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LIMITING ABSORPTION PRINCIPLE FOR DIRAC OPERATOR WITH CONSTANT MAGNETIC FIELD AND LONG-RANGE POTENTIAL

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1. Introduction

The Dirac Hamiltonian with magnetic vector potential $\mathbf{a} = (a_j(x))_{j=1,\dots,d}$ is expressed by the following form

$$H(\mathbf{a}) = \sum_{j=1}^{d} \gamma_j (P_j - a_j) + m\gamma_{d+1} + V, \tag{1.1}$$

where $P_j = 1/i\partial_{x_j}$, V is a multiplication of an Hermitian matrix $V(x)$. m is the mass of electron. The matrices $\{\gamma_j\}$ satisfy the following relations

$$\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk}\mathbf{1} \quad (j, k = 1, \dots, d+1). \tag{1.2}$$

Here δ_{jk} is Kronecker's delta and $\mathbf{1}$ is an identity matrix. We assume that the speed of the light $c = 1$. When $V \equiv 0$, the square of $H(\mathbf{a})$ has the form

$$H(\mathbf{a})^2 = \sum_{j=1}^{d} (P_j - a_j)^2 + m^2 + \frac{1}{i} \sum_{1 \le j < k \le d} b_{jk}(x)\gamma_j\gamma_k, \tag{1.3}$$

where

$$b_{jk}(x) = \partial_{x_k} a_j(x) - \partial_{x_j} a_k(x). \tag{1.4}$$

It is called Pauli's Hamiltonian. The skew symmetric matrix $(b_{jk}(x))$ is the magnetic field associated with $\mathbf{a}$. We say the magnetic field is asymptotically constant if it satisfies the following conditions as $|x| \to \infty$:

$$b_{jk}(x) \to {}^{\exists}\Lambda_{jk} \quad (1 \le j, k \le d), \tag{1.5}$$

where $(\Lambda_{jk})_{j,k}$ is a constant matrix.

The aim of this paper is to prove the limiting absorption principle for $H(\mathbf{a})$ with a constant magnetic field $(b_{jk}(x))$ and a long-range electric potential $V(x)$ when $d = 3$.

Let us recall some known facts about the Dirac Hamiltonian with a constant magnetic field for $d = 2, 3$. As can be infered from (1.3), the spectrum of $H(\mathbf{a})$ is closely related with that of magnetic Schrödinger operator appearing in the right hand side of (1.3), which depends largely on the space dimension. Suppose $d = 2$ at first. For simplicity we consider the case that the magnetic field $b(x) = \partial_{x_2} a_1(x) - \partial_{x_1} a_2(x) = \lambda > 0$. In this case, the Dirac Hamiltonian $h(\lambda)$ is represented by

$$h(\lambda) = \sigma_1 \left(P_1 + \frac{\lambda}{2} x_2 \right) + \sigma_2 \left(P_2 - \frac{\lambda}{2} x_1 \right) + m\sigma_3, \tag{1.6}$$

with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

They are called Pauli's spin matrices. Obviously $\{\sigma_j\}$ satisfy the relation (1.2) and by an elementary calculus we have

$$h(\lambda)^2 = \left(P_1 + \frac{\lambda}{2} x_2 \right)^2 + \left(P_2 - \frac{\lambda}{2} x_1 \right)^2 + m^2 - \lambda\sigma_3. \tag{1.7}$$

The right hand side is a de-coupled 2 dimensional magnetic Schödinger operator. So it suggests that the spectrum of $h(\lambda)$ is discrete and

$$\sigma(h(\lambda)) \subset \left\{ \pm\sqrt{2\lambda n + m^2} \quad | \ n = 0, 1, 2 \dots \right\}.$$

In fact we have

$$\sigma(h(\lambda)) = \left\{ \sqrt{2\lambda n + m^2}, \quad -\sqrt{2\lambda(n+1) + m^2} \ | \ n = 0, 1, 2 \dots \right\}$$

by using Foldy-Wouthuysen transform. (See 7.1.3 in [10].) Therefore the spectrum of $h(\lambda)$ is of pure point with infinite multiplicities.

Next we consider the case of $d = 3$. We assume

$$\mathbf{a}_0(x) = \left(-\frac{\lambda x_2}{2}, \frac{\lambda x_1}{2}, 0 \right) \quad (\lambda > 0).$$

Then the associated magnetic field is constant along x_3-axis :

$$B(x) = (b_{32}(x), b_{13}(x), b_{21}(x)) = (0, 0, \lambda).$$

We denote the associated Dirac Hamiltonian as $H_0(\lambda)$. It is the following operator acting on $\mathbb{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$:

$$H_0(\lambda) = \alpha_1 \left(P_1 + \frac{\lambda x_2}{2} \right) + \alpha_2 \left(P_2 - \frac{\lambda x_1}{2} \right) + \alpha_3 P_