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Author(s)	Shimada, Nobuo; Wu, Ching-Mu
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Osaka University

## BORDISM ALGEBRAS OF PERIODIC TRANSFORMATIONS

NOBUO SHIMADA AND CHING-MU WU\*

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For the equivariant bordism groups of  $C^\infty$ -manifolds with differentiable actions of  $S^1=U(1)$  and its subgroups  $Z_n$ , the cases of free actions have been studied by Conner-Floyd [3], Conner [2], Su [11], Uchida [13], Kamata [5, 6] and others.

The purpose of this note is to study the ring structure of bordism for the cases of semi-free actions (cf. Alexander [1], Miščenko [8]).

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### 1. The ring structure of $\mathcal{M}_*(S^i)$ ( $i=1, 3$ ).

It was shown by Conner-Floyd [3] and Uchida [12] that the following sequences are exact (and also split):

$$(1.1) \quad 0 \rightarrow \mathcal{I}_*(Z_2) \xrightarrow{\nu} \mathcal{M}_*(Z_2) \xrightarrow{\partial} \mathcal{N}_*(Z_2) \rightarrow 0,$$

$$(1.2) \quad 0 \rightarrow \mathcal{O}_*(S^1) \xrightarrow{\nu} \mathcal{M}_*(S^1) \xrightarrow{\partial} \Omega_*(S^1) \rightarrow 0,$$

$$(1.3) \quad 0 \rightarrow \mathcal{O}_*(S^3) \xrightarrow{\nu} \mathcal{M}_*(S^3) \xrightarrow{\partial} \Omega_*(S^3) \rightarrow 0,$$

where  $\mathcal{I}_*(Z_2)$  is the bordism group of unoriented manifolds with involution and  $\mathcal{O}_*(S^i)$  ( $i=1, 3$ ) are the bordism groups of oriented manifolds with semi-free  $S^i$ -action. Corresponding to these bordism groups, the cases of free involution and free  $S^i$ -action are denoted by  $\mathcal{N}_*(Z_2)$  and  $\Omega_*(S^i)$  respectively. And  $\mathcal{M}_*(Z_2)=\sum_{k \geq 0} \mathcal{N}_*(BO(k))$ ,  $\mathcal{M}_*(S^1)=\sum_{k \geq 0} \Omega_*(BU(k))$  and  $\mathcal{M}_*(S^3)=\sum_{k \geq 0} \Omega_*(BSp(k))$ .

The above three exact sequences are apparently analogous, and in fact we can study them under a uniform argument.

Let  $F$  denote either one of the fields of real numbers  $R$ , complex numbers  $C$ , or quaternions  $H$ . Let  $d=\dim_R F$ , and let  $FP(n)$  denote the  $n$ -dimensional projective space.

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**Proposition (1.4)** (cf. Ossa [9]) (1) *The bordism groups  $\mathcal{N}_*(Z_2)$  and  $\Omega_*(S^{d-1})$  ( $d-1=1, 3$ ) are free modules over  $\mathcal{N}_*$  and  $\Omega_*$  respectively with generating set  $\{\alpha_{d_{n-1}}=[S^{d_{n-1}}, T_0]; n \geq 1\}$  where  $T_0: S^{d-1} \times S^{d_{n-1}} \rightarrow S^{d_{n-1}}$ , ( $S^{d_{n-1}} \subset F^n$ , the unit sphere), is the usual scalar multiplication.*

(2) *The bordism groups  $\mathcal{M}_*(Z_2) \approx \mathcal{N}_*[\theta_0, \theta_1, \dots]$  and  $\mathcal{M}_*(S^1) \approx \Omega_*[\theta_0, \theta_1, \dots]$  ( $i=1, 3$ ) are the polynomial algebras in  $\theta_0, \theta_1, \dots$  over the Thom bordism rings  $\mathcal{N}_*$  and  $\Omega_*$  respectively, where  $\theta_n=[\eta_n \rightarrow FP(n)]$ , and  $\eta_n$  is the canonical line bundle.*

