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## LIMITING ABSORPTION PRINCIPLE FOR DIRAC OPERATOR WITH CONSTANT MAGNETIC FIELD AND LONG-RANGE POTENTIAL

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#### 1. Introduction

The Dirac Hamiltonian with magnetic vector potential  $\mathbf{a} = (a_j(x))_{j=1,...,d}$  is expressed by the following form

(1.1) 
$$H(\mathbf{a}) = \sum_{j=1}^{d} \gamma_j (P_j - a_j) + m \gamma_{d+1} + V,$$

where  $P_j = 1/i\partial_{x_j}$ , V is a multiplication of an Hermitian matrix V(x). m is the mass of electron. The matrices  $\{\gamma_i\}$  satisfy the following relations

(1.2) 
$$\gamma_j \gamma_k + \gamma_k \gamma_j = 2\delta_{jk} \mathbf{1} \quad (j, k = 1, \dots, d+1).$$

Here  $\delta_{jk}$  is Kronecker's delta and **1** is an identity matrix. We assume that the speed of the light c = 1. When  $V \equiv 0$ , the square of  $H(\mathbf{a})$  has the form

(1.3) 
$$H(\mathbf{a})^2 = \sum_{j=1}^d (P_j - a_j)^2 + m^2 + \frac{1}{i} \sum_{1 \le j < k \le d} b_{jk}(x) \gamma_j \gamma_k,$$

where

(1.4) 
$$b_{jk}(x) = \partial_{x_k} a_j(x) - \partial_{x_j} a_k(x).$$

It is called Pauli's Hamiltonian. The skew symmetric matrix  $(b_{jk}(x))$  is the magnetic field associated with **a**. We say the magnetic field is asymptotically constant if it satisfies the following conditions as  $|x| \to \infty$ :

(1.5) 
$$b_{jk}(x) \to {}^{\exists} \Lambda_{jk} \quad (1 \le j, k \le d),$$

where  $(\Lambda_{jk})_{j,k}$  is a constant matrix.

The aim of this paper is to prove the limiting absorption principle for  $H(\mathbf{a})$  with a constant magnetic field  $(b_{ik}(x))$  and a long-range electric potential V(x) when d = 3.

К. Үокоуама

Let us recall some known facts about the Dirac Hamiltonian with a constant magnetic field for d = 2, 3. As can be infered from (1.3), the spectrum of  $H(\mathbf{a})$  is closely related with that of magnetic Schrödinger operator appearing in the right hand side of (1.3), which depends largely on the space dimension. Suppose d = 2 at first. For simplicity we consider the case that the magnetic field  $b(x) = \partial_{x_2}a_1(x) - \partial_{x_1}a_2(x) = \lambda > 0$ . In this case, the Dirac Hamiltonian  $h(\lambda)$  is represented by

(1.6) 
$$h(\lambda) = \sigma_1 \left( P_1 + \frac{\lambda}{2} x_2 \right) + \sigma_2 \left( P_2 - \frac{\lambda}{2} x_1 \right) + m \sigma_3,$$

with  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ ,  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

They are called Pauli's spin matrices. Obviously  $\{\sigma_j\}$  satisfy the relation (1.2) and by an elementary calculus we have

(1.7) 
$$h(\lambda)^{2} = \left(P_{1} + \frac{\lambda}{2}x_{2}\right)^{2} + \left(P_{2} - \frac{\lambda}{2}x_{1}\right)^{2} + m^{2} - \lambda\sigma_{3}.$$

The right hand side is a de-coupled 2 dimensional magnetic Schödinger operator. So it suggests that the spectrum of  $h(\lambda)$  is discrete and

$$\sigma(h(\lambda)) \subset \left\{ \pm \sqrt{2\lambda n + m^2} \mid n = 0, 1, 2 \dots \right\}.$$

In fact we have

$$\sigma(h(\lambda)) = \left\{ \sqrt{2\lambda n + m^2}, \quad -\sqrt{2\lambda(n+1) + m^2} \mid n = 0, 1, 2 \dots \right\}$$

by using Foldy-Wouthuysen transform. (See 7.1.3 in [10].) Therefore the spectrum of  $h(\lambda)$  is of pure point with infinite multiplicities.

Next we consider the case of d = 3. We assume

$$\mathbf{a}_0(x) = \left(-\frac{\lambda x_2}{2}, \frac{\lambda x_1}{2}, 0\right) \quad (\lambda > 0).$$

Then the associated magnetic field is constant along  $x_3$ -axis :

$$B(x) = (b_{32}(x), b_{13}(x), b_{21}(x)) = (0, 0, \lambda).$$

We denote the associated Dirac Hamiltonian as  $H_0(\lambda)$ . It is the following operator acting on  $\mathbb{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ :

(1.8) 
$$H_0(\lambda) = \alpha_1 \left( P_1 + \frac{\lambda x_2}{2} \right) + \alpha_2 \left( P_2 - \frac{\lambda x_1}{2} \right) + \alpha_3 P_3 + m\beta,$$

where  $\{\alpha_i\}$  and  $\beta$  are  $4 \times 4$  Hermitian matrices such that

(1.9) 
$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can easily see that these matrices also satisfy the relation (1.2). It is known that  $H_0(\lambda)$  is essentially self-adjoint on  $C_0^{\infty}(\mathbb{R}^3) \otimes \mathbb{C}^4$ . (See Theorem 4.3 in [10].) Now we consider the spectrum of  $H_0(\lambda)$ . At first we rewrite  $H_0(\lambda)$  as follows.

(1.10) 
$$H_0(\lambda) = Q_0 + m\beta = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix},$$

with  $D_0 = \sigma \cdot (P - \mathbf{a}_0)$  and  $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ .

By using Foldy-Wouthuysen transform, explained in detail in the following section,  $H_0(\lambda)$  can be diagonalized by a unitary operator  $U_{FW}$ .

(1.11) 
$$U_{FW}H_0(\lambda)U_{FW}^{-1} = \begin{pmatrix} \sqrt{D_0^2 + m^2} & 0\\ 0 & -\sqrt{D_0^2 + m^2} \end{pmatrix}.$$

From the commutation relation (1.2) we have

(1.12) 
$$D_0^2 = \left(P_1 + \frac{\lambda x_2}{2}\right)^2 + \left(P_2 - \frac{\lambda x_1}{2}\right)^2 + P_3^2 - \lambda\beta.$$

We can easily see that  $\sigma(D_0^2) = [0, \infty)$ . So we have

$$\sigma(H_0(\lambda)) = (-\infty, -m] \cup [m, \infty).$$

Therefore in the 3 dimensional case, the spectrum of  $H_0(\lambda)$  is absolutely continuous.

