Tsukamoto, I.
Osaka J. Math.
40 (2003), 595–620

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF $x'' = e^{\alpha\lambda t}x^{1+\alpha}$ WHERE $-1 < \alpha < 0$

ICHIRO TSUKAMOTO

(Received February 5, 2002)

1. Introduction

Let us consider second order nonlinear differential equations

$$(1.1)_\pm \qquad x'' = \pm t^\beta x^{1+\alpha}$$

$$(1.2)_\pm \qquad x'' = \pm e^{\sigma t} x^{1+\alpha}$$

where $' = d/dt$, the double signs correspond in the same order in every equation and α, β, σ are parameters. Using Chapter 7 of [1], we can state value of solving these. First these can be derived from an important second order nonlinear differential equation

$$(1.3) \qquad \frac{d}{dt}\left(t^\rho \frac{du}{dt}\right) \pm t^\sigma u^n = 0,$$

ρ, σ, n being parameters, which contains the Emden equation of astrophysics and the Fermi-Thomas equation of atomic physics and so has several interesting physical applications. Second (1.3) is mathematically interesting, because (1.3) is nontrivial, nonlinear and has a large class of solutions whose behavior can be ascertained with astonishing accuracy nevertheless these cannot be generally obtained explicitly. In addition $(1.1)_\pm$, $(1.2)_\pm$ are examples of differential equations positive radial solutions of a nonlinear elliptic partial differential equation satisfy (cf. [17]).

Actually many authors have considered $(1.1)_\pm$, $(1.2)_\pm$ and (1.3) in more general form in [2], [5] through [9], [13], [20] and so on. In these papers they mainly discussed asymptotic behavior of the solution continuable to ∞. On the other hand, initial value problems of $(1.1)_+$, $(1.2)_+$, $(1.2)_-$ and $(1.1)_-$ were considered in [10], [11], in [14], [16], in [15], [16] and in [17] respectively in case of $\alpha > 0$ and asymptotic behavior of all the solutions was studied.

In the case $\alpha < 0$, the initial value problems of $(1.1)_\pm$, $(1.2)_\pm$ are not considered yet, while in [5], [8], [20] etc. this case was already considered for differential equations with more general form than $(1.1)_\pm$, $(1.2)_\pm$ and for the solutions continuable to ∞. So in this paper, we shall consider $(1.2)_+$ where $-1 < \alpha < 0$ as a first step. Since it is convenient to put $\sigma = \alpha\lambda$, the equation to be considered has the form

$$x'' = e^{\alpha\lambda t} x^{1+\alpha} \tag{E}$$

where $-1 < \alpha < 0$, $\lambda < 0$. It is noteworthy that the case $\lambda > 0$ can be reduced to our case if we replace t with $-t$. A domain where (E) will be considered is given as

$$-\infty < t < \infty, \quad 0 < x < \infty. \tag{1.4}$$

Notice that if p is a positive number and r is a real number, then throughout this paper p^r always takes its positive branch.

The initial condition given to (E) is

$$x(t_0) = a, \quad x'(t_0) = b \tag{I}$$

where

$$-\infty < t_0 < \infty, \quad a > 0, \quad -\infty < b < \infty.$$

t_0 will be fixed arbitrary and a suitably. For every b, we shall study asympototic behavior of all solutions of an initial value problem (E), (I).

For this, we shall use the method which follows the arguments originally done in [10], [11] and applied in [14] through [19]. In this method, we adopt a transformation

$$y = \psi(t)^{-\alpha}\phi(t)^{\alpha}, \quad z = y' \tag{T}$$

where $\psi(t) = \lambda^{2/\alpha}e^{-\lambda t}$ $(\lambda^{2/\alpha} = (\lambda^2)^{1/\alpha})$ is a particular solution of (E) and $\phi(t)$ is a solution of (E). This transforms (E) into a first order rational differential equation

$$\frac{dz}{dy} = \frac{(\alpha - 1)z^2 + 2\alpha\lambda yz - \alpha^2\lambda^2(y^2 - y^3)}{\alpha yz}. \tag{R}$$

Using a parameter s, we rewrite this as a 2-dimensional dynamical system

$$\begin{aligned} \frac{dy}{ds} &= \alpha yz \\ \frac{dz}{ds} &= (\alpha - 1)z^2 + 2\alpha\lambda yz - \alpha^2\lambda^2(y^2 - y^3). \end{aligned} \tag{D}$$

Graphs of solutions of (R) have the same shape as orbits of solutions of (D) except on the y and z axes. Since from (1.4) only a positive solution of (E) is considered, y is always positive. Finally we note that

$$z = \alpha y\left(\lambda + \frac{\phi'(t)}{\phi(t)}\right) \tag{1.5}$$

got from (T) will be often used.

2. On solutions of (E) obtained from orbits of (D) connecting its two singularities

The singularities of (D) are points $(0, 0)$, $(1, 0)$. From orbits of (D) connecting these points, we get the following through (T):

Theorem I. *Let $\phi(t)$ be a solution of the initial value problem* (E), (I) *and suppose*

$$0 < a < \psi(t_0).$$

Then there exist b_1, $b_2 (b_1 < b_2)$ such that
(i) *if $b = b_1$, $\phi(t)$ is defined for $-\infty < t < \infty$ so that in the neighborhood of $t = \infty$, $\phi(t)$ is represented as*

$$\phi(t) = \lambda^{2/\alpha}e^{-\lambda t}\left[1 + Ce^{(\mu_1/\alpha)t} + \sum_{n=2}^{\infty} a_n\left\{Ce^{(\mu_1/\alpha)t}\right\}^n\right] \tag{2.1}$$

where C, a_n are constants and

$$\mu_1 = (1 + \sqrt{1+\alpha})\alpha\lambda,$$

and as $t \to -\infty$,

$$\phi(t) = ct + d + \frac{(ct)^{1+\alpha}}{\alpha^2\lambda^2}e^{\alpha\lambda t}(1 + o(1)) \tag{2.2}$$

where $c(< 0)$, d are constants,
(ii) *if $b = b_2$, $\phi(t)$ is defined for $-\infty < t < \infty$ so that in the neighborhood of $t = \infty$, $\phi(t)$ is represented as*

$$\begin{aligned}\phi(t) \;=\; \lambda^{2/\alpha}e^{-\lambda t}\Big[&1 + (A + gB^N t)e^{(\mu_1/\alpha)t} + Be^{(\mu_2/\alpha)t} \\ &+ \sum_{m+n\geq 2} a_{mn}\left\{(A + gB^N t)e^{(\mu_1/\alpha)t}\right\}^m\left\{Be^{(\mu_2/\alpha)t}\right\}^n\Big]\end{aligned} \tag{2.3}$$

where A, B, g, a_{mn} are constants, $B \neq 0$, $N = \mu_1/\mu_2$,

$$\mu_2 = (1 - \sqrt{1+\alpha})\alpha\lambda$$

and $g \neq 0$ only if μ_1/μ_2 is a positive integer, and in the neighborhood of $t = -\infty$, $\phi(t)$ is represented as

$$\phi(t) = \lambda^{2/\alpha}C^{1/\alpha}\left\{1 + \sum_{n=1}^{\infty} a_n(Ce^{\alpha\lambda t})^n\right\} \tag{2.4}$$

where C, a_n *are constants,*
(iii) *if* $b_1 < b < b_2$, $\phi(t)$ *is defined for* $-\infty < t < \infty$ *and represented as* (2.3) *in the neighborhood of* $t = \infty$ *and* (2.2) *in the neighborhood of* $t = -\infty$.

