



Title	$1/1^{2n} + (1+1/2)/2^{2n} + (1+1/2+1/3)/3^{2n} + \dots$ ナ ル無限級数ヲ $1/1^n + 1/2^n + 1/3^n + \dots$ ノイ ツカニヨリ表ハス関係式
Author(s)	祐乗坊, 瑞満
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$$10. \quad \frac{1}{1^{2n}} + \frac{1+\frac{1}{2}}{2^{2n}} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3^{2n}} + \dots + 1$$

無限級数 $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \dots$ / イツカニヨリ表ハス關係式

名大理学部数学科学生

祐乘坊瑞満

(昭和20年3月10日受付)

$\int \log(1-t) t^{s-1} dt \quad (s>0)$ ヲ次ノ如キ積分路ニテ積分シタル。
 先ヅ單位円ヲ1ヨリ出発ニテ時計ト逆ノ向キニ一周シ次ニ1ヨリ原
 点ノ傍迄、ソレニテ微小円ニテ原点ヲ負ノ向キニ一周ニ再ビ1ニ戻
 ル積分路ナル。(但シ1ニ微小円ニヨリ回避セル) ノトキ
 微小円ヨリノ影響ハ零ナル。又0カラ1マデノ積分ハ往復ノ

、差が

$$(1 - e^{2\pi s^2}) \int_0^1 \log(1-t) t^{s-1} dt$$

又単位円 C 上での積分は

$$\begin{aligned} & \int_0^{2\pi} \log(1 - \cos \theta - i \sin \theta) e^{i(s-1)\theta} e^{i\theta} i d\theta \\ &= i \int_0^{2\pi} \log 2 \sin \frac{\theta}{2} + i \frac{\theta - \pi}{2} e^{is\theta} d\theta \end{aligned}$$

両者、和は Cauchy の定理により零に等しいから

$$\begin{aligned} & i \int_0^{2\pi} \left(\log 2 \sin \frac{\theta}{2} + i \frac{\theta - \pi}{2} \right) \frac{e^{is\theta}}{e^{2\pi si} - 1} d\theta \\ &= \int_0^1 \log(1-t) t^{s-1} dt \\ &= - \sum_{n=0}^{\infty} (-1)^n C_{n+2} s^n \dots \dots \dots (1) \end{aligned}$$

$$\text{故に} \quad C_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \dots$$

$$\text{然るに} \quad \frac{e^{i\theta s}}{e^{2\pi si} - 1} = \sum_{n=0}^{\infty} \frac{B_n(\frac{\theta}{2\pi})}{n!} (2\pi si)^n$$

故に (1) の両辺を s の冪級数に展開し、 s^{n-1} の係数を比較すれば

$$(-1)^n C_{n+1} = i \int_0^{2\pi} \left(\log 2 \sin \frac{\theta}{2} + i \frac{\theta - \pi}{2} \right) \frac{B_n(\frac{\theta}{2\pi})}{n!} (2\pi i)^{n-1} d\theta$$

左辺は実数であり、 n が偶数であれば

$$C_{2n+1} = \frac{(2\pi)^{2n-1}}{(2n)!} (-1)^n \int_0^{2\pi} B_{2n}(\frac{\theta}{2\pi}) \log 2 \sin \frac{\theta}{2} d\theta$$

又 n が奇数のときは

$$\begin{aligned} C_{2n} &= \int_0^{2\pi} \frac{\theta - \pi}{2} \frac{B_{2n-1}(\frac{\theta}{2\pi})}{(2n-1)!} (2\pi)^{2n-2} (-1)^{n-1} d\theta \\ &= (-1)^{n-1} \frac{(2\pi)^{2n-2}}{2} \frac{1}{(2n-1)!} \int_0^{2\pi} (\theta - \pi) B_{2n-1}(\frac{\theta}{2\pi}) d\theta \end{aligned}$$

次に $\int_0^1 \{\log(1-t)\}^2 t^{s-1} dt$ を求む。これは同様に扱われる。

$$\begin{aligned} & t \int_0^{2\pi} \left\{ \left(\log 2 \sin \frac{\theta}{2} \right)^2 - \frac{(\theta - \pi)^2}{4} + i(\theta - \pi) \log 2 \sin \frac{\theta}{2} \right\} \frac{e^{i\theta s}}{e^{2\pi si} - 1} d\theta \\ &= \int_0^1 \{\log(1-t)\}^2 t^{s-1} dt = 2 \sum_{n=0}^{\infty} (-1)^n C'_{n+2} s^n \\ &= \sum_{n=0}^{\infty} i \int_0^{2\pi} \left\{ \left(\log 2 \sin \frac{\theta}{2} \right)^2 - \frac{(\theta - \pi)^2}{4} + i(\theta - \pi) \log 2 \sin \frac{\theta}{2} \right\} \frac{B_n(\frac{\theta}{2\pi})}{n!} (2\pi si)^{n-1} d\theta \end{aligned}$$

$$42 = 2 \sum_{n=0}^{\infty} (-1)^n C_{n+2} 5^n,$$

$$\text{茲} = C_n' = \frac{1}{2^n} + \frac{1+\frac{1}{2}}{3^n} + \frac{1+\frac{1}{2}+\frac{1}{3}}{4^n} + \dots$$