Let us consider the perturbation of  $H_0(\lambda)$ : We put

(1.13) 
$$H(\lambda) = H_0(\lambda) + V.$$

Our aim is to show the so-called limiting absorption principle, namely the existence of the boundary value of the resolvent  $(z - H(\lambda))^{-1}$  on the real axis. The precise assumption on V will be given in Section 3. It is closely related to the absence of singular continuous spectrum of the operator and the asymptotic completeness of the wave operator associated with  $H_0(\lambda)$  and  $H(\lambda)$ . To prove the limiting absorption principle, we use Mourre's commutator method, which makes great progress for various Schrödinger operators. (For example, see [8].)

Suppose we are given a self-adjoint operator H on a separable Hilbert space. For a closed interval  $I \subset \mathbb{R}$  we denote the spectral measure, corresponding to the interval

as  $E_I(H)$ . Once we find some self-adjoint operator A satisfying the following inequality, we obtain many informations about H:

(1.14) 
$$E_I(H)i[H,A]E_I(H) \ge \alpha E_I(H) + K,$$

where  $\alpha$  is a positive number and K is a compact operator. To be accurate, we can see the following properties hold.

(i)  $\sigma_{pp}(H) \cap I$ , the eigenvalues of H in I, are discrete.

(ii) The boundary value of the resolvent on  $I \setminus \sigma_{pp}(H)$  exists in some weighted Hilbert space. (**limiting absorption principle**)

For example, we consider a usual Schrödinger operator

$$H = -\Delta + V(x).$$

Here V(x) is a real valued function, which is decaying as  $|x| \to \infty$ . In this case we choose  $A = 1/(2i)\{x \cdot \nabla_x + \nabla_x \cdot x\}$  as the conjugate operator. Then the Mourre's inequality holds for any compact interval  $I \subset \mathbb{R} \setminus \{0\}$ . As a result one can show that  $\sigma_{pp}(H)$  is discrete with no accumulation point except  $\{0\}$ . We denote  $\langle \cdot \rangle = (|\cdot|^2 + 1)^{1/2}$ . Then we can also see the boundary values

$$\langle x \rangle^{-s} (H - \mu \mp i0)^{-1} \langle x \rangle^{-s}$$

exist in the operator norm for s > 1/2 and  $\mu \in \mathbb{R} \setminus (\{0\} \cup \sigma_{pp}(H))$ .

As for the Schrödinger operator with constant magnetic field, Iwashita [6] shows the limiting absorption principle for long-range potential by using commutator method. In [6] the following self-adjoint operator is considered.

(1.15) 
$$\tilde{H} = \left(P_1 + \frac{\lambda x_2}{2}\right)^2 + \left(P_2 - \frac{\lambda x_1}{2}\right)^2 + P_3^2 + V(x).$$

A self-adjoint operator  $A = 1/2(P_3 \cdot x_3 + x_3 \cdot P_3)$  is used as the conjugate operator. As a result, the existence of the boundary values

$$\langle x_3 \rangle^{-s} (\tilde{H} - \mu \mp i0)^{-1} \langle x_3 \rangle^{-s}$$

is proved for s > 1/2 and  $\mu \in \mathbb{R} \setminus (\{\lambda(2n+1) | n = 0, 1, 2, ...\} \cup \sigma_{pp}(\tilde{H})).$ 

Commutator method is also used for the free Dirac Hamiltonian and that with a scalar potential, which is decaying as  $|x| \to \infty$ . (See [2].) As for the electromagnetic Dirac Hamiltonian, asymptotic behavior of the solution of the Dirac equation is investigated in [3]. In their paper, the time dependent electromagnetic field  $a_j(x, t)$  is required to satisfy the following properties.

(i) Each  $a_j(x, t)$  satisfies the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) a_j(x,t) = 0.$$

(ii) The initial data  $a_j(x, 0)$  and  $\partial_t a_j(x, 0)$  are compactly supported in  $\mathbb{R}^3$ . Hachem [5] showed the limiting absorption principle for the following electromagnetic Dirac Hamiltonian with a short-range potential V(x).

$$H = \alpha_1 P_1 + \alpha_2 (P_2 + \lambda x_1) + \alpha_3 P_3 + m\beta + V(x).$$

His idea is roughly as follows. First let us consider the case  $V \equiv 0$ . By passing to the Fourier transformation with respect to  $x_2, x_3$ -variables we denote  $F_0HF_0^*$  as D(p)  $(p = (p_2, p_3))$ . Then we have

(1.16) 
$$D(p)^2 = A_+(p) \oplus A_-(p) \oplus A_+(p) \oplus A_-(p),$$

where  $A_{\pm}(p)$  are harmonic oscillators defined as follows.

(1.17) 
$$A_{\pm}(p) = -\frac{d^2}{dx_1^2} + (\lambda x_1 + p_2)^2 \pm \lambda + p_3^2 + m^2$$

He then switched on the short-range potential V(x) by perturbative argument. Roughly speaking, his assumption means that the absolute value of each component of V is dominated from above by  $C\langle x'\rangle^{-1-\epsilon}\langle x\rangle^{-\epsilon}$   $(x' = (x_2, x_3))$  for sufficiently large x. We remark that  $\epsilon > 0$  is used as a sufficiently small parameter throughout this paper. To be accurate,  $\langle x'\rangle^{1+\epsilon}V(x)$  is required to be a  $H_0(\lambda)$ -compact operator.

In this paper we treat directly the following operator

(1.18) 
$$H(\lambda) = \alpha_1 \left( P_1 + \frac{\lambda}{2} x_2 \right) + \alpha_2 \left( P_2 - \frac{\lambda}{2} x_1 \right) + \alpha_3 P_3 + m\beta + V(x),$$

where V(x) is a matrix potential. Our strategy is to apply Mourre's commutator method directly to this operator, which enables us to include the long-range diagonal components for V(x). In this case it seems that an appropriate choice of the conjugate operator is

$$\frac{P_3}{\langle P_3 \rangle} \cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3 \rangle},$$

which is inspired by [11], when we proved the limiting absorption principle for timeperiodic Schrödinger operator. In fact the method of the proof shares many ideas in common with [11]. Namely we rewrite  $H_0(\lambda)$  by a direct integral and the conjugate operator A acts on each space of fiber. Our main results are Theorem 3.4 and Corollary 3.7.

