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# SATELLITE KNOTS IN 1-GENUS 1-BRIDGE POSITIONS

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## 1. Introduction

Let $(V_i, t_i)$ be a pair of a solid torus $V_i$ and an unknotted arc $t_i$ properly imbedded in $V_i$ for $i = 1$ and 2. Summing these pairs we obtain a pair $(M, K)$ of the 3-dimensional sphere or a lens space $M$ and a knot $K$. We call such a knot $K$ a *1-genus 1-bridge knot.* We determine in this paper when a 1-genus 1-bridge knot is a satellite knot.

A set of mutually disjoint arcs $T = \{t_1, \cdots, t_n\}$ properly imbedded in a handlebody $V$ is *trivial* if there is a set of mutually disjoint discs $D = \{D_1, \cdots, D_n\}$ such that $t_i \subset \partial D_i$, $t_i \cap D_j = \emptyset$ and $\partial D_i - t_i \subset \partial V$ for $1 \leq i, j \leq n$ and $i \neq j$. These discs are called *cancelling discs* of $T$.

Let $M$ be a closed orientable 3-dimensional manifold. A closed surface $H$ imbedded in $M$ is called a *Heegaard splitting surface* of $M$ if $H$ splits $M$ into two handlebodies $V_1$ and $V_2$. A link $L$ in $M$ is said to be in *$g$-genus $n$-bridge position* with respect to $H$ if $L$ is transeverse to $H$ and $L \cap V_i$ consists of trivial $n$ arcs in $V_i$ for $i = 1$ and 2. We say also that $H$ is a *$g$-genus $n$-bridge decomposition* of $L$. A 0-genus $n$-bridge link is usually called an *$n$-bridge link.*

Note that if a link is in $g$-genus 1-bridge position, then it is a knot. Such a knot is very important in light of some results and conjectures on Dehn surgery on knots. For example, see [3], [4] by D. Gabai, [1] by J. Berge, [9], [10], [11] by Y-Q. Wu. It is well-known that 2-bridge knots are 1-genus 1-bridge knots.

Let $M$ be the 3-dimensional sphere or a lens space (not homeomorphic to $S^2 \times S^1$). Let $H$ be a genus 1 Heegaard surface of $M$. This surface $H$ divides $M$ into two solid tori $V_1$ and $V_2$. Suppose a knot $K$ is in 1-genus 1-bridge position with respect to $H$. Then $K \cap V_i$ consists of a trivial arc, say $t_i$, in $V_i$ for $i = 1$ and 2.

The decomposition $H$ is said to be *$K$-reducible* if $V_i$ contains a meridian disc $D_i$ and $V_j$ contains a cancelling disc $D_j$ of $t_j$ such that $D_i \cap t_i = \emptyset$ and $\partial D_i \cap \partial D_j = \emptyset$ for $i = 1$, $j = 2$ or $i = 2$, $j = 1$. If $H$ is $K$-reducible, then $K$ is the trivial knot bounding a disc composed of two cancelling discs as shown in [Lemma 2.2, 6].

The decomposition $H$ is said to be *weakly $K$-reducible* if $V_i$ contains a meridian disc $D_i$ and $V_j$ contains a cancelling disc $D_j$ of $t_j$ such that $\operatorname{int} D_i$ intersects $t_i$ transversely in one point and $\partial D_i \cap \partial D_j = \emptyset$ for $i = 1$, $j = 2$ or $i = 2$, $j = 1$. If $H$ is weakly $K$-reducible, then $K$ is the trivial knot or a 2-bridge knot when $M = S^3$, and $K$ is a core knot or a composite knot of a core knot and a 2-bridge knot when $M$ is a lens space. This will be shown in Lemma 2.1.

Note that weak $K$-reducibility is not a generalized notion of $K$-reducibility. These definitions are motivated by [7] by T. Kobayashi and O. Saeki.

A knot $K$ is a *trivial knot* if it bounds a disc imbedded in $M$. A non-trivial knot $K$ in a lens space $M$ is a *core knot* if the exterior $E(K) = M - \mathrm{int}N(K)$ is homeomorphic to a solid torus. A knot $K$ in $M$ is *split* if $M$ contains a sphere $S$ which decomposes $M$ into a punctured lens space and a ball containing $K$ in its interior. This sphere $S$ is called a *splitting sphere*. A knot $K$ in $M$ is *composite* if $M$ contains a 2-sphere $S$ which intersects $K$ transeversely in 2 points and $S \cap E(K)$ is $\partial$-incompressible in $E(K)$. We call this 2-sphere $S$ a *decomposing sphere*. A knot is said to be *prime* if it is not composite. A knot is said to be *satellite* if $E(K)$ contains an incompressible torus $T$ which is not parallel to $\partial E(K)$. It is well-known that a composite knot is a satellite knot.

In this paper we determine when a 1-genus 1-bridge knot is a satellite knot because very much is known about Dehn surgery on satellite knots [5] by C.McA. Gordon. In the course of the proof, we obtain two theorems, Theorems I and II, which are already shown by H. Doll in [2]. For Theorem I, see Conjecture 1.3, the sentence right after the proof of 1.1' from 1.3 and "the proof of 1.3 from 1.6 for $g = 0, 1$ or $M$ is irreducible and non-Haken of genus $g$" in section 5 in [2]. For Theorem II, see the latter half of Theorem 1.6 in [2].

**Theorem I.** (H. Doll, [2]) *Let $M$, $K$, $H$, $V_i$, $t_i$ be as above, especially $K$ is in 1-genus 1-bridge position with respect to $H$. Suppose that $K$ is a split knot. Then the decomposition $H$ is $K$-reducible and $K$ is the trivial knot.*

**Theorem II.** (H. Doll, [2]) *Let $M$, $K$, $H$, $V_i$, $t_i$ be as above, especially $K$ is in 1-genus 1-bridge position with respect to $H$. Suppose that $H$ is not $K$-reducible and $K$ is composite. Then $M$ is a lens space rather than the 3-dimensional sphere, and $H$ is weakly $K$-reducible. Moreover, $K$ is a sum of two 1-string tangles $(B_i, T_i)$, $(i = 1, 2)$ as below:*

(1)$B_1$ *is a once punctured lens space, and $cl(B_1 - N(T_1))$ is a solid torus, and*

(2)$B_2$ *is a ball, and $cl(B_2 - N(T_2))$ is homeomorphic to the exterior of a non-trivial 2-bridge knot.*

The next is the main theorem of this paper.

**Theorem III.** *Let $M$, $K$, $H$, $V_i$, $t_i$ be as above, especially $K$ is in 1-genus 1-bridge position with respect to $H$. Suppose that $H$ is neither $K$-reducible nor weakly $K$-reducible. If $K$ is a satellite knot, then there is an annulus $Z$ on $H$ such that there* *is a cancelling disc $C_i$ of $t_i$ with $(\partial C_i \cap H) \subset Z$ for $i = 1$ and 2. Moreover,the incompressible torus is isotopic to $\partial N(C_1 \cup Z \cup C_2)$ in $E(K)$.*

It is well-known that a 1-genus 1-bridge knot is of tunnel number one. K. Morimoto and M.Sakuma showed in [(2.1)Theorem,8] that a companion knot of a satellite knot of tunnel number one in $S^3$ is a torus knot.

