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11. Rademacher ノ定理ノ解析的証明

東大　伊関兼四郎

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1. 自然数 n ノ無制限ナ分割ノ数 $p(n)$ ノ生成函数ノ一次変換ニ関聯シテ 1 ノ 24k 乗根(トハ或ル自然数)ヲ決定スル問題ガ生ズル。此ノ問題ハ本集楕円 Modul 函数ノ変換ノ理論ニ属スルモノデ、Hermite 以来何人カノ数学者ニ依ッテ研

完サレタ、其ノ後 Hans Rademacher ハ同題ノ24k乗根ヲ表スーツノ新公式ヲ得テ之ヲ算術的ニ証明シタ 以下筆者ハ Rademacher ノ公式ノ解析的証明ヲ述ベテ見タイ、其ノタメニ先ヅ Hermite ノ公式, Rademacher ノ公式ノ何レトモ異ナルーツノ公式ヲ導キ, 然ル後此ノ公式ガ後者ト同等デアルヲヲ証明スル

2. $p(n)$ ノ生成函数ヲ $f(x)$ トスレバ明ラカニ

$$f(x)=\prod_{n=1}^{\infty}\frac{1}{1-x^n}=1+\sum_{n=1}^{\infty}p(n)x^n \qquad (|x|<1)$$

今 $x=\exp\left(\frac{2\pi i h}{k}-\frac{2\pi z}{k}\right)$ ト置ケバ $f(x)$ ノ変換公式ハ

$$f(x)=\omega_{h,k}\sqrt{z}\,\exp\left(\frac{\pi}{12kz}-\frac{\pi z}{12k}\right)f\left\{\exp\left(\frac{2\pi i l}{k}-\frac{2\pi}{kz}\right)\right\}$$

トナル. 茲ニ z ハ $\Re(z)>0$, $\Re(\sqrt{z})>0$ ナル任意ノ複素数, k ハ任意ノ自然数, h ハ $(h,k)=1$; $1\leqq h\leqq k$ ナル任意ノ自然数, l ハ合同式 $hl\equiv -1 \pmod k$ ノ任意ノ有理整数解, $\omega_{h,k}$ ハ1ノ24k乗根デ其ノ値ハ h ガ奇数ナル時ハ

$$\omega_{h,k}=\left(\frac{-k}{h}\right)\exp\left(-\left\{\frac{1}{4}(2-hk-h)+\frac{1}{12}\left(k-\frac{1}{k}\right)(2h-l+h^2l)\right\}\pi i\right),$$

k ガ奇数ナルトキハ

$$\omega_{h,k}=\left(\frac{-h}{k}\right)\exp\left(-\left\{\frac{1}{4}(k-1)+\frac{1}{12}\left(k-\frac{1}{k}\right)(2h-l+h^2l)\right\}\pi i\right)$$

デ表サレル. 茲ニ $\left(\frac{a}{b}\right)$ ハ Legendre-Jacobi ノ記号デアル。勿論 h,k ガ共ニ奇数ナルトキハ上ノ二公式ハ等シイ値ヲ与ヘル。之等二公式ハ Hermite ガ得タモノデアル.

Rademacher ノ公式ハ Hermite ノ公式トハ全ク異ナル形ヲテアッテ

$$\omega_{h,k}=\exp\left\{\pi i\sum_{\mu=1}^{k-1}\frac{\mu}{k}\left(\frac{h\mu}{k}-\left[\frac{h\mu}{k}\right]-\frac{1}{2}\right)\right\}\quad(k\geqq 2)$$

ト書カレル。茲ニ $[u]$ ハ実数 u ヲ超エナイ最大ノ有理整数ヲ表ハス. (Gauss ノ記号).

筆数ハ先ヅ

$$\omega_{h,k} = \exp\left(\frac{\pi i}{4k}\sum_{\mu=1}^{k-1}\cot\frac{\pi\mu}{k}\cot\frac{\pi h\mu}{k}\right) \quad (k\geq 2)$$

ヲ導カル，以下 k ハ正数トシテ

$$\rho = \exp\left(\frac{2\pi i h}{k}\right) \qquad \gamma = \exp\left(-\frac{2\pi z}{k}\right)$$

トオケバ，$x=\rho\gamma$，　$0<\gamma<1$ トナル．前述ノ変換公式ノ両辺ノ対数ヲ取レバ（α ハ有理整数トシテ）

$$\sum_{n=1}^{\infty}\log\frac{1}{1-x^n} = \log\omega_{h,k} + \frac{1}{2}\log z + \frac{\pi}{12kz} - \frac{\pi z}{12k} + \log f\left\{\exp\left(\frac{2\pi i l}{k} - \frac{2\pi}{kz}\right)\right\} + 2\pi i\alpha.$$