### 2. Conjugate operator

Let us recall

(2.1) 
$$Q_0 = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix}, \quad D_0 = \sigma(P - \mathbf{a}_0),$$

with

(2.2) 
$$\mathbf{a}_0(x) = \left(-\frac{\lambda x_2}{2}, \frac{\lambda x_1}{2}, 0\right).$$

The Dirac Hamiltonian  $Q_0+m\beta$  can be diagonalized by sandwiching it between a unitary operator U and  $U^* = U^{-1}$ . In the beginning of this section we introduce a unitary operator which diagonalizes the self-adjoint operator  $H_0(\lambda)$ . Secondly we give a conjugate operator associated with the diagonalized Dirac Hamiltonian. Finally we show Mourre's inequality for original Hamiltonians  $H_0(\lambda)$  and  $H(\lambda)$ .

Let  $Q_0$  be the self-adjoint operator as in (2.1) and  $|Q_0| = \sqrt{Q_0^2}$ ,  $|H_0(\lambda)| = \sqrt{H_0(\lambda)^2}$ . We define a unitary operator  $U_{FW}$ , which diagonalizes  $H_0(\lambda)$ , in the following way.

DEFINITION 2.1. (i) At first we define a signature function associated with  $Q_0$  by

(2.3) 
$$\operatorname{sgn} Q_0 = \begin{cases} \frac{Q_0}{|Q_0|} , & \text{on } (\ker Q_0)^{\perp} \\ 0 , & \text{on } (\ker Q_0) \end{cases}$$

We note that sgn  $Q_0$  is isometory on  $(\ker Q_0)^{\perp}$ .

(ii) We can easily see that  $m/|H_0(\lambda)| \le 1$ . So we denote the square root of  $1/2(1 \pm m/|H_0(\lambda)|)$  as  $a_{\pm}$ . i.e.

(2.4) 
$$a_{\pm} = \frac{1}{\sqrt{2}} \sqrt{1 \pm \frac{m}{|H_0(\lambda)|}}.$$

(iii) Combining these operators we define the operator  $U_{FW}$  as

(2.5) 
$$U_{FW} = a_+ + \beta (\operatorname{sgn} Q_0) a_-.$$

**Lemma 2.2.** (i)  $U_{FW}$  is a unitary operator on  $L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ . Further,

(2.6) 
$$U_{FW}^* = U_{FW}^{-1} = a_+ - \beta (\operatorname{sgn} Q_0) a_-.$$

(ii)  $H_0(\lambda)$  can be diagonalized by  $U_{FW}$  as follows.

(2.7) 
$$U_{FW}H_0(\lambda)U_{FW}^{-1} = |H_0(\lambda)|\beta = \begin{pmatrix} \sqrt{D_0^2 + m^2} & 0\\ 0 & -\sqrt{D_0^2 + m^2} \end{pmatrix}.$$

Proof. See 5.6.1 in [10].

We denote the diagonalized Dirac Hamiltonian as  $\hat{H}_0(\lambda)$ . i.e.

$$\hat{H}_0(\lambda) = U_{FW} H_0(\lambda) U_{FW}^{-1}$$

We rewrite (1.12) as follows.

$$D_0^2 = \left(\begin{array}{cc} D_- & 0\\ 0 & D_+ \end{array}\right).$$

Here  $D_{\pm}$  are the operators acting on  $L^2(\mathbb{R}^3)$  such that

$$D_{\pm} = \left(P_1 + \frac{\lambda}{2}x_2\right)^2 + \left(P_2 - \frac{\lambda}{2}x_1\right)^2 + P_3^2 \pm \lambda.$$

It is well-known that  $(P_1 + \lambda/2x_2)^2 + (P_2 - \lambda/2x_1)^2$  has eigenvalues

$$\{\lambda(2n+1) \mid n=0,1,2,\ldots\}.$$

We denote the eigenprojection on each eigenspace as  $\Pi_n$ . With these projections,  $\sqrt{D_0^2 + m^2}$  can be rewritten as follows.

(2.8) 
$$\sqrt{D_0^2 + m^2} = \sum_{n=0}^{\infty} \begin{pmatrix} d_n \otimes \Pi_n & 0\\ 0 & d_{n+1} \otimes \Pi_n \end{pmatrix},$$

with  $d_n = d_n(P_3) = \sqrt{2\lambda n + P_3^2 + m^2}$ .

Combining (2.7) and (2.8), we have

$$\begin{split} f(\hat{H}_0(\lambda)) &= \\ &\sum_{n=0}^{\infty} \left( \begin{array}{cc} f(d_n) \otimes \Pi_n & & \\ & f(d_{n+1}) \otimes \Pi_n & \\ & & f(-d_n) \otimes \Pi_n \\ & & & f(-d_{n+1}) \otimes \Pi_n \end{array} \right), \end{split}$$

for any Borel function f.

Now we define the conjugate operator. At first we define

(2.9) 
$$\hat{A} = \frac{1}{2} \left\{ \frac{P_3}{\langle P_3 \rangle} \cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3 \rangle} \right\}.$$

We note that  $\hat{A}$  is essentially self-adjoint operator on  $D(|x_3|)$ . (It is obtained by use of Nelson's commutator theorem [9].) The conjugate operator for the Dirac Hamiltonian

associated with constant magnetic field is defined by sandwiching  $\hat{A}\beta$  between  $U_{FW}^{-1}$ and  $U_{FW}$ :

(2.10) 
$$A = U_{FW}^{-1}(\hat{A}\beta)U_{FW}.$$

Letting F be a Fourier transformation with respect to  $x_3$ -variable. We define the selfadjoint operator  $A_F$  by  $F\hat{A}F^{-1}$ . Then we have

(2.11) 
$$(e^{itA_F}\phi)(x_1, x_2, p_3) = \left|\frac{\partial\Gamma_t}{\partial p_3}(p_3)\right|^{1/2} \phi(x_1, x_2, \Gamma_t(p_3)),$$

for  $\phi \in L^2(\mathbb{R}^2_x \times \mathbb{R}_p)$ . Here  $\Gamma_t$  is a solution of the following equation.

(2.12) 
$$\begin{cases} \frac{d}{dt}\Gamma_t(p_3) = \langle \Gamma_t(p_3) \rangle^{-1}\Gamma_t(p_3) \\ \Gamma_0(p_3) = p_3 \end{cases}$$

For the proof, see Appendix 1 in [8]. Therefore the unitary group  $e^{it\hat{A}\beta}$  is rewritten

$$(2.13) \qquad (Fe^{it\hat{\lambda}\beta}F^{-1}\phi)(x_1, x_2, p_3) = \begin{pmatrix} \left|\frac{\partial\Gamma_t}{\partial p_3}(p_3)\right|^{1/2}\phi_1(x_1, x_2, \Gamma_t(p_3)) \\ \left|\frac{\partial\Gamma_t}{\partial p_3}(p_3)\right|^{1/2}\phi_2(x_1, x_2, \Gamma_t(p_3)) \\ \left|\frac{\partial\Gamma_{-t}}{\partial p_3}(p_3)\right|^{1/2}\phi_3(x_1, x_2, \Gamma_{-t}(p_3)) \\ \left|\frac{\partial\Gamma_{-t}}{\partial p_3}(p_3)\right|^{1/2}\phi_4(x_1, x_2, \Gamma_{-t}(p_3)) \end{pmatrix}.$$

Before we compute the commutator  $i[H(\lambda), A]$ , we have to care that the following matters hold.