In order to prove the above theorems, we need the next theorem. Let $X$ be a 3-manifold, and $T$ a 1-dimensional manifold properly imbedded in $X$. Let $F$ be a connected surface properly imbedded in $X$ such that it is transverse to $T$. Then $F$ is called *$T$-incompressible* if for any disc $D$ such that $D \cap F = \partial D$ and $D \cap T = \emptyset$ there is a disc $D'$ on $F$ such that $\partial D' = \partial D$ and $D' \cap T = \emptyset$. We call $F$ is *meridionally incompressible* in $(X, T)$ if for any disc $D$ such that $D \cap F = \partial D$ and $|D \cap T| = 1$, there is a disc $D'$ on $F$ such that $\partial D' = \partial D$ and $|D' \cap T| = 1$. The surface $F$ is *$T$-$\partial$-incompressible* if for any disc $D$ such that $\alpha = D \cap F$ is a subarc of $\partial D$, $\partial D - \alpha \subset \partial X$ and $D \cap T = \emptyset$, there is a disc $D'$ on $F$ such that $D' \cap F = \partial D' \cap F = \alpha$, $\partial D' - \alpha \subset \partial F$ and $D' \cap T = \emptyset$.

**Theorem IV.** *Let $M$, $K$, $H$, $V_i$, $t_i$ be as above, especially $K$ is in 1-genus 1-bridge position with respect to $H$. Suppose that $H$ is neither $K$-reducible nor weakly $K$-reducible. Let $F$ be a $K$-incompressible and meridionally incompressible closed surface imbedded in $M$ with $|F \cap K| \leq 2$. Suppose that $F_0 = F \cap E(K)$ is $\partial$-incompressible in $E(K)$. Then one of the following two conditions holds.*

(1)*We can move $F$ by an isotopy of the pair $(M, K)$ so that $F \cap V_1$ consists of a parallel set of peripheral discs each of which cuts off a ball containing $t_1$ from $V_1$.*

(2)*We can deform $F$ so that $F \cap V_1$ is a peripheral disc which intersects $t_1$ in two points, cuts off a rational tangle $(B, T)$ and is $(t_1 - T)$-$\partial$-compressible in $cl(V_1 - B)$.*

## 2. Preliminaries

**Lemma 2.1.** *Let $H$ be a 1-genus 1-bridge decomposition of a knot $K$. Suppose that $H$ is weakly $K$-reducible. Then $K$ is a sum of two 1-string tangles $(B_k, T_k)$, $(k = 1, 2)$ as below:*

(1)*$B_1$ is a ball or a once punctured lens space, and the exterior of $T_1$ is homeomorphic to a solid torus, and*

(2)*$B_2$ is a ball, and $cl(B_2 - N(T_2))$ is homeomorphic to the exterior of a (possibly trivial) 2-bridge knot.*

*When $M$ is the 3-sphere, $K$ is the trivial knot if $(B_2, T_2)$ is the trivial tangle, and $K$ is a 2-bridge knot if $(B_2, T_2)$ is non-trivial. When $M$ is a lens space, $K$ is a core knot if $(B_2, T_2)$ is the trivial tangle, and $K$ is a composite knot if $(B_2, T_2)$ is non-trivial.*

Proof. Let $D_i$ be a meridian disc with $|D_i \cap t_i| = 1$, and $D_j$ a cancelling disc as in the definition of weak $K$-reducibility. Let $B_1 = N(D_i) \cup \text{cl}(V_j - N(D_j))$, $B_2 = \text{cl}(M - B_1)$ and $T_i = K \cap B_i$. Since $(B_2, T_2)$ is the sum of the ball $V_i - N(D_i)$ containing two subarcs of $t_i$ and the ball $N(D_j)$ containing the arc $t_j$, its exterior is homeomorphic to that of a 2-bridge knot. See Figure 2.1. $\square$

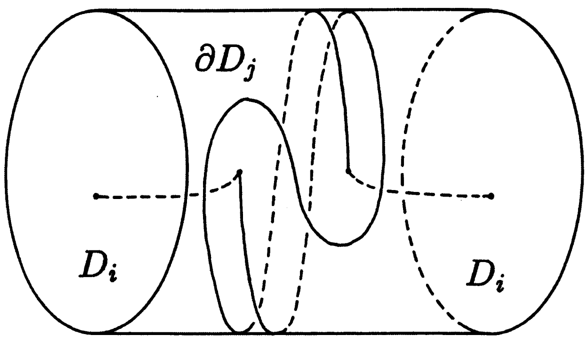


Figure 2.1

**Lemma 2.2.** *Let $H$ be a 1-genus 1-bridge decomposition of a knot $K$. The decomposition $H$ is weakly $K$-reducible if and only if there is a 2-sphere $S$ satisfying the following two conditions for $i = 1$, $j = 2$ or $i = 2$, $j = 1$.*

(1)*$S \cap V_i$ is a peripheral disc which intersects $t_i$ in two points, cuts off a rational tangle $(B, T)$ and is $(t_i - T)$-$\partial$-compressible in cl$(V_i - B)$.*

(2)*$S \cap V_j$ is a peripheral disc which is disjoint from $t_j$, cuts off a ball $B'$ containing $t_j$ from $V_j$.*

Proof. First we assume that there is a 2-sphere $S$ as above. The disc $D = S \cap V_i$ has a $\partial$-compressing disc in $\text{cl}(V_i - B)$. Along this disc compressing $D$, we obtain two meridian discs $R_1$, $R_2$ each of which intersects $t_i$ in one point. We can take a cancelling disc $C'$ of $t_j$ in $B'$ to be disjoint from $S$. Since $(C' \cap H) \subset (B' \cap H) = (B \cap H)$, we have $\partial C' \cap \partial R_1 = \emptyset$. Then the discs $R_1$ and $C'$ imply that $H$ is weakly $K$-reducible.

Conversely, we assume that $H$ is weakly $K$-reducible. Then $V_i$ contains a meridian disk $D_i$ and $V_j$ contains a cancelling disc $D_j$ of $t_j$ such that $D_i$ intersects $t_i$

transversely in one point and $\partial D_i \cap \partial D_j = \emptyset$. Let $D_i'$ be a parallel copy of $D_i$ such that $D_i' \cap D_j = \emptyset$. The boundary loops $\partial D_i$ and $\partial D_i'$ together divide the torus $H$ into two annuli, one of which, say $A$, contains $\partial t_i$. We can take an essential arc $\alpha$ on $A$ so that $\alpha \cap \partial D_j = \emptyset$. We perform a band sum operation on $D_i$ and $D_i'$ along $\alpha$, to obtain a peripheral disc $Q_1$ satisfying the condition (1) above. We can take a neighbourhood $N(D_j)$ in $V_j$ so that $\partial(N(D_j) \cap H) = \partial Q_1$. Then $Q_1 \cup (\partial N(D_j) - H)$ is a desired 2-sphere. □

**Lemma 2.3.** *Let $V$ be a solid torus, $t$ a trivial arc properly imbedded in $V$, and $C$ a cancelling disc of $t$. Then there is a meridian disc $Q$ of $V$ containing $C$.*

Proof. A standard cut and paste argument allows us to take a meridian disc $Q'$ of $V$ to be disjoint from $C$. We take an arbitrary arc $\alpha$ connecting $\partial C$ and $\partial Q'$ on $\partial V$ and perform a band sum operation on these discs along $\alpha$ to obtain a new cancelling disc, say $C'$ of $t$. This disc $C'$ can be isotoped slightly so that $C' \cap C = t$. Then $Q = C \cup C'$ is a desired meridian disc of $V$. □

## 3. Formation of Graphs

Our goal of sections 3,4,5 and 6 is Theorem IV$'$ below. Let $M$, $H$, $V_i$, $K$, $t_i$ be as in section 1. We consider the three situations below simultaneously.