但シ対数ハ偏角ガ $-\pi$ ヨリ偏角ガ π ニ至ル主値ヲ取ル．此ノ等式ノ両辺ノ虚部ヲ取レバ．

$$J\left(\sum_{n=1}^{\infty}\log\frac{1}{1-x^n}\right) = J\log\omega_{h,k} + J\log f\left\{\exp\left(\frac{2\pi i l}{k} - \frac{2\pi}{kz}\right)\right\} + 2\pi\alpha,$$

$$\lim_{z\to 0} J\left(\sum_{n=1}^{\infty}\log\frac{1}{1-x^n}\right) = J\log\omega_{h,k} + 2\pi\alpha$$

$$\omega_{h,k} = \exp\left\{\lim_{\gamma\to 1-0} J\left(\sum\log\frac{1}{1-x^n}\right)\right\}$$

故ニ次ノ式ヲ証明スレバヨイ．

$$\lim_{\gamma\to 1-0} J\left(\sum_{n=1}^{\infty}\log\frac{1}{1-x^n}\right) = \frac{\pi}{4k}\sum_{\mu=1}^{k-1}\cot\frac{\pi\mu}{k}\cot\frac{\pi h\mu}{k}.$$

サテ

$$\sum_{n=1}^{\infty}\log\frac{1}{1-x^n} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^{mn}}{m} = \sum_{n=1}^{\infty}\frac{1}{n}\frac{x^n}{1-x^n},$$

$$J\left(\sum_{n=1}^{\infty}\log\frac{1}{1-x^n}\right) = \sum_{n=1}^{\infty}\frac{1}{n}J\left(\frac{x^n}{1-x^n}\right) = \sum_{n=1}^{\infty}\frac{1}{n}J\left(\frac{1}{1-x^n}\right)$$

$$= \sum_{\substack{n=1\\ n\not\equiv 0\,(\mathrm{mod}\,k)}}^{\infty}\frac{1}{n}J\left(\frac{1}{1-x^n}\right)$$

然 n ニ最後ノ級数ハ後ニ証明スル如ク $0<\gamma<1$ ニ於テ一様ニ収斂スルカラ

$$\lim_{\gamma\to 1-0}\left(\sum_{\substack{n=1\\ n\not\equiv 0\,(\mathrm{mod}\,k)}}^{\infty}\frac{1}{n}J\left(\frac{1}{1-x^n}\right)\right) = \sum_{\substack{n=1\\ n\not\equiv 0\,(\mathrm{mod}\,k)}}^{\infty}\frac{1}{n}J\lim_{\gamma\to 1-0}\frac{1}{1-x^n}$$

筆算ハ先ヅ

$$\omega_{h,k} = \exp\left(\frac{\pi i}{4k}\sum_{\mu=1}^{k-1}\cot\frac{\pi\mu}{k}\cot\frac{\pi h\mu}{k}\right) \quad (k\geq 2)$$

ヲ導カル，以下各ハ正数トシテ

$$\rho = \exp\left(\frac{2\pi i h}{k}\right) \qquad \gamma = \exp\left(-\frac{2\pi z}{k}\right)$$

トオケバ，$x=\rho\gamma$，$0<\gamma<1$トナル．前述ノ変換公式ノ両辺ノ対数ヲ取レバ（αハ有理整数ニシテ）

$$\sum_{n=1}^{\infty}\log\frac{1}{1-x^n} = \log\omega_{h,k} + \frac{1}{2}\log z + \frac{\pi}{12kz} - \frac{\pi z}{12k} + \log f\left\{\exp\left(\frac{2\pi i l}{k} - \frac{2\pi}{kz}\right)\right\} + 2\pi i\alpha.$$

但シ対数ハ偏角が$-\pi$ヨリ偏角$\leq\pi$ナル主値ヲ取ル．此ノ等式ノ両辺ノ虚部ヲ取レバ．

$$J\left(\sum_{n=1}^{\infty}\log\frac{1}{1-x^n}\right) = J\log\omega_{h,k} + J\log f\left\{\exp\left(\frac{2\pi i l}{k} - \frac{2\pi}{kz}\right)\right\} + 2\pi\alpha,$$

$$\lim_{z\to 0} J\left(\sum_{n=1}^{\infty}\log\frac{1}{1-x^n}\right) = J\log\omega_{h,k} + 2\pi\alpha$$

$$\omega_{h,k} = \exp\left\{\lim_{\gamma\to 1-0} J\left(\Sigma\log\frac{1}{1-x^n}\right)\right\}$$

故ニ次ノ式ヲ証明スレバヨイ．

$$\lim_{\gamma\to 1-0} J\left(\sum_{n=1}^{\infty}\log\frac{1}{1-x^n}\right) = \frac{\pi}{4k}\sum_{\mu=1}^{k-1}\cot\frac{\pi\mu}{k}\cot\frac{\pi h\mu}{k}.$$