*Proof.* (1) is well known (cf. [3]). For (2), we shall only show the case of  $\mathcal{M}_*(S^1)$ . The other cases are analogous. The weak direct sum  $\sum_{k \geq 0} H_*(BU(k); Z)$  and  $\mathcal{M}_*(S^1) = \sum_{k \geq 0} \Omega_*(BU(k))$  can be given the structure of graded rings. The multiplications are given by

$$H_i(BU(k); Z) \otimes H_j(BU(l); Z) \rightarrow H_{i+j}(BU(k+l); Z)$$

and

$$\Omega_i(BU(k)) \otimes \Omega_j(BU(l)) \rightarrow \Omega_{i+j}(BU(k+l))$$

which are induced by the Whitney sum map:  $BU(k) \times BU(l) \rightarrow BU(k+l)$ .

The natural map  $\mu: \Sigma \Omega_*(BU(k)) \rightarrow \Sigma H_*(BU(k); Z)$  (Conner Floyd [3]) is then a ring homomorphism.

Let  $a_n = \mu(\theta_n) = \{CP(n)\} \in H_{2n}(CP(\infty); Z) = H_{2n}(BU(1); Z)$  for  $\theta_n = [\eta_n \rightarrow CP(n)] \in \Omega_{2n}(BU(1))$ . Then  $\{a_n; n \geq 0\}$  is an additive base of  $H_*(BU(1); Z)$ .

To show  $\mathcal{M}_*(S^1) \approx \Omega_*[\theta_0, \theta_1, \dots]$ , it suffices to show that  $\{\theta_{i_1} \cdots \theta_{i_k}; 0 \leq i_1 \leq \cdots \leq i_k\}$  is an  $\Omega_*$ -base of  $\Omega_*(BU(k))$ . To see this, it is only necessary to show that  $\{\mu(\theta_{i_1} \cdots \theta_{i_k}) = a_{i_1} \cdots a_{i_k}\}$  is an additive base of  $H_*(BU(k); Z)$  (Conner-Floyd [3], Theorem 18.1). On the other hand, the map  $f: (CP(\infty))^k \rightarrow BU(k)$  induces a monomorphism  $f^*: H^*(BU(k); Z) \rightarrow H^*(CP(\infty))^k; Z$  whose image is the ring of symmetric polynomials in  $x_1, \dots, x_k$  where  $H^*(CP(\infty))^k; Z = Z[x_1, \dots, x_k]$  with  $\deg x_i = 2$ . Let  $s_\omega = \sum x_1^{i_1} \cdots x_k^{i_k}$  (the symmetric sum without repetition) where  $\omega = (i_1, i_2, \dots, i_k)$ ,  $0 \leq i_1 \leq \cdots \leq i_k$ . Then  $\{s_\omega; \omega = (i_1, \dots, i_k), 0 \leq i_1 \leq \cdots \leq i_k\}$  is an additive base of  $f^*(H^*(BU(k); Z) \approx H^*(BU(k); Z)$ . Since  $a_\omega = a_{i_1} \cdots a_{i_k} = f_*\{CP(i_1) \times \cdots \times CP(i_k)\}$ , we can easily obtain the following (Milnor [7]):

$$\begin{aligned} \langle s_{\omega'}, a_\omega \rangle &= \langle x_1^{i_1}, \{CP(i_1)\} \rangle \cdots \langle x_k^{i_k}, \{CP(i_k)\} \rangle \\ &= \begin{cases} 0 & (\omega' \neq \omega), \\ 1 & (\omega' = \omega). \end{cases} \end{aligned}$$

The assertion thus follows.

## 2. The ring structure of $\mathcal{O}_*(S^i)$ ( $i=1, 3$ )

Alexander [1] studied the ring structure of  $\mathcal{J}_*(Z_2)$  by using the exact sequence

(1.1). If we make further use of Proposition (1.4), the ring structures of  $\mathcal{J}_*(\mathbb{Z}_2)$  and  $\mathcal{O}_*(S^i)$  may be determined in a definite form. We shall treat the case of  $\mathcal{O}_*(S^1)$  in the following.