(1) "(i)" The knot $K$ is split and $F$ is a splitting sphere.

(2) "(ii)" The knot $K$ is non-split and composite, and $F$ is a decomposing sphere.

(3) "(iii)" The knot $K$ is non-split and prime. Let $F$ be a $K$-incompressible and meridionally incompressible connected closed surface imbedded in $M$ such that $|F \cap K| \leq 2$ and $F_0 = F \cap E(K)$ is $\partial$-incompressible in $E(K)$.

In the following *deforming* $F$ means rechoosing $F$ in cases (i) and (ii), and moving $F$ by an isotopy of the pair $(M, K)$ in case (iii).

**Theorem IV$'$.** *Let $F$ be of type (i), (ii) or (iii) as above. Then one of the following three conditions holds.*

(1) *The decomposition $H$ is $K$-reducible in case (i) or weakly $K$-reducible in cases (ii) and (iii).*

(2) *We can deform $F$ so that $F \cap V_1$ consists of a parallel set of peripheral discs each of which cuts off a ball containing $t_1$ from $V_1$.*

(3) *We can deform $F$ so that $F \cap V_1$ is a peripheral disc which intersects $t_1$ in two points, cuts off a rational tangle $(B, T)$ and is $(t_1 - T)$-$\partial$-compressible in $cl(V_1 - B)$.*

**Lemma 3.1.** *We can deform $F$ so that $F \cap V_1$ consists of meridian discs disjoint from $t_1$ and at most two meridian discs intersecting $t_1$ transversely in one point.*

Proof. By Lemma 2.3 there is a meridian disc $Q$ of $V_1$ which contains $t_1$. Let $Q'$ be a meridian disc of $V_1$ which is disjoint form $Q$. We isotope $F$ in $(M, K)$ so that it is transverse to $Q$ and disjoint from $Q'$. Then every curve of $(F \cap V_1) \cap Q$ intersects $t_1$ at most once since $F_0$ is $\partial$-incompressible in $E(K)$ in cases (ii) and (iii). We can isotope the intersection arcs of $(F \cap V_1) \cap Q$ disjoint from $t_1$ out of $V_1$ along the subdiscs of $Q$. Then $F \cap Q$ consists of only proper arcs intersecting $t_1$ precisely once.

Let $B$ be the ball obtained by cutting $V_1$ along $Q$, and $A$ the annulus $B \cap \partial V_1$. The arc $t_1$ divides each copy of $Q$ into two subdiscs. Suppose that there is an arc $\alpha$ of $F \cap A$ such that $\alpha$ is inessential on $A$ and has both endpoints in the same subdisc of a copy of $Q$. Then $A$ contains an inessential arc $\beta$ which is isotopic rel. $\partial\beta$ into a subarc of $\partial Q - \partial t_1$ (see Figure 3.1). Hence we can isotope fixing $F \cap K$ in $M$ so that every inessential arc of $F \cap A$ connects the two subdiscs of a copy of $Q$.

We can deform $F$ so that $F \cap B$ is incompressible. (For example, in case (i), let $D$ be a compressing disc of $F \cap B$, where $\partial D$ divides $F$ into two discs $D_1$ or $D_2$. Then $D \cup D_1$ or $D \cup D_2$ is a splitting sphere of $K$.) Then $F \cap B$ consists of discs. The boundaries of these discs meet a copy of $Q$ at most once. Recall that $F_{\cap}Q' = \phi$. When we recover $V_1$ by attaching two copies of $Q$ on $\partial B$, we obtain three kinds of discs: meridian discs which do not intersect $Q$, meridian discs which intersect $Q$ in an arc, and peripheral discs which intersect $Q$ in at most one arc. We can push the peripheral discs out of $V_1$ by an isotopy of the pair $(M, K)$, and obtain the desired conclusion. □

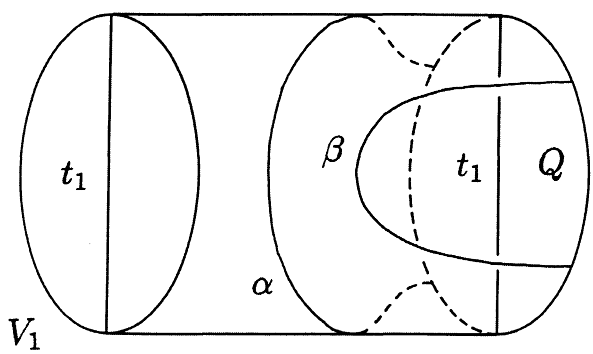


Figure 3.1

Moreover we take $F$ so that the number of meridian discs of $F \cap V_1$ is minimal over all deformations of $F$. Let $P = F \cap V_2$, $E(V_2, t_2)$ =cl$(V_2 - N(t_2))$ and $P_0 = F \cap E(V_2, t_2)$. Then $P$ is $t_2$-incompressible and meridionally incompressible in $(V_2, t_2)$, but is possibly $t_2$-$\partial$-compressible. Let $C$ be a cancelling disc of $t_2$ in $V_2$. We take $C$

and $F$ so that $|\partial C \cap \partial P|$ is minimal over all cancelling discs of $t_2$ and all deformations of $F$. Moreover, we can isotope $C$ in $V_2$ fixing $\partial C$ so that $C \cap P$ consists of arcs only.

Let $Q_1, \cdots, Q_m$ be the meridian discs of $F \cap V_1$ which do not intersect $t_1$ and appear in $V_1$ in this order so that $t_1$ is between $Q_m$ and $Q_1$. Let $R_1, \cdots, R_n$, $(n \leq 2)$ be the meridian discs of $F \cap V_1$ which intersect $t_1$ transeversely in one point appearing in $V_1$ in this order. We assume that $Q_m$ and $R_1$ are adjacent in $V_1$.

Clearly $m + n > 0$. Since $M$ is the 3-sphere or a lens space, $M$ does not contain a non-separating closed surface, and hence $m + n$ is even. We study in this and next sections the case (I) of $m > 0$ and $n > 0$. If $m = 0$ or $n = 0$, then either (II) $n = 2$ and both meridian discs intersect $t_1$ in one point, or (III) $m \geq 2$ and all the meridian discs are disjoint from $t_1$. We will study case (II) in section 5 and case (III) in section 6. In sections 3, 4 and 5, $F$ intersects $K$, and hence we consider the cases (ii) and (iii) only.

The four discs $Q_1$, $Q_m$, $R_1$ and $R_n$ together divide the solid torus $V_1$ into four balls $B_1$, $B_2$, $B_3$ and $B_4$, where $B_1$ is between $Q_1$ and $Q_m$, $B_2$ is between $Q_m$ and $R_1$, $B_3$ is between $R_1$ and $R_n$, $B_4$ is between $R_n$ and $Q_1$. Let $A_i$ be the annulus $H \cap \partial B_i$ for $i = 1, 2, 3$ and 4. We regard $B_3$ as the disc $R_1$, and $A_3$ as the loop $\partial R_1$ when $n = 1$.Similar for $B_1$ and $A_1$.

