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ON REGULAR CATEGORIES

Dedicated to Professor Keizo Asano for his 60th birthday

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Gabriel and Oberst have defined a C_3-spectral category in [6] and we shall consider spectral categories which are not necessarily C_3 in this note. We call a category "*spectral*", if the category is abelian and every morphism splits. We shall define a regular category and show that the concept of regularity and spectrality are equivalent if a category is amenable, (see the below for definitions of regular category and amenable category.) and give some characterizations of a special regular category. Finally as an application of the above argument, we shall show that $\mathfrak{A}/\mathfrak{J}(\mathfrak{A})$ is a completely reducible abelian category if $\mathfrak{A}$ is artinian and noetherian abelian category, where $\mathfrak{J}(\mathfrak{A})$ is the radical of $\mathfrak{A}$ (see the below for definitions of radical of category and $\mathfrak{A}/\mathfrak{J}(\mathfrak{A})$.). We shall give further applications in the forthcoming paper jointed with M. Harada. The author would like to express his thank to Professor M. Harada for his suggestion of the problem. We shall make use of the notations and definitions in [8].

Let $\mathfrak{A}$ be an additive category and $\mathfrak{A}_m$ the totality of morphisms in $\mathfrak{A}$. For $\alpha\in\mathfrak{A}_m$, $\alpha\mathfrak{A}_m$ means the class of all morphisms of $\alpha\beta$, where $\beta\in\mathfrak{A}_m$ and $\alpha\beta$ is defined. Furthermore, we can define a right (resp. left) ideal $\mathfrak{C}$ in $\mathfrak{A}$ which are analogous to the case of rings. Let $\mathfrak{C}$ be a sub-class of $\mathfrak{A}_m$ which satisfies the following two conditions;

1 For every α, $\beta\in\mathfrak{C}$, $\alpha\pm\beta\in\mathfrak{C}$, whenever $\alpha\pm\beta$ is defined.

2 $\beta\alpha\in\mathfrak{C}$ (resp. $\alpha\beta\in\mathfrak{C}$) for any $\alpha\in\mathfrak{A}_m$, $\beta\in\mathfrak{C}$ such that $\beta\alpha$ (resp. $\alpha\beta$) is defined.

We call $\mathfrak{C}$ a *right* (resp. *left*) *ideal* of $\mathfrak{A}$. We denote the quotient category of $\mathfrak{A}$ with respect to $\mathfrak{C}$ by $\mathfrak{A}/\mathfrak{C}$; namely the objects of $\mathfrak{A}/\mathfrak{C}$ are the same as the objects of $\mathfrak{A}$, and for A, B in $\mathfrak{A}/\mathfrak{C}$, $[A, B]_{\mathfrak{A}/\mathfrak{C}}$ is equal to $[A, B]/[A, B]\cap\mathfrak{C}$. We call a two sided ideal $\mathfrak{R}$ of $\mathfrak{A}$ "*radical*," if $\mathfrak{R}\cap[A, A]$ is equal to the Jacobson radical of $[A, A]$ for every object A in $\mathfrak{A}$.

Let $\mathfrak{A}$ be an additive category with finite coproduct. We call $\mathfrak{A}$ "*regular*," if the ring $[A, A]$ is regular in the sense of Von Neumann for every object A in $\mathfrak{A}$ (cf. [9]), and we call $\mathfrak{A}$ "*amenable*," if every idempotent morphism in $[A, A]$ has the kernel for every $A\in\mathfrak{A}$ ([4]).

Proposition 1. *Let $\mathfrak{A}$ be an additive category with finite coproduct. Then the following statements are equivalent*

1) *$\mathfrak{A}$ is regular.*
2) *For every morphism α in $\mathfrak{A}$, there is a morphism x such that $\alpha=\alpha x\alpha$.*
3) *There are idempotent morphisms e, e' suct that $\mathfrak{A}_m\alpha=\mathfrak{A}_m e$, $\alpha\mathfrak{A}_m=e'\mathfrak{A}_m$.*

Proof. 1)$\Rightarrow$2) Let α be in $[A, B]$, then we put $\alpha'=\begin{pmatrix}0 & 0\\ \alpha & 0\end{pmatrix}$: $A\oplus B\to A\oplus B$. Then there exists $x=\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}$; $A\oplus B\to A\oplus B$ such that $\alpha'=\alpha' x\alpha'$. Therefore $\alpha=\alpha x_{12}\alpha$.