In the exact sequence (1.2)

$$0 \rightarrow \mathcal{O}_*(S^1) \xrightarrow{\nu} \mathcal{M}_*(S^1) \xrightarrow{\partial} \Omega_{*-1}(S^1) \rightarrow 0,$$

$\nu$  is a ring homomorphism defined by  $\nu[M^n, T] = \sum_i [\nu_{F_i} \rightarrow F_i]$  where  $F_i$  is a connected component of the fixed point set  $F_T$  of  $T$  and itself an oriented closed submanifold of  $M^n$ , and  $\nu_{F_i}$  is the normal bundle to  $F_i$  in  $M^n$ . Also  $\partial[\xi \rightarrow M^n] = [S(\xi), T]$  where  $S(\xi) = \partial D(\xi)$  is the boundary of the disk bundle  $D(\xi)$ , and  $T$  is the standard fibre-preserving  $U(1)$ -action. We then have

$$(2.1) \quad \partial\theta_n = [S^{2n+1}, T_0] = \alpha_{2n+1}, \quad \partial\theta_0 = \alpha_{2n-1}.$$

Now, let

$$\sigma_n = [CP(n+1), T], \quad n \geq 1, \quad \text{with } T(s, (z_0, \dots, z_{n+1})) = (sz_0, \dots, sz_n, z_{n+1}).$$

Then

$$(2.2) \quad \nu(\sigma_n) = \theta_n - \theta_0^{n+1} \quad (n \geq 1)$$

holds.

Next, we define an  $\Omega_*$ -map

$$(2.3) \quad \Gamma: \mathcal{O}_*(S^1) \rightarrow \mathcal{O}_{*+2}(S^1)$$

as follows (Conner-Floyd [3], p.119). If  $T_0$  is the standard  $S^1$ -action on  $D^2$ , then for a manifold  $(M^n, T)$  with a semi-free  $S^1$ -action  $T$ , we form a manifold  $(\tilde{M}^{n+2}, \tilde{T})$  from  $(-D^2 \times M^n, T_0 \times 1)$  and  $(D^2 \times M^n, T_0 \times T)$  by identifying the boundaries via the equivariant diffeomorphism  $\varphi: (S^1 \times M^n, T_0 \times 1) \rightarrow (S^1 \times M^n, T_0 \times T)$  which is defined by  $\varphi(s, x) = (s, sx)$ . We then define  $\Gamma$  by

$$(2.4) \quad \Gamma(M^n, T) = (\tilde{M}^{n+2}, \tilde{T}) = (-D^2 \times M^n, T_0 \times 1) \cup_{\varphi} (D^2 \times M^n, T_0 \times T)_2.$$

Since the fixed point set of  $\tilde{T}$  is  $F_{\tilde{T}} = ((0) \times M^n)_1 \cup ((0) \times F_T)_2$ , we have

$$(2.5) \quad \nu[\tilde{M}^{n+2}, \tilde{T}] = \nu[M^n, T] \cdot \theta_0 - [M^n] \cdot \theta_0.$$

By using the following notations

$$\begin{aligned} \iota: \mathcal{M}_n(S^1) &\rightarrow \mathcal{M}_{n+2}(S^1), \quad \iota(x) = x \cdot \theta_0, \\ \varepsilon: \mathcal{O}_n(S^1) &\rightarrow \Omega_n, \quad \varepsilon[M^n, T] = [M^n], \\ \tau: \Omega_n &\rightarrow \mathcal{M}_{n+2}(S^1), \quad \tau[M^n] = [M^n] \cdot \theta_0. \end{aligned}$$

we can express (2.5) in the form

$$(2.6) \quad \nu\Gamma = \iota\nu - \tau\varepsilon.$$

**Lemma (2.7).** (cf. Alexander [1])

$$\begin{aligned}\Gamma(ab) &= \Gamma(a)b + \varepsilon(a)\Gamma(b), \\ &= a\Gamma(b) + \varepsilon(b)\Gamma(a) \text{ for } a, b \in \mathcal{O}_*(S^1).\end{aligned}$$

*Proof.* It follows easily from (1.2) and (2.6).

