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ON AN EXPONENTIAL CHARACTER OF THE SPECTRAL DISTRIBUTION FUNCTION OF A RANDOM DIFFERENCE OPERATOR

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1. Let H^0 be a second order difference operator

$$(H^{0}u)(a) = \frac{\sigma^{2}}{2} \{u(a-1)-2u(a)+u(a+1)\}, a \in \mathbb{Z},$$

u being a function on the space Z of all integers. We then consider a random difference operator H^{ω} defined by

$$(H^{\omega}u)(a) = -(H^{\circ}u)(a) + q(a, \omega)u(a), \quad a \in \mathbb{Z},$$

where $\{q(a, \omega)\}_{a \in \mathbb{Z}}$ is a family of random variables defined on a probability space (Ω, \mathcal{B}, P) .

We assume that $\{q(a, \omega)\}_{a \in \mathbb{Z}}$ forms a non-negative valued stationary Markov process with one step transition function P(x, A) and absolute probability $\mu(A)$:

$$P(q(a_1) \in A_1, q(a_2) \in A_2, \dots, q(a_n) \in A_n)$$

= $\int_{A_1 \dots A_n} \mu(dx_1) P^{(a_2 - a_1)}(x_1, dx_2) P^{(a_3 - a_2)}(x_2, dx_3) \dots P^{(a_n - a_{n-1})}(x_{n-1}, dx_n)$

for integers $a_1 < a_2 < \cdots < a_n$ and Borel set A_1, A_2, \cdots, A_n of $[0, \infty)$. Here $P^{(k)}(x, A)$ denotes the k-th iterate of P(x, A).

Denote by $L^2(Z)$ the Hilbert space consisting of all square summable functions with inner product $(u, v) = \sum_{a \in Z} u(a)v(a)$. For each $\omega \in \Omega$, H^{ω} determines a selfadjoint operator A^w by

Let $\{E_{\lambda}^{\omega}, \lambda \in \mathbb{R}^{1}\}$ be the resolution of the identity associated with A^{ω} . Then $(E_{\lambda}^{\omega}I_{0}, I_{0})$ is measurable in ω and we can define the spectral distribution function ρ of $\{H^{\omega}\}$ by

$$\rho(\lambda) = E((E_{\lambda}^{\omega}I_{0}, I_{0}))$$

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where E is the expectation in ω with respect to P and $I_0(a) = \delta_{0a}$, $a \in \mathbb{Z}$ ([1]). $\rho(\lambda)$ vanishes for $\lambda < 0$. Our present aim is to prove the following theorem.

Theorem.

- (i) If $P(0, \{0\}) = b > 0$ and $\mu(\{0\}) > 0$, then $\lim_{x \neq 0} \sqrt{x} \log \rho(x) > -\infty$. (ii) If $\int_0^\infty \frac{1}{1+y} P(x, dy) < c$ μ -a.e. x for some c < 1, then $\overline{\lim_{x \neq 0}} \sqrt{x} \log \rho(x) < 0$.

A similar result has been obtained by M. Fukushima ([1]) when q(a), $a \in \mathbb{Z}$, are non-negative valued independent identically distributed random variables. We further mention the works of L. A. Pastur ([2]) and S. Nakao ([3]) for related results on the one dimensional Schrödinger operators with random potentials. The present novelty is to make use of a Markovian character of the local time (cf. M. L. Silverstein [4]).

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At first we collect some lemmas for the proof of our theorem. 2.

We introduce the continuous Markov process $M=(\dot{\Omega}, \dot{\mathcal{B}}, \dot{X}_t, \dot{P}_a)$ on Z with the generator H^0 . Denoting by \dot{E}_0 the expectation with respect to \dot{P}_0 , we have Kac representation as follows.

Lemma 1 ([1]).

$$\int_0^\infty e^{-t\lambda} d\rho(\lambda) = E \times \dot{E}_0 \Big[\exp \Big(- \int_0^t q(\dot{X}_s, \omega) ds \Big); \ \dot{X}_t = 0 \Big]$$

The proof of our theorem reduces to finding how fast $E imes \dot{E}_0 \Big[\exp \Big(- \Big(- \Big) \Big] \Big]$ $\int_{0}^{t} q(\dot{X}_{s}, \omega) ds); \ \dot{X}_{t} = 0 \] \ tends \ to \ zero \ as \ t \to \infty \ because \ of \ Lemma \ 1 \ and \ the \ follow$ ing Tauberian theorem.

Lemma 2 ([1]). Let $\phi(\lambda)$ be non-decreasing function on $[0, \infty)$ with $\phi(0)=0$ and $\psi(t)$ be its Laplace transform:

$$\psi(t)=\int_0^\infty e^{-t\lambda}d\phi(\lambda)$$

- (i) If $\lim_{t \to \infty} \frac{1}{t^{\gamma}} \log \psi(t) > -\infty$ then $\lim_{x \to 0} x^{\gamma/1-\gamma} \log \phi(x) > -\infty$.
- (ii) If $\lim_{t \to \infty} \frac{1}{t^{\gamma}} \log \psi(t) < 0$ then $\lim_{x \to 0} x^{\gamma/1-\gamma} \log \phi(x) < 0$.

For the investigation of asymptotic behavior of $E \times \dot{E}_0 \left[\exp \left(- \int_{\alpha}^t q(\dot{X}_s, \omega) \right); \right]$

 $\dot{X}_t=0$ as $t\to\infty$, the following two lemmas are of great use.

Lemma 3 ([4]). Put $\dot{L}(t, x) = \int_0^t I_x(\dot{X}_s) ds$, t > 0, $x \in Z$, $\sigma_x(s) = \sup \{t; \dot{L}(t, x) \leq s, \dot{T}_{x,y}(s) = \dot{L}(\sigma_x(s), y)$, then it holds that

(i) $\{\dot{T}_{y,z}(s), y, z \ge x\}$ and $\{\dot{T}_{y,z}(s), y, z \le x\}$ are multually independent for each $x \in Z$ and $s \ge 0$,

(ii)
$$\dot{E}_0[\exp(-\alpha \dot{T}_{x,x-y-z}(s))|\dot{T}_{x,x-y}(s)=l]=\frac{1}{\alpha z+1}\exp\left(\frac{-l\alpha}{\alpha z+1}\right)$$
 for each $\alpha > 0$,

x > 0 (<0) and y, $z \ge 0$ such that $y + z \le x (\le -x)$,

(iii) $\{\dot{T}_{x,x^{-}y}(s)\}$ is Markovian in $y \ge 0$ for fixed $s \ge 0$ and x > 0 (<0).

Corollary Put $\mathscr{B}_{x-u}^{x}(s) = \sigma[\dot{T}_{x,x-v}(s), v=0, 1, \dots, u] \ s \ge 0$, then we have

$$\dot{E}_0\left[\exp\left(-\alpha \dot{T}_{x,x-y-z}(s)\right) \mid \mathcal{B}^x_{x-y}(s)\right] \leq \frac{1}{\alpha z+1}$$

for each x > 0 (<0) and y, $z \ge 0$ such that $y + z \le x$ ($y + z \le -x$).

Lemma 4 ([1]). Let \dot{R}_t be the number of states where \dot{X}_s visits during the interval [0, t), then we have

- (i) $\lim_{t \neq \infty} t^{-1/3} \log \dot{E}_0[e^{-\beta_1 \dot{R}_t}; \dot{X}_t = 0] > -\infty$ for any positive constant $\beta_1 > 0$,
- (ii) $\lim_{t\to\infty} t^{-1/3} \log \dot{E}_0[e^{-\beta_2 R_t}] < 0$

for any positive constant $\beta_2 > 0$.

3. Now we give the proof of our theorem.

Put
$$k(t) = E \times \dot{E}_0 \left[\exp\left(-\int_0^t q(\dot{X}_s, \omega) ds\right); \ \dot{X}_t = 0 \right]$$
, then
 $k(t) = \sum_{k=0}^{\infty} \sum_{\substack{m+n=k \ m, n \ge 0}} E \times \dot{E}_0 \left[\exp\left(-\sum_{x=-n}^m \dot{L}(t, x)q(x, \omega)\right); \ \dot{M}_t = m, \ \dot{m}_t = -n, \ \dot{X}_t = 0 \right]$

where $\dot{M}_t = \sup \{\dot{X}_s: 0 \le s \le t\}$ $\dot{m}_t = \inf \{X_s: 0 \le s \le t\}$. Taking the expectation of $\exp(-\sum_{x=-n}^{m} \dot{L}(t, x)q(x, \omega))$ with respect to P, we have by stationarity and Markov property of $(\Omega, \mathcal{B}, P, q)$

$$E\left[\exp\left(-\sum_{x=-n}^{m}L(t, x)q(x, \omega)\right)\right] \ge E\left[\prod_{x=-n}^{m}I_{0}(q(x, \omega))\right] = \mu(\{0\})b^{m+n}$$

therefore

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$$\begin{split} k(t) &\geq \sum_{k=0}^{\infty} \sum_{\substack{m+n=k \\ m,n \geq 0}} \dot{E}_{0}[\mu(\{0\})b^{k}; \ \dot{M}_{t} = \dot{m}, \ m_{t} = -n, \ \dot{X}_{t} = 0] \\ &= \frac{\mu(\{0\})}{b} \sum_{k=1}^{\infty} b^{k} \dot{P}_{0}(\dot{R}_{t} = k, \ \dot{X}_{t} = 0) = \frac{\mu(\{0\})}{b} \dot{E}_{0}[e^{-\beta_{1}\dot{R}_{t}}; \ \dot{X}_{t} = 0], \\ \beta_{1} = -\log b \,. \end{split}$$

Because of Lemma 4

$$\lim_{t\to\infty}t^{-1/3}\log k(t)>-\infty.$$

We get the first assertion (i) of our theorem by Lemma 1 and Lemma 2.

Turning to the proof of the second assertion (ii), we put

$$k_1(t) = E \times \dot{E}_0 \left[\exp\left(-\int_0^t q(\dot{X}_s, \omega) ds\right); \ \dot{X}_t = 0, \ \dot{R}_t < t \right],$$

$$k_2(t) = E \times \dot{E}_0 \left[\exp\left(-\int_0^t q(\dot{X}_s, \omega) ds\right); \ \dot{X}_t = 0, \ \dot{R}_t \ge t \right],$$

then we get

Putting $\tau_i = t \wedge \inf \{s; \dot{X}_s = i\}, i \in \mathbb{Z}$, it is clear that

$$\dot{L}(\tau_m, x) \leq \dot{L}(t, x)$$
, $\dot{L}(\tau_{-n}, x) \leq \dot{L}(t, x)$ and
 $\dot{L}(\tau_m, x) = \dot{T}_{m,x}(0)$, $\dot{L}(\tau_{-n}, x) = \dot{T}_{-n,x}(0)$.

Therefore

$$\exp\left(-2\sum_{x=-n}^{m}\dot{L}(t, x)q(x, \omega)\right)$$

$$\leq \exp\left(-2\sum_{x=0}^{m}\dot{T}_{m,x}(0)q(x, \omega)-2\sum_{x=-n}^{-1}\dot{T}_{-n,x}(0)q(x, \omega)\right).$$

Taking the expectation with respect to $P \times \dot{P}_0$, we get

$$E \times \dot{E}_0 [\exp(-2\sum_{x=-n}^m \dot{L}(t, x)q(x, \omega))] \leq E \left[\prod_{x=-n}^m \frac{1}{1+2q(x, \omega)}\right]$$

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because of Lemma 3, (i) (iii) and its Corollary. From Markov property of $(\Omega, \mathcal{B}, P, q)$ and the assumption in our theorem, it follows that

$$E\left[\prod_{x=-n}^{m}\frac{1}{1+2q(x,\omega)}\right] = E\left[\prod_{x=-n}^{m-1}\frac{1}{1+2q(x,\omega)}E\left[\frac{1}{1+2q(1,\omega)}\middle|q(0)\right]\right]$$
$$< cE\left[\prod_{x=-n}^{m-1}\frac{1}{1+2q(x,\omega)}\right] < c^{m+n+1}.$$

Now we have

$$\begin{split} k_{1}(t)^{2} &\leq \frac{[t]([t]+1)}{2} \sum_{k=0}^{[t]-1} \sum_{\substack{m+n=k \\ m,n \geq 0}} c^{m+n+1} \dot{P}_{0}[\dot{X}_{t}=0, \dot{M}_{t}=m, \dot{m}_{t}=-n] \\ &= \frac{[t]([t]+1)}{2} \sum_{k=1}^{[t]} c^{k} \dot{P}_{0}(\dot{X}_{t}=0, \dot{R}_{t}=k) \\ &\leq \frac{[t]([t]+1)}{2} \dot{E}_{0}(e^{-\beta_{2} \dot{R}_{t}}; \dot{X}_{t}=0), \beta_{2}=-\log c \;. \end{split}$$

On the other hand

$$\begin{split} k_{2}(t) &\leq E \times \dot{E}_{0} \bigg[\exp \left(-\int_{0}^{t} q(\dot{X}_{s}, \omega) ds; \ \dot{X}_{t} = 0, \ \dot{M}_{t} \geq \frac{t}{2} \right) \bigg] \\ &+ E \times \dot{E}_{0} \bigg[\exp \left(-\int_{0}^{t} q(\dot{X}_{s}, \omega); \ \dot{X}_{t} = 0, \ \dot{m}_{t} \leq -\frac{t}{2} \right) \bigg] \\ &\leq E \times \dot{E}_{0} \bigg[\exp \left(-\sum_{x=0}^{\lfloor t/2 \rfloor} \dot{L}(\tau_{\lfloor t/2 \rfloor}, x) q(x, \omega) \right) \bigg] \\ &+ E \times \dot{E}_{0} \bigg[\exp \left(-\sum_{x=-1}^{\lfloor t/2 \rfloor} \dot{L}(\tau_{-\lfloor t/2 \rfloor}, x) q(x, \omega) \right) \bigg] \\ &\leq E \bigg[\prod_{x=0}^{\lfloor t/2 \rfloor} \frac{1}{1+q(x, \omega)} \bigg] + E \bigg[\prod_{x=-1}^{-\lfloor t/2 \rfloor} \frac{1}{1+q(x, \omega)} \bigg] < 2c^{\lfloor t/2 \rfloor} \\ &= 2e^{-\beta_{2} \lfloor t/2 \rfloor} . \end{split}$$

As a result

$$k(t) = k_1(t) + k_2(t) \leq \sqrt{\frac{[t]([t]+1)}{2}} \dot{E}_0(e^{-\beta_2 \dot{k}_1}) + 2e^{-\beta_2 [t/2]},$$

which, combined with Lemma 4, leads us to

$$\overline{\lim_{t\uparrow^{\infty}}} t^{-1/3} \log k(t) < 0.$$

Hence we arrive at the second assertion of our theorem.

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