If we can take $C$ so that $\partial C \cap A_1 = \emptyset$, then $H$ is $K$-reducible, which contradicts that $K$ is not trivial. If we can take $C$ so that $\partial C \cap A_3 = \emptyset$, then $H$ is weakly $K$-reducible, which is a desired conclusion of Theorem IV$'$. Hence we can assume that $\partial C \cap A_1$ and $\partial C \cap A_3$ are non-empty sets. Since $|\partial C \cap \partial P|$ is minimal, it is clear that

(1) $\partial C \cap A_1$ consists of essential arcs in the annulus $A_1$ such that each of them intersects every meridian loop $\partial Q_i$ just once for $i = 1, \cdots, m$, and

(2) $\partial C \cap A_3$ consists of essential arcs in the annulus $A_3$ such that each of them intersects every meridian loop $\partial R_i$ just once for $i = 1, \cdots, n$, and

(3) For $i = 2$ and 4, $\partial C \cap A_i$ consists of essential arcs in the annulus $A_i$, inessential arcs which are essential on $A_i - \partial t_1$, and an arc connecting $\partial t_1$ and $\partial A_i$.

We form a graph $G$ on the disc $C$. The vertices of $G$ are "long vertices" $\partial C \cap A_1$ and $\partial C \cap A_3$. The edges of $G$ are the arcs $C \cap P$. The edges of $G$ are classified into two classes: we regard an arc of $C \cap P$ an *interior edge* if its endpoints are in long vertices of $G$, a *$\partial$-edge* if one of its endpoints is in a long vertex and the other is in $t_2$. Note that there is not an edge whose both endpoints are in $t_2$ since $F_0$ is $\partial$-incompressible in $E(K)$. A long vertex contains $m$ or $n$ endpoints of edges.

## 4. Case (I)

**Lemma 4.1.** *Let $e$ be an edge of $G$ which is outermost on $C$. Then $e$ is not a $\partial$-edge.*

Proof. Suppose for a contradiction that $e$ is a $\partial$-edge. We can assume without loss of generality that $e$ has an endpoint in $\partial Q_1$ or $\partial R_1$. Let $O$ ($\subset C$) be the outermost disc of $e$, that is, $O$ is surrounded by $e$ and a subarc of $\partial C$ and $O$ does not contain another edge of $G$. We isotope a regular neighbourhood of $e$ on $P$ along the outermost disc $O$.

When $e$ has an endpoint in $\partial Q_1$, $Q_1$ is deformed to a meridian disc intersecting $t_1$ in one point. This operation does not change the number $|F \cap V_1|$ but decreases the number of intersection $|\partial C \cap \partial P|$. This is a contradiction.

When $e$ has an endpoint in $\partial R_1$, $R_1$ is deformed to a meridian disc $D$ intersecting $t_1$ in 2 points, and $F_0$ is $\partial$-compressible. This is a contradiction. □

An edge is called a *loop edge* if its two endpoints are in the same long vertex.

**Lemma 4.2.** *The graph $G$ does not contain a loop edge.*

Proof. Suppose that $G$ contains a loop edge $e$, and we can take $e$ to be outermost on the disc $C$. Let $X$ be the long vertex which $e$ is incident to, and $O$ the disc face surrounded by $e$ and a subarc of $X$. This face $O$ does not contain an edge, and endpoints of $e$ are in adjacent meridian discs, say $Q_x$ and $Q_{x+1}$ or $R_1$ and $R_2$. In both cases, we push regular neighbourhood of $e$ on $P$ into $V_1$ along the disc $O$, and these meridian discs are connected by a band and deformed into a peripheral disc $D$ in $V_1$. If $D \cap t_1 = \emptyset$, we push this peripheral disc into $V_2$ and decrease the number of the meridian discs of $F \cap V_1$, which is a contradiction. If $D \cap t_1 \neq \emptyset$, then $F_0$ is $\partial$-compressible, and we obtain a contradiction. □

Let $E$ be a set of edges of $G$. An edge $e$ of $E$ is said to be *outermost away from* $t_2$ among $E$ if there is an *outermost disc* $O$ ($\subset C$) surrounded by $e$ and a subarc $\alpha$ of $\partial C - t_2$ such that $O$ does not contain another edge of $E$.

**Lemma 4.3.** *Let $e$ be an edge of $G$ which is outermost away from $t_2$. Then $e$ does not connect a vertex of $\partial C \cap A_1$ and a vertex of $\partial C \cap A_3$.*

Proof. Suppose that $e$ is such an outermost edge. We can assume without loss of generality that $e$ connects $Q_m$ and $R_1$. We push a regular neighbourhood of $e$ on $P$ into $V_1$ along the outermost disc. Then two meridian discs $Q_m$ and $R_1$ are connected by a band, and are deformed into a peripheral disc $D$ intersecting $t_1$ in one point. We can push $D$ into $V_2$ by an isotopy of the pair $(M, K)$, which contradicts the minimality of the number of the meridian discs of $F \cap V_1$. □

**Lemma 4.4.** *Let $e$ be an edge of $G$ which is outermost away from $t_2$. Then $e$ does not connect two long vertices of $\partial C \cap A_1$.*

Proof. Suppose that $e$ is such an outermost edge. We can assume without loss of generality that $e$ has both endpoints in $\partial Q_m$. We push a regular neighbourhood of $e$ on $P$ into $V_1$ along the outermost disc. Then a band is attached to $Q_m$, and $Q_m$ is deformed into an annulus $A$ one of whose boundary components bounds a disc $D$ $(\subset H)$ intersecting $K$ precisely once. In case (ii), $\partial D$ divides $F$ into two discs $D_1$ and $D_2$, and either $D \cup D_1$ or $D \cup D_2$, say $D \cup D_1$ is a decomposing sphere. In case (iii), the loop $\partial D$ bounds a disc $D'$ on $F$ such that $D'$ intersects $K$ precisely once because $F$ is meridionally incompressible. We can isotope $F$ so that $D'$ is replaced by $D$. If $A \subset D_2$ or $A \subset D'$, then we have a contradiction to the minimality of $|F \cap V_1|$. Thus $A$ is deformed to a meridian disc intersecting $K$ in one point. In all cases, either $|F \cap V_1|$ or $|\partial C \cap \partial P|$ decreases. This is a contradiction. □

**Lemma 4.5.** *Let $e$ be an edge of $G$ which is outermost away from $t_2$. Then $e$ does not connect two long vertices of $\partial C \cap A_3$.*

Proof. Suppose that $e$ is such an outermost edge. We can assume without loss of generality that $e$ has both endpoints in $\partial R_1$. We push a regular neighbourhood of $e$ on $P$ into $V_1$ along the outermost disc. Then a band is attached to $R_1$, and $R_1$ is deformed into an annulus $A$ one of whose boundary components bounds a disc $D$ $(\subset H)$ intersecting $K$ precisely once. In case (ii), $\partial D$ divides $F$ into two discs $D_1$ and $D_2$, and either $D \cup D_1$ or $D \cup D_2$, say $D \cup D_1$ is a decomposing sphere. In case (iii), the loop $\partial D$ bounds a disc $D'$ on $F$ such that $|D' \cap K| = 1$. We can isotope $F$ so that $D'$ is deformed onto $D$. If $A \subset D_2$ or $A \subset D'$, then we have a contradiction to the minimality of $|F \cap V_1|$. Hence, in both cases, the deformed $F$ contains the meridian disc $A \cup D$ intersecting $t_1$ in 2 points, and $F_0$ is $\partial$-compressible. This is a contradiction. □