サテ

$$\sum_{n=1}^{\infty}\log\frac{1}{1-x^n} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^{mn}}{m} = \sum_{n=1}^{\infty}\frac{1}{n}\frac{x^n}{1-x^n},$$

$$J\left(\sum_{n=1}^{\infty}\log\frac{1}{1-x^n}\right) = \sum_{n=1}^{\infty}\frac{1}{n}J\left(\frac{x^n}{1-x^n}\right) = \sum_{n=1}^{\infty}\frac{1}{n}J\left(\frac{1}{1-x^n}\right)$$

$$= \sum_{\substack{n=1\\ n\not\equiv 0\,(\mathrm{mod}\,k)}}^{\infty}\frac{1}{n}J\left(\frac{1}{1-x^n}\right)$$

然ルニ最後ノ級数ハ後ニ証明スル如ク$0<\gamma<1$ニ於テ一様収斂スルカラ

$$\lim_{\gamma\to 1-0}\left(\sum_{\substack{n=1\\ n\not\equiv 0\,(\mathrm{mod}\,k)}}^{\infty}\frac{1}{n}J\left(\frac{1}{1-x^n}\right)\right) = \sum_{\substack{n=1\\ n\not\equiv 0\,(\mathrm{mod}\,k)}}^{\infty}\frac{1}{n}J\lim_{\gamma\to 1-0}\frac{1}{1-x^n}$$

$$= \sum_{\substack{n=1 \\ m\not\equiv 0 \pmod k}}^{\infty} \frac{1}{n} J\left(\frac{1}{1-\rho^n}\right) = \sum_{\substack{n=1 \\ m\not\equiv 0 \pmod k}}^{\infty} \frac{1}{2n} \cot m\theta \quad \left(\theta = \frac{\pi h}{k}\right)$$

$$= \lim_{n\to\infty} \sum_{m=1}^{k-1} \frac{1}{2} \cot m\theta \sum_{s=0}^{n} \frac{1}{m+ks}$$

サテ自然数 n ニ対シテ

$$\sum_{s=0}^{n} \frac{1}{m+ks} = \frac{1}{m} + \frac{1}{k} \sum_{s=1}^{n} \frac{1}{s+m/k} = \frac{1}{m} + \frac{1}{k} \sum_{s=1}^{n} \left(\frac{1}{s+\frac{m}{k}} - \frac{1}{s}\right) + \frac{1}{k} \sum_{s=1}^{n} \frac{1}{s}$$

$$= \frac{1}{k}\left\{\log n - \frac{\Gamma'}{\Gamma}\left(\frac{m}{k}\right) + o(1)\right\} \quad (n\to\infty)$$

故ニ $\sum_{m=1}^{k-1} \frac{1}{2} \cot m\theta \sum_{s=0}^{n} \frac{1}{m+ks} = \frac{1}{2k} \sum_{m=1}^{k-1} \left\{-\frac{\Gamma'}{\Gamma}\left(\frac{m}{k}\right)\right\} \cot m\theta$

$$+ \frac{\log n}{2k} \sum_{m=1}^{k-1} \cot m\theta + o(1) \to \frac{1}{2k} \sum_{m=1}^{k-1} \left\{-\frac{\Gamma'}{\Gamma}\left(\frac{m}{k}\right)\right\} \cot m\theta \ (n\to\infty)$$

何トナレバ明ラカニ $\sum_{m=1}^{k-1} \cot m\theta = 0$ ダカラデアル．然ルニ

$$\frac{1}{2k} \sum_{m=1}^{k-1} \left\{-\frac{\Gamma'}{\Gamma}\left(\frac{m}{k}\right)\right\} \cot m\theta = \frac{1}{4k} \sum_{m=1}^{k-1} \left\{\frac{\Gamma'}{\Gamma}\left(1-\frac{m}{k}\right) - \frac{\Gamma'}{\Gamma}\left(\frac{m}{k}\right)\right\} \cot m\theta$$

$$= \frac{\pi}{4k} \sum_{m=1}^{k-1} \cot\frac{\pi m}{k} \cot\frac{\pi h m}{k}, \qquad \text{Q. E. D.}$$

3. 前述ノ一様収斂性ヲ証明スル．其ノタメニハ $0<\gamma<1$ ニ於イテ $\sum_{s=0}^{\infty} A_s$ ガ一様ニ収斂スルヲ示セバヨイ，但シ

$$A_s = \sum_{m=1}^{k-1} \frac{1}{m+ks} J\left(\frac{1}{1-\rho^m \gamma^{m+ks}}\right)$$