2)$\Rightarrow$3). The proof is completely similar to the case of ring. Let α be in $[A, B]$ then there exists x; $B\to A$ such that $\alpha=\alpha x\alpha$. Let we put $x\alpha=e$ then e is idempotent, $\alpha=xe$ and $e=x\alpha=x\alpha x\alpha$. Therefore $\mathfrak{A}_m\alpha=\mathfrak{A}_m e$. Similarly, we have $\alpha\mathfrak{A}_m=e'\mathfrak{A}_m$.

3)$\Rightarrow$1) Let α be in $[A, A]$, then we can denote $\alpha=xe$, $e=y\alpha$ for some $x, y\in\mathfrak{A}_m$. Therefore, $\alpha y\alpha=\alpha e=xe^2=xe=\alpha$.

Theorem 2. *Let $\mathfrak{A}$ be an amenable category. Then $\mathfrak{A}$ is regular if and only if $\mathfrak{A}$ is spectral.*

Proof. We assume that $\mathfrak{A}$ is regular. Then we shall show first that $\mathfrak{A}$ is abelian. Let α: $A\to B$ be any morphism in $\mathfrak{A}$.

i) We shall show that α has the kernel and the cokernel. From the assumption, we can denote that $\alpha=xe$, $e=y\alpha$ for some $x, y, e\in\mathfrak{A}_m$, where e is an idempotent. From the assumption (cf. [8], p. 31), every idempotent e in $[A, A]$ has its image and we denote it by eA. Let i_{1-e}: $(1-e)A\to A$ be the inclusion. Then we shall show that i_{1-e} is the kernel of α. We have $\alpha i_{1-e}=xei_{1-e}=0$. Let β be any morphism such that $\alpha\beta=0$, then $\beta=(i_e p_e+i_{1-e}p_{1-e})\beta=i_{1-e}p_{1-e}\beta$, since $i_e p_e\beta=e\beta=y\alpha\beta=0$, where p_e and p_{1-e} are the projections of A to eA and $(1-e)A$, respectively. Therefore $(1-e)A=\mathrm{Ker}\,\alpha$. Similarly, we have $(1-e')B=\mathrm{coker}\,\alpha$, where $\alpha=e'x'$, $e'=\alpha y'$.

ii) Next, we shall show that $\mathfrak{A}$ is normal and conormal. We note first that $\mathfrak{A}$ is balanced. Let f: $A\to B$ be monomorphic and epimorphic. From the assumption we have $f=xe$ for some $x\in\mathfrak{A}_m$. Then e is monomorphic since f is monomorphic. Hence $\mathrm{Ker}\,f=(1-e)A=0$ from i), and $e=1_A$. Therefore, there exists $g\in\mathfrak{A}_m$ such that $gf=1_A$ since $\mathfrak{A}_m f=\mathfrak{A}_m e=\mathfrak{A}_m 1_A$. From the duality we have $fg'=1_B$ for some $g'\in\mathfrak{A}_m$. Hence f is isomorphic. Let α be monomorphic, then $\mathrm{coker}\,\alpha=(1-e')B$, $\ker(1-e')=e'B$ from i), where $\alpha=e'x'$, $e'=\alpha y'$ for some x' $y'\in\mathfrak{A}_m$ and e' is idempotent. We put $\beta=p_{e'}x'$, then $i_{e'}\beta=i_{e'}p_{e'}x'=e'x'=\alpha$ .We shall show that β is isomorphic. β is monomorphic since α is monomorphic. We have $p_{e'}=e'\,x'y'$, since $i_{e'}p_{e'}=e'=\alpha y'=e'x'y'=i_{e'}p_{e'}x'y'$ and $i_{e'}$ is monomorphic. Let z be any morphism such that $z\beta=0$, then $0=z\beta=zp_{e'}x'=zp_{e'}x'y'=zp_{e'}$. Hence $z=0$ since $p_{e'}$ is epimorphic, which implies that β is epimorphic, Therefore, β is isomorphic from the above argument. Hence $\mathfrak{A}$ is normal, and we can show that $\mathfrak{A}$ is conormal from the duality. Let α be in $[A, B]$, then $A=eA\oplus(1-e)A=eA\oplus\ker\alpha$, $B=e'B\oplus(1-e')B=\operatorname{im}\alpha\oplus(1-e')B$ from i). Therefore $\mathfrak{A}$ is spectral. Conversely, We assume that $\mathfrak{A}$ is spectral. For any $\alpha\colon A\to B$, we have $A=\ker\alpha\oplus A'$, $B=\operatorname{im}\alpha\oplus B$, and $\alpha|A'=x\colon A'\to\operatorname{im}\alpha$ is isomorphic. We put $e=i_{A'}p_{A'}$, then e is idempotent and $\alpha=xe$. Furthermore, $e=i_{A'}x^{-1}\alpha$. Hence $\mathfrak{A}_m\alpha=\mathfrak{A}_m e$. Similarly we have $\alpha\mathfrak{A}_m=e'\mathfrak{A}_m$.

