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A THEOREM OF PAINLEVÉ ON PARAMETRIC SINGULARITIES OF ALGEBRAIC DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

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0. Introduction

Consider an algebraic differential equation of the first order $F(y,y')=0$ over an algebraically closed ordinary differential field k of characteristic 0, where F is an irreducible polynomial over k. Recently Matsuda [3] presented a differential-algebraic definition for $F=0$ to be free from parametric singularities and gave a purely algebraic proof of the following theorem essentially due to Fuchs [2] and Poincaré [8]: Suppose that $F=0$ is free from parametric singularities. Then it is reduced to a Riccati equation or a defining equation of elliptic function by a birational transformation over k if the genus of $F=0$ is 0 or 1 respectively. The author [4] proved that under the above assumption it is reduced to an equation of Clairaut type by a birational transformation over k if the genus is greater than 1. This theorem is essentially due to Poincaré [8], Painlevé [5] and Picard [6].

Here a differential-algebraic formulation and its proof of the following theorem which is essentially due to Painlevé [5], [6] will be given: The general solution η of $F=0$ depends algebraically upon an arbitrary constant over some differential extension field of k if and only if there exists an algebraic differential equation of the first order $G=0$ over k such that it is free from parametric singularities and the general solution of $G=0$ is a rational function of η and η' over k. Here, we assume that k contains non-constants.

Let k be an algebraically closed ordinary differential field of characteristic 0, and Ω be a universal differential extension field of k. Suppose that K is a differential subfield of Ω and it is an algebraic function field of one variable over k. Let P be a prime divisor of K and K_P be the completion of K with respect to P. Then K_P is a differential extension field of K and the differentiation is continuous in the metric of K_P (cf. [1, p. 114]). Let ν_P and t_P denote respectively the normalized valuation belonging to P and a prime element in P. The following definition is due to Matsuda [3]: K is said to be *free from parametric singularities* over k if we have $\nu_P(t'_P)\geqq 0$ for each prime divisor P of

K. Here, we do not set the assumption that K takes the form $k(y,y')$ with some element y of K, which is done in [3]. In this general situation is the author's paper [4] which will be quoted later.

Let k^* be a differential extension field of k in Ω; we take for granted that the field of constants of k^* is the same as that of k and K,k^* are independent over k. Since k is algebraically closed, K and k^* are linearly disjoint over k (cf. [11, p. 19]). K^*, k_0 and K_0^* will indicate $k^*(K)$, the fields of constants of k and K^* respectively; k_0 is algebraically closed.

DEFINITION. K will be said to be of *Painlevé type* over k if there exists such k^* that $K_0^* \neq k_0$.

If K_1 is of Painlevé type over k and K_2 is an algebraic extension field of K_1 of finite degree, then K_2 is so over k: For K_2 and k^* are independent over k.

Theorem. *K is of Painlevé type over k if and only if there exists a differential subfield of K which is free from parametric singularities over k.*

The "if" part is known: For, a differential subfield M is of Painlevé type over k in our sense if M is free from parametric singularities over k (cf. [4]). Suppose that K is of Painlevé type over k. Let Γ be the totality of those prime divisors P of K satisfying $\nu_P(t_P') < 0$. Assume that K is not free from parametric singularities over k. Then Γ is not empty. Let P be an element of Γ. Then the number n_P defined by $n_P = 1 - \nu_P(t_P')$ does not depend on the choice of a prime element t_P in P. It is an integer greater than 1. We define G_P as the group of all differential k-automorphisms of K_P that are continuous in the metric of K_P. By a theorem of Rosenlicht [9, Th. 3] we have the following:

Lemma. *G_P is a cyclic group of order n_P.*

Let L denote the totality of those element of K each of which is left invariant under G_P for any P in Γ. Then L is a differential extension field of k. It is proved to be free from parametric singularities over k. Thus Theorem is obtained. If k contains non-constants then L takes the form $k(y,y')$ with some element y of L.

In case $\Gamma = \phi$, we set $L = K$.

Proposition. *Suppose that K is of Painlevé type over k. Then, every differential subfield of K which is free from parametric singularities over k is contained in L.*

Lemma, Theorem and Proposition will be proved in §1, §2 and §3 respectively. In the last §4 some examples will be given.

REMARK 1. Suppose that K is of Painlevé type over k. Then, there exists such k_1^* that $(K_1^*)_0 \neq k_0$ and $[K^* : k^*(K_0^*)] = [K_1^* : k_1^*((K_1^*)_0)]$ if $k^* \supset k_1^*$, where $K_1^*=k_1^*(K)$ and $(K_1^*)_0$ denotes the field of constants of K_1^*.