サテ s ヲ自然数トスレバ $0<\gamma<e^{-\frac{1}{\sqrt{s}}}$ ナルトキ

$$|A_s| \leqq \sum_{m=1}^{k-1} \frac{1}{ks} \frac{\gamma^{ks}}{|1-\rho^m \gamma^{m+ks}|^2} \leqq \frac{K}{s} e^{-k\sqrt{s}}$$

茲ニ $K = K(h) > 0$ ハ h, m, s, γ ニ無関係ニテ h, m, s, γ $(0<\gamma<1)$ ノ如何ナル許サレタル組合セニ対シテモ

$$|1-\rho^m \gamma^{m+ks}|^2 \geqq \frac{1}{K}$$

ナルモノトスル．カヽル K ガ存在スルヿハ明ラカデアル．更ニ，$e^{-\frac{1}{\sqrt{s}}} \leqq \gamma < 1$ ナルトキハ

$$\left|A_s - \sum_{m=1}^{k-1} \frac{1}{ks} J\left(\frac{1}{1-\rho^m \gamma^{m+ks}}\right)\right|$$

$$= \left|\sum_{m=1}^{\infty} \frac{m}{ks(m+ks)} J\left(\frac{1}{1-\rho^m \gamma^{m+\nu}}\right)\right| \leqq \frac{K}{s^2}$$

故 $= \frac{K}{s^2} + \left| \sum_{m=1}^{k-1} \frac{1}{ks} \mathfrak{I}\left(\frac{1}{1-\rho^m \gamma^{m+ks}} \right) \right|$

$\leqq \frac{K}{s^2} + \left| \frac{1}{ks} \sum_{m=1}^{k} \mathfrak{I}\left(\frac{1}{1-\rho^m \gamma^{ks}} \right) \right|$

$= \frac{K}{s^2} + \left| \sum_{m=1}^{k-1} \frac{1}{ks} \mathfrak{I}\left(\frac{1}{1-\rho^m \gamma^{m+ks}} - \frac{1}{1-\rho^m \gamma^{ks}} \right) \right|$

$= \frac{K}{s^2} + \left| \sum_{m=1}^{k-1} \frac{1}{ks} \mathfrak{I}\left\{ \frac{\rho^m \gamma^{ks}(\gamma^m - 1)}{(1-\rho^m \gamma^{m+ks})(1-\rho^m \gamma^{ks})} \right\} \right|$

$\leqq \frac{K}{s^2} + \frac{K}{s}(1-\gamma^k) \leqq \frac{K}{s^2} + \frac{K}{s}\left(1 - e^{-\frac{k}{\sqrt{s}}}\right) \leqq \frac{K}{s^2} + \frac{kK}{s\sqrt{s}}$

斯クテ自然数 s 及ビ正数 $\gamma < 1$ ニ対シテ

$$|A_s| \leqq \frac{K}{s} e^{-k\sqrt{s}} + \frac{K}{s^2} + \frac{kK}{s\sqrt{s}},$$

$$\sum_{s=1}^{\infty} |A_s| \leqq \sum_{s=1}^{\infty} \left(\frac{K}{s} e^{-k\sqrt{s}} + \frac{K}{s^2} + \frac{kK}{s\sqrt{s}} \right) < \infty \qquad \text{Q.E.D.}$$

4. 最後ニ Rademacher ノ公式ヲ証明スル

$$\pi \sum_{\mu=1}^{k-1} \frac{\mu}{k} \left(\frac{h\mu}{k} - \left[\frac{h\mu}{k} \right] - \frac{1}{2} \right) = -\pi \sum_{\mu=1}^{k-1} \frac{\mu}{k} \sum_{n=1}^{\infty} \frac{\sin 2n\mu\theta}{n\pi}$$

$$= -\sum_{n=1}^{\infty} \frac{1}{kn} \sum_{\mu=1}^{k-1} \mu \sin 2n\mu\theta \qquad \left(\theta = \frac{\pi h}{k}\right)$$

$$= -\sum_{n=1}^{\infty} \frac{1}{kn} \frac{k \sin\{2(k-1)n\theta\}}{(2\sin n\theta)^2} = \sum_{n=1}^{\infty} \frac{1}{2n} \cot n\theta$$

$n \not\equiv 0 \pmod k$ $\qquad$ $n \not\equiv 0 \pmod k$

$$= \frac{\pi}{4k} \sum_{m=1}^{k-1} \cot \frac{\pi m}{k} \cot \frac{\pi h m}{k}$$

$$\exp\left\{ \pi i \sum_{\mu=1}^{k-1} \frac{\mu}{k} \left(\frac{h\mu}{k} - \left[\frac{h\mu}{k} \right] - \frac{1}{2} \right) \right\}$$

$$= \exp\left(\frac{\pi i}{4k} \sum_{m-1}^{k-1} \cot \frac{\pi m}{k} \cot \frac{\pi h m}{k} \right)$$

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