REMARK 1. Let $\mathfrak{A}$ be a regular category, then we can imbed $\mathfrak{A}$ into a spectral category $\mathfrak{A}^*$. Let $\mathfrak{A}^*$ be the category whose objects are pairs (A, e) where $A\in\mathfrak{A}$ and e is an idempotent in $[A, A]$. We define the morphisms of (A, e) to (B, f) as follows: first we consider the subset C of t in $[A, B]$ such that $te=ft$, and define a congruent relation among them; $t\equiv t'$ if and only if $ft=ft'$. Then we put $[(A, e), (B, f)]=$the congruent classes $\bar{t}$ of C. Next we shall show that $\mathfrak{A}^*$ is a regular category. For any $\bar{\alpha}\colon(A, e)\to(A, e)$, there exists a morphism x such that $\alpha=\alpha x\alpha$ from the assumption. We put $x'=exe$, then $ex'=x'e$, and $\overline{\alpha x'\alpha}=\bar{\alpha}$, which implies that $\mathfrak{A}^*$ is regular. $\mathfrak{A}$ is imbeded into $\mathfrak{A}^*$ by a natural imbedding functor T such that $T(A)=(A, 1_A)$.

REMARK 2. A special case of Theorem 2 was obtained by Harada (unpublished.). We shall give some interesting applications to a case of category of injective modules in the forthcoming paper.

We shall give some characterizations of special regular categories.

Proposition 3. *Let $\mathfrak{A}$ be an abelian category, then we consider the following conditions*

1). *$\mathfrak{A}$ is regular.*

2). *For any epimorphism $\alpha\colon A\to B$, we have $[B, B]\alpha\subseteq\alpha[A, A]$*

3). *The projection functor $T\colon\mathfrak{A}\to\mathfrak{A}/\mathfrak{C}$ is epi-preserve for every ideal $\mathfrak{C}$. then we have* 1)$\Leftrightarrow$2) *and* 1)$\Rightarrow$3). *Furthermore we have* 3)$\Rightarrow$1) *if $\mathfrak{A}$ is a locally small C_3-category.*

Proof. 1)$\Leftrightarrow$2). If $\mathfrak{A}$ is regular, then every object in $\mathfrak{A}$ is projective, and hence $[B, B]\alpha\subseteqq\alpha[A, A]$ for an epimorphism α; $A\to B$. Conversely, let

$$\begin{array}{ccccc} B & \xrightarrow{x} & C & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle y} & & \\ & & A & & \end{array}$$

be a diagram with x epimorphic. We define an epimorphism $x'\colon A\oplus B\oplus C\to A\oplus C$ by setting $x'=\begin{pmatrix}1 & 0 & 0\\ 0 & x & 0\end{pmatrix}$. Put $y'=\begin{pmatrix}0 & 0\\ y & 0\end{pmatrix}\in[A\oplus C, A\oplus C]$. Then from the assumption, there exists z in $[A\oplus B\oplus C,\ A\oplus B\oplus\ C]$ such that $y'x'=x'z$. Hence $y=zx_{21}$, where $z=(z_{ij})$. Which means that every object A is projective in $\mathfrak{A}$, and hence, $\mathfrak{A}$ is regular.