Let us start the proof. First we consider $(1, 0)$. Putting

$$y = 1 + \eta, \quad z = \zeta,$$

we get from (D)

$$\begin{aligned} \frac{d\eta}{ds} &= \alpha\zeta + \cdots \\ \frac{d\zeta}{ds} &= \alpha^2\lambda^2\eta + 2\alpha\lambda\zeta + \cdots \end{aligned} \tag{2.5}$$

where $\cdots$ denotes terms whose degrees are greater than the previous terms. The coefficient matrix of the linear terms of (2.5) has eigenvalues μ_1, μ_2. Since $-1 < \alpha < 0$, we get

$$\mu_1 > \mu_2 > 0. \tag{2.6}$$

A linear transformation

$$\begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} \alpha & \alpha \\ \mu_1 & \mu_2 \end{pmatrix} \begin{pmatrix} \tilde{\eta} \\ \tilde{\zeta} \end{pmatrix}$$

transforms (2.5) into

$$\frac{d\tilde{\eta}}{ds} = \mu_1\tilde{\eta} + \cdots , \quad \frac{d\tilde{\zeta}}{ds} = \mu_2\tilde{\zeta} + \cdots .$$

Owing to Theorem A of [3] and its proof, a transformation

$$\tilde{\eta} = w_1 + \cdots , \quad \tilde{\zeta} = w_2 + \cdots$$

holomorphic in the neighborhood of $(w_1, w_2) = (0, 0)$ transforms this into

$$\frac{dw_1}{ds} = \mu_1 w_1 + g w_2^N, \quad \frac{dw_2}{ds} = \mu_2 w_2 \tag{2.7}$$

where g is a constant such that $g \neq 0$ only if μ_1/μ_2 is a positive integer and $N = \mu_1/\mu_2$. That is

$$w_1 = (A + gB^N s)e^{\mu_1 s}, \quad w_2 = Be^{\mu_2 s} \tag{2.8}$$

where A, B are arbitrary constants. Therefore an arbitrary solution of (D) converging to $(1, 0)$ is given as

$$y = 1 + \alpha w_1 + \alpha w_2 + \sum_{m+n\geq 2} a_{mn} w_1^m w_2^n \tag{2.9}$$

$$z = \mu_1 w_1 + \mu_2 w_2 + \sum_{m+n\geq 2} b_{mn} w_1^m w_2^n \tag{2.10}$$

in the neighborhood of $(w_1, w_2) = (0, 0)$, namely of $s = -\infty$. From (2.7), (2.9) we get

$$z = y' = \{\alpha(\mu_1 w_1 + g w_2^N) + \alpha\mu_2 w_2 + \cdots\}s'.$$

If we fix s, s' and vary A, B and compare this with (2.10), then since

$$\frac{\partial(w_1, w_2)}{\partial(A, B)} = e^{(\mu_1+\mu_2)s} \neq 0$$

and hence w_1, w_2 can attain arbitrary values, we conclude

$$s = \frac{t}{\alpha} + C$$

where C is an arbitrary constant. So if we replace $(A + gB^N C)e^{\mu_1 C}$, g/α and $Be^{\mu_2 C}$ with A, g and B respectively, then since

$$B^N e^{\mu_1 C} = (Be^{\mu_2 C})^N$$

in case of $g \neq 0$, we get

$$y = 1 + \alpha(A + gB^N t)e^{(\mu_1/\alpha)t} + \alpha Be^{(\mu_2/\alpha)t} + \sum_{m+n\geq 2} a_{mn}((A + gB^N t)e^{(\mu_1/\alpha)t})^m (Be^{(\mu_2/\alpha)t})^n.$$

Because $t \to \infty$ as $s \to -\infty$, this is valid in the neighborhood of $t = \infty$. Consequently we get the following through (T):

Lemma 2.1. *From* (2.9), (2.10) *we obtain a solution* $\phi(t)$ *of* (E) *with the form* (2.3) *valid in the neighborhood of* $t = \infty$.

Note that B of (2.3) is not always nonzero in Lemma 2.1. If $B \neq 0$, then from (2.9), (2.10) we have

$$\lim_{s\to-\infty} \frac{z}{y-1} = \frac{\mu_2}{\alpha}. \tag{2.11}$$

However if $B = 0$, then we get

$$\lim_{s\to-\infty} \frac{z}{y-1} = \frac{\mu_1}{\alpha}. \tag{2.12}$$

Therefore there exists uniquely a solution of (R) satisfying (2.12). Indeed from (2.9), (2.10) and $B = 0$ this is represented as

$$z = \frac{\mu_1}{\alpha}(y-1) + \sum_{n=2}^{\infty} c_n (y-1)^n \tag{2.13}$$

in the neighborhood of $y = 1$. Conversely we get (2.10) from substituting (2.9) into (2.13). In this way we get (2.9), (2.10) from (2.13). Hence if we take $A = C$, $B = 0$ in Lemma 2.1, we obtain

Lemma 2.2. *From* (2.13) *we get a solution* $\phi(t)$ *of* (E) *with the form* (2.1) *in the neighborhood of* $t = \infty$.

For further discussions, we examine the sign of dz/ds in (D). So we put $dz/ds = 0$. Then we get $z = Z_{\pm}(y)$ where

$$Z_{\pm}(y) = \frac{\alpha\lambda}{1-\alpha} y \left\{ 1 \pm \sqrt{\alpha - (\alpha - 1)\, y} \right\}.$$

$Z_{\pm}(y)$ are defined for $y \geq -\alpha/(1-\alpha)$ and

$$Z_{\pm}\left(-\frac{\alpha}{1-\alpha}\right) = -\frac{\alpha^2\lambda}{(1-\alpha)^2}, \quad Z_{+}(y) > Z_{-}(y) \quad \text{for} \quad y > -\frac{\alpha}{1-\alpha}.$$

$Z_{+}(y)$ is monotone increasing. If $\alpha \leq -1/3$, then $Z_{-}(y)$ is monotone decreasing. If $-1/3 < \alpha < 0$, then $Z_{-}(y)$ has a minimal value and a maximal value and is monotone increasing in the interval between these extremums and monotone decreasing outside this interval. Moreover we get

$$\lim_{y \to \infty} Z_{\pm}(y) = \pm\infty.$$

Lemma 2.3. *If* $y < -\alpha/(1-\alpha)$ *or* $z > Z_{+}(y)$ *or* $z < Z_{-}(y)$, *then*

$$\frac{dz}{ds} < 0,$$

if $z = Z_{\pm}(y)$, *then*

$$\frac{dz}{ds} = 0,$$

and if $Z_{-}(y) < z < Z_{+}(y)$, *then*

$$\frac{dz}{ds} > 0.$$

Let us consider how the solution (2.9), (2.10) of (D) behaves asymptotically as $s \to \infty$, if $B = 0$. For this we introduce a curve

$$z = f(y) = \alpha\lambda(y - y^2).$$

Owing to the proof of Proposition 4 of [14], we get

$$\frac{d}{ds}(z - f(y)) = (\alpha + 1)\alpha^2\lambda^2 y^3(1 - y) > 0 \tag{2.14}$$

when a solution (y, z) of (D) passes the curve $z = f(y)$ where $0 < y < 1$. Moreover we have

$$\frac{\mu_1}{\alpha} = -\frac{\alpha\lambda}{1 - \sqrt{1 + \alpha}} < -\alpha\lambda = f'(1).$$