**Lemma 2.3.** (i) A is a self-adjoint operator on  $\mathbb{H}$ . (ii)  $e^{-itA}$  leaves  $D(H_0(\lambda))$  invariant, i.e.

(2.14) 
$$\sup_{|t|\leq 1} \|H_0(\lambda)e^{itA}(H_0(\lambda)+i)^{-1}\phi\|_{\mathbb{H}} < \infty \quad for \quad \phi \in \mathbb{H},$$

where  $\|\cdot\|_{\mathbb{H}}$  denotes the operator norm on  $\mathbb{H}$ .

Proof. The self-adjointness of A is easily obtained from that of  $\hat{A}\beta$ . To see the invariance of  $D(H_0(\lambda))$ , it is sufficient to show the following

(2.15) 
$$\sup_{|t|\leq 1} \|\hat{H}_0(\lambda)e^{it\hat{A}\beta}(\hat{H}_0(\lambda)+i)^{-1}\phi\|_{\mathbb{H}} < \infty \quad \text{for} \quad \phi \in \mathbb{H}.$$

From the arguements (2.11) and (2.13), we have

$$F\hat{H}_{0}(\lambda)e^{it\hat{A}\beta}(\hat{H}_{0}(\lambda)+i)^{-1}F^{-1}\phi$$

$$=\sum_{n=0}^{\infty}\begin{pmatrix} D_{n,t}^{+} \left| \frac{\partial\Gamma_{t}}{\partial p_{3}}(p_{3}) \right|^{1/2}\phi_{1}(x_{1},x_{2},\Gamma_{t}(p_{3})) \\ D_{n+1,t}^{+} \left| \frac{\partial\Gamma_{t}}{\partial p_{3}}(p_{3}) \right|^{1/2}\phi_{2}(x_{1},x_{2},\Gamma_{t}(p_{3})) \\ D_{n,-t}^{-} \left| \frac{\partial\Gamma_{-t}}{\partial p_{3}}(p_{3}) \right|^{1/2}\phi_{3}(x_{1},x_{2},\Gamma_{-t}(p_{3})) \\ D_{n+1,-t}^{-} \left| \frac{\partial\Gamma_{-t}}{\partial p_{3}}(p_{3}) \right|^{1/2}\phi_{4}(x_{1},x_{2},\Gamma_{-t}(p_{3})) \end{pmatrix}$$

where  $D_{n,\alpha}^{\pm} = d_n(p_3)(d_n(\Gamma_{\alpha}(p_3)) \pm i)^{-1} \otimes \Pi_n$ . By integrating (2.12), we have

(2.16)  $|\Gamma_{\alpha}(p_3) - p_3| \le 1 \quad (|\alpha| \le 1).$ 

So (2.14) is obtained from the fact that  $D_{n,\alpha}^{\pm}$  is bounded uniformly for  $n \in \mathbb{N}$  and  $|\alpha| \leq 1$ .

Before we show Mourre's inequality, we introduce the usual functional calculus, started by Helffer and Sjöstrand.

Suppose that  $f \in C^{\infty}(\mathbb{R})$  satisfies the following condition for some  $m_0 \in \mathbb{R}$ .

(2.17) 
$$|f^{(k)}(t)| \le C_k (1+|t|)^{m_0-k}, \quad \forall k \in \mathbb{N} \cup \{0\}.$$

Then we can construct an almost analytic extension  $\tilde{f}(z)$  of f(t) having the following properties

$$\tilde{f}(t) = f(t), \quad t \in \mathbb{R},$$
  
 $\operatorname{supp} \tilde{f} \subset \{z; |\operatorname{Im} z| \le 1 + |\operatorname{Re} z|\},$ 

(2.18) 
$$|\partial_{\bar{z}}\tilde{f}(z)| \leq C_N |\operatorname{Im} z|^N \langle z \rangle^{m_0 - 1 - N}, \quad \forall N \in \mathbb{N}.$$

Then for all f, satisfying (2.17) for  $m_0 < 0$  and a self-adjoint operator H, we have

(2.19) 
$$f(H) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z)(z-H)^{-1} dz \wedge d\bar{z}.$$

With this form, we can compute the commutator of an operator P and g(A) in the following way.

For operators P and Q, we define  $ad_Q^0(P) = P$  and inductively  $ad_Q^m(P) = [ad_Q^{m-1}(P), Q]$  for  $m \in \mathbb{N}$ .

K. Yokoyama

**Lemma 2.4.** Let A and P be self-adjoint operators on  $\mathbb{H}$ . Suppose that  $ad_A^m(P)(A+i)^{-m}$  extends to a bounded operator for  $1 \leq m \leq n$ . Then for any  $g \in C^{\infty}(\mathbb{R})$  satisfying (2.17) with  $m_0 < 0$ , we have

(2.20) 
$$Pg(A) = \sum_{m=0}^{n-1} \frac{g^{(m)}(A)}{m!} a d_A^m(P) + \frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{g}(z) R_{n,A,P}^r(z) dz \wedge d\bar{z}$$

where  $R_{n,A,P}^{r}(z) = (z - A)^{-n} a d_{A}^{n}(P)(z - A)^{-1}$ , and

(2.21) 
$$g(A)P = \sum_{m=0}^{n-1} ad_A^m(P) \frac{(-1)^m}{m!} g^{(m)}(A) + \frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{g}(z) R_{n,A,P}^l(z) dz \wedge d\bar{z}$$

where  $R_{n,A,P}^{l}(z) = (z - A)^{-1} a d_{A}^{n}(P)(z - A)^{-n}$  and  $\tilde{g}(z)$  denotes an almost analytic extension of g(t).

For the proof of above results, see [4].

#### 3. Limiting absorption principle for long-range potentials

Now we show the Mourre's inequality for the Dirac Hamiltonian by choosing *A* defined in the previous section as the conjugate operator.