1)$\Rightarrow$3). It is clear since every morphism splits

Next we assume that $\mathfrak{A}$ is a locally small C_3-category.

3)$\Rightarrow$1). Let A be a subobject of B. From the assumption on $\mathfrak{A}$, there exists a complement A^C of A and $A\oplus A^C$ is essential in B. Let, $\alpha\colon B\to B/(A\oplus A^c)=C$ be the natural epimorphism. Put $\mathfrak{C}=\mathfrak{A}_m\alpha\mathfrak{A}_m$, then $T(\alpha)=0$ and hence $[C, C]\subseteq\mathfrak{A}_m\alpha\mathfrak{A}_m$. Therefore, there exist $x\colon C\to B$, $y\colon C\to C$ such that $1_C=y\alpha x$, hence $B=\operatorname{im} x\oplus\ker y\alpha$. Since $\ker y\alpha\supseteq A\oplus A^C$, and A is essential in B, $\operatorname{im} x=0$, and hence $C=0$, which implies that $\mathfrak{A}$ is regular.

Proposition 4. *Let $\mathfrak{A}$ be a C_3-category with a generator U, then $\mathfrak{A}$ is regular if and only if for any injective subobjects M_i $i=1, 2$ in M, $M_1\cap M_2$, is also injective.*

Proof. Only if part is clear. Let $R=[U, U]$, and $\mathfrak{M}_R$ be the category of right R-modules. Let $S=[U,\ ]\colon \mathfrak{A}\to\mathfrak{M}_R$ and $T=U\otimes\colon \mathfrak{M}_R\to\mathfrak{A}$ be the usual ajoint functors. Then for any object A in $\mathfrak{A}$, $S(\hat{A})=\widehat{S(A)}$ where $\hat{A}$ and $\widehat{S(A)}$ is an injective envelope of A and $S(A)$, respectively, by [10]. Furthermore we can express $S(A)=N_1\cap N_2$ for some N_i such that $N_i\approx\widehat{S(A)}$ (cf. [3] p. 63). Therefore $A=TS(A)=T(N_1\cap N_2)=T(N_1)\cap T(N_2)$, since T is kernel preserving (cf. [8] p. 55 6–5), and $T(N_i)\approx\hat{A}$. Hence $\mathfrak{A}$ is regular.

Harada has defined a semi-simple category in [7]. We shall give the following proposition related to a semi-simple category

Proposition 5. *Let $\mathfrak{A}$ be a small abelian category. Then $\mathfrak{A}$ is semi-simple artinian if and only if additive functor category $(\mathfrak{A}, Ab)$ is completely reducible, where Ab is the category of all abelian groups.*

Proof. From the assumption, $\mathfrak{A}$ is semi-simple artinian if and only if $\mathfrak{A}$ is artinian completely reducible ([7], 1.2). Let $\mathfrak{A}$ be semi-simple artinian, then we shall show that if A is irreducible in $\mathfrak{A}$, then $[A,\ ]$ is irreducible in $(\mathfrak{A}, Ab)$. Let any $F\subsetneqq[A,\ ]$, then $F(A)\subseteq[A, A]$, which implies $F(A)=[A, A]$, or $F(A)=0$, since $F(A)$ is a left ideal of $[A, A]$ which is a division ring. If $F(A)=[A, A]$, then for any $B\in\mathfrak{A}$, we can express $B=\bigoplus_{i=1}^{n}B_i$, where every B_i is irreducible in $\mathfrak{A}$ from the assumption, we have $F(B)=\bigoplus_{i=1}^{n}F(B_i)$ and $[A, B]=\bigoplus_{i=1}^{n}[A, B_i]$. If $A\approx B_i$, then $F(A)\approx F(B_i)$, and if $A\not\approx B_j$, then $F(B_j)=[A, B_j]=0$. Hence $F(B)=\bigoplus_{i=1}^{n}F(B)=\bigoplus_{i=1}^{n}[A, B_i]=[A, B]$ which is a contradiction.
Hence $F(A)=0$. We have easily that $F(B)=0$ for any $B\in\mathfrak{A}$, which implies