Proof of Theorem IV$'$ in case (I). In this case, we ultimately obtain a contradiction. Let $b$ be a boundary edge which is outermost among all the boundary edges of $G$ on $C$. Let $O$ be the outermost disc of $b$. By Lemma 4.1, $O$ contains an interior edge of $G$. Hence $G$ contains an interior edge which is outermost away from $t_2$. On the other hand, by Lemmas 4.2, 4.3, 4.4 and 4.5, $G$ cannot contain such an edge. This is a contradiction. □

## 5. Case (II)

In this section we assume that $F \cap V_1$ consists of parallel meridian discs $R_1$ and $R_2$ each of which intersects $t_1$ transversely in one point. The two discs $R_1$ and $R_2$ together divide the solid torus $V_1$ into two balls $B_1$, $B_2$, where $B_2$ contains $\partial t_1$. Let $A_i$ be the annulus $H \cap \partial B_i$ for $i = 1$ and 2.

If we can take $C$ so that $\partial C \cap A_1$ is the empty set, then $H$ is a weakly $K$-reducible

1-genus 1-bridge decomposition, which is a desired conclusion of Theorem IV$'$. Hence we can assume that $\partial C \cap A_1 \neq \emptyset$.

Since we took $C$ and $F$ so that $|\partial C \cap \partial P|$ is minimal, it is clear that

(1) $\partial C \cap A_1$ consists of essential arcs in the annulus $A_1$, and

(2) $\partial C \cap A_2$ consists of essential arcs in the annulus $A_2$, inessential arcs each of which separates the two points $\partial t_1$, and two arcs connecting $\partial t_1$ and $\partial A_2$.

We form a graph $G$ on the disc $C$. The vertices of $G$ are "long vertices" $\partial C \cap A_1$. The edges of $G$ are the arcs $C \cap P$. Since $|F \cap K| \leq 2$ and $|(F \cap V_1) \cap K| = 2$, the edges of $G$ are all interior edges. A long vertex contains 2 endpoints of edges. We give an orientation to $C$ arbitrarily. Then $\partial C$ has the induced orientation and the long vertices also do. A long vertex is *positive* if its orientation goes from $R_1$ to $R_2$, and otherwise, *negative*.

The next lemma is proved by the same argument as in the proof of Lemma 4.2.

**Lemma 5.1.** *The graph $G$ does not contain a loop edge.*

**Lemma 5.2.** *Let $e$ be an edge of $G$ which is outermost away from $t_2$. Then $e$ does not connect vertices of mutually opposite signs.*

Proof. Suppose that $e$ is such an outermost edge. We can assume without loss of generality that $e$ has both endpoints in $\partial R_1$. We push a regular neighbourhood of $e$ on $P$ into $V_1$ along the outermost disc. Then a band is attached to $R_1$, and $R_1$ is deformed into an annulus $A$ one of whose boundary components bounds a disc $D$ ($\subset H$) intersecting $K$ precisely once. In case (ii), $\partial D$ divides $F$ into two discs $D_1$ and $D_2$, and either $D \cup D_1$ or $D \cup D_2$, say $D \cap D_1$ is a decomposing sphere. In case (iii), the loop $\partial D$ bounds a disc $D'$ on $F$ such that $|D' \cap K| = 1$. We can isotope $F$ so that $D'$ is replaced by $D$. If $A \subset D_2$ or $A \subset D'$, then we have a contradiction to the minimality of $|F \cap V_1|$. Since $D$ intersects $t_1$ in one point, $R_2 \subset D_2$ or $R_2 \subset D'$, and the annulus $A$ is deformed into a meridian disc $D_3$ which intersects $t_2$ in two points. Moreover, $F \cap V_1 = D_3$, which implies that $F$ is a non-separating surface in $M$. This is a contradiction. □

**Lemma 5.3.** *If $G$ contains an edge $e$ which is outermost away from $t_2$ and connect vertices of the same sign, then we can deform $F$ so that $F \cap V_1$ is a peripheral disc which intersects $t_2$ in two points, cuts off a rational tangle $(B, T)$ and is $(t_1 - T)$-$\partial$-compressible in $cl(V_1 - B)$.*

Proof. The edge $e$ connects $R_1$ and $R_2$. We push a regular neighbourhood of $e$ on $P$ into $V_1$ along the outermost disc. Then two meridian discs $R_1$ and $R_2$ are connected by a band, and are deformed into a peripheral disc $D$ intersecting $t_1$ in 2 points as desired. □

Proof of Theorem IV′ in case (II). The graph $G$ contains an interior edge $e$ which is outermost away from $t_2$. By Lemmas 5.1 and 5.2, $e$ connects vertices of the same sign. Then by Lemma 5.3 we obtain a desired conclusion of Theorem IV′. □

## 6. Case (III)

In this section we assume that $F \cap V_1$ consists of meridian discs $Q_1, \cdots, Q_m$ which are disjoint from $t_1$. It is clear that $m$ is positive and even. We consider cases (i), (ii) and (iii) simultaneously in this section. The two discs $Q_1$ and $Q_m$ together divide the solid torus $V_1$ into two balls $B_1$, $B_2$, where $B_2$ contains $t_1$. Let $A_i$ be the annulus $H \cap \partial B_i$ for $i = 1$ and 2.

If we can take $C$ so that $\partial C \cap A_1$ is the empty set, then $H$ is a $K$-reducible 1-genus 1-bridge decomposition, which is a desired conclusion of Theorem IV′ in case (i), and which contradicts that $K$ is not trivial in cases (ii) and (iii). Hence we can assume that $\partial C \cap A_1 \neq \emptyset$. Since we took $C$ and $F$ so that $|\partial C \cap \partial P|$ is minimal, it is clear that

(1) $\partial C \cap A_1$ consists of essential arcs in the annulus $A_1$ such that each of them intersects every meridian loop $\partial Q_i$ just once for $i = 1, \cdots, m$, and

(2) $\partial C \cap A_2$ consists of essential arcs in the annulus $A_2$, inessential arcs each of which separates the two points $\partial t_1$, and two arcs connecting $\partial t_1$ and $\partial A_2$.

We form a graph $G$ on the disc $C$. The vertices of $G$ are "long vertices" $\partial C \cap A_1$. The edges of $G$ are the arcs $C \cap P$. The edges of $G$ are classified into two classes: interior edges and $\partial$-edges as in section 3. A long vertex contains $m$ endpoints of edges. We give an orientation to $C$ arbitrarily, and long vertices are classified into positive ones and negative ones as in section 5.

The next three lemmas are proved by the same arguments as in the proofs of Lemmas 4.1, 4,2 and 4.4. We omit them.

**Lemma 6.1.** *Let $e$ be an edge of $G$ which is outermost on $C$. Then $e$ is not a $\partial$-edge.*

**Lemma 6.2.** *The graph $G$ does not contain a loop edge.*

**Lemma 6.3.** *Let $e$ be an edge of $G$ which is outermost away from $t_2$. Then $e$ does not connect vertices of mutually opposite signs.*

Let $e$ be an interior edge connecting vertices $X$, $Y$ of $G$. The edge $e$ is a *diagonal* edge if the two vertices $X$ and $Y$ are not adjacent on the arc $\partial C - t_2$. A vertex $X$ of $G$ is *nice* if there are neither boundary edges nor diagonal edges incident to $X$.