**Lemma 3.1.** Let  $\mathbb{R}_{\mathbb{N}}$  be the following discrete subset of  $\mathbb{R}$ 

$$\mathbb{R}_{\mathbb{N}} = \{\pm \sqrt{2\lambda n + m^2} \mid n = 0, 1, 2, \ldots\} \subset \mathbb{R}.$$

We take a compact interval  $I \subset \mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$  arbitrarily. Then there exists  $\alpha > 0$  such that the following inequality holds for any real valued  $f \in C_0^{\infty}(I)$ 

(3.1) 
$$f(H_0(\lambda))i[H_0(\lambda), A]f(H_0(\lambda)) \ge \alpha f(H_0(\lambda))^2.$$

Proof. By the relations (2.7) and (2.10), it is sufficient to show the inequality

(3.2) 
$$f(\hat{H}_0(\lambda))i[\hat{H}_0(\lambda), \hat{A}\beta]f(\hat{H}_0(\lambda)) \ge \alpha f(\hat{H}_0(\lambda))^2.$$

We rewrite the commutator as follow.

(3.3) 
$$i\left[\hat{H}_{0}(\lambda),\hat{A}\beta\right] = \begin{pmatrix} i\left[\sqrt{D_{0}^{2}+m^{2}},\hat{A}\right]\\ i\left[\sqrt{D_{0}^{2}+m^{2}},\hat{A}\right] \end{pmatrix}.$$

We proceed the calculus more precisely to see that

(3.4) 
$$i\left[\sqrt{D_0^2+m^2},\hat{A}\right] = \sum_{n=0}^{\infty} \left(i\left[d_n,\hat{A}\right] \otimes \Pi_n i\left[d_{n+1},\hat{A}\right] \otimes \Pi_n\right)$$

by (2.8). From (3.3) and (3.4) the left hand side of (3.2) is rewritten as

(3.5) 
$$f(\hat{H}_0(\lambda))i[\hat{H}_0(\lambda),\hat{A}\beta]f(\hat{H}_0(\lambda)) = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \\ & & & I_4 \end{pmatrix}$$

where

$$I_{1} = \sum_{n=0}^{\infty} f(d_{n})i[d_{n}, \hat{A}]f(d_{n}) \otimes \Pi_{n},$$

$$I_{2} = \sum_{n=0}^{\infty} f(d_{n+1})i[d_{n+1}, \hat{A}]f(d_{n+1}) \otimes \Pi_{n},$$

$$I_{3} = \sum_{n=0}^{\infty} f(-d_{n})i[d_{n}, \hat{A}]f(-d_{n}) \otimes \Pi_{n},$$

$$I_{4} = \sum_{n=0}^{\infty} f(-d_{n+1})i[d_{n+1}, \hat{A}]f(-d_{n+1}) \otimes \Pi_{n}.$$

We note that all the sum in  $I_1, \ldots, I_4$  are finite since f is a compactly supported function. By an elementary caluculus, we have

(3.6) 
$$i[d_l, \hat{A}] = \frac{P_3^2}{\sqrt{2\lambda l + P_3^2 + m^2} \langle P_3 \rangle} \quad (l \in \mathbb{N} \cup \{0\}).$$

Since supp  $f \subset I \subset \mathbb{R} \setminus \mathbb{R}_N$ ,  $P_3$  is away from zero when  $P_3 \in \text{supp } f(d_l(P_3))$  or  $P_3 \in \text{supp } f(-d_l(P_3))$ . So there exist  $C_l > 0$  such that

$$f(d_l)i[d_l, \hat{A}]f(d_l) \otimes \Pi_l \ge C_l f(d_l)^2 \otimes \Pi_l,$$
  
$$f(-d_l)i[d_l, \hat{A}]f(-d_l) \otimes \Pi_l \ge C_l f(-d_l)^2 \otimes \Pi_l.$$

Since only a finite number of  $l = l_j$  (j = 1, ..., N) is concerned, we have (3.2) with  $\alpha = \inf_{j=1,...,N} C_{l_j}$ .

Now we give the assumption for the potential, which is necessary to prove Mourre's inequality associated to  $H(\lambda)$ . After that we give an example of V satisfying this assumption. The potential V consists of a sum of long-range part and short-range part. In our case short-range potential means  $V(x) = O(\langle x \rangle^{-\epsilon} \langle x_3 \rangle^{-1-\epsilon})$  as  $|x| \to \infty$ . And long-range part is a multiplication of a real valued function  $\varphi(x)$  such that  $\varphi(x) = O(\langle x \rangle^{-\epsilon})$  as  $|x| \to \infty$ . More precisely we assume that V satisfies the following.

ASSUMPTION 3.2. V = V(x) is a multiplicative operator of a  $4 \times 4$  Hermitian matrix satisfying the following properties.

- (i) V is a  $H_0(\lambda)$ -compact operator.
- (ii) The form [V, A] can be extended to a  $H_0(\lambda)$ -compact operator.

For example a  $4 \times 4$  matrix V(x) satisfying the following inequality is  $H_0(\lambda)$ -compact.

$$|V(x)| \le C \langle x \rangle^{-\epsilon} \quad (x \in \mathbb{R}^3).$$

It is owing to the fact that  $V(x)(-\Delta_x + 1)^{-1}$  is compact. (It is due to Theorem 2.6 in [1].) Under this assumption we show Mourre's inequality for  $H(\lambda)$ .

**Lemma 3.3.** Suppose V satisfies Assumption 3.2. (i) We take  $\mu \in \mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$  and  $\delta > 0$  so that the closed interval  $I \equiv [\mu - \delta, \mu + \delta] \subset \mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$ . There exist  $\alpha > 0$  and a compact operator K such that the following inequality holds for all  $f \in C_0^{\infty}(I)$ .

(3.8) 
$$f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \ge \alpha f(H(\lambda))^2 + K.$$

(ii) There is no accumulation point of  $\sigma_{pp}(H(\lambda))$  in  $\mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$ . For  $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$ , there exist  $\delta_0 > 0$  and  $\alpha_0 > 0$  such that the following inequality holds for all  $f \in C_0^{\infty}([\mu - \delta_0, \mu + \delta_0])$ .

(3.9) 
$$f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \ge \alpha_0 f(H(\lambda))^2.$$

Proof. From (2.19) we have

$$f(H(\lambda)) - f(H_0(\lambda)) = \frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} f(z) (z - H(\lambda))^{-1} V(z - H_0(\lambda))^{-1} dz \wedge d\bar{z}$$

for  $f \in C_0^{\infty}(\mathbb{R})$ . We can easily see that  $f(H(\lambda)) - f(H_0(\lambda))$  is a compact operator since  $V(H_0(\lambda) + i)^{-1}$  is compact. Combing this fact and (3.1), we have (3.8) by replacing  $f(H_0(\lambda))$  in (3.1) by  $f(H(\lambda))$ . As for the non-existence of the accumulation point of  $\sigma_{pp}(H(\lambda))$ , see Theorem 2.2 in [7]. (3.9) follows from the argument in [8]

With this inequality we have the limiting absorption principle for the Dirac Hamiltonian. **Theorem 3.4.** Let s > 1/2. Suppose V satisfies Assumption 3.2. Then for  $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$ , the following limits

(3.10) 
$$R^{\pm}(\mu) = \lim_{\epsilon \downarrow 0} \langle x_3 \rangle^{-s} (H(\lambda) - \mu \mp i\epsilon)^{-1} \langle x_3 \rangle^{-s}$$

exist and  $R^{\pm}(\mu)$  are continuous with respect to  $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$ .