$F=0$. Furthermore, for any $F\in(\mathfrak{A}, Ab)$, we have an exact sequence, $\bigoplus_{\mathfrak{A}\in A}[A, \ ]\to F\to 0$, and hence, $(\mathfrak{A}, Ab)$ is completely reducible. Conversely we assume that $(\mathfrak{A}, Ab)$ is completely reducible. Then for any $A\in\mathfrak{A}$, we have, $[A, \ ]=\bigoplus_i F_i$, where every F_i is irreducible in $(\mathfrak{A}, Ab)$. Since F_i is small projective in $(\mathfrak{A}, Ab)$, we can find $F_i=[A_i, \ ]$ for some $A_i\in\mathfrak{A}$ ([4], p. 229), and we shall show that A_i is irreducible in $\mathfrak{A}$. For any $B\subsetneqq A_i$, we have the natural transformation $f=[i, \ ]$: $[A_i, \ ]\to[B, \ ]$, then we have $\ker f=[A_i, \ ]$ or $\ker f=0$ in $(\mathfrak{A}, Ab)$, since $F_i=[A_i, \ ]$ is irreducible in $(\mathfrak{A}, Ab)$, where i is the inclusion. If $\ker f=0$, then $f_{A_i/B}=[i, A_i/B]: [A, A_i/B]\to[B, A_i/B]$ is monomorphic and $f_{A_i/B}(g)=B\xrightarrow{i}A_i\xrightarrow{g}A_i/B=0$ for the natural epimorphism g and hence $g=0$, which is a contradiction. Hence $\ker f=[A_i, \ ]$ and $f_{A_i}(1_{A_i})=B\xrightarrow{i}A_i\xrightarrow{1_{Ai}}A_i$ $=0$, which implies $B=0$, and A_i is irreducible in $\mathfrak{A}$. Also since $[A, A]$ $=\bigoplus_i[A_i, A]$, we can express $[A, A]=\bigoplus_{i=1}^{n}[A_i, A]$ for some integer n, which implies $[A, \ ]=\bigoplus_{i=1}^{n}[A_i, \ ]=[\bigoplus_{i=1}^{n}A, \ ]$ from the naturality of the functor. Hence $A=\bigoplus_{i=1}^{n}A_i$, and $\mathfrak{A}$ is semi-simple artinian.

Proposition 6. *Let $\mathfrak{A}$ be a small artinian abelian category. Then, $(\mathfrak{A}, Ab)$ is completely reducible if and only if $(\mathfrak{A}, Ab)$ is regular.*

Proof. Only if part is clear. We assume that $(\mathfrak{A}, Ab)$ is regular, then from the assumption and ([4], p. 119), for any sub-functor F in $[A, \ ]$, F is a direct summand of $[A, \ ]$, and we can find $F=[A', \ ]$ for some direct summand A' of A. Hence $[A, \ ]$ is artinian in $(\mathfrak{A}, Ab)$, and so, $[A, \ ]$ is completel reducible which implies that $(\mathfrak{A}, Ab)$ is completely reducible since $\{[A, \ ], A\in\mathfrak{A}\}$, is a family of generators for $(\mathfrak{A}, Ab)$.

Finally, we shall give following theorem as an application of Theorem 2, which is due to Harada (unpublished).

Theorem 7. *Let $\mathfrak{A}$ be an artinian and noethrian abelian category. Then $\mathfrak{A}/\mathfrak{J}(\mathfrak{A})$ is a completely reducible abelian category, where $\mathfrak{J}(\mathfrak{A})$ is the radical of $\mathfrak{A}$.*

Proof. From the assumption, every object M is a coproduct of finite directly indecomposable sub-objects of M, namely $M=\bigoplus_{i=1}^{n}M_i$. It is well known in the case of modules that $R=[M, M]$ is a semi-primary ring such that R contains the nilpotent radical N and R/N is an artinian semi-simple ring, therefore, R/N is a regular ring. This fact is also valid in the case of abelian category, (cf. [5]). Hence $\mathfrak{A}/\mathfrak{J}(\mathfrak{A})$ is a regular category. Let α be in R and by $\bar{\alpha}$ we denote the class of α in R/N. If $\bar{\alpha}$ is idempotent, then there exists an idempotent e in R such that $\bar{e}=\bar{\alpha}$ ([1] p. 545, 77. 4). Since $1=(1-e)+e$ on R, $\bar{\alpha}$ has the kernel $\overline{i_{1-e}}$: $(1-e)A\to A$. Hence $\mathfrak{A}/\mathfrak{J}(\mathfrak{A})$ is amenable, therefore $\mathfrak{A}/\mathfrak{J}(\mathfrak{A})$ is spectral from Theorem 2 and is completely reducible from artinian and noetherian.

REMARK 3. In the above theorem, if $\mathfrak{A}$ is only noetherian abelian, then this theorem is not valid. For example, Let $\mathfrak{A}$ be the category of all finitely generated abelian groups, which is noetherian abelian and is not artinian. We shall show that $\mathfrak{A}/\mathfrak{J}(\mathfrak{A})$ is not abelian. Let α be any map of Z to $Z/(P)$, where Z is the ring of rational integers, (P) is the abelian group generated by a integer P. Then $\alpha\in\mathfrak{J}(\mathfrak{A})$, because $[Z\oplus Z/(P),\ Z\oplus Z/(P)]\cong\begin{pmatrix} Z & 0 \\ [Z, Z/(P)] & Z/(P)\end{pmatrix}=R_{Z\oplus Z/(P)}$, and $R_{Z\oplus Z/(P)}\begin{pmatrix} 0 & 0 \\ \alpha & 0\end{pmatrix}$ is nilpotent left ideal, hence $\begin{pmatrix} 0 & 0 \\ \alpha & 0\end{pmatrix}\in\mathfrak{J}(R_{Z\oplus Z/(P)})$, which implies $\alpha\in\mathfrak{J}(\mathfrak{A})$ since $\mathfrak{J}(R_{Z\oplus Z/(P)})\cap[Z, Z/(P)]=\mathfrak{J}(\mathfrak{A})\cap[Z, Z/(P)]$. Let $f\colon Z\to Z$ is any non zero map of Z to Z, then we shall show that f is monomorphic and epimorphic in $\mathfrak{A}/\mathfrak{J}(\mathfrak{A})$. $g\colon G\to Z$ is any map such that $fg=0$, then since we can express $G=G_1\oplus G_2$ and $g=(g_1, g_2)\colon G_1\oplus G_2\to Z$, where G_1 and G_2 are the torsion sub-group and a free sub-group of G, respectively, we can show easily that $g_1=g_2=0$, hence f is monomorphic. Next let $g'\colon Z\to G'$ is any map such that $g'f=0$, then as above we can express $G'=G'_1\oplus G'_2$, and $g'=\begin{pmatrix} g'_1 \\ g'_2\end{pmatrix}$. We have $g'_1\in\mathfrak{J}(\mathfrak{A})$ from the above argument, and we can show easily that $g'_2=0$, which also implies that f is an epimorphism in $\mathfrak{A}/\mathfrak{J}(\mathfrak{A})$. If $\mathfrak{A}/\mathfrak{J}(\mathfrak{A})$ is abelian, then f is isomorphic in $\mathfrak{A}/\mathfrak{J}(\mathfrak{A})$, and so there exists g'' such that $g''f=1$ (mod $\mathfrak{J}([Z, Z])$). Since $\mathfrak{J}([Z, Z])=0$, $f=\pm 1$, which is a contradiction.

REMARK 4. We shall generalize this argument in the forthcoming paper.

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