**Theorem (2.8).** *The bordism group  $\mathcal{O}_*(S^1)$  is a free  $\Omega_*$ -module with generating set  $\{\Gamma^l(\sigma_{j_1} \cdots \sigma_{j_k}); l \geq 0, 1 \leq j_1 \leq \cdots \leq j_k\} \cup \{1\}$ . Its ring structure is then given as the quotient of the polynomial algebra  $\Omega_*[\Gamma^l(\sigma_j); l \geq 0, j \geq 1]$  by the ideal generated by*

$$\begin{aligned}\Gamma^l(\sigma_j) \cdot \Gamma^{m+1}(\sigma_k) - \Gamma^m(\sigma_k) \cdot \Gamma^{l+1}(\sigma_j) - \varepsilon \Gamma^l(\sigma_j) \cdot \Gamma^{m+1}(\sigma_k) \\ + \varepsilon \Gamma^m(\sigma_k) \cdot \Gamma^{l+1}(\sigma_j) \quad (l, m \geq 0; j, k \geq 1).\end{aligned}$$

*Similarly for  $\mathcal{O}_*(S^3)$ .*

*Proof.* Turning our attention to (2.6), we see that if the monomials of  $\mathcal{M}_*(S^1)$  are given a suitable order, then

$$\nu(\Gamma^l(\sigma_{j_1} \cdots \sigma_{j_k})) = \theta_0^l \theta_{j_1} \cdots \theta_{j_k} + \text{lower terms}$$

holds. Since  $\{\theta_0^l \theta_{j_1} \cdots \theta_{j_k}\}$  is an  $\Omega_*$ -base of  $\mathcal{M}_*(S^1)$ , the first half of the theorem follows.

For the second half, we first observe that  $\{\Gamma^l(\sigma_j); l \geq 0, j \geq 1\}$  forms a generating set of the  $\Omega_*$ -algebra  $\mathcal{O}_*(S^1)$  in virtue of (2.7). Then the assertion of the theorem can be verified easily by comparing the ranks of the associated graded modules of the suitably filtered modules in question.

### 3. On the ring structure of $\mathcal{O}_*(Z_3)$

Let  $p$  be an odd prime. We treat here the case of  $Z_p$ -action, particularly the case  $p=3$ , in this section. We already have the following exact sequence (Conner [2], Wu [14]):

$$(3.1) \quad 0 \rightarrow \Omega_* \xrightarrow{i_*} \mathcal{O}_*(Z_p) \xrightarrow{\nu} \mathcal{M}_*(Z_p) \xrightarrow{\partial} \tilde{\Omega}_*(Z_p) \rightarrow 0$$

where  $\mathcal{O}_*(Z_p)$ , similarly in the previous section, denotes the bordism group of semi-free  $Z_p$ -action, and  $\mathcal{M}_*(Z_p) = \mathcal{M}^*(S^1) \otimes_{\Omega_*} \cdots \otimes_{\Omega_*} \mathcal{M}_*(S^1)$  ( $(p-1)/2$ -fold tensor product over  $\Omega_*$ ). The reduced group  $\tilde{\Omega}^*(Z_p) = \text{Ker } \bar{\varepsilon}$ , where  $\Omega_*(Z_p)$  is the bordism group of free  $Z_p$ -action, and  $\bar{\varepsilon}: \Omega_*(Z_p) \rightarrow \Omega_*$  is defined by  $\bar{\varepsilon}[M^n, \tau] = [M^n/\tau]$ . The homomorphism  $i_*: \Omega_* \rightarrow \mathcal{O}_*(Z_p)$  is defined by  $i_*[M^n] = [M^n] \cdot \mu_0 = [M^n \times Z_p, 1 \times \sigma]$ , where  $\mu_0 = [Z_p, \sigma] \in \mathcal{O}_0(Z_p)$ . And the homomorphisms  $\nu$  and  $\partial$  are to be analogously defined as in the previous section.

