be an element of Γ. We have a prime element t in P such that $t'=ct^{1-n}$ (cf. §1). There exists uniquely a prime divisor P^* of K^* such that the restriction of $\nu^*_{P^*}$ to K is ν_P, where $\nu^*_{P^*}$ is the normalized valuation belonging to P^*. The completion K_P of K with respect to P is a subfield of the completion $K^*_{P^*}$ of K^* with respect to P^*.

We shall show that K^* and K_P are linearly disjoint over K. They are so if and only if k^* and K_P are linearly disjoint over k, since $K^*=k^*(K)$ and K, k^* are linearly disjoint over k. Assume that m elements $a_1, \cdots, a_m$ of k^* are linearly dependent over K_P: $\sum_{i=1}^m a_i u_i=0$ with $u_i \in K_P$ and $u_i \neq 0$ for some i. We may suppose that $\nu_P(u_1) \leqq \nu_P(u_i)$ for any i. For each i let b_i be an element of k such that $\nu_P(u_i/u_1-b_i)>0$. Then $b_1=1$ and $\sum_{i=1}^m a_i b_i=0$. Hence $a_1, \cdots, a_m$ are linearly dependent over k. Thus k^*, K_P are linearly disjoint over k and K^*, K_P are linearly disjoint over K.

For each element σ of G_P there exists uniquely a continuous differential k^*-automorphism σ^* of $K^*_{P^*}$ whose restriction to K_P is σ. Set $G^*_P=\{\sigma^*; \sigma \in G_P\}$. Let us define a subset $L^*(P)$ of K^* as the totality of those elements of K^* each of which is left invariant under G^*_P. Put $L^*=\cap L^*(P)$, $P \in \Gamma$. Then, L^* is a differential extension field of k^*.

We shall prove that K^*_0 is contained in L^*. Let γ be a constant of K^* that is transcendental over k. Take an element P of Γ. In $K^*_{P^*}$ we have

$$\gamma = \sum_{i=p}^{\infty} a_i t^i, \; a_i \in k^*, \; a_p \neq 0 .$$

Differentiation of this expression of γ gives us

$$0 = \gamma' = \sum_{i=p}^{\infty} [ica_i + a'_{i-n}] t^{i-n};$$

here we assume that $a_i=0$ if $i<p$. This implies

$$ica_i + a'_{i-n} = 0 \; (p \leqq i) . \tag{2}$$

In particular, $pca_p + a'_{p-n} = pca_p = 0$. Hence $p=0$. We shall show that $i \equiv 0 \pmod{n}$ if $a_i \neq 0$. To the contrary assume that there exists an index i such that $a_i \neq 0$ with $i \not\equiv 0 \pmod{n}$. Let j be the minimum of those indices. Then we get $a_{j-n}=0$, and $a_j=0$ by (2). Hence our assertion is true. Since $\sigma^* t^n = t^n$, γ is contained in $L^*(P)$. Hence $\gamma \in L^*$.

Put $L(P)=L^*(P) \cap K$ and $L=L^* \cap K = \cap L(P)$, $P \in \Gamma$. By the definition of L^* and L, $L^* \supset k^*(L)$. We prove that

$$L^* = k^*(L) . \tag{3}$$

Let x be an element of L^*. Since $L^* \subset K^* = k^*(K)$, we have

$$\sum_{i=1}^{r} a_i u_i - x \sum_{j=1}^{s} b_j v_j = 0;$$

here $a_i, b_j \in k^*$ and $u_i, v_j \in K$. Among those expressions of x pick one with a minimal s. We may assume that $a_1, \cdots, a_r$ are linearly independent over K and $v_s = 1$. Then

$$(4) \qquad a_1, \cdots, a_r, xb_1, \cdots, xb_{s-1}$$

are linearly independent over K by the minimality of s. Let P be an element of Γ. Then the members of (4) are linearly independent over K_P, since K^* and K_P are linearly disjoint over K. Take an element σ^* of G_P^*. Then,

$$\begin{aligned} 0 &= \sigma^*(\sum_{i=1}^{r} a_i u_i - x \sum_{j=1}^{s} b_j v_j) \\ &= \sum_{i=1}^{r} a_i \sigma u_i - x \sum_{j=1}^{s} b_j \sigma v_j , \end{aligned}$$

since $\sigma^* x = x$. We have $\sigma v_s = v_s$ by $v_s = 1$. Hence,

$$\sum_{i=1}^{r} a_i (u_i - \sigma u_i) - x \sum_{j=1}^{s-1} b_j (v_j - \sigma v_j) = 0 .$$

Since each of $u_i - \sigma u_i$ and $v_j - \sigma v_j$ is in K_P, we obtain $u_i = \sigma u_i$ and $v_j = \sigma v_j$ ($1 \leqq i \leqq r$, $1 \leqq j \leqq s$). Hence, $u_i, v_j \in L(P)$. They are in L and $x \in k^*(L)$. Therefore (3) holds.