When $G \cup t_2$ is not connected, $G$ contains a connected component which does not contain a $\partial$-edge. A connected component $G'$ of $G$ is *outermost away form* $t_2$ if there is a subarc $\gamma$ of $\partial C - t_2$ such that $G' \cap \partial C \subset \gamma$ and $(G - G') \cap \gamma = \emptyset$. We take an arbitrary connected component $G'$ of $G$ among outermost ones away from $t_2$. When $G \cup t_2$ is connected, we put $G' = G$. Let $n'$ be the number of vertices of $G'$. If $n' \leq 2$, then there cannot be a diagonal edge in $G'$.

**Lemma 6.4.** *Let $b$ be a $\partial$-edge which is outermost on $C$ among all the $\partial$-edges of $G'$. Then the outermost disc $O$ of $b$ contains a diagonal edge.*

Proof. Since $G'$ contains a $\partial$-edge, $G' = G$. Let $X$ be the vertex which $b$ is incident to. If $O$ does not contain an edge incident to $X$ except for $b$, then Lemma 6.1 implies that there is another connected component of $G$ in $O$, which is a contradiction. Hence $O$ contains $x$ interior edges incident to $X$ except for $b$ where $0 < x < m$. Suppose for a contradiction that $O$ contains no diagonal edges, then all the vertices on $O$ are nice except for $X$. Then $O$ contains a vertex $Y$ adjacent to $X$ on the arc $\partial O \cap \partial C$, and the above $x$ edges connect $X$ and $Y$. Since the vertex $Y$ is nice, there are $m - x$ interior edges of $G'$ connecting $Y$ and the vertex($\neq X$) adjacent to $Y$. Similar arguments show that $\partial O$ contains infinite number of vertices, which is a contradiction. □

**Lemma 6.5.** $n' \geq 2$.

Proof. Suppose that $n' = 1$. Then the graph $G'$ has only one vertex $X$. The edges of $G'$ are $\partial$-edges because $G'$ has no loop edges by Lemma 6.2. Then $G'$ contains a $\partial$-edge which is outermost on $C$, which contradicts Lemma 6.1. □

**Lemma 6.6.** *If $G'$ does not contain a diagonal edge, then $n' = 2$ and $G'$ consists of two vertices $X$ and $Y$ of the same sign and a parallel set of $m$ interior edges connect $X$ and $Y$.*

Proof. The graph $G'$ does not contain a $\partial$-edge by Lemma 6.4. Hence all the vertices of $G'$ are nice. Let $X$ be a vertex of $G'$ which is the nearest to $t_2$ on $\partial C$, and $Y$ the vertex of $G'$ which is adjacent to $X$ on $\partial C - t_2$. Then $X$ and $Y$ are connected by $m$ interior edges, and $n' = 2$. The vertices $X$ and $Y$ are of the same sign by Lemma 6.3. □

**Lemma 6.7.** *If $G'$ contains a diagonal edge, then there is one, say $e$, satisfying the conditions below. Let $O$ be the subdisc of $C$ surrounded by $e$ and a subarc $\delta$ of $\partial C - t_2$, and $X$ and $Y$ the vertices which $e$ is incident to.*

(1) *All the long vertices which meet $\delta$ are of the same sign, and*

(2) *all the long vertices contained in $\delta$ are nice (except for $X$ and $Y$).*

Proof. Among all the diagonal edges of $G'$ we take one, say $e$, which is outermost away from $t_2$. Let $O$ be the outermost disc of $e$. The disc $O$ does not contain a $\partial$-edge. Hence all the vertices of $G'$ contained in $\partial O \cap \partial C$ are nice (except for $X$ and $Y$). By Lemma 6.3, all the vertices incident to $\partial O$ are of the same sign. □

**Lemma 6.8.** *We can deform $F$ so that $F \cap V_1$ consists of a parallel set of peripheral discs each of which cuts off a ball containing $t_1$ from $V_1$.*

Proof. By lemmas 6.6 and 6.7, $G'$ contains a pair of vertices $X$ and $Y$ such that

(1) $X$ and $Y$ are adjacent on the arc $\partial C - t_2$, and

(2) $X$ and $Y$ are of the same sign, and

(3) $X$ and $Y$ are connected by a parallel set of $m/2$ edges one of which is outermost away from $t_2$ among all the edges of $G'$.

We push regular neighbourhoods of these edges on $F$ into $V_1$. Then the discs $Q_1$ and $Q_m$, $Q_2$ and $Q_{m-1}$, $\cdots$, $Q_{m/2}$ and $Q_{(m/2)+1}$ are connected by bands, and deformed into peripheral discs in $V_1$. □

This lemma completes the proof of Theorem IV$'$, which includes Theorem IV.

## 7. When $F \cap V_1$ consists of peripheral discs

Let $M$, $H$, $V_i$, $K$, $t_i$, $F$ and $F_0$ be as in section 3. The surface $F$ is of type (i), (ii) or (iii). Our goal of this section is the next proposition.

**Proposition 7.1.** *Either the condition (1) of Theorem IV$'$ holds, (2) we can deform $F$ so that $F \cap V_1$ consists of parallel peripheral annuli such that they are disjoint from $t_1$ and are not peripheral in $V_1 - t_1$, and boundary loops of these annuli are essential and non-meridional on $\partial \mathbf{V}_1$, or the condition (3) of Theorem IV$'$ holds.*

We begin with the condition (2) of Theorem IV$'$. Then $F \cap V_1$ consists of parallel peripheral discs each of which cuts off a ball containing $t_1$ from $V_1$. Moreover we take $F$ so that the number of peripheral discs of $F \cap V_1$ is minimal over all deformations of $F$. Let $P = F \cap V_2$, $E(V_2, t_2) = \mathrm{cl}(V_2 - N(t_2))$ and $P_0 = P \cap E(V_2, t_2)$. Then $P$ is $t_2$-incompressible and meridionally incompressible, but is possibly $t_2$-$\partial$-compressible in $V_2$. Let $C$ be a cancelling disc of $t_2$ in $V_2$. We take $C$ and $F$ so that $|\partial C \cap \partial P|$ is minimal over all cancelling discs of $t_2$ and all deformations of $F$. Moreover, we can isotope $C$ in $V_2$ fixing $\partial C$ so that $C \cap P$ consists of arcs only.

Let $Q_1, \cdots, Q_m$ be the peripheral discs of $F \cap V_1$ which appear in $V_1$ in this order so that $Q_1$ is the nearest to $t_1$. Clearly $m > 0$. Let $B$ be the ball which is cut off from $V_1$ by $Q_1$, and $E$ the disc $B \cap H$. Let $R$ be the annulus on $H$ surrounded by $\partial Q_1$ and $\partial Q_m$.

If we can take $C$ so that $\partial C \cap R$ is the empty set, then $H$ is a $K$-reducible 1-genus 1-bridge decomposition, which is a desired conclusion of Proposition 7.1 in case (i), and is a contradiction in cases (ii) and (iii). Hence we can assume that $\partial C \cap R \neq \emptyset$. Since we take $C$ and $F$ so that $|\partial C \cap \partial P|$ is minimal, it is clear that

(1) $\partial C \cap R$ consists of essential arcs in the annulus $R$, and each of them intersects $\partial Q_i$ just once for $i = 1, \cdots, m$, and

(2) $\partial C \cap E$ consists of arcs separating the two points $\partial t_1$ and two arcs connecting $\partial t_1$ and $\partial Q_1$, and

(3) $\partial C \cap (H - \mathrm{int}(E \cup R))$ consists of essential arcs.