### Sketch of proof

From (3.9) and Theorem 2.2 in [7], we can see that the boundary value  $\langle A \rangle^{-s}(H(\lambda) - \mu \mp i0)^{-1} \langle A \rangle^{-s}$  exist for  $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$ . To see the existence of (3.10), it is sufficient to show the boundedness of  $\langle A \rangle^s \langle x_3 \rangle^{-s}$ . Since  $\langle \hat{A} \rangle^s \langle x_3 \rangle^{-s}$  is bounded, it is sufficient to show  $\langle x_3 \rangle^s U_{FW} \langle x_3 \rangle^{-s}$  is bounded. We prove it in the following Lemma. Before that we introduce smooth functions. Let  $\chi(t) \in C^{\infty}(\mathbb{R})$  such that

(3.11) 
$$\chi(t) = \begin{cases} \frac{1}{\sqrt{2}} & \left(t > -\frac{m^2}{3}\right) \\ 0 & \left(t < -\frac{2m^2}{3}\right). \end{cases}$$

With this function we define  $F_{\pm}(t)$  and  $F_{\chi,\pm}$  as follows.

$$F_{+}(t) = \chi(t)\sqrt{1 + \frac{m}{\sqrt{t + m^{2}}}}$$

$$F_{-}(t) = \chi(t)\left(\sqrt{1 + \frac{m}{\sqrt{t + m^{2}}}}\right)^{-1} \frac{1}{\sqrt{t + m^{2}}}$$

$$F_{\chi,+}(t) = F_{+}(t) - \chi(t)$$

$$F_{\chi,-}(t) = \sqrt{t + m^{2}}F_{-}(t) - \chi(t)$$

Then we can easily verify that

$$a_{+} = F_{+}(Q_{0}^{2}),$$
  
 $a_{-} \operatorname{sgn} Q_{0} = F_{-}(Q_{0}^{2})Q_{0} = Q_{0}F_{-}(Q_{0}^{2}).$ 

Obviously  $[Q_0, F_-(Q_0^2)] = 0$ . By the construction of these functions, we can also see that  $F_{\chi,\pm}(t)$  satisfy (2.17) with  $m_0 < 0$ . So we apply the functional calculus in Section 2 to  $F_{\chi,\pm}(t)$  and see the following properties hold.

**Lemma 3.5.** Suppose  $0 \le s \le 2$  and  $z \in \mathbb{C} \setminus \mathbb{R}$ . Then (i) For  $0 < s \le 1$ , there exists  $C_s > 0$  such that

(3.12) 
$$\|\langle x \rangle^{s} (z - Q_{0}^{2})^{-1} \langle x \rangle^{-s} \|_{\mathbb{H}} \leq C_{s} (|\operatorname{Im} z|^{-1} + |\operatorname{Im} z|^{-2} \langle z \rangle).$$

(ii) For  $1 < s \le 2$ , there exists  $C'_s > 0$  such that

$$(3.13) ||\langle x\rangle^{s}(z-Q_{0}^{2})^{-1}\langle x\rangle^{-s}||_{\mathbb{H}} \le C_{s}'(|\operatorname{Im} z|^{-1}+|\operatorname{Im} z|^{-2}\langle z\rangle+|\operatorname{Im} z|^{-3}\langle z\rangle^{2}).$$

(iii)  $\langle x \rangle^{s} F_{+}(Q_{0}^{2}) \langle x \rangle^{-s}$  and  $\langle x \rangle^{s} F_{-}(Q_{0}^{2}) Q_{0} \langle x \rangle^{-s}$  are bounded operators.

Proof. For the proof of (i) and (ii), we use the resolvent equation. Suppose  $0 < s \le 1$ . Then

(3.14) 
$$\langle x \rangle^{s} (z - Q_{0}^{2})^{-1} \langle x \rangle^{-s} = (z - Q_{0}^{2})^{-1} + (z - Q_{0}^{2})^{-1} (Q_{0}^{2} + 1)$$

$$(3.15) (Q_0^2+1)^{-1} [\langle x \rangle^3, Q_0^2] (z-Q_0^2)^{-1} \langle x \rangle^{-3}.$$

From the boundedness of  $(Q_0^2 + 1)^{-1}[\langle x \rangle^s, Q_0^2]$  and the following estimate

(3.16) 
$$\|(z-Q_0^2)^{-1}(Q_0^2+1)\|_{\mathbb{H}} \le C(|\operatorname{Im} z|^{-1}\langle z\rangle+1),$$

we obtain (i). As for the case  $1 < s \leq 2$ , we rewrite the last term  $(z - Q_0^2)^{-1}[\langle x \rangle^s, Q_0^2](z - Q_0^2)^{-1}\langle x \rangle^{-s}$  as

(3.17) 
$$(z - Q_0^2)^{-1} (Q_0^2 + 1) (Q_0^2 + 1)^{-1} [\langle x \rangle^s, Q_0^2] \langle x \rangle^{-s+1}$$

(3.18) 
$$\times \langle x \rangle^{s-1} (z - Q_0^2)^{-1} \langle x \rangle^{-s+1} \langle x \rangle^{-1}.$$

By using the result for  $0 < s \le 1$ , we have the inequality for  $1 < s \le 2$ .

With these estimates, we prove (iii). Since  $\chi(Q_0^2) \equiv 1$ , we can easily see that

(3.19) 
$$\langle x \rangle^s F_+(Q_0^2) \langle x \rangle^{-s} = \langle x \rangle^s F_{\chi,+}(Q_0^2) \langle x \rangle^{-s} + I.$$

Since  $F_{\chi,+}(t)$  satisfies (2.18) for  $m_0 = -1/2$ ,  $F_{\chi,+}(Q_0^2)$  can be rewritten as follows.