By (3) we get $L \neq k$, since $L^* \supset K_0^* \supsetneqq k_0$. We shall prove that L is free from parametric singularities over k. Let Q, $\bar{\nu}_Q$ and τ be a prime divisor of L, the normalized valuation belonging to Q and a prime element in Q respectively. Suppose that P is an extension of Q to K. Then, $L_Q \subset K_P$. Let e be the ramification exponent of P with respect to L. Take a prime element t_1 in P such that $\tau = t_1^e$. Then,

$$e\bar{\nu}_Q(\tau'/\tau) = \nu_P(et_1'/t_1) = \nu_P(t_1') - 1 ,$$

and

$$(5) \qquad e\bar{\nu}_Q(\tau') = \nu_P(t_1') + e - 1 .$$

If $P \notin \Gamma$, we have $\nu_P(t_1') \geqq 0$ and $\bar{\nu}_Q(\tau') \geqq 0$. Let us assume that $P \in \Gamma$. Then each element of L_Q is left invariant under G_P. By (5), $e\bar{\nu}_Q(\tau') = e - n$. For each σ of G_P, $\sigma t = \mathcal{E} t$, $\mathcal{E}^n = 1$. There exists an element σ of G_P such that $\mathcal{E}$ is a primitive n-th root of 1. Since $\nu_P(t^e/\tau) = 0$, there exists a nonzero element a of k such that $\nu_P(t^e/\tau - a) > 0$. We have

$$\nu_P(\sigma(t^e/\tau - a)) = \nu_P(\mathcal{E}^e t^e/\tau - a) > 0 ,$$

since $\sigma t = \mathcal{E} t$ and $\sigma\tau = \tau$. Hence,

$$\nu_P((\mathcal{E}^e - 1)t^e/\tau) \geqq \text{Min}\, \{\nu_P(\mathcal{E}^e t^e/\tau - a),\, \nu_P(t^e/\tau - a)\} > 0 .$$

We have $\varepsilon^e=1$, because $\nu_P(t^e/\tau)=0$. Therefore, n divides e, since ε is a primitive n-th root of 1: $e\bar{\nu}_Q(\tau')=e-n\geqq 0$. Thus L is free from parametric singularities over k.

3. Proof of Proposition

Suppose that M is a differential subfield of K which is free from parametric singularities over k. Then $M^*=k^*(M_0^*)$ for some k^*, where $M^*=k^*(M)$ and M_0^* is the field of constants of M^* (cf. [4]). Since $K_0^*\supset M_0^*\supsetneqq k_0, K_0^*$ contains k_0 properly. We may suppose that k^* is algebraically closed. Since L^* includes K_0^*, $M_0^*\subset K_0^*\subset L^*$. Hence, $M^*=k^*(M_0^*)\subset L^*$. Thus,

$$M\subset K\cap M^*\subset K\cap L^* = L.$$

4. Examples

EXAMPLE 1. Suppose that $K=k(t)$ and $t'=t^2-1$. Then a k-automorphism σ of K defined by $\sigma t=1/t$ is a differential one. Let P be a prime divisor of K determined by $\nu_P(t)=1$. Then σ is not continuous in the metric of K_P.

EXAMPLE 2. Rosenlicht [10] proved the following theorem: Suppose that $k=k_0$ and $K=k(y)$: If K is of Painlevé type over k in our sense, then either $y'=af/f_y$ or $y'=1/g_y$ with $f, g\in k(y)$ and $a\in k$.

EXAMPLE 3. Suppose that $K=k(t)$ and $2tt'=1$. Let P be the prime divisor of K determined by $\nu_P(t)=1$. Then $n_P=2$, and $\Gamma=\{P\}$. The generator σ of G_P satisfies $\sigma t=-t$, and $L=k(t^2)$.

References

[1] C. Chevalley: Introduction to the theory of algebraic functions of one variable, Math. Surveys VI, Amer. Math. Soc., 1951.

[2] L. Fuchs: *Über Differentialgleichungen, deren Integrale feste Verzweibungspunkte besitzen,* S.-B. der Königl. Preuss. Akad. Wiss. Berlin, **32** (1884), 699–710.

[3] M. Matsuda: *Algebraic differential equations of the first order free from parametric singularities from the differential-algebraic standpoint,* J. Math. Soc. Japan **30** (1978), 447–455.

[4] K. Nishioka: *Algebraic differential equations of Clairaut type from the differential-algebraic standpoint,* J. Math. Soc. Japan **31** (1979), 553–559.

[5] P. Painlevé: *Mémoire sur les équations différentielles du premier order,* Ann. Sci. École Norm. Sup. 3^e Série **8** (1891), 9–58, 103–140, 201–226, 267–284; **9** (1892), 9–30, 101–144, 283–308.

[6] ———: Leçons sur la théorie analytique des équations différentielles, professées à Stockholm (1895), Hermann, Paris, 1897; Oeuvres, Tome I, 205–798, Centre National de la Recherche Scientifique, Paris, 1972.

[7] E. Picard: Traité d'analyse, Tome III, 2e Édition, Gauthier-Villars, Paris, 1908.
[8] H. Poincaré: *Sur un théorème de M. Fuchs,* Acta Math. **7** (1885), 1–32.
[9] M. Rosenlicht: *Canonical forms for local derivations,* Pacific J. Math. **42** (1972), 721–732.
[10] ———: *Nonminimality of the differential closure,* Pacific J. Math. **52** (1974), 529–537.
[11] A. Weil: Foundations of algebraic geometry, Amer. Math. Soc. Colloq. Publ. Vol. 29, Providence, 1946.

Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan