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ON AN EXPONENTIAL CHARACTER OF THE SPECTRAL DISTRIBUTION FUNCTION OF A RANDOM DIFFERENCE OPERATOR

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1. Let H^0 be a second order difference operator

$$(H^0u)(a) = \frac{\sigma^2}{2} \{u(a-1) - 2u(a) + u(a+1)\}, \quad a \in Z,$$

u being a function on the space Z of all integers. We then consider a random difference operator H^ω defined by

$$(H^\omega u)(a) = -(H^0u)(a) + q(a, \omega)u(a), \quad a \in Z,$$

where $\{q(a, \omega)\}_{a \in Z}$ is a family of random variables defined on a probability space (Ω, \mathcal{B}, P) .

We assume that $\{q(a, \omega)\}_{a \in Z}$ forms a non-negative valued stationary Markov process with one step transition function $P(x, A)$ and absolute probability $\mu(A)$:

$$\begin{aligned} &P(q(a_1) \in A_1, q(a_2) \in A_2, \dots, q(a_n) \in A_n) \\ &= \int_{A_1 \dots A_n} \mu(dx_1) P^{(a_2 - a_1)}(x_1, dx_2) P^{(a_3 - a_2)}(x_2, dx_3) \dots P^{(a_n - a_{n-1})}(x_{n-1}, dx_n) \end{aligned}$$

for integers $a_1 < a_2 < \dots < a_n$ and Borel set A_1, A_2, \dots, A_n of $[0, \infty)$. Here $P^{(k)}(x, A)$ denotes the k -th iterate of $P(x, A)$.

Denote by $L^2(Z)$ the Hilbert space consisting of all square summable functions with inner product $(u, v) = \sum_{a \in Z} u(a)v(a)$. For each $\omega \in \Omega$, H^ω determines a selfadjoint operator A^ω by

$$\begin{cases} \mathcal{D}(A^\omega) = \{u \in L^2(Z); H^\omega u \in L^2(Z)\} \\ A^\omega u = H^\omega u \quad u \in \mathcal{D}(A^\omega). \end{cases}$$

Let $\{E_\lambda^\omega, \lambda \in R^1\}$ be the resolution of the identity associated with A^ω . Then $(E_\lambda^\omega I_0, I_0)$ is measurable in ω and we can define the spectral distribution function ρ of $\{H^\omega\}$ by

$$\rho(\lambda) = E((E_\lambda^\omega I_0, I_0))$$

where E is the expectation in ω with respect to P and $I_0(a) = \delta_{0a}$, $a \in Z$ ([1]). $\rho(\lambda)$ vanishes for $\lambda < 0$. Our present aim is to prove the following theorem.

Theorem.

- (i) If $P(0, \{0\}) = b > 0$ and $\mu(\{0\}) > 0$, then $\lim_{x \downarrow 0} \sqrt{x} \log \rho(x) > -\infty$.
- (ii) If $\int_0^\infty \frac{1}{1+y} P(x, dy) < c$ μ -a.e. x for some $c < 1$, then $\overline{\lim}_{x \downarrow 0} \sqrt{x} \log \rho(x) < 0$.

A similar result has been obtained by M. Fukushima ([1]) when $q(a)$, $a \in Z$, are non-negative valued independent identically distributed random variables. We further mention the works of L. A. Pastur ([2]) and S. Nakao ([3]) for related results on the one dimensional Schrödinger operators with random potentials. The present novelty is to make use of a Markovian character of the local time (cf. M. L. Silverstein [4]).

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2. At first we collect some lemmas for the proof of our theorem.

We introduce the continuous Markov process $M = (\dot{\Omega}, \mathcal{B}, \dot{X}_t, \dot{P}_a)$ on Z with the generator H^0 . Denoting by \dot{E}_0 the expectation with respect to \dot{P}_0 , we have Kac representation as follows.

Lemma 1 ([1]).

$$\int_0^\infty e^{-\lambda t} d\rho(\lambda) = E \times \dot{E}_0 \left[\exp \left(- \int_0^t q(\dot{X}_s, \omega) ds \right); \dot{X}_t = 0 \right].$$

The proof of our theorem reduces to finding how fast $E \times \dot{E}_0 \left[\exp \left(- \int_0^t q(\dot{X}_s, \omega) ds \right); \dot{X}_t = 0 \right]$ tends to zero as $t \rightarrow \infty$ because of Lemma 1 and the following Tauberian theorem.

Lemma 2 ([1]). Let $\phi(\lambda)$ be non-decreasing function on $[0, \infty)$ with $\phi(0) = 0$ and $\psi(t)$ be its Laplace transform:

$$\psi(t) = \int_0^\infty e^{-t\lambda} d\phi(\lambda)$$

- (i) If $\lim_{t \uparrow \infty} \frac{1}{t^\gamma} \log \psi(t) > -\infty$ then $\lim_{x \downarrow 0} x^{\gamma/1-\gamma} \log \phi(x) > -\infty$.
- (ii) If $\overline{\lim}_{t \uparrow \infty} \frac{1}{t^\gamma} \log \psi(t) < 0$ then $\overline{\lim}_{x \downarrow 0} x^{\gamma/1-\gamma} \log \phi(x) < 0$.

For the investigation of asymptotic behavior of $E \times \dot{E}_0 \left[\exp \left(- \int_0^t q(\dot{X}_s, \omega) \right); \right]$

$\dot{X}_t=0$] as $t \rightarrow \infty$, the following two lemmas are of great use.

Lemma 3 ([4]). *Put $\dot{L}(t, x) = \int_0^t I_x(\dot{X}_s) ds$, $t > 0$, $x \in Z$, $\sigma_x(s) = \sup\{t; \dot{L}(t, x) \leq s$, $\dot{T}_{x,y}(s) = \dot{L}(\sigma_x(s), y)$, then it holds that*

(i) $\{\dot{T}_{y,z}(s), y, z \geq x\}$ and $\{\dot{T}_{y,z}(s), y, z \leq x\}$ are mutually independent for each $x \in Z$ and $s \geq 0$,

(ii) $\dot{E}_0[\exp(-\alpha \dot{T}_{x,x-y-z}(s)) | \dot{T}_{x,x-y}(s) = l] = \frac{1}{\alpha z + 1} \exp\left(\frac{-l\alpha}{\alpha z + 1}\right)$ for each $\alpha > 0$,

$x > 0 (< 0)$ and $y, z \geq 0$ such that $y + z \leq x (\leq -x)$,

(iii) $\{\dot{T}_{x,x-y}(s)\}$ is Markovian in $y \geq 0$ for fixed $s \geq 0$ and $x > 0 (< 0)$.

Corollary *Put $\mathcal{B}_{x-u}^x(s) = \sigma[\dot{T}_{x,x-u}(s), v = 0, 1, \dots, u]$ $s \geq 0$, then we have*

$$\dot{E}_0[\exp(-\alpha \dot{T}_{x,x-y-z}(s)) | \mathcal{B}_{x-y}^x(s)] \leq \frac{1}{\alpha z + 1}$$

for each $x > 0 (< 0)$ and $y, z \geq 0$ such that $y + z \leq x (y + z \leq -x)$.

Lemma 4 ([1]). *Let \dot{R}_t be the number of states where \dot{X}_s visits during the interval $[0, t]$, then we have*

(i) $\liminf_{t \uparrow \infty} t^{-1/3} \log \dot{E}_0[e^{-\beta_1 \dot{R}_t}; \dot{X}_t = 0] > -\infty$

for any positive constant $\beta_1 > 0$,

(ii) $\limsup_{t \uparrow \infty} t^{-1/3} \log \dot{E}_0[e^{-\beta_2 \dot{R}_t}] < 0$

for any positive constant $\beta_2 > 0$.

3. Now we give the proof of our theorem.

Put $k(t) = E \times \dot{E}_0 \left[\exp\left(-\int_0^t q(\dot{X}_s, \omega) ds\right); \dot{X}_t = 0 \right]$, then

$$k(t) = \sum_{k=0}^{\infty} \sum_{\substack{m+n=k \\ m, n \geq 0}} E \times \dot{E}_0 \left[\exp\left(-\sum_{x=-n}^m \dot{L}(t, x) q(x, \omega)\right); \dot{M}_t = m, \dot{m}_t = -n, \dot{X}_t = 0 \right]$$

where $\dot{M}_t = \sup\{\dot{X}_s; 0 \leq s \leq t\}$ $\dot{m}_t = \inf\{\dot{X}_s; 0 \leq s \leq t\}$. Taking the expectation of $\exp\left(-\sum_{x=-n}^m \dot{L}(t, x) q(x, \omega)\right)$ with respect to P , we have by stationarity and Markov property of $(\Omega, \mathcal{B}, P, q)$

$$E \left[\exp\left(-\sum_{x=-n}^m \dot{L}(t, x) q(x, \omega)\right) \right] \geq E \left[\prod_{x=-n}^m I_0(q(x, \omega)) \right] = \mu(\{0\}) b^{m+n}.$$

therefore

$$\begin{aligned}
k(t) &\geq \sum_{k=0}^{\infty} \sum_{\substack{m+n=k \\ m, n \geq 0}} \dot{E}_0[\mu(\{0\})b^k; \dot{M}_i = \dot{m}, m_i = -n, \dot{X}_i = 0] \\
&= \frac{\mu(\{0\})}{b} \sum_{k=1}^{\infty} b^k \dot{P}_0(\dot{R}_i = k, \dot{X}_i = 0) = \frac{\mu(\{0\})}{b} \dot{E}_0[e^{-\beta_1 \dot{R}_i}; \dot{X}_i = 0], \\
\beta_1 &= -\log b.
\end{aligned}$$

Because of Lemma 4

$$\lim_{t \uparrow \infty} t^{-1/3} \log k(t) > -\infty.$$

We get the first assertion (i) of our theorem by Lemma 1 and Lemma 2.

Turning to the proof of the second assertion (ii), we put

$$\begin{aligned}
k_1(t) &= E \times \dot{E}_0 \left[\exp \left(- \int_0^t q(\dot{X}_s, \omega) ds \right); \dot{X}_i = 0, \dot{R}_i < t \right], \\
k_2(t) &= E \times \dot{E}_0 \left[\exp \left(- \int_0^t q(\dot{X}_s, \omega) ds \right); \dot{X}_i = 0, \dot{R}_i \geq t \right],
\end{aligned}$$

then we get

$$\begin{aligned}
k_1(t)^2 &= \left\{ \sum_{k=0}^{[t]-1} \sum_{\substack{m+n=k \\ m, n \geq 0}} E \times \dot{E}_0 \left[\exp \left(- \sum_{x=-n}^m \dot{L}(t, x) q(x, \omega) \right); \dot{M}_i = m, \dot{m}_i = -n, \dot{X}_i = 0 \right] \right\}^2 \\
&\leq \frac{[t]([t]+1)}{2} \sum_{k=0}^{[t]-1} \sum_{\substack{m+n=k \\ m, n \geq 0}} \{ E \times \dot{E}_0 \left[\exp \left(- \sum_{x=-n}^m \dot{L}(t, x) q(x, \omega) \right); \right. \\
&\quad \left. \dot{M}_i = m, \dot{m}_i = -n, \dot{X}_i = 0 \right] \}^2 \\
&\leq \frac{[t]([t]+1)}{2} \sum_{k=0}^{[t]-1} \sum_{\substack{m+n=k \\ m, n \geq 0}} E \times \dot{E}_0 \left[\exp \left(-2 \sum_{x=-n}^m \dot{L}(t, x) q(x, \omega) \right) \right. \\
&\quad \left. \dot{P}_0[\dot{M}_i = m, \dot{m}_i = -n, \dot{X}_i = 0] \right].
\end{aligned}$$

Putting $\tau_i = t \wedge \inf \{s; \dot{X}_s = i\}$, $i \in Z$, it is clear that

$$\begin{aligned}
\dot{L}(\tau_m, x) &\leq \dot{L}(t, x), & \dot{L}(\tau_{-n}, x) &\leq \dot{L}(t, x) & \text{and} \\
\dot{L}(\tau_m, x) &= \dot{T}_{m,x}(0), & \dot{L}(\tau_{-n}, x) &= \dot{T}_{-n,x}(0).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\exp \left(-2 \sum_{x=-n}^m \dot{L}(t, x) q(x, \omega) \right) \\
&\leq \exp \left(-2 \sum_{x=0}^m \dot{T}_{m,x}(0) q(x, \omega) - 2 \sum_{x=-n}^{-1} \dot{T}_{-n,x}(0) q(x, \omega) \right).
\end{aligned}$$

Taking the expectation with respect to $P \times \dot{P}_0$, we get

$$E \times \dot{E}_0 \left[\exp \left(-2 \sum_{x=-n}^m \dot{L}(t, x) q(x, \omega) \right) \right] \leq E \left[\prod_{x=-n}^m \frac{1}{1 + 2q(x, \omega)} \right]$$

because of Lemma 3, (i) (iii) and its Corollary. From Markov property of $(\Omega, \mathcal{B}, P, q)$ and the assumption in our theorem, it follows that

$$\begin{aligned} E\left[\prod_{x=-n}^m \frac{1}{1+2q(x, \omega)}\right] &= E\left[\prod_{x=-n}^{m-1} \frac{1}{1+2q(x, \omega)} E\left[\frac{1}{1+2q(1, \omega)} \middle| q(0)\right]\right] \\ &< c E\left[\prod_{x=-n}^{m-1} \frac{1}{1+2q(x, \omega)}\right] < c^{m+n+1}. \end{aligned}$$

Now we have

$$\begin{aligned} k_1(t)^2 &\leq \frac{[t]([t]+1)}{2} \sum_{\substack{k=0 \\ m+n=k \\ m, n \geq 0}}^{[t]-1} c^{m+n+1} \dot{P}_0[\dot{X}_t=0, \dot{M}_t=m, \dot{m}_t=-n] \\ &= \frac{[t]([t]+1)}{2} \sum_{k=1}^{[t]} c^k \dot{P}_0(\dot{X}_t=0, \dot{R}_t=k) \\ &\leq \frac{[t]([t]+1)}{2} \dot{E}_0(e^{-\beta_2 \dot{R}_t}; \dot{X}_t=0), \beta_2 = -\log c. \end{aligned}$$

On the other hand

$$\begin{aligned} k_2(t) &\leq E \times \dot{E}_0 \left[\exp\left(-\int_0^t q(\dot{X}_s, \omega) ds; \dot{X}_t=0, \dot{M}_t \geq \frac{t}{2}\right) \right] \\ &\quad + E \times \dot{E}_0 \left[\exp\left(-\int_0^t q(\dot{X}_s, \omega); \dot{X}_t=0, \dot{m}_t \leq -\frac{t}{2}\right) \right] \\ &\leq E \times \dot{E}_0 \left[\exp\left(-\sum_{x=0}^{[t/2]} \dot{L}(\tau_{[t/2]}, x) q(x, \omega)\right) \right] \\ &\quad + E \times \dot{E}_0 \left[\exp\left(-\sum_{x=-1}^{-[t/2]} \dot{L}(\tau_{-[t/2]}, x) q(x, \omega)\right) \right] \\ &\leq E \left[\prod_{x=0}^{[t/2]} \frac{1}{1+q(x, \omega)} \right] + E \left[\prod_{x=-1}^{-[t/2]} \frac{1}{1+q(x, \omega)} \right] < 2c^{[t/2]} \\ &= 2e^{-\beta_2 [t/2]}. \end{aligned}$$

As a result

$$k(t) = k_1(t) + k_2(t) \leq \sqrt{\frac{[t]([t]+1)}{2} \dot{E}_0(e^{-\beta_2 \dot{R}_t})} + 2e^{-\beta_2 [t/2]},$$

which, combined with Lemma 4, leads us to

$$\overline{\lim}_{t \uparrow \infty} t^{-1/3} \log k(t) < 0.$$

Hence we arrive at the second assertion of our theorem.

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