It follows from (2.12) and $Z'_-(1) = -\alpha\lambda/2$ that (2.13) lies between $z = f(y)$ and $z = Z_+(y)$ in the yz plane. Hence from Lemma 2.3 and Poincaré-Bendixon's theorem, (2.9), (2.10) where $B = 0$ tend to $(0, 0)$ as $s \to \infty$. Since (2.13) is obtained uniquely from (2.9), (2.10), we get

Lemma 2.4. *There exists the unique solution* $z = z_1(y)$ *of* (R) *defined for* $0 \le y \le 1$ *such that* (2.13) *holds. Moreover we obtain*

$$\lim_{y \to +0} z_1(y) = \lim_{y \to 1-0} z_1(y) = 0.$$

Next let us consider the singularity $(0, 0)$ of (D). As preparation of this, we show

Lemma 2.5. *If there exists a solution* $y = y(x)$ *of a Briot-Bouquet differential equation*

$$x\frac{dy}{dx} = f(x, y) \tag{2.15}$$

where $f(x, y)$ *is a holomorphic function in the neighborhood of* $(x, y) = (0, 0)$ *with the form*

$$f(x, y) = \lambda y + ax + \sum_{j+k \ge 2} a_{jk}x^j y^k, \quad \lambda < 0 \tag{2.16}$$

and if the accumulation points of $y(x)$ *contain* 0 *as* x *tends to* 0 *with bounded* $\arg x$, *then* $y(x)$ *is the unique holomorphic solution.*

Proof. It is known that if λ is not a positive integer, then there exists the unique holomorphic solution $y = h(x)$ of (2.15) such that $h(0) = 0$. So we put $z = y - h(x)$.

Then we get

$$x\frac{dz}{dx} = zg(x, z)$$

where $g(x, z)$ is a holomorphic function in the neighborhood of $(x, z) = (0, 0)$ and $g(0, 0) = \lambda$. Hence it suffices to show that the solution $y(x)$ of (2.15) satisfying the assumption of this lemma is identically zero, when we get

$$f(x, y) = \lambda y \left(1 + \sum_{j+k \geq 1} a_{jk} x^j y^k\right) \tag{2.17}$$

instead of (2.16).

Suppose the contrary. Then $y(x)$ is not identically zero. Now if $c(\neq 0)$ is an accumulation point of $y(x)$ as x tends to 0 with bounded $\arg x$, then there exists a compact neighborhood U of 0 such that $c \notin U$. Since $y(x)$ intersects the boundary of U infinitely many times and the boundary of U is compact, $y(x)$ has its accumulation point on the boundary of U as x tends to 0 with bounded $\arg x$. Since U can be taken sufficiently small, we may suppose that c is so small that $f(x, c) \neq 0$. Hence we get from (2.15)

$$\frac{dx}{dy} = \frac{x}{f(x, y)}$$

which implies the contradiction $x \equiv 0$ from Painlevé's theorem (cf. Theorem 3.2.1 of [4]) and the uniqueness theorem. Consequently $y(x)$ converges to 0 as x tends to 0 with bounded $\arg x$.

Substituting $y = y(x)$ into (2.15), we have from (2.17)

$$x\frac{dy}{dx} = \lambda y(1 + o(1))$$

as x tends to 0 with bounded $\arg x$. Therefore for some x_0 in the neighborhood of 0 and $y_0 = y(x_0)$ we get

$$\int_{y_0}^{y} \frac{dy}{y} = \lambda \int_{x_0}^{x} \frac{1 + o(1)}{x}\, dx$$

where $|x| < |x_0|$ and $y = y(x)$. From Cauchy's theorem we take Γ as a path of integration of the righthand side so that if $x \in \Gamma$, then $|x|$ and $\arg x$ vary monotonously along Γ. Then we get

$$\int_{x_0}^{x} \frac{1 + o(1)}{x}\, dx = \int_{x_0}^{x} \frac{dx}{x}(1 + o(1))$$

and thus

$$\log \frac{y}{y_0} = \lambda \left(\log \frac{x}{x_0} \right) (1 + o(1))$$

whose real parts deduce a contradiction as x tends to 0 with bounded $\arg x$. Hence $y(x)$ is identically zero. □

Lemma 2.6. *Let $z = z(y)$ be a solution of* (R) *such that*

$$\lim_{y\to 0} z(y) = 0.$$

Then we have

$$\lim_{y\to 0} y^{-1}z = \alpha\lambda. \tag{2.18}$$

Proof. Since Lemma 5 of [14] is valid also in our case, we get

$$\lim_{y\to 0} y^{-1}z = \alpha\lambda, \quad \pm\infty.$$

If $\lim_{y\to 0} y^{-1}z = \pm\infty$, we put $w = yz^{-1}$. Then as in [14] we obtain

$$y\frac{dw}{dy} = \frac{w}{\alpha} - 2\lambda w^2 + \alpha\lambda^2(1 - y)w^3.$$

Since $1/\alpha < 0$, it follows from Lemma 2.5 that this has the unique holomorphic solution $w \equiv 0$. This is a contradiction and (2.18) is valid.

Concerning a solution $z = z(y)$ of (R) converging to 0 as $y \to 0$ and satisfying (2.18), we shall get some lemmas. □

Lemma 2.7. *There exists uniquely a solution $z = z_2(y)$ of* (R) *such that* (2.18) *holds and*

$$\lim_{y\to 0} y^{-1}v = \lambda \tag{2.19}$$

where

$$v = y^{-1}z - \lambda.$$

Furthermore in the neighborhood of $y = 0$ we get

$$z_2(y) = \alpha\lambda y + \lambda y^2 + \cdots . \tag{2.20}$$

Proof. Putting $w = y^{-1}v - \lambda$, we get

$$w \to 0 \quad \text{as } y \to 0$$

from (2.19) and

$$\frac{dw}{dy} = \frac{-(\alpha+1)y(w+\lambda)^2 - \alpha^2\lambda w}{\alpha y(yw+\lambda y+\alpha\lambda)}.$$

This is (27) of [14]. Hence if we follow discussion of [14] after (27), then the proof is completed. □

Using (2.20) and (T), we have

Lemma 2.8. *From $z_2(y)$ we obtain a solution $\phi(t)$ of* (E) *with the form* (2.4) *in the neighborhood of $t = -\infty$.*

Here in the same way as in [16] we conclude

Lemma 2.9. *A solution of* (R) *satisfying* (2.18) *and not satisfying* (2.19) *is given as*

$$z = \alpha\lambda y + \frac{\alpha^2\lambda y}{\log|y|}\left(1 + O\left(\frac{\log|\log|y||}{\log|y|}\right)\right) \quad \textit{as } y \to 0. \tag{2.21}$$

In the neighborhood of $y = 0$, we get

$$(2.21) > \alpha\lambda y > f(y) > z_2(y)$$

since $-1 < \alpha < 0$. However a solution of (D) satisfies

$$\frac{dy}{ds} = 0, \quad \frac{dz}{ds} = -\alpha^2\lambda^2(y^2 - y^3) < 0$$

on the segment $0 < y < 1$, $z = 0$. Therefore from (2.14) and Poincaré-Bendixon's theorem we have

$$(y, z_2(y)) \to (1, 0) \quad \text{as } s \to -\infty.$$

Namely $z_2(y)$ is defined for $0 \le y \le 1$ and

$$\lim_{y\to 1-0} z_2(y) = 0.$$

Because only $z_1(y)$ satisfies (2.12) and only $z_2(y)$ satisfies (2.18) and (2.19), we conclude

Lemma 2.10. *On $0 < y < 1$, we get*

$$z_1(y) > z_2(y) > 0$$

and $z_1(y)$ *is represented as* (2.21) *in the neighborhood of* $y = 0$ *and* $z_2(y)$ *as* (2.9), (2.10) *where* $B \neq 0$ *in the neighborhood of* $y = 1$.

