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ON A GENERALIZATION OF THE “DIV-CURL LEMMA”

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Abstract

We present a generalization of the div-curl lemma to a Banach space framework which is not included in the almost existing generalizations. An example is shown where this generalization is needed.

1. The div-curl lemma and its generalization

In this note we present a generalization of the famous “div-curl lemma”, which was first formulated by [1] in a Hilbert space setting. This lemma is widely used in the analysis of nonlinear partial differential equations. In [5] the result was generalized to a Banach space framework.

Here we present a further generalization to a setting, where one allows every component $v^k_i$, $w^k_i$ of the vectors $v^k$, $w^k$ to lie in different $L^{p_i}$ spaces (for details see below). This is of special interest in problems arising from limiting procedures in the hydrodynamic equations for plasmas and semiconductors [3], where both the original version and the version presented in [5] are not sufficient for the analysis.

We denote by

\[(1) \quad (\text{curl } w)_{i,j} = w_{i,x_j} - w_{j,x_i}\]

the curl (matrix) of a vector field. The superscript $(\cdot)'$ indicates the conjugate index with $1 = 1/p + 1/p'$. Then, the generalized version of the “div-curl lemma” reads

**Theorem 1.1.** Let $U \subset \mathbb{R}^n$ be a bounded, open, smooth domain. Let $1 < p_i < \infty$ for $i = 1, \ldots, n$ and let us denote $p_{\min} = \min_{1 \leq i \leq n} p_i$ and $p_{\max} = \max_{1 \leq i \leq n} p_i$. Let $v^k(x), w^k(x) \in \mathbb{R}^n$ for $k \in \mathbb{N}$ satisfying

- $\{v^k_i\}_{k=1}^\infty$ and $\{w^k_i\}_{k=1}^\infty$ are bounded sequences in $L^{p_i}(U)$ and $L^{p_i}(U)$, respectively, with $1/p_{\min} - 1/n < 1/p_{\max}$.
- $\{\text{div } v^k\}_{k=1}^\infty$ lies in a compact set of $W^{-1,t}(U)$, where $t \geq \max_{1 \leq i \leq n} (p'_i) = (p_{\min})'$.
- $\{(\text{curl } w^k)_{i,j}\}_{k=1}^\infty$ lies in a compact set of $W^{-1,s_{i,j}}(U)$ for $1 \leq i, j \leq n$, where $\min_{1 \leq j \leq n} s_{i,j} \geq p_i$ for $1 \leq i \leq n$.

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Then, the following convergence holds in $\mathcal{D}'(U)$
\begin{equation}
\nu^k \cdot w^k \rightharpoonup v \cdot w,
\end{equation}
where $v, w$ denote the weak limits (of subsequences) of \( \{v^k\}_{k=1}^{\infty} \) and \( \{w^k\}_{k=1}^{\infty} \), respectively.

**Remark 1.1.** Note that in the case $n \leq p_{\min}$ the condition $1/p_{\min} - 1/n < 1/p_{\max}$ is always satisfied.

**Remark 1.2.** The classical “div-curl lemma” is obtained by setting $p_i = p'_i = 2$ for $1 \leq i \leq n$. Then \( \{\text{div } v^k\}_{k=1}^{\infty} \) and \( \{(\text{curl } w^k)_{i,j}\}_{k=1}^{\infty} \) have to be in a compact set of $H^{-1}(U)$.

**Remark 1.3.** As already mentioned in [5] a Banach space framework of the “div-curl” lemma is given. There, the case of $p_i = p, 1 \leq i \leq n$ is covered.

**Proof.** The proof follows the ideas of the proof of the “div-curl lemma” given in [2].

In a first step we define functions $u^k_i, 1 \leq i \leq n$ as (unique) solutions of the problem
\begin{equation}
-\Delta u^k_i = w^k_i \quad \text{in } U, \quad u^k_i = 0 \quad \text{on } \partial U.
\end{equation}
The $u^k_i$ are uniformly bounded in $W^{2,p}_i(U)$ for $1 \leq i \leq n$.

In the second step we define the functions
\begin{equation}
z^k = -\text{div } u^k, \quad y^k = u^k - \nabla z^k
\end{equation}
with
\begin{equation}
y^k_i = u^k_i - z^k_i = (u^k_{i,j,x_j} - u^k_{i,x_j})_{x_j} = ((\text{curl } u^k)_{j,i})_{x_j}.
\end{equation}
Therefore, \( \{z^k\}_{k=1}^{\infty} \) is bounded in $W^{1,p_{\min}}(U)$ and compact in $L'(U)$ with $1/p_{\min} - 1/n < 1/r \leq 1$. The sequence \( \{y^k\}_{k=1}^{\infty} \) is on one hand bounded in $L^p_{\min}(U)$ (second derivatives of $u^k$), on the other hand compact in $L^{\min_{1 \leq j \leq n}(s,j)}(U)$. Indeed, according to the assumptions \( (\text{curl } u^k)_{j,i} \) is compact in $W^{-1,s,j}(U)$. Therefore, \( (\text{curl } u^k)_{j,i} \) lies compactely in $W^{1,s,j}(U)$ according to the results in [4] (at this point the smooth boundary \( C^\infty \) is required).

Then, the limits $z, y, u$ of $z^k, y^k, u^k$ satisfy $z = -\text{div } u, y = w - \nabla z$ and
\begin{equation}
-\Delta u_i = w_i \quad \text{in } U, \quad u_i = 0 \quad \text{on } \partial U.
\end{equation}

Finally, using a testfunction $\Phi \in C_0^\infty(U)$ we obtain for $\min_{1 \leq j \leq n}(s,j,i) \geq p_i, 1 \leq i \leq n$
\begin{equation}
\int_U \nu^k \cdot y^k \Phi \, dx \to \int_U v \cdot y \Phi \, dx.
\end{equation}
Similarly, for \( t \geq (p_{\text{min}}) = \max_{1 \leq i \leq n}(p'_i) \) we have

\[
\int_U \text{div} v^k z^k \Phi \, dx \to \int_U \text{div} v z \Phi \, dx.
\]  

Due to the assumption \( 1/p_{\text{min}} - 1/n < 1/p_{\text{max}} \) there exist always values of \( r \) such that (where \( r \) is the parameter used above)

\[
\frac{1}{p_{\text{min}}} - \frac{1}{n} < \frac{1}{r} < \frac{1}{p_{\text{max}}}.
\]  

Then, the second inequality (in (9)) garantuees the convergence

\[
\int_U v^k \cdot \nabla \Phi z^k \, dx \to \int_U v \cdot \nabla \Phi z \, dx.
\]  

Combining (7)–(10) we obtain

\[
\int_U v^k \cdot w^k \Phi \, dx = \int_U v^k \cdot y^k \Phi \, dx - \int_U \text{div} v^k z^k \Phi \, dx - \int_U v^k \cdot \nabla \Phi z^k \, dx
\]

\[
\to \int_U v \cdot y \Phi \, dx - \int_U \text{div} v z \Phi \, dx - \int_U v \cdot \nabla \Phi z \, dx
\]

\[
= \int_U v \cdot w \Phi \, dx.
\]  

This ends the proof.

**Example.** Suppose \( n = 2 \) and \( p_1 < 2 \ (p'_1 > 2), \ p_2 > 2 \ (p'_2 < 2). \) The curl matrix has two (nonvanishing) elements (curl \( w^k \))_1,2 = -(curl \( w^k \))_2,1. Then \( t > p'_1 \) and \( s_{1,2} > p_2 \) are required in order to apply the theorem. This is an examples where the results in [5] do not apply.

**References**


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