We form a graph $G$ on the disc $C$. The vertices of $G$ are "long vertices" $\partial C \cap R$. The edges of $G$ are the arcs $C \cap P$. The edges of $G$ are classified into two classes: interior edges and $\partial$-edges as in section 3. A long vertex contains $m$ endpoints of edges.

**Lemma 7.2.** *Let $e$ be an edge of $G$ which is outermost on $C$. Then $e$ is not a $\partial$-edge.*

Proof. Suppose that $e$ is a $\partial$-edge. Then $e$ has an endpoint in $\partial Q_1$. Let $O$ ($\subset C$) be the outermost disc of $e$. We isotope a regular neighbourhood of $e$ on $P$ along the outermost disc $O$. Then $Q_1$ is deformed to a peripheral disc $D$ intersecting $t_1$ precisely once. We can push $D$ into $V_2$ by an isotopy of the pair $(M, K)$, which contradicts the minimality of $|F \cap V_1|$. □

A similar proof as that of Lemma 4.2 shows the next lemma, and we omit it.

**Lemma 7.3.** *The graph $G$ does not contain a loop edge.*

**Lemma 7.4.** *Let $e$ be an edge of $G$ which is outermost away from $t_2$. Then $e$ does not connect vertices which are adjacent on the arc $\partial C - t_2$ and connected by an arc of $\partial C \cap E$.*

Proof. Suppose that $e$ is such an outermost edge. Then $e$ has both endpoints in $\partial Q_1$. We push a regular neighbourhood of $e$ on $P$ into $V_1$ along the outermost disc. Then a band is attached to $Q_1$, and $Q_1$ is deformed into an annulus $A$ which cuts off a regular neighbourhood of $t_1$ from $V_1$. This regular neighbourhood contains a disc

$D$ such that $D \cap A = \partial D$ and $|D \cap t_1| = 1$. This is a contradiction in case (i). In case (ii), $\partial D$ divides $F$ into two discs $D_1$ and $D_2$, and either $D \cup D_1$ or $D \cup D_2$ is a decomposing sphere. In case (iii), since $F$ is meridionally incompressible, there is a disc $D'$ on $F$ such that $\partial D' = \partial D$ and $|D' \cap t_1| = 1$. We isotope $F$ so that $D'$ is deformed onto $D$. Thus, in cases (ii) and (iii), $A$ is deformed to a peripheral disc intersecting $t_1$ once. We can push this peripheral disc into $V_2$ by an isotopy of the pair $(M, K)$, This contradicts the minimality of the number $|F \cap V_1|$. □

**Lemma 7.5.** *The graph $G$ contains two vertices $X$ and $Y$ which are adjacent on the arc $\partial C - t_2$, connected by an arc of $\partial C \cap (H - int(E \cup R))$ and a parallel set of $m$ interior edges.*

Proof. We apply the same arguments as in section 6, and obtain a pattern as in Lemma 6.6 or 6.7. Note that the condition "the vertices are of the same sign" is substituted by the condition "the adjacent vertices are connected by arcs of $\partial C \cap (H - \mathrm{int}(E \cup R))$". On $\partial C - t_2$, there appear arcs of $\partial C \cap E$ and arcs of $\partial C \cap (H - \mathrm{int}(E \cup R))$ alternately between long vertices. Hence patterns as in Lemma 6.7 is impossible, and we obtain a pattern as in Lemma 6.6. □

Proof of Proposition 7.1. We push regular neighbourhoods of the parallel set of $m$ edges in Lemma 7.5 on $F$ into $V_1$. Then bands are attached to the discs $Q_1, \cdots Q_m$, and they are deformed into parallel peripheral annuli in $V_1$. If their boundary loops are meridional ,then we can deform the innermost annulus to a meridian disc by a deformation of F, which contradicts that F is separating. □

## 8. Proof of Theorem I

We prove Theorem I in this section. The surface $F$ is a splitting sphere. We apply Theorem IV$'$. When the conclusion (1) holds, $H$ is $K$-reducible, and we obtain the desired conclusion. The conclusion (3) is impossible because $F \cap K = \emptyset$. Hence we assume that the conclusion (2) of Theorem IV$'$ holds to obtain a contradiction. We denote the graph $G$ in section 7 by $G_C$ in this section for clearness. We form a graph $G_F$ on the sphere $F$. The vertices of $G_F$ are "fat" vertices $Q_1, \cdots, Q_m$. The edges of $G_F$ are arcs of $C \cap P$. We thus obtain two graphs $G_C$ and $G_F$, whose edges are in one to one correspondence.

An edge $e$ of $G_F$ is a *loop edge* if its endpoints are both in the same fat vertex. If a loop edge $e$ of $G_F$ cuts off a disc from the punctured sphere $P$, then we call $e$ a *trivial loop edge.*

**Lemma 8.1.** *The graph $G_F$ does not contain a trivial loop edge.*

Proof. Suppose for a contradiction that $G_F$ contains a trivial loop edge $e$. Let $O$ be the disc which $e$ cuts off from $P$. We can take $e$ to be outermost on $P$, that is, so that int$O$ does not contain an edge. The edge $e$ divides the cancelling disc $C$ into two discs $C_1$ and $C_2$, one of which, say $C_1$ contains $t_2$. Then the disc $C' = C_1 \cup O$ is a cancelling disc of $t_2$, and after a small isotopy of $C'$ we have $|\partial C' \cap \partial P| < |\partial C \cap \partial P|$, which is a contradiction. □

Proof of Theorem I. Each of the parallel $m$ edges of $G_C$ in the conclusion of Lemma 7.5 corresponds to loop edges of $G_F$. Since $F$ is a sphere, there is a trivial loop edge among them, which contradicts Lemma 8.1. □

## 9. Proof of Theorem II

We prove Theorem II in this section. Let $F$ be a decomposing sphere of $K$. We apply Proposition 7.1. When the conclusion (1) holds, $H$ is weakly $K$-reducible, which is the desired conclusion of Theorem II. The conclusion (3) implies that $H$ is weakly $K$-reducible by Lemma 2.2. Hence we can assume that the conclusion (2) holds. Then $F \cap V_1$ consists of parallel peripheral annuli, say $R_1, \cdots, R_m$, whose boundary components are essential on $H$. Among these loops, there are at least two loops which are innermost on the sphere $F$. At least one of the innermost discs, say $D$, intersects $K$ at most one point. The disc $D$ is a meridian disc of $V_2$ since $\partial D$ is essential in $H$. We can take a cancelling disc $C$ of $t_1$ in $V_1$ so that $C \cap (R_1 \cup \cdots \cup R_m) = \emptyset$. Then the discs $D$ and $C$ imply that $H$ is $K$-reducible or weakly $K$-reducible. In the former case, we obtain a contradiction. In the latter case, we obtain the desired conclusion.

In addition, $M$ is not the 3-dimensional sphere by Lemma 2.1. This completes the proof of Theorem II.