REMARK 2. By a result (3) in §2 we have the following: If $K^*=k^*(K_0^*)$ for some k^*, then K is free from parametric singularities over k. Hence, K is free from parametric singularities over k if K^* is so over some algebraically closed k^*.

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1. Proof of Lemma

Let P be an element of Γ and n be n_P. By a theorem of Rosenlicht [9, Th. 3] there exists a prime element t in P such that $t'=ct^{1-n}$ with a nonzero element c of k. Suppose that σ is an element of G_P. Then, we have $\nu_P(\sigma x)=\nu_P(x)$ for each x in K_P, since σ is continuous in the metric of K_P. We shall prove that $\sigma t=\mathcal{E}t$ with $\mathcal{E}^n=1$, $\mathcal{E}\in k_0$. In K_P, $\sigma t=\sum_{k=1}^{\infty}a_i t^{i+1}$; $a_i\in k$, $a_0\neq 0$. Differentiation of this expression of σt gives us

$$\begin{aligned}(\sigma t)' &= \textstyle\sum_{i=0}^{\infty}(i+1)a_i t^i t'+\sum_{i=0}^{\infty}a_i' t^{i+1}\\ &= t^{1-n}\textstyle\sum_{i=0}^{\infty}[(i+1)ca_i+a_{j-n}']t^i;\end{aligned}$$

here we assume that $a_i=0$ if $i<0$. On the other hand

$$\sigma(t')=c\sigma(t^{1-n})=ct^{1-n}(\textstyle\sum_{i=0}^{\infty}a_i t^i)^{1-n}.$$

Hence,

$$(\textstyle\sum_{i=0}^{\infty}a_i t^i)^{n-1}\sum_{i=0}^{\infty}[(i+1)ca_i+a_{i-n}']t^i=c. \tag{1}$$

Comparing the constant terms of both sides, we have $ca_0^n=c$ and $a_0^n=1$; a_0 is a constant. Let us show that $a_i=0$ for any $i\geqq 1$. To the contrary assume that there exists an index $i\geqq 1$ with $a_i\neq 0$. Let j be the minimum of those indices. The coefficient of t^j on the left hand side of (1) is $(n+j)ca_0^{n-1}a_j$. Hence $a_j=0$. This is a contradiction. Thus $a_i=0$ for any $i\geqq 1$. Therefore $\sigma t=\mathcal{E}t$ with $\mathcal{E}=a_0$. Conversely let $\mathcal{E}$ be an n-th root in k_0 of 1 and θ be a mapping of K_P to itself defined by

$$\theta(x)=\textstyle\sum b_i(\mathcal{E}t)^i,\ x=\sum b_i t^i,\ b_i\in k.$$

Then θ is a continuous k-automorphism of K_P. It is a differential one: For,

$$\begin{aligned}\theta(x') &= \theta(\textstyle\sum icb_i t^{i-n}+\sum b_i' t^i)\\ &= \textstyle\sum icb_i(\mathcal{E}t)^{i-n}+\sum b_i'(\mathcal{E}t)^i\\ &= \textstyle\sum icb_i\mathcal{E}^i t^{i-n}+\sum b_i'\mathcal{E}^i t^i\\ &= (\textstyle\sum b_i\mathcal{E}^i t^i)'=(\theta x)'.\end{aligned}$$

2. Proof of Theorem

Suppose that K is of Painlevé type over k. We may assume that k^* is algebraically closed, since K and the algebraic closure of k^* are independent over k. If Γ is empty, then K is free from parametric singularities over k. We assume that Γ is not empty. Let P be an element of Γ. We have a prime element t in P such that $t'=ct^{1-n}$ (cf. §1). There exists uniquely a prime divisor P^* of K^* such that the restriction of $\nu^*_{P^*}$ to K is ν_P, where $\nu^*_{P^*}$ is the normalized valuation belonging to P^*. The completion K_P of K with respect to P is a subfield of the completion $K^*_{P^*}$ of K^* with respect to P^*.

We shall show that K^* and K_P are linearly disjoint over K. They are so if and only if k^* and K_P are linearly disjoint over k, since $K^*=k^*(K)$ and K, k^* are linearly disjoint over k. Assume that m elements $a_1, \cdots, a_m$ of k^* are linearly dependent over K_P: $\sum_{i=1}^m a_i u_i=0$ with $u_i \in K_P$ and $u_i \neq 0$ for some i. We may suppose that $\nu_P(u_1) \leqq \nu_P(u_i)$ for any i. For each i let b_i be an element of k such that $\nu_P(u_i/u_1-b_i)>0$. Then $b_1=1$ and $\sum_{i=1}^m a_i b_i=0$. Hence $a_1, \cdots, a_m$ are linearly dependent over k. Thus k^*, K_P are linearly disjoint over k and K^*, K_P are linearly disjoint over K.