For the sake of simplicity, we consider only the case of  $p=3$  in the following. We have the following commutative diagram (Wu [14]):

$$(3.2) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathcal{O}_*(S^1) & \xrightarrow{\nu} & \mathcal{M}_*(S^1) & \xrightarrow{\partial} & \Omega_*(S^1) \rightarrow 0 \\ & & \downarrow \lambda & & \approx \downarrow \bar{\lambda} & & \downarrow \bar{\bar{\lambda}} \\ 0 & \rightarrow & \Omega_* \xrightarrow{i_*} \mathcal{O}_*(Z_3) & \xrightarrow{\nu} & \mathcal{M}_*(Z_3) & \xrightarrow{\partial} & \tilde{\Omega}_*(Z_3) \rightarrow 0 \end{array}$$

where each vertical map  $\lambda$  is the natural map obtained by restricting the corresponding  $S^1$ -action to its subgroup  $Z_3$ . We then have the following results (Conner-Floyd [3], §36 and §46):

- 1) The map  $\bar{\bar{\lambda}}: \Omega_*(S^1) \rightarrow \tilde{\Omega}_*(Z_3)$  is an epimorphism.
- 2)  $\Omega_*(S^1) \approx \Omega_{*-1}(CP(\infty))$  is a free  $\Omega_*$ -module generated by  $\{\alpha_{2n-1} = [S^{2n-1}, T]; n \geq 1\}$ .
- 3) There is a sequence of oriented closed manifolds,  $M^4, M^8, \dots, M^{4k}, \dots$  such that if we put

$$(3.3) \quad \beta_{2n-1} = 3\alpha_{2n-1} + [M^4]\alpha_{2n-5} + [M^8]\alpha_{2n-9} + \dots, (n \geq 1),$$

then  $\{\beta_{2n-1}; n \geq 1\}$  constitutes a generating set for  $K = \text{Ker } \bar{\bar{\lambda}}$ .

Now, put

$$(3.4) \quad \bar{\beta}_n = 3\theta_0^n + [M^4]\theta_0^{n-2} + [M^8]\theta_0^{n-4} + \dots, (n \geq 1),$$

and identify  $\mathcal{M}_*(Z_3)$  with  $\mathcal{M}_*(S^1)$  by the isomorphism  $\bar{\lambda}$ . Then  $\bar{\beta}_n$  is in the kernel of  $\partial: \mathcal{M}_*(Z_3) \rightarrow \tilde{\Omega}_*(Z_3)$  for each  $n \geq 1$ :

$$(3.5) \quad \partial(\bar{\beta}_n) = 0.$$

Therefore, from the exact sequence (3.1), there exists  $\mu_n \in \mathcal{O}_*(Z_3)$  such that  $\nu(\mu_n) = \bar{\beta}_n$  for each  $n \geq 1$ . We thus obtain the following theorem (Wu [14]).

**Theorem (3.6).**  $\mathcal{O}_*(Z_3)$  is isomorphic as a free  $\Omega_*$ -module to the direct sum of  $\Omega_*\{\mu_0, \mu_1, \dots\}$  and  $\lambda(\mathcal{O}_*(S^1))$ .