## 10. Proof of Theorem III

Let $M$, $H$, $V_i$, $K$, $t_i$, $F$ and $F_0$ be as in section 3. Suppose that the surface $F$ is of type (iii) such that $F$ is disjoint from $K$ in this section. First, we will prove Proposition 10.1 below. We apply Proposition 7.1. The conclusion (3) of Proposition 7.1 is now absurd since $F$ is disjoint from $K$. We begin with the situation of the conclusion (2) of Proposition 7.1. Then $F \cap V_1$ consists of parallel peripheral annuli $R_1, \cdots, R_m$ each of which cuts off a solid torus containing the trivial arc $t_1$ from $V_1$. Moreover, the boundary loops $\partial R_i$ are essential on $H$. We take $F$ so that the number of peripheral annuli of $F \cap V_1$ is minimal up to isotopy of $F$ in $(M, K)$. We assume that $R_1, \cdots, R_m$ appear in this order in $V_1$ and $R_1$ is the nearest to $t_1$. We can take a cancelling disc $C_1$of $t_1$so that $C_1 \cap R_1 = \phi$. Let $P = F \cap V_2$, $E(V_2, t_2) =$cl$(V_2 - N(t_2))$ and $P_0 = P \cap E(V_2, t_2)$. Then $P$ is $t_2$-incompressible and meridionally incompressible, but is possibly $t_2$-$\partial$-compressible. Let $C$ be a cancelling disc of $t_2$ in $V_2$. We take $C$ and $F$ so that $|\partial C \cap \partial P|$ is minimal over all cancelling discs of $t_2$ and up to isotopy of $F$ in $(M, K)$. Moreover, we can isotope $C$ in $V_2$ so that $C \cap P$ consists of arcs only.

Clearly $m > 0$. Let $V$ be the solid torus which is cut off from $V_1$ by $R_1$, and $A_2$ the annulus $V \cap H$. Let $A_1$ be the annulus $H - \mathrm{int} A_2$.

**Proposition 10.1.** *Let $M$, $H$, $V_i$, $K$, $t_i$, $F$ and $F_0$ be as in section 3. The surface $F$ is of type (iii) and is disjoint from $K$. Then one of the following three conditions holds.*

(1) *The conclusion (1) of Theorem IV′ holds.*

(2) (a) *There is a cancelling disc $C$ of $t_2$ such that $\partial C \cap A_1 = \emptyset$, or*
(b) *the graph $G$ which will be formed in this section contains a pattern as in Lemma 10.4.*

We assume that the conclusions (1) and (2)(a) do not hold to prove that the conclusion (2)(b) holds. Since we take $C$ and $F$ so that $|\partial C \cap \partial P|$ is minimal, it is clear that

(1) $\partial C \cap A_1$ consists of essential arcs in the annulus $A_1$, and each of them intersects each component of $\partial R_i$ just once for $i = 1, \cdots, m$, and

(2) $\partial C \cap A_2$ consists of essential arcs in the annulus $A_2$, inessential arcs each of which separates the two points $\partial t_1$, and two arcs connecting $\partial t_1$ and $\partial R_1$.

We form a graph $G$ on the disc $C$. The vertices of $G$ are "long vertices" $\partial C \cap A_1$. The edges of $G$ are the arcs $C \cap P$. The edges of $G$ are all interior edges since $F$ is disjoint from $K$. A long vertex contains $2m$ endpoints of edges.

**Lemma 10.2.** *The graph $G$ does not contain a loop edge.*

Proof. Suppose that $G$ contains a loop edge $e$, and we can take $e$ to be an outermost one on the disc $C$. Then either (1) endpoints of $e$ are in adjacent peripheral annuli, say $R_x$ and $R_{x+1}$, or (2) $e$ connects two boundary loops of $R_m$. We push a regular neighbourhood of $e$ on $P$ into $V_1$ along the outermost disc of $e$. Then two loops are deformed into a loop $\ell$ bounding a disc $D$ disjoint from $\partial t_1$ on $H$. In case (1), $R_x$ and $R_{x+1}$ are deformed to an annulus $R$ with a hole. In case (2), $R_m$ is deformed into a torus $T$ with a hole.

The $K$-incompressible surface $F$ contains a disc $D'$ such that $\partial D = \partial D'$ and $D' \cap K = \emptyset$. We can move $F$ by an isotopy of the pair $(M, K)$ so that $D'$ is deformed onto $D$. In case (1), $R_x$ and $R_{x+1}$ are deformed to a peripheral annulus or disc $R'$, and we can push $R'$ into $V_2$ by an isotopy of the pair $(M, K)$. This contradicts to the minimality of the number of the peripheral annuli of $F \cap V_1$. In case (2), $T$ is deformed to a torus $T' = T \cup D$, and $F = T'$. We can push this torus to be entirely contained in int $V_1$, and hence $F$ is $K$-compressible, which is a contradiction. □

**Lemma 10.3.** *Let $e$ be an edge of $G$ which is outermost away from $t_2$. Then $e$ does not connect vertices which are adjacent on the arc $\partial C - t_2$ and connected by an inessential arc of $\partial C \cap A_2$.*

Proof. Suppose that $e$ is such an outermost edge. Then $e$ has both endpoints in $\partial R_1$. We push a regular neighbourhood of $e$ on $P$ into $V_1$ along the outermost disc. Then a band is attached to $R_1$, and $R_1$ is deformed into a pair of pants, one of whose boundary components bounds a disc $D$ intersecting $K$ at a single point on $H$. Since $F$ is meridionally incompressible, $\partial D$ bounds a disc intersecting $K$ at a single point on $F$. This contradicts that $F$ is disjoint from $K$. □

**Lemma 10.4.** *The graph $G$ contains two vertices $X$ and $Y$ such that*

(1) *$X$ and $Y$ are adjacent on the arc $\partial C - t_2$, and*

(2) *$X$ and $Y$ are connected by a parallel set of $m$ edges one of which is outermost away from $t_2$ among all the edges of $G$, and*

(3) *$X$ and $Y$ are connected by an essential arc of $\partial C \cap A_2$.*

Proof. We apply the same arguments as in section 6, and obtain a pattern as in Lemma 6.6 or 6.7. Hence $G$ contains a pair of vertices $X$ and $Y$ as in the conclusion of this lemma. □

Proof of Theorem III. Let $T$ be an incompressible torus which is not parallel to $\partial E(K)$. If $T$ is meridionally compressible, then $K$ is composite, and $H$ is weakly $K$-reducible by Theorem II, which is a contradiction. Hence $T$ is meridionally incompressible.

We apply Proposition 10.1 setting $F = T$. The conclusion (1) is that $H$ is weakly $K$-reducible, which is a contradiction. When the conclusion (2)(a) holds, we can isotope int P so that $P \cap C = \phi$. Hence $\mathrm{cl}(M - N(C_1 \cup A_2 \cup C))$ is a Seifert fibred manifold over a disc with two exceptional fibres, and T is isotopic to its boundary. Hence we can assume that (2)(b) holds, and the graph $G$ contains edges as in Lemma 10.4. We push regular neighbourhoods of these edges on $T$ into $V_1$. Then bands are attached to the annuli $R_1, \cdots R_m$, and they are deformed into tori with a hole in $V_1$. Since $T$ is of genus one, $m = 1$ and $T \cap V_2$ is a peripheral disc which cuts off a ball containing $t_2$.

Then the desired conclusion easily follows. □

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