For each element σ of G_P there exists uniquely a continuous differential k^*-automorphism σ^* of $K^*_{P^*}$ whose restriction to K_P is σ. Set $G^*_P=\{\sigma^*; \sigma \in G_P\}$. Let us define a subset $L^*(P)$ of K^* as the totality of those elements of K^* each of which is left invariant under G^*_P. Put $L^*=\cap L^*(P)$, $P \in \Gamma$. Then, L^* is a differential extension field of k^*.

We shall prove that K^*_0 is contained in L^*. Let γ be a constant of K^* that is transcendental over k. Take an element P of Γ. In $K^*_{P^*}$ we have

$$\gamma = \textstyle\sum_{i=p}^{\infty} a_i t^i,\ a_i \in k^*,\ a_p \neq 0\,.$$

Differentiation of this expression of γ gives us

$$0 = \gamma' = \textstyle\sum_{i=p}^{\infty} [ica_i + a'_{i-n}] t^{i-n};$$

here we assume that $a_i=0$ if $i<p$. This implies

$$(2) \qquad ica_i + a'_{i-n} = 0 \ (p \leqq i)\,.$$

In particular, $pca_p+a'_{p-n}=pca_p=0$. Hence $p=0$. We shall show that $i\equiv 0 \pmod n$ if $a_i\neq 0$. To the contrary assume that there exists an index i such that $a_i\neq 0$ with $i \not\equiv 0 \pmod n$. Let j be the minimum of those indices. Then we get $a_{j-n}=0$, and $a_j=0$ by (2). Hence our assertion is true. Since $\sigma^* t^n=t^n$, γ is contained in $L^*(P)$. Hence $\gamma \in L^*$.

Put $L(P)=L^*(P)\cap K$ and $L=L^*\cap K=\cap L(P)$, $P\in\Gamma$. By the definition of L^* and L, $L^* \supset k^*(L)$. We prove that

$$(3) \qquad L^* = k^*(L)\,.$$

Let x be an element of L^*. Since $L^* \subset K^* = k^*(K)$, we have

$$\sum_{i=1}^r a_i u_i - x \sum_{j=1}^s b_j v_j = 0;$$

here $a_i, b_j \in k^*$ and $u_i, v_j \in K$. Among those expressions of x pick one with a minimal s. We may assume that $a_1, \cdots, a_r$ are linearly independent over K and $v_s = 1$. Then

$$(4) \qquad a_1, \cdots, a_r, xb_1, \cdots, xb_{s-1}$$

are linearly independent over K by the minimality of s. Let P be an element of Γ. Then the members of (4) are linearly independent over K_P, since K^* and K_P are linearly disjoint over K. Take an element σ^* of G_P^*. Then,

$$\begin{aligned} 0 &= \sigma^*(\textstyle\sum_{i=1}^r a_i u_i - x \sum_{j=1}^s b_j v_j) \\ &= \textstyle\sum_{i=1}^r a_i \sigma u_i - x \sum_{j=1}^s b_j \sigma v_j , \end{aligned}$$

since $\sigma^* x = x$. We have $\sigma v_s = v_s$ by $v_s = 1$. Hence,

$$\sum_{i=1}^r a_i(u_i - \sigma u_i) - x \sum_{j=1}^{s-1} b_j(v_j - \sigma v_j) = 0 .$$

Since each of $u_i - \sigma u_i$ and $v_j - \sigma v_j$ is in K_P, we obtain $u_i = \sigma u_i$ and $v_j = \sigma v_j$ ($1 \leqq i \leqq r$, $1 \leqq j \leqq s$). Hence, $u_i, v_j \in L(P)$. They are in L and $x \in k^*(L)$. Therefore (3) holds.

By (3) we get $L \neq k$, since $L^* \supset K_0^* \supsetneqq k_0$. We shall prove that L is free from parametric singularities over k. Let Q, $\bar{\nu}_Q$ and τ be a prime divisor of L, the normalized valuation belonging to Q and a prime element in Q respectively. Suppose that P is an extension of Q to K. Then, $L_Q \subset K_P$. Let e be the ramification exponent of P with respect to L. Take a prime element t_1 in P such that $\tau = t_1^e$. Then,