Now we continue the argument of [16] used for obtaining Lemma 2.9 and have

$$\phi(t) \sim ct \quad \text{as } t \to -\infty.$$

Moreover integrating both sides of (E) twice, we conclude

Lemma 2.11. *From* (2.21) *we get a solution* $\phi(t)$ *of* (E) *with the form* (2.2) *as* $t \to -\infty$.

Finally, note that from (T) and (1.5) the initial condition (I) gives an initial condition

$$z(y_0) = z_0 \tag{2.22}$$

to (R). Here

$$y_0 = \psi(t_0)^{-\alpha}a^{\alpha}, \quad z_0 = \alpha y_0\left(\lambda + \frac{b}{a}\right). \tag{2.23}$$

Since $z_1(y)$, $z_2(y)$ are defined for $0 < y < 1$, we take

$$0 < y_0 < 1.$$

This is equivalent to

$$0 < a < \psi(t_0). \tag{2.24}$$

Fix t_0 arbitrarily and a so as to satisfy (2.24). Then varying b, y_0 is fixed and z_0 varies. Suppose that from (R) and (2.22) we get $z_1(y)$ if $b = b_1$ and $z_2(y)$ if $b = b_2$. Then from Lemma 2.10, we obtain $b_1 < b_2$ since z_0 is monotone decreasing in b. If $b_1 < b < b_2$, then we have a solution $z(y)$ of (R) and (2.22) such that

$$z_1(y) > z(y) > z_2(y).$$

Thus the unique existence of $z_1(y)$ and $z_2(y)$ implies

Lemma 2.12. *Let* $z(y)$ *be a solution of* (R), (2.22) *with* $b_1 < b < b_2$. *Then* $z(y)$ *is defined for* $0 \leq y \leq 1$ *and represented as* (2.21) *in the neighborhood of* $y = 0$ *and as* (2.9), (2.10) *where* $B \neq 0$ *in the neighborhood of* $y = 1$.

Consequently in Theorem I, (i) follows from Lemmas 2.2, 2.4, 2.10, 2.11 and (ii) from Lemmas 2.1, 2.8, 2.10 and (iii) from Lemmas 2.1, 2.11, 2.12.

3. Preliminaries for the further discussions

Before considering the cases not treated yet, we need the following discussions. If we put $y = 1/\eta$ in (R), we get

$$\frac{dz}{d\eta} = -\frac{(\alpha-1)\eta^3 z^2 + 2\alpha\lambda\eta^2 z - \alpha^2\lambda^2(\eta-1)}{\alpha\eta^4 z}. \tag{3.1}$$

If we put $z = 1/\zeta$ in (R), then

$$\frac{d\zeta}{dy} = -\frac{(\alpha-1)\zeta + 2\alpha\lambda y\zeta^2 - \alpha^2\lambda^2(y^2-y^3)\zeta^3}{\alpha y}. \tag{3.2}$$

Furthermore if we put $y = 1/\eta$, z $= 1/\zeta$, then we have

$$\frac{d\zeta}{d\eta} = \frac{(\alpha-1)\eta^3\zeta + 2\alpha\lambda\eta^2\zeta^2 - \alpha^2\lambda^2(\eta-1)\zeta^3}{\alpha\eta^4}. \tag{3.3}$$

Moreover if we put

$$w = \eta^{-3/2}\zeta, \quad \xi = \eta^{1/2},$$

then we obtain a Briot-Bouquet differential equation

$$\xi\frac{dw}{d\xi} = -\frac{\alpha+2}{\alpha}w + 4\lambda\xi w^2 - -\alpha\lambda^2(\xi^2-1)w^3. \tag{3.4}$$

Let $z = z(y)$ be a solution of (R). Then through (T) we obtain a solution $x = \phi(t)$ of (E). If (ω_-, ω_+) denotes a domain of $\phi(t)$ and if y is a function obtained from $\phi(t)$ through (T), then we get

Lemma 3.1. *$y \to \infty$ as $t \to \omega_\pm$ imply that $\omega_\pm$ are finite respectively.*

Proof. If a solution z of (R) is bounded, then (3.1) implies a contradiction $\eta \equiv 0$. Hence z is unbounded.

So we consider (3.4). If $\xi = 0$, then the righthand side of (3.4) vanishes if and only if $w = 0, \pm\rho$ where

$$\rho = \frac{1}{\alpha\lambda}\sqrt{\frac{\alpha+2}{2}}.$$

Here let c be an accumulation point of a solution w of (3.4) as $\xi \to 0$, namely $y \to \infty$. Suppose that $c \neq 0, \pm\rho, \pm\infty$. Then from (3.4) we get a contradiction $\xi \equiv 0$. Hence $c = 0, \pm\rho, \pm\infty$.

If $c = 0$, then we get from (3.4)

$$w = C\xi^{-(1+2/\alpha)}\left[1 + \sum_{m+n\geq 1} a_{mn}\xi^m\{C\xi^{-(1+2/\alpha)}\}^n\right]$$

where C is an arbitrary constant and the power series converges in the neighborhood of $\xi = 0$, since $-(\alpha+2)/\alpha > 0$ and the righthand side of (3.4) is divisible by w. Returning the variables, we have

$$y^{1/\alpha}\left\{1 + \sum_{m+n\geq 1} b_{mn}y^{-m/2+((\alpha+2)/2\alpha)n}\right\} = \frac{t-\omega_-}{\alpha C} \quad \text{or} \quad \frac{t-\omega_+}{\alpha C} \tag{3.5}$$

where $-\infty < \omega_- < \omega_+ < \infty$.

If $c = \pm\rho$, we put $\theta = w - c$. Then we have

$$\xi\frac{d\theta}{d\xi} = \frac{2(\alpha+2)}{\alpha^2\lambda}\xi + \frac{2(\alpha+2)}{\alpha}\theta + \cdots .$$

Since $2(\alpha+2)/\alpha < 0$ and θ is real so that $\arg\theta$ is bounded, Lemma 2.5 implies that θ is holomorphic and represented as

$$\theta = \sum_{n=1}^{\infty} a_n\xi^n.$$

Here, return the variables. Then we get

$$-2cy^{-1/2} - \sum_{n=1}^{\infty}\frac{2a_n}{n+1}y^{-(n+1)/2} = t - \omega_- \quad \text{or} \quad t - \omega_+ \tag{3.6}$$

where $-\infty < \omega_- < \omega_+ < \infty$.