(3.20) 
$$\frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{F}_{\chi,+}(z) \langle x \rangle^{s} (z - Q_{0}^{2})^{-1} \langle x \rangle^{-s} dz \wedge d\bar{z}.$$

From this formula and (i) (ii) we have

$$\begin{aligned} \|\partial_{\bar{z}}\tilde{F}_{\chi,+}(z)\langle x\rangle^{s}(z-Q_{0}^{2})^{-1}\langle x\rangle^{-s}\|_{\mathbb{H}} \\ &\leq C|\partial_{\bar{z}}\tilde{F}_{\chi,+}(z)|(|\operatorname{Im} z|^{-1}+|\operatorname{Im} z|^{-2}\langle z\rangle+|\operatorname{Im} z|^{-3}\langle z\rangle^{2}). \end{aligned}$$

From (2.18) we have

$$\|\partial_{\bar{z}}\tilde{F}_{\chi,+}(z)\langle x\rangle^{s}(z-Q_{0}^{2})^{-1}\langle x\rangle^{-s}\|_{\mathbb{H}}\leq C\langle z\rangle^{-5/2}.$$

This implies the boundedness of  $\langle x \rangle^{s} F_{+}(Q_{0}^{2}) \langle x \rangle^{-s}$ .

In a similar way, we rewrite  $\langle x \rangle^s F_-(Q_0^2) Q_0 \langle x \rangle^{-s}$  as

(3.21) 
$$\langle x \rangle^{s} F_{\chi,-}(Q_{0}^{2}) \langle x \rangle^{-s} \langle x \rangle^{s} \frac{Q_{0}}{\sqrt{Q_{0}^{2}+m^{2}}} \langle x \rangle^{-s} + \langle x \rangle^{s} \frac{Q_{0}}{\sqrt{Q_{0}^{2}+m^{2}}} \langle x \rangle^{-s}.$$

It is sufficient to show the boundedness of  $\langle x \rangle^s Q_0 / \sqrt{Q_0^2 + m^2} \langle x \rangle^{-s}$ . To see this, we denote  $\chi(t)/\sqrt{t+m^2} \in C^{\infty}(\mathbb{R}^3)$  as S(t) and its almost analytic extension as  $\tilde{S}(z)$ . We can easily see that  $S(Q_0^2)\langle x \rangle^s Q_0 \langle x \rangle^{-s}$  is bounded. So we obtain the boundedness of  $\langle x \rangle^s F_-(Q_0^2)Q_0 \langle x \rangle^{-s}$  if we show that  $[\langle x \rangle^s, S(Q_0^2)]Q_0 \langle x \rangle^{-s}$  is bounded. We rewrite it as follows.

$$\frac{1}{2\pi i}\int_{\mathbb{C}}\partial_{\bar{z}}\tilde{S}(z)\frac{Q_0^2+1}{z-Q_0^2}(Q_0^2+1)^{-1}[\langle x\rangle^s,Q_0^2]Q_0\langle x\rangle^{-s}\langle x\rangle^s(z-Q_0^2)^{-1}\langle x\rangle^{-s}dz\wedge d\bar{z}.$$

By an elementary calculus, we have  $(Q_0^2+1)^{-1}[\langle x \rangle^s, Q_0^2]Q_0 \langle x \rangle^{-s}$  bounded. Combining (i) and (ii), we have

$$\begin{split} \|[\langle x\rangle^s,S(Q_0^2)]Q_0\langle x\rangle^{-s}\|_{\mathbb{H}} &\leq C\int_{\mathbb{C}}|\partial_{\bar{z}}\tilde{S}(z)|\{1+|\operatorname{Im} z|^{-1}\}\\ &\times\{|\operatorname{Im} z|^{-1}+|\operatorname{Im} z|^{-2}\langle z\rangle+|\operatorname{Im} z|^{-3}\langle z\rangle^2\}dz\wedge d\bar{z}<\infty. \end{split}$$

This implies the boundedness of  $\langle x \rangle^s F_{-}(Q_0^2) Q_0 \langle x \rangle^{-s}$ .

Next we give an example of V. It requires smoothness, but allows long-range part in its diagonal components.

**Lemma 3.6.** Let V be a  $4 \times 4$  Hermitian matrix of the form

(3.22) 
$$V(x) = (v_{ij}(x)) + \varphi(x)I_4 \equiv V_s(x) + V_l(x)$$

where  $V_s(x) = (v_{ij}(x))$  is an Hermitian matrix and  $I_4$  is an identity matrix. Suppose the following conditions hold. Then V(x) satisfies Assumption 3.2.

There exist  $\delta > 0$  such that the following inequalities hold for all multi-index  $\alpha$ .

$$(3.23) |\partial_x^{\alpha} v_{ij}(x)| \le C_{\alpha} \langle x \rangle^{-\delta - |\alpha|} \langle x_3 \rangle^{-1} (1 \le i, j \le 4).$$

 $\varphi(x) \in C^{\infty}(\mathbb{R}^3)$  is real valued and satisfies

(3.24) 
$$|\partial_x^{\alpha}\varphi(x)| \leq C_{\alpha}' \langle x \rangle^{-\delta - |\alpha|}.$$

The relatively compactness of V(x) itself is clear since V satisfies (3.7). So we only have to show the relatively compactness of [V, A]. We prove the relatively compactness of  $[V_s, A] = [V_s, U_{FW}^{-1} \hat{A} \beta U_{FW}]$  at first. From the boundedness of  $\langle x_3 \rangle^{-1} \hat{A} \beta$ 

К. Үокоуама

and the relatively compactness of  $V_s\langle x_3\rangle$ , it is sufficient to show that  $\langle x_3\rangle U_{FW}\langle x_3\rangle^{-1}$ and  $\langle x_3\rangle U_{FW}^{-1}\langle x_3\rangle^{-1}$  are bounded operators in  $\mathbb{H}$ . We have already proved it in Lemma 3.5.

Next we treat the long-range term. The conjugate operator A can be decomosed into the sum of  $J_1, \ldots, J_4$  where

$$J_{1} = F_{+}(Q_{0}^{2})\hat{A}\beta F_{+}(Q_{0}^{2}),$$
  

$$J_{2} = F_{+}(Q_{0}^{2})\hat{A}\beta^{2}F_{-}(Q_{0}^{2})Q_{0},$$
  

$$J_{3} = \beta F_{-}(Q_{0}^{2})Q_{0}\hat{A}\beta F_{+}(Q_{0}^{2}),$$
  

$$J_{4} = \beta F_{-}(Q_{0}^{2})Q_{0}\hat{A}\beta^{2}F_{-}(Q_{0}^{2})Q_{0}$$

We prove that the  $H_0(\lambda)$  - compactness holds for each of  $[V_l, J_1], \ldots, [V_l, J_4]$ . To see this we use the functional calculus again and rewrite  $J_1$  as follows.

$$F_{+}(Q_{0}^{2})\hat{A}\beta F_{+}(Q_{0}^{2}) = \hat{A}\beta F_{+}(Q_{0}^{2})^{2} + [F_{\chi,+}(Q_{0}^{2}), \hat{A}\beta]F_{+}(Q_{0}^{2})$$
$$\equiv J_{1}' + J_{1}''$$

At first we prove the boundedness of  $J_1''$  and consequently the relatively compactness of  $[V_l, J_1'']$ . By using (2.19), we rewrite  $[F_{\chi,+}(Q_0^2), \hat{A}\beta]$  as follows.