$$e\bar{\nu}_Q(\tau'/\tau) = \nu_P(et_1'/t_1) = \nu_P(t_1') - 1 ,$$

and

$$(5) \qquad e\bar{\nu}_Q(\tau') = \nu_P(t_1') + e - 1 .$$

If $P \notin \Gamma$, we have $\nu_P(t_1') \geqq 0$ and $\bar{\nu}_Q(\tau') \geqq 0$. Let us assume that $P \in \Gamma$. Then each element of L_Q is left invariant under G_P. By (5), $e\bar{\nu}_Q(\tau') = e - n$. For each σ of G_P, $\sigma t = \mathcal{E} t$, $\mathcal{E}^n = 1$. There exists an element σ of G_P such that $\mathcal{E}$ is a primitive n-th root of 1. Since $\nu_P(t^e/\tau) = 0$, there exists a nonzero element a of k such that $\nu_P(t^e/\tau - a) > 0$. We have

$$\nu_P(\sigma(t^e/\tau - a)) = \nu_P(\mathcal{E}^e t^e/\tau - a) > 0 ,$$

since $\sigma t = \mathcal{E} t$ and $\sigma\tau = \tau$. Hence,

$$\nu_P((\mathcal{E}^e - 1)t^e/\tau) \geqq \mathrm{Min}\,\{\nu_P(\mathcal{E}^e t^e/\tau - a),\ \nu_P(t^e/\tau - a)\} > 0 .$$

We have $\varepsilon^e=1$, because $\nu_P(t^e/\tau)=0$. Therefore, n divides e, since ε is a primitive n-th root of 1: $e\bar{\nu}_Q(\tau')=e-n\geqq 0$. Thus L is free from parametric singularities over k.

3. Proof of Proposition

Suppose that M is a differential subfield of K which is free from parametric singularities over k. Then $M^*=k^*(M_0^*)$ for some k^*, where $M^*=k^*(M)$ and M_0^* is the field of constants of M^* (cf. [4]). Since $K_0^*\supset M_0^*\supsetneqq k_0, K_0^*$ contains k_0 properly. We may suppose that k^* is algebraically closed. Since L^* includes K_0^*, $M_0^*\subset K_0^*\subset L^*$. Hence, $M^*=k^*(M_0^*)\subset L^*$. Thus,

$$M\subset K\cap M^*\subset K\cap L^* = L\,.$$

4. Examples

EXAMPLE 1. Suppose that $K=k(t)$ and $t'=t^2-1$. Then a k-automorphism σ of K defined by $\sigma t=1/t$ is a differential one. Let P be a prime divisor of K determined by $\nu_P(t)=1$. Then σ is not continuous in the metric of K_P.

EXAMPLE 2. Rosenlicht [10] proved the following theorem: Suppose that $k=k_0$ and $K=k(y)$: If K is of Painlevé type over k in our sense, then either $y'=af/f_y$ or $y'=1/g_y$ with $f, g\in k(y)$ and $a\in k$.

EXAMPLE 3. Suppose that $K=k(t)$ and $2tt'=1$. Let P be the prime divisor of K determined by $\nu_P(t)=1$. Then $n_P=2$, and $\Gamma=\{P\}$. The generator σ of G_P satisfies $\sigma t=-t$, and $L=k(t^2)$.

References

[1] C. Chevalley: Introduction to the theory of algebraic functions of one variable, Math. Surveys VI, Amer. Math. Soc., 1951.

[2] L. Fuchs: *Über Differentialgleichungen, deren Integrale feste Verzweibungspunkte besitzen,* S.-B. der Königl. Preuss. Akad. Wiss. Berlin, **32** (1884), 699–710.

[3] M. Matsuda: *Algebraic differential equations of the first order free from parametric singularities from the differential-algebraic standpoint,* J. Math. Soc. Japan **30** (1978), 447–455.

[4] K. Nishioka: *Algebraic differential equations of Clairaut type from the differential-algebraic standpoint,* J. Math. Soc. Japan **31** (1979), 553-559.

[5] P. Painlevé: *Mémoire sur les équations différentielles du premier order,* Ann. Sci. École Norm. Sup. 3e Série **8** (1891), 9–58, 103–140, 201–226, 267–284; **9** (1892), 9–30, 101–144, 283–308.

[6] ———: Leçons sur la théorie analytique des équations différentielles, professées à Stockholm (1895), Hermann, Paris, 1897; Oeuvres, Tome I, 205–798, Centre National de la Recherche Scientifique, Paris, 1972.

[7] E. Picard: Traité d'analyse, Tome III, 2e Édition, Gauthier-Villars, Paris, 1908.
[8] H. Poincaré: *Sur un théorème de M. Fuchs,* Acta Math. **7** (1885), 1–32.
[9] M. Rosenlicht: *Canonical forms for local derivations,* Pacific J. Math. **42** (1972), 721–732.
[10] ———: *Nonminimality of the differential closure,* Pacific J. Math. **52** (1974), 529–537.
[11] A. Weil: Foundations of algebraic geometry, Amer. Math. Soc. Colloq. Publ. Vol. 29, Providence, 1946.

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