Now we suppose $c = \pm\infty$. Putting $w = 1/\theta$, we have

$$\theta \to 0 \quad \text{as } \xi \to 0,$$
$$\frac{d\xi}{d\theta} = \frac{\alpha\xi\theta}{(\alpha+2)\theta^2 - 4\alpha\lambda\xi\theta + 2\alpha^2\lambda^2(\xi^2-1)}.$$

These imply a contradiction $\xi \equiv 0$. Consequently $c \neq \pm\infty$. □

Corollary 3.2. *If $c = 0$, we get*

$$\begin{aligned}\phi(t) = {} & \frac{\lambda^{2/\alpha}e^{-\lambda\omega_-}}{\alpha C}(t-\omega_-) \\ & \times\left\{1 + \sum_{l+m+n\geq 1} d_{lmn}(t-\omega_-)^l(t-\omega_-)^{-\alpha m/2}(t-\omega_-)^{(\alpha+2)n/2}\right\}\end{aligned} \tag{3.7}$$

in the neighborhood of $t=\omega_-$ *and*

$$\phi(t) = \frac{\lambda^{2/\alpha}e^{-\lambda\omega_+}}{\alpha C}(\omega_+ - t) \times \left\{1 + \sum_{l+m+n\geq 1} d_{lmn}(\omega_+ - t)^l(\omega_+ - t)^{-\alpha m/2}(\omega_+ - t)^{(\alpha+2)n/2}\right\} \tag{3.8}$$

in the neighborhood of $t=\omega_+$ *where* C *is an arbitrary constant and* d_{lmn} *are constants. Moreover if* $c=\pm\rho$, *we get*

$$\phi(t) = \left\{\frac{2(\alpha+2)}{\alpha^2}\right\}^{1/\alpha} e^{-\lambda\omega_-}(t-\omega_-)^{-2/\alpha}\left\{1+\sum_{n=1}^{\infty}c_n(t-\omega_-)^n\right\} \tag{3.9}$$

in the neighborhood of $t=\omega_-$ *and*

$$\phi(t) = \left\{\frac{2(\alpha+2)}{\alpha^2}\right\}^{1/\alpha} e^{-\lambda\omega_+}(\omega_+-t)^{-2/\alpha}\left\{1+\sum_{n=1}^{\infty}c_n(\omega_+-t)^n\right\} \tag{3.10}$$

in the neighborhood of $t=\omega_+$ *where* c_n *are constants.*

Furthermore if $c\neq 0, \pm\rho$, *then the solution* $\phi(t)$ *of* (E) *cannot be obtained.*

Proof. In the proof of Lemma 3.1, we get $c=0, \pm\rho$. If $c=0$, then we put

$$y=\frac{1}{\eta}, \quad \eta^{1/2}=\xi$$

and get from (3.5)

$$\xi^{-2/\alpha}\left\{1+\sum_{m+n\geq 1}\tilde{a}_{mn}\xi^{m-((\alpha+2)/\alpha)n}\right\} = \frac{t-\omega_-}{\alpha C} \quad \text{or} \quad \frac{t-\omega_+}{\alpha C}.$$

Here if we put

$$\theta=\xi^{-2/\alpha}, \quad \tau=\frac{t-\omega_-}{\alpha C} \quad \text{or} \quad \frac{t-\omega_+}{\alpha C}$$

then

$$\theta\left\{1+\sum_{m+n\geq 1}\tilde{a}_{mn}\theta^{-(\alpha/2)m}\theta^{((\alpha+2)/2)n}\right\}=\tau.$$

Hence we have

$$\tau^{-\alpha/2} = \theta^{-\alpha/2}\left\{1+\sum_{m+n\geq 1}b_{mn}\theta^{-(\alpha/2)m}\theta^{((\alpha+2)/2)n}\right\} \tag{3.11}$$

$$\tau^{(\alpha+2)/2} = \theta^{(\alpha+2)/2}\left\{1 + \sum_{m+n\geq 1} c_{mn}\theta^{-(\alpha/2)m}\theta^{((\alpha+2)/2)/n}\right\}. \tag{3.12}$$

Applying the inverse function theorem to (3.11) and (3.12), we obtain

$$\theta^{-\alpha/2} = \tau^{-\alpha/2}\left\{1 + \sum_{m+n\geq 1} \hat{a}_{mn}\tau^{-(\alpha/2)m}\tau^{((\alpha+2)/2)n}\right\}.$$

Therefore we get

$$y^{1/\alpha} = \tau\left\{1 + \sum_{m+n\geq 1} \hat{b}_{mn}\tau^{-(\alpha/2)m}\tau^{((\alpha+2)/2)n}\right\}$$

and so (3.7) and (3.8) through (T).

If $c = \pm\rho$, then from (3.6) and (T) we have (3.9) and (3.10). □

4. On the other solutions of (E)

Let $z_3(y)$ be a solution of (R) which exists in $y > 1$ and is represented as (2.13) in the neighborhood of $y = 1$. Moreover, suppose that we get $z_3(y)$ as a solution of the initial value problem (R), (2.22), if $a > \psi(t_0)$ and $b = b_3$. Then if $\phi(t)$ denotes a solution of the initial value problem (E), (I) as in Section 2, we have

Theorem II. *If $0 < a < \psi(t_0)$ and $b > b_2$, then $\phi(t)$ is defined for $\omega_- < t < \infty$ where $\omega_- > -\infty$ and represented as* (3.7) *in the neighborhood of $t = \omega_-$ and* (2.3) *where $B \neq 0$ in the neighborhood of $t = \infty$.*

If $a = \psi(t_0)$ and $b = -a\lambda$, then $\phi(t) \equiv \psi(t)$ and if $a = \psi(t_0)$ and $b > -a\lambda$, then the conclusion of the case $0 < a < \psi(t_0)$, $b > b_2$ follows.

If $a > \psi(t_0)$, then there exists b_3 such that

(i) *if $b = b_3$, $\phi(t)$ is defined for $\omega_- < t < \infty$ where $\omega_- > -\infty$ and represented as* (3.7) *in the neighborhood of $t = \omega_-$ and* (2.1) *in the neighborhood of $t = \infty$,*

(ii) *if $b > b_3$, the conclusion of the case $0 < a < \psi(t_0)$, $b > b_2$ follows.*

For starting the proof, recall (2.14). Then

$$\frac{d}{ds}(z - f(y)) = (\alpha + 1)\alpha^2\lambda^2 y^3(1 - y) < 0, \tag{4.1}$$

when a solution (y, z) of (D) passes the curve $z = f(y)$ where $y > 1$. Therefore

$$z_3(y) < f(y),$$

since $f'(1) > \mu_1/\alpha$. Thus there exists y_1 $(1 < y_1 \leq \infty)$ such that

$$\lim_{y \to y_1} z_3(y) = -\infty.$$

However if y_1 is finite, then putting $z = 1/\zeta$ we obtain (3.2) and a contradiction $\zeta \equiv 0$. Consequently we conclude

$$y_1 = \infty, \quad \lim_{y \to \infty} z_3(y) = -\infty.$$

Furthermore we suppose

$$b > b_2 \quad \text{or} \quad b \geq b_3.$$

If $z_0 > 0$, then the solution $z_+(y)$ of (R), (2.22) satisfies

$$0 \leq z_+(y) < z_2(y). \tag{4.2}$$

Moreover in the yz plane, $z_+(y)$ connects at some point $(\tilde{y}, 0)$ with a solution $z_-(y)$ of (R) satisfying

$$z_-(y) \leq 0 \quad \text{if} \quad \tilde{y} < y \leq 1, \quad z_-(y) < z_3(y) \leq 0 \quad \text{if} \quad y > 1. \tag{4.3}$$

On the other hand, if $z_0 < 0$, then the solution $z_-(y)$ of (R), (2.22) satisfies (4.3) and connects with a solution $z_+(y)$ of (R) satisfying (4.2) at $(\tilde{y}, 0)$. If $z_0 = 0$, then the solution of (R), (2.22) is given as $z_+(y)$ and $z_-(y)$ which satisfy (4.2) and (4.3) respectively and connect mutually at $(y_0, 0)$. So let $z(y)$ be a many-valued function such that

$$z(y) = z_+(y) \quad \text{if} \quad z(y) \geq 0, \quad z(y) = z_-(y) \quad \text{if} \quad z(y) \leq 0. \tag{4.4}$$

Then the same discussion as was done for $z_3(y)$ shows

$$\lim_{y \to \infty} z(y) = -\infty. \tag{4.5}$$

Here we state Lemma 4 of [15] as follows:

Lemma 4.1. *Let $z_\pm(y)$ be solutions of* (R) *such that*

$$z_+(\tilde{y}) = z_-(\tilde{y}) = 0$$

for some $\tilde{y}$ and

$$z_+(y) > 0, \quad z_-(y) < 0 \quad \textit{for} \quad y \neq \tilde{y}$$

(i) *If* $z_0 > 0$ *and* $z_+(y)$ *satisfies* (R), (2.22) *and* $y(t)$ *is a solution of*

$$\frac{dy}{dt} = z_+(y), \quad y(t_0) = y_0,$$

then there exists t_1 *such that*

$$\lim_{t\to t_1+0} y(t) = \tilde{y}$$

and $y(t)$ *can be continued in the interval* $t < t_1$ *uniquely by*

$$\frac{dy}{dt} = z_-(y), \quad y(t_1) = \tilde{y}$$

(ii) *If* $z_0 < 0$, *we get the similar conclusion.*
(iii) *If* $z_0 = 0$ *and* $z_\pm(y)$ *satisfy* (R), (2.22), *then* $y(t)$ *can be defined uniquely by*

$$\begin{aligned} \frac{dy}{dt} &= z_+(y) \quad \text{if } t > t_0, \quad \frac{dy}{dt} = 0 \quad \text{if } t = t_0, \\ \frac{dy}{dt} &= z_-(y) \quad \text{if } t < t_0, \quad y(t_0) = y_0. \end{aligned}$$

Proof. If $(y, z) = (p(t), q(t))$ is a solution of

$$\begin{aligned} &\frac{dy}{dt} = z \\ &\frac{dz}{dt} = \frac{(\alpha-1)z^2 + 2\alpha\lambda yz - \alpha^2\lambda^2(y^2 - y^3)}{\alpha y} \\ &y(t_0) = y_0, \quad z(t_0) = z_0, \end{aligned} \tag{4.6}$$

then it suffices to put $y(t) = p(t)$.

Return to our discussion. Then for $z(y)$ defined as (4.4) we get $y(t)$ and a solution $\phi(t)$ of (E) through Lemma 4.1 and (T). Recall that (ω_-, ω_+) denotes a domain of ϕ(t).

Since $(y(t), z(y(t)))$ is a solution of (4.6), we have

$$\lim_{t\to\omega_-} y(t) = \infty$$

from (4.2), (4.3), (4.4), (4.5) and Poincaré-Bendixon's theorem. Hence Lemma 3.1 implies $\omega_- > -\infty$. Because $z_3(y) < f(y)$, we obtain

$$\lim_{y\to\infty} y^{-3/2}z(y) = -\infty.$$

Hence if we put

$$y = \frac{1}{\eta}, \quad z = \frac{1}{\zeta}, \quad w = \eta^{-3/2}\zeta,$$

then we get

$$\lim_{\eta \to 0} w = 0$$

and a solution $\phi(t)$ of (E) represented as (3.7) from Corollary 3.2.

Let us now consider the case $t \to \omega_+$. Then if $b = b_3$, we have $\phi(t)$ represented as (2.1) from Lemma 2.2, since $z_3(y)$ is represented as (2.13). In this case, we get $\omega_+ = \infty$. Moreover if $b > b_2$ or $b > b_3$, then since $(y(t), z(y(t)))$ is a solution of (4.6), Poincaré-Bendixon's theorem implies

$$y \to 1, \quad z_+(y) \to 0 \quad \text{as } t \to \omega_+.$$

Hence y, $z(y)$ are given as (2.9), (2.10) and from Lemma 2.1 we get $\phi(t)$ represented as (2.3) where $B \neq 0$ and $\omega_+ = \infty$. Now the proof of Theorem II is completed. □

Next, suppose

$$b < b_1 \quad \text{if} \quad 0 < a < \psi(t_0), \qquad b < -a\lambda \quad \text{if} \quad a = \psi(t_0), \qquad b < b_3 \quad \text{if} \quad a > \psi(t_0).$$

Then the solution $z(y)$ of (R), (2.22) satisfies

$$z(y) > z_1(y), \quad z(y) > z_3(y). \tag{4.7}$$

In this case, we get

Theorem III. *If $a > \psi(t_0)$, then there exist b_4, $b_5 (b_5 < b_4 < b_3)$ such that*
(i) *if $b_4 < b < b_3$, then $\phi(t)$ is defined for $\omega_- < t < \infty$ where $\omega_- > -\infty$ and represented as* (3.7) *in the neighborhood of $t = \omega_-$ and* (2.3) *where $B \neq 0$ in the neighborhood of $t = \infty$,*
(ii) *if $b = b_4$, then $\phi(t)$ is defined for $\omega_- < t < \infty$ where $\omega_- > -\infty$ and represented as* (3.9) *in the neighborhood of $t = \omega_-$ and* (2.3) *where $B \neq 0$ in the neighborhood of $t = \infty$,*
(iii) *if $b_5 < b < b_4$, then $\phi(t)$ is defined for $-\infty < t < \infty$ and represented as* (2.2) *as $t \to -\infty$ and* (2.3) *where $B \neq 0$ in the neighborhood of $t = \infty$,*
(iv) *if $b = b_5$, then $\phi(t)$ is defined for $-\infty < t < \omega_+$ where $\omega_+ < \infty$ and represented as* (2.2) *as $t \to -\infty$ and* (3.10) *in the neighborhood of $t = \omega_+$,*
(v) *if $b < b_5$, then $\phi(t)$ is defined for $-\infty < t < \omega_+$ where $\omega_+ < \infty$ and represented as* (2.2) *as $t \to -\infty$ and* (3.8) *in the neighborhood of $t = \omega_+$.*

If $a = \psi(t_0)$, then there exists $b_5 (< -a\lambda)$ such that replacing b_4 with $-a\lambda$ we get (iii), (iv), (v) *and if $0 < a < \psi(t_0)$, then replacing $-a\lambda$ with b_1 the conclusion of the case $a = \psi(t_0)$ follows.*

Proof. Now in (R) we put

$$y^{-1/2} = \eta, \quad z^{-1} = \eta^3(-\rho + u). \tag{4.8}$$

Then from (21) of [14] we have

$$\eta\frac{du}{d\eta} = \frac{2(\alpha+2)}{\alpha^2\lambda}\eta + \left(2 + \frac{4}{\alpha}\right)u + \cdots . \tag{4.9}$$

In the proof of Lemma 3.1, we put

$$y = \frac{1}{\eta}, \quad z = \frac{1}{\zeta}, \quad \xi = \eta^{1/2}, \quad w = \eta^{-3/2}\zeta, \quad \theta = w - c \tag{4.10}$$

where $c = \pm\rho$ and obtained the differential equation similar to (4.9). Since $2 + 4/\alpha < -2$, Lemma 2.5 implies that there exists the unique solution $u(\eta)$ of (4.9) such that $u(0) = 0$. Moreover $u(\eta)$ is holomorphic in the neighborhood of $\eta = 0$. Hence we get a solution of (R) such as

$$z = y^{3/2}\left(-\rho^{-1} + \sum_{n=1}^{\infty}\tilde{a}_n y^{-n/2}\right). \tag{4.11}$$

Since $y \to \infty$ as $\eta \to 0$, this is valid in the neighborhood of $y = \infty$. Moreover from the uniqueness of $u(\eta)$, (4.11) is uniquely determined. So we denote (4.11) as $z_4(y)$. Owing to

$$\lim_{y\to\infty}\frac{z_4(y)}{f(y)} = 0$$

and (4.1), we have

$$z_3(y) < f(y) < z_4(y). \tag{4.12}$$

Furthermore through (4.11) and (T) we get a solution $\phi(t)$ of (E) and $\omega_- > -\infty$ from Lemma 3.1. Using the notation of (4.10), we obtain

$$w \to c\#(c = -\rho) \quad \text{as} \quad \eta \to 0.$$

Hence from Corollary 3.2, $\phi(t)$ is represented as (3.8) in the neighborhood of $t = \omega_-$.