(3.25) 
$$\frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{F}_{\chi,+}(z) (z - Q_0^2)^{-1} [Q_0^2, \hat{A}\beta] (z - Q_0^2)^{-1} dz \wedge d\bar{z}.$$

From (3.16) we have  $[Q_0^2, \hat{A}\beta](z - Q_0^2)^{-1}$  is dominated from above by  $C\{1 + |\operatorname{Im} z|\}$ . So we have

(3.26) 
$$\|[F_{\chi,+}(Q_0^2), \hat{A}\beta]\| \le C \int_{\mathbb{C}} |\partial_{\bar{z}} \tilde{F}_{\chi,+}(z)| \{ |\operatorname{Im} z|^{-1} + |\operatorname{Im} z|^{-2} \langle z \rangle \} dz \wedge d\bar{z}.$$

Since the almost analytic extension  $\tilde{F}_{\chi,+}(z)$  satisfies

$$(3.27) |\partial_{\bar{z}}\tilde{F}_{+}(z)| \le C_N |\operatorname{Im} z|^N \langle z \rangle^{-3/2-N} \quad (^{\forall} N \in \mathbb{N}),$$

we have  $[F_{\chi,+}(Q_0^2), \hat{A}\beta]$  is bounded and inductively  $[V_l, J_1'']$  is  $H_0(\lambda)$ -compact. So we only have to show the relatively compactness of  $[V_l, J_1']$ .

(3.28) 
$$[V_l, J_1'] = [V_l, \hat{A}\beta]F_+(Q_0^2)^2 + \hat{A}\beta[V_l, F_+(Q_0^2)^2].$$

Clearly  $[V_l, \hat{A}\beta]F_+(Q_0^2)$  is  $H_0(\lambda)$ -compact. Again we rewrite the commutator in the second term, by use of (2.19). Then we have  $\langle x \rangle^{1+\delta}[V_l, F_+(Q_0^2)^2]$  is bounded. Combing these facts, we have the relatively compactness of  $[V_l, J_1]$ .

As for the commutator  $[V_l, J_2], \ldots, [V_l, J_4]$  we also replace  $F_{\pm}$  by  $F_{\chi,\pm}$  and use the functional calculus. The proof of relatively compactness of  $[V_l, J_2]$  and  $[V_l, J_3]$  are

almost the same. We only give the proof for  $J_2$ . We also estimate the 'principle' part before we compute the commutator with  $V_l$ .

(3.29) 
$$J_2 = \hat{A}F_+(Q_0^2)F_-(Q_0^2)Q_0 + [F_+(Q_0^2), \hat{A}]F_-(Q_0^2)Q_0$$

It is sufficient to show that  $[V_l, \hat{A}F_+(Q_0^2)F_-(Q_0^2)Q_0]$  is a  $H_0(\lambda)$ -compact operator. We decompose it into the following sum.

$$\begin{split} & [V_l, \hat{A}] F_+(Q_0^2) F_-(Q_0^2) Q_0 \\ & + \hat{A} [V_l, F_+(Q_0^2) F_-(Q_0^2)] Q_0 \\ & + \hat{A} F_+(Q_0^2) F_-(Q_0^2) [V_l, Q_0]. \end{split}$$

We can easily see that the first and the third term is relatively compact since  $\langle x \rangle^{1+\delta}[V_l, Q_0]$  is bounded. As for the second term, we can also see the relatively compactness in the same argument as we have done in the proof of Lemma 3.5 (iii).

As for  $J_4$ , the proof is similar. We rewrite it as

(3.30) 
$$\hat{A}F_{-}(Q_{0}^{2})^{2}Q_{0}^{2} + [F_{-}(Q_{0}^{2})Q_{0}, \hat{A}]F_{-}(Q_{0}^{2})Q_{0}$$

We can also obtain the relatively compactness by estimating the term  $[V, AF_{-}(Q_0^2)^2 Q_0^2]$ .

**Corollary 3.7.** Let V be a  $4 \times 4$  Hermitian matrix and s > 1/2. Suppose V satisfies the condition in Lemma 3.6. Then the following limits

(3.31) 
$$R^{\pm}(\mu) = \lim_{\epsilon \downarrow 0} \langle x_3 \rangle^{-s} (H(\lambda) - \mu \mp i\epsilon)^{-1} \langle x_3 \rangle^{-s}$$

exist for  $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$  and  $\mathbb{R}^{\pm}(\mu)$  are continuous with respect to  $\mu$ .

#### References

- J. Avron, I. Herbst and B. Simon: Schrödinger operators with magnetic fields. 1. General interactions, Duke Math. J., 45 (1978), 847–883.
- [2] A. Boutet de Monvel-Berthier, D. Manda and R. Purice: *Limiting absorption principle for the Dirac operator*, Ann. Inst. Henri Poincaré, 58 (1993), 413–431.
- [3] A. Boutet de Monvel-Berthier, R. Purice: *The Dirac evolution equation in the presence of an electromagnetic wave*, Helv. Phys. Acta, **67** (1994), 167–187.
- [4] C. Gérard: Asymptotic completeness for 3-particle long-range systems, Invent. Math., 114 (1993), 333–397.
- [5] G. Hachem: Effet Zeeman pour un électron de Dirac, Ann. Inst. Henri Poincaré, 58 (1993), 105–123.
- [6] H. Iwashita: On the long-range scattering for one- and two- particle Schrödinger operators with constant magnetic fields, Tsukuba J. Math., 19 (1995), 369–376.

#### K. Yokoyama

- [7] A. Jensen, E. Mourre and P. Perry: Multiple commutator estimates and resolvent smoothness in quantum scattering theory, Ann. Inst. Henri Poincaré, 41 (1984), 207–225.
- [8] E. Mourre: Absence of singular continuous spectrum for certain self-adjoint operators, Commun. Math. Phys., 78 (1981), 391–408.
- [9] M. Reed, B. Simon: Methods of Modern Mathematical Physics 2, Academic Press, New York San Francisco London.
- [10] B. Thaller: The Dirac Equation, Texts Monographs Phys. Springer-Verlag, 1992.
- [11] K. Yokoyama: Mourre theory for time-periodic systems, Nagoya Math. J., 149 (1998), 193–210.

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