We go on to study the multiplicative structure of  $\mathcal{O}_*(Z_3)$ . It is evident from the previous arguments that  $\{\mu_k (k \geq 0), \Gamma^l(\sigma_j) (l \geq 0, j \geq 1)\}$  can be taken as a generating set of  $\Omega_*$ -algebra  $\mathcal{O}_*(Z_3)$  where  $\lambda(\Gamma^l(\sigma_j))$  is simply denoted by  $\Gamma^l(\sigma_j)$ . The map  $\lambda: \mathcal{O}_*(S^1) \rightarrow \mathcal{O}_*(Z_3)$  is a ring isomorphism of  $\mathcal{O}_*(S^1)$  into  $\mathcal{O}_*(Z_3)$ , and  $\text{Im } \lambda \approx \mathcal{O}_*(S^1)$  is a subalgebra of  $\mathcal{O}_*(Z_3)$ . Also,  $\text{Im } i_* = \text{Ker } \nu = \Omega_* \cdot \mu_0$  ( $\mu_0 = [Z_3, \sigma]$ ) is an ideal of  $\mathcal{O}_*(Z_3)$ . In fact, if we let  $\varepsilon: \mathcal{O}_*(Z_3) \rightarrow \Omega_*$  be defined by  $\varepsilon[M^n, \tau] = [M^n]$ , we have

$$(3.7) \quad \mu_0 \cdot a = \varepsilon(a)\mu_0 \quad \text{for } a \in \mathcal{O}_*(Z_3).$$

We next have to appropriately choose and fix  $\mu_n (n \geq 1)$  so as to study the relations among them. First, let

$$(3.8) \quad \mu_1 = [M^2, \tau_1],$$

where  $M^2$  is the algebraic curve  $z_0^3 + z_1^3 + z_2^3 = 0$  in  $CP(2)$  which is non-singular and of genus 1. The action  $\tau_1$  is defined by  $\tau_1(z_0, z_1, z_2) = (z_0, z_1, \rho z_2)$  with  $\rho = \exp(2\pi i/3)$ . The fixed points of  $\tau_1$  are  $(1, -1, 0)$ ,  $(\rho, -1, 0)$  and  $(\rho^2, -1, 0)$ , (Conner [2]), so we have

$$(3.9) \quad \nu(\mu_1) = 3\theta_0.$$

Next, let

$$(3.10) \quad \mu_2 = [CP(2), \tau_2]$$

where  $\tau_2$  is defined by  $\tau_2(z_0, z_1, z_2) = (z_0, \rho z_1, \rho^2 z_2)$ .

Since the fixed points of  $\tau_2$  are  $(1, 0, 0)$ ,  $(0, 1, 0)$  and  $(0, 0, 1)$ , we have

$$(3.11) \quad \nu(\mu_2) = 3\theta_0^2.$$

Before determining appropriate  $\mu_n (n \geq 3)$ , we need some preparation. First note that for an oriented closed manifold  $(M^n, T)$  with any  $S^1$ -action  $T$ , not necessarily semi-free, we can define also the  $\Gamma$ -operation just as in §2;

$$(3.12) \quad \Gamma[M^n, T] = [\tilde{M}^{n+2}, \tilde{T}]$$

which is an operation on the bordism group of all  $S^1$ -actions.

We may also define the manifold with  $Z_3$ -action  $\lambda(T)$  (the restriction of an  $S^1$ -action  $T$ ) as follows:

$$(3.13) \quad \lambda(M^n, T) = (M^n, \lambda(T)).$$

Since the fixed point set of  $\lambda(\tilde{T})$  for  $(\tilde{M}^{n+2}, \lambda(\tilde{T}))$  is  $F_{\lambda(\tilde{T})} = ((0) \times M^n)_1 \cup ((0) \times F_{\lambda(T)})_2$ , we have

$$(3.14) \quad \nu[\tilde{M}^{n+2}, \lambda(\tilde{T})] = \nu[M^n, \lambda(T)] \cdot \theta_0 - [M^n] \cdot \theta_0,$$

or equivalently,

$$(3.15) \quad \nu\lambda\Gamma = \nu\lambda - \tau\epsilon\lambda.$$

We thus have, by induction, the formula

$$(3.16) \quad \nu\lambda\Gamma^n = \nu\lambda - \sum_{i=0}^{n-1} \nu\lambda - i\tau\epsilon\lambda\Gamma^i.$$