Moreover since

$$\lim_{y\to\infty} z_4(y) = -\infty,$$

we get

$$\lim_{y\to 1+0} z_4(y) = 0$$

from Lemma 2.3 and (4.12). Hence we may suppose that $z_4(y)$ is obtained from (R), (2.22) if $a > \psi(t_0)$ and $b = b_4$. Furthermore from (4.12) we have $b_3 > b_4$.

If $b_4 < b < b_3$, then the solution $z(y)$ of (R), (2.22) satisfies

$$z_3(y) < z(y) < z_4(y). \tag{4.13}$$

Hence we obtain

$$\lim_{y\to\infty} z(y) = -\infty.$$

Therefore defining y and $\phi(t)$ through (T) and noting that $(y, z(y))$ is a solution of (4.6), we get an alternative as $t \to \omega_-$ from Lemma 3.1 as follows:

$$\omega_- > -\infty, \quad \lim_{t\to\omega_-} \phi(t) = 0 \tag{4.14}$$

$$\omega_- > -\infty, \quad \lim_{t\to\omega_-} \phi(t) = \infty. \tag{4.15}$$

In case of (4.15) we get a contradiction

$$\lim_{t\to\omega_-} y = \lim_{t\to\omega_-} \lambda^{-2} e^{\alpha\lambda t}\phi(t)^{\alpha} = 0.$$

Next we consider the case (4.14). For this we use

$$\lim_{t\to\omega_-} y^{-3/2} z = \lim_{t\to\omega_-} \frac{\alpha\phi'(t)}{y^{1/2}\phi(t)}. \tag{4.16}$$

Since $\phi''(t) > 0$, $\phi(\omega_-) = 0$, we obtain

$$0 \le \phi'(\omega_-) < \infty.$$

In the case $0 < \phi'(\omega_-) < \infty$, we get

$$\lim_{t\to\omega_-} \frac{\phi'(t)^2}{y\phi(t)^2} = \lim_{t\to\omega_-} \frac{\phi'(t)^2}{\lambda^{-2}e^{\alpha\lambda t}\phi(t)^{\alpha+2}} = \infty.$$

Therefore from (4.16) we have

$$\lim_{t\to\omega_-} w = \lim_{t\to\omega_-} y^{3/2} z^{-1} = 0.$$

This implies that $\phi(t)$ is represented as (3.7) from Corollary 3.2. On the other hand, in the case $\phi'(\omega_-) = 0$ l'Hospital's theorem implies

$$\lim_{t\to\omega_-} \frac{\phi'(t)^2}{y\phi(t)^2} = \frac{2\lambda^2}{\alpha+2}$$

and from (4.16) we obtain

$$\lim_{t\to\omega_-} y^{-3/2}z = -\rho^{-1}.$$

Hence if we put $u = \theta$ where θ is defined as (4.10), then u is a solution of (4.9) with $u(0) = 0$. Since u exists uniquely, we get a contradiction $z(y) \equiv z_4(y)$.

Suppose $b_4 \le b < b_3$. Then we have

$$z_3(y) < z(y) \le z_4(y),$$
$$y \to 1, \quad z(y) \to 0, \quad \frac{z(y)}{y-1} \to \frac{\mu_2}{\alpha} \quad \text{as } t \to \omega_+,$$

since only $z_3(y)$ satisfies

$$\frac{z(y)}{y-1} \to \frac{\mu_1}{\alpha} \quad \text{as } y \to 1.$$

Therefore y, $z(y)$ are represented as (2.9), (2.10) and from Lemma 2.1 we obtain a solution $\phi(t)$ of (E) expressed as (2.3) where $B \neq 0$. Moreover $\omega_+ = \infty$. Now we conclude (i), (ii) of Theorem III.

Next, suppose $b < b_4$. Then we get

$$z(y) > z_4(y). \tag{4.17}$$

Here we consider the case $t \to \omega_-$. If $y \to \infty$, $z(y) \to -\infty$ as $t \to \omega_-$, then in the neighborhood of $y = \infty$

$$y^{-3/2}z(y) < 0.$$

So if we put

$$\eta = \frac{1}{y}, \quad \zeta = \frac{1}{z}, \quad w = \eta^{-3/2}\zeta, \quad \xi = \eta^{1/2}, \tag{4.18}$$

then we have (3.4). Supposing that c is an accumulation point of w as $\xi \to 0$, we obtain

$$c \le -\rho$$

from (4.17). However if $c = -\rho$, then we conclude a contradiction $z(y) \equiv z_4(y)$. Hence we get

$$c < -\rho.$$

From Corollary 3.2, this implies that there exists no solution of (3.4) whose accumulation points contain c as $y \to \infty$ and that

$$y \to \infty, \quad z(y) \to -\infty$$

does not occur.

Therefore from Lemma 2.3, we have $z(y) > 0$ or $z(y)$ becomes a many-valued function such that

$$z(y) = z_+(y) \quad \text{if} \quad z(y) \geq 0, \quad z(y) = z_-(y) \quad \text{if} \quad z(y) \leq 0$$

where for some $\tilde{y}$, $z_-(y)$ is defined on $1 \leq y \leq \tilde{y}$ and $z_+(y)$ on $0 \leq y \leq \tilde{y}$ so that

$$z_+(0) = z_+(\tilde{y}) = z_-(1) = z_-(\tilde{y}) = 0, \quad z_+(y) \geq 0, \quad z_-(y) \leq 0.$$

Indeed if $z \to \gamma$ ($\gamma \neq \pm\infty$) as $y \to \infty$, then from (3.1) we get a contradiction $\eta = 1/y \equiv 0$.

Since $z_\pm(y)$ just defined satisfy the assumption of Lemma 4.1, we define $y(t)$ as in Lemma 4.1. If (ω_-, ω_+) denotes a domain of $y(t)$, then we have

$$\lim_{t \to \omega_-} (y(t), z(y(t))) = (0, 0),$$

since $(y(t), z(y(t)))$ satisfies (4.6). Hence it follows from Lemma 2.9 that (2.21) is obtained for $(y(t), z(y(t)))$. Therefore from Lemma 2.11 we get $\omega_- = -\infty$ and $\phi(t)$ is represented as (2.2) as $t \to -\infty$. Similarly in the case $z(y) > 0$ we have

$$\lim_{y \to 0} z(y) = 0$$

from Lemma 2.3 and hence (2.2) as $t \to -\infty$.

Next we consider the case $t \to \omega_+$. Then there exist the following possibilities:

$$\omega_+ < \infty, \quad \lim_{t \to \omega_+} \phi(t) = 0 \tag{4.19}$$

$$\omega_+ < \infty, \quad \lim_{t \to \omega_+} \phi(t) = \infty \tag{4.20}$$

$$\omega_+ = \infty, \quad \lim_{t \to \omega_+} \phi(t) = 0 \tag{4.21}$$

$$\omega_+ = \infty, \quad 0 < \lim_{t \to \omega_+} \phi(t) < \infty \tag{4.22}$$

$$\omega_+ = \infty, \quad \lim_{t \to \omega_+} \phi(t) = \infty. \tag{4.23}$$

Here we define y and z through (T).