両辺、 5^{2n} 、係数ヲ等シト置テ

$$2 C_{2n+2} = \int_0^{2\pi} \frac{B_{2n+1}(\frac{\theta}{2\pi})}{(2n+1)!} (2\pi)^{2n} (-1)^{n-1} (\theta - \pi) \log 2 \sin \frac{\theta}{2} d\theta$$

$$C_{2n+2} = (-1)^{n-1} \frac{(2\pi)^{2n}}{2} \frac{1}{(2n+1)!} \int_0^{2\pi} B_{2n+1}(\frac{\theta}{2\pi}) (\theta - \pi) \log 2 \sin \frac{\theta}{2} d\theta$$

n 、代リ $= n-1$ ト書キ 変換スルハ

$$C_{2n}' = (-1)^n \frac{(2\pi)^{2n-1}}{2} \frac{1}{(2n-1)!} \int_0^{2\pi} B_1(\frac{\theta}{2\pi}) B_{2n-1}(\frac{\theta}{2\pi}) d\theta$$

$$\text{然ル} = B_1(x) B_n(x)$$

$$= B_{n+1}(x) + \frac{1}{n+1} \binom{n+1}{2} B_1 B_{n-1}(x) - \frac{1}{n+1} \binom{n+1}{4} B_2 B_{n-3}(x) \\ + \dots + (-1)^{k+1} B_k \frac{\binom{n+1}{2k}}{n+1} B_{n-2k+1}(x) + \dots$$

コノ展開ハアラユル多項式カソリ、次数以下、次数、Bernoulli 多項式、一次結合トシテ一意的ニアラハサレルヲ及ビ $B_n(1-x) = (-1)^n B_n(x)$ カラ $B_n(x)$, $B_{n-2}(x)$, $\dots$ ノ項ハ上式ニハ不要ト更ニ $= B_n(1+x) = B_n(x) + nx^{n-1}$ カラ未定係数ヲ決定スルヲ得ラレル

$$\text{故} = C_{2n}' = (-1)^n \frac{(2\pi)^{2n-1}}{2} \frac{1}{(2n-1)!} \left(\int_0^{2\pi} B_{2n}(\frac{\theta}{2\pi}) \log 2 \sin \frac{\theta}{2} d\theta \right. \\ \left. + \sum \frac{(-1)^{k+1}}{2n} B_k \binom{2n}{2k} \int_0^{2\pi} B_{2n-2k}(\frac{\theta}{2\pi}) \log 2 \sin \frac{\theta}{2} d\theta \right)$$

$$\text{然ル} = \int_0^{2\pi} B_{2n-2k}(\frac{\theta}{2\pi}) \log 2 \sin \frac{\theta}{2} d\theta = \frac{(-1)^n k! (2n-2k)!}{(2\pi)^{2n-2k-1}} C_{2n-2k+1}$$

コレ等ヲ代入シテ

$$C_{2n}' = (-1)^n \frac{(2\pi)^{2n-1}}{2} \frac{1}{(2n-1)!} \left\{ \frac{(2n)! (-1)^n}{(2\pi)^{2n}} C_{2n+1} \right. \\ \left. + \sum_{k=1}^{n-1} \frac{(-1)^{k+1} B_k \binom{2n}{2k}}{2n} \frac{(-1)^{n-1} (2n-2k)!}{(2\pi)^{2n-2k-1}} C_{2n-2k+1} \right\} \\ = n C_{2n+1} - \sum_{k=1}^{n-1} \frac{(2\pi)^{2k}}{2} \frac{B_k}{(2k)!} C_{2n-2k+1}$$

$$C_{2n}' \text{ の代り} = \frac{1}{1^{2n}} + \frac{1+\frac{1}{2}}{2^{2n}} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3^{2n}} + \dots (= C_{2n}' + C_{2n+1}') \\ = (n+1) C_{2n+1}' - \sum_{k=1}^{n-1} \frac{(2\pi)^{2k}}{2} \frac{B_k}{2k!} C_{2n-2k+1}'$$

$$\text{然} = C_{2k}' = \frac{(2\pi)^{2k}}{2} \frac{B_k}{2k!}$$

$$\therefore \frac{1}{1^{2n}} + \frac{1+\frac{1}{2}}{2^{2n}} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3^{2n}} + \dots$$

$$= (n+1) C_{2n+1}' - \sum_{k=1}^{n-1} C_{2k}' C_{2n-2k+1}'$$

$$= (n+1) C_{2n+1}' - \sum_{k=1}^{n-1} C_{2k+1}' C_{2n-2k}$$

$$\text{又 } C_{2n} = \frac{(2\pi)^{2n}}{2} \frac{B_n}{2n!} \text{ 此公式ハ前出}$$

$$C_{2n} = (-1)^{n-1} \frac{(2\pi)^{2n-2}}{2} \frac{1}{(2n-1)!} \int_0^{2\pi} (\theta - \pi) B_{2n-1}\left(\frac{\theta}{2\pi}\right) d\theta$$

ト同様の積分法を行フタ等式カラ出ル $\int_0^{2\pi} B_{2n}\left(\frac{\theta}{2\pi}\right) d\theta$ トカラ計算スベ
出ル。先 = 得タ公式、特別、場合トシテ

$$\frac{1}{1^2} + \frac{1+\frac{1}{2}}{2^2} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3^2} + \dots = 2 \left(\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots \right)$$

$\log(1-x)$ の代り = $\log(1+x) \tan^{-1} x$ を入レテ積分ヲ用ヒル事ヨリ

先 = 得タ公式ト同様、関係ガ類似、級数 = 対シ成立スル事
ガワカル。