Consider now the manifold

$$(3.17) \quad (CP(2), T_2)$$

where  $T_2$  is an  $S^1$ -action defined by  $T_2(e^{i\theta}, (z_0, z_1, z_2)) = (z_0, e^{i\theta}z_1, e^{2i\theta}z_2)$ . It is seen that

$$(3.18) \quad \lambda[CP(2), T_2] = [CP(2), \tau_2] = \mu_2.$$

We then define

$$(3.19) \quad \mu_n = \lambda\Gamma^{n-2}[CP(2), T_2], (n \geq 2).$$

It now follows from (3.16) that

$$(3.20) \quad \begin{aligned} \nu(\mu_n) &= 3\theta_0^n - \sum_{i=0}^{n-3} \varepsilon(\mu_{i+2})\theta_0^{n-2-i} \\ &= 3\theta_0^n - \sum_{i=2}^{n-1} \varepsilon(\mu_i)\theta_0^{n-i}. \end{aligned}$$

This may a little differ from the condition  $\nu(\mu_n) = \bar{\beta}_n$  of (3.6). However, it does not matter, because  $\{\bar{\beta}_n; n \geq 1\}$  is still a free base of the kernel of  $\partial: \Omega_*[\theta_0] \rightarrow \bar{\Omega}_*(Z_3)$  when we take for  $\bar{\beta}_n$  the right-hand side of (3.20) instead of (3.4).

From the above definition of  $\mu_n$ , we first obtain the following relations,

$$(3.21) \quad \begin{aligned} \mu_1^2 &= 3\mu_2 - \varepsilon(\mu_2)\mu_0, \quad \mu_1\mu_2 = 3\mu_3 + \varepsilon(\mu_2)\mu_1, \\ \mu_1\mu_n &= 3\mu_{n+1} + \varepsilon(\mu_n)\mu_1 - \varepsilon(\mu_{n+1})\mu_0, \quad (n \geq 1), \\ \mu_2\mu_n &= 3\mu_{n+2} + \varepsilon(\mu_n)\mu_2 + \varepsilon(\mu_{n+1})\mu_1 - \varepsilon(\mu_{n+2})\mu_0, \quad (n \geq 2) \end{aligned}$$

which can be proved by operating  $\nu$  and  $\varepsilon$  on both sides of the equations. Here notice that  $\varepsilon|_{\text{Ker } \nu}$  is a monomorphism,  $\varepsilon(\mu_1) = 0$  and  $\varepsilon(\mu_3) = 0$ .

Moreover, we have

$$(3.22) \quad \begin{aligned} \mu_1\Gamma^l(\sigma_i) &= 3\Gamma^{l+1}(\sigma_i) + \varepsilon(\Gamma^l(\sigma_i))\mu_1 - \varepsilon(\Gamma^{l+1}(\sigma_i))\mu_0, \\ \mu_2\Gamma^l(\sigma_i) &= 3\Gamma^{l+2}(\sigma_i) + \varepsilon(\Gamma^l(\sigma_i))\mu_2 + \varepsilon(\Gamma^{l+2}(\sigma_i))\mu_1 - \varepsilon(\Gamma^{l+2}(\sigma_i))\mu_0 \end{aligned}$$

We have finally the relations among  $\mu_n$  for  $n \geq 3$  and  $\Gamma^l(\sigma_i)$  as follows:

$$(3.23) \quad \begin{aligned} \mu_n\mu_m &= \mu_{n-1}\mu_{m+1} + \varepsilon(\mu_m)\mu_n - \varepsilon(\mu_{n-1})\mu_{m+1}, \\ \mu_n\Gamma^l(\sigma_i) &= \mu_{n-1}\Gamma^{l+1}(\sigma_i) + \varepsilon(\Gamma^l(\sigma_i))\mu_n - \varepsilon(\mu_{n-1})\Gamma^{l+1}(\sigma_i), \quad (n \geq 3). \end{aligned}$$

The relations (3.23) could be also derived from (2.7), if we formally put  $\mu_n = \Gamma(\mu_{n-1})$ . Hence the multiplicative structure of  $\mathcal{O}_*(Z_3)$  is essentially ruled by (2.7), except for (3.7), (3.21) and (3.22).



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