In the case (4.19) we get

$$\lim_{t \to \omega_+} y = \infty.$$

Suppose that

$$z \to \gamma \, (\gamma \neq \pm\infty) \quad \text{as } t \to \omega_+.$$

Then from (3.1) we have a contradiction. Therefore

$$z \to \infty \quad \text{as } t \to \omega_+$$

since if $z < 0$, then $dy/dt < 0$ and $y \to \infty$ as $t \to \omega_+$ is impossible. Now we use (4.18) and transform (R) into (3.4). If c denotes an accumulation point of a solution w of (3.4), then it follows from Corollary 3.2 that in the neighborhood of $t = \omega_+$ we get a solution $\phi(t)$ of (E) represented as (3.8) for $c = 0$ and (3.10) for $c = \rho$. Moreover since we get $z > 0$ and $c \geq 0$, we do not obtain a solution of (E) for $c \neq 0$, ρ. As is shown in the proof of Lemma 3.1 and Corollary 3.2, (3.10) is obtained from the unique holomorphic solution

$$w = \rho + \sum_{n=1}^{\infty} a_n \xi^n$$

of (3.4), namely from a solution

$$z = \rho^{-1} y^{3/2} \left(1 + \sum_{n=1}^{\infty} \tilde{a}_n y^{-n/2}\right)$$

of (R). Existence of this is unique and so we denote this as $z_5(y)$. Furthermore from the unique existence of $z_5(y)$, existence of (3.10) is also unique. So for (3.10) we put

$$\phi'(t_0) = b_5.$$

If (3.8) is got from a solution $z(y)$ of (R), then from $0 < \rho$, we get

$$z_5(y) < z(y).$$

Therefore if we put

$$\phi'(t_0) = b$$

for (3.8), then from (2.23) we have

$$b_5 > b.$$

If the case (4.20), we obtain

$$\lim_{t \to \omega_+} y = 0$$

which is impossible. Moreover if the cases (4.21) and (4.22) occur, we get

$$\lim_{t \to \omega_+} y = \infty.$$

This contradicts Lemma 3.1.

Finally we suppose (4.23). Then if $\phi'(t)$ is bounded as $t \to \omega_+$, we have

$$\lim_{t\to\omega_+} \frac{\phi(t)}{e^{-\lambda t}} = \lim_{t\to\omega_+} \frac{\phi'(t)}{-\lambda e^{-\lambda t}} = 0$$

and

$$\lim_{t\to\omega_+} y = \lim_{t\to\omega_+} \lambda^{-2} \left(\frac{\phi(t)}{e^{-\lambda t}}\right)^{\alpha} = \infty.$$

This implies a contradiction

$$\omega_+ < \infty.$$

If $\phi'(t)$ is unbounded as $t \to \omega_+$, then from l'Hospital's theorem we get

$$\lim_{t\to\omega_+} \frac{\phi(t)}{e^{-\lambda t}} = \lim_{t\to\omega_+} \frac{\phi''(t)}{\lambda^2 e^{-\lambda t}} = \lim_{t\to\omega_+} \frac{1}{\lambda^2} \left(\frac{\phi(t)}{e^{-\lambda t}}\right)^{1+\alpha}.$$

On the other hand, since the orbit of the solution (y, z) of (D) cannot cross the y axis twice and $z(= dy/dt)$ does not vanish twice,

$$\lim_{t\to\omega_+} y$$

exists and hence

$$\lim_{t\to\omega_+} \frac{\phi(t)}{e^{-\lambda t}}$$

does. Therefore this is equal to 0, $\lambda^{2/\alpha}$, ∞ and we get

$$\lim_{t\to\omega_+} y = \infty,\ \ 1,\ \ 0$$

respectively. However

$$\lim_{t\to\omega_+} y = \infty,\ \ 0$$

cannot occur as above. Thus we have

$$\lim_{t\to\omega_+} y = 1.$$

This occurs only if the orbit $z = z(y)$ of (D) gets into the region where $dz/ds > 0$ in the yz plane (cf. Lemma 2.3). Hence we obtain

$$z(y) < z_5(y), \quad b_5 < b(= \phi'(t_0))$$

and (2.3) where $B \neq 0$ from $z(y)$ as above.

Now the case

$$0 < a < \psi(t_0), \quad b < b_1 \text{ or } a = \psi(t_0), \quad b < -a\lambda$$

is left. However from (3.2) the solution $z(y)$ of (R), (2.22) cannot diverge to ∞ as y tends to a finite value. Moreover if $z(y)$ converges to 0 as $y \to 1-0$, then we get the alternative of (2.11) and (2.12) and therefore a contradiction

$$z(y) \leq z_1(y).$$

Thus the present case is reduced to the case

$$a > \psi(t_0), \quad b < b_4$$

and the proof of Theorem III is completed. □

References

[1] R. Bellman: Stability Theory of Differential Equations, McGraw-Hill, New York, 1953.

[2] G.J. Butler: *An integral criterion for the oscillation of a second order sublinear ordinary differential equation*, Indian J. Math. **24** (1982), 1–7.

[3] H. Dulac: *Solutions d'un système d'équations différentielles dans le voisinage de valeurs singulières*, Bull. Soc. Math. Fr. **40** (1912), 324–363.

[4] E. Hille: Ordinary Differential Equations in the Complex Domain, John-Wiley, 1976.

[5] K. Kamo: *Classification of proper solutions of some Emden-Fowler equations*, Hiroshima Math. J. **29** (1999), 459–477.

[6] T. Kusano and C.A. Swanson: *Asymptotic theory of singular semilinear elliptic equations*, Canad. Math. Bull. **27** (1984), 223–232.

[7] M. Naito: *Asymptotic behavior of solutions of second order differential equations with integrable coefficients*, Trans. Amer. Math. Soc. **282** (1984), 577–588.

[8] M. Naito: *Nonoscillatory solutions of second order differential equations with integrable coefficients*, Proc. Amer. Math. Soc. **109** (1990), 769–774.

[9] Z. Nehari: *On a class of nonlinear second-order differential equations*, Trans. Amer. Math. Soc. **95** (1960), 101–123.

[10] T. Saito: *On bounded solutions of $x'' = t^{\beta}x^{1+\alpha}$*, Tokyo J. Math. **1** (1978), 57–75.

[11] T. Saito: *Solutions of $x'' = t^{\alpha\lambda-2}x^{1+\alpha}$ with movable singularity*, Tokyo J. Mah. **2** (1979), 262–283.

[12] R.A. Smith: *Singularities of solutions of certain plain autonomous systems*, Proc. Roy. Soc. Edinburgh Sect. A, **72** (1973/74), 307–315.

[13] S.D. Taliaferro: *Asymptotic behavior of solutions of $y'' = \phi(t)y^{\lambda}$*, J. Math. Anal. Appl. **66** (1978), 95–134.

[14] I. Tsukamoto, T. Mishina and M. Ono: *On solutions of $x'' = e^{\alpha\lambda t}x^{1+\alpha}$*, Keio Sci. Tech. Rep. **35** (1982), 1–36.

[15] I. Tsukamoto: *On solutions of $x'' = -e^{\alpha\lambda t}x^{1+\alpha}$*, Tokyo J. Math. **12** (1989), 181–203.

[16] I. Tsukamoto: *Remarks on the papers "On solutions of $x'' = \pm e^{\alpha\lambda t}x^{1+\alpha}$"*, Funkcial. Ekvac. **37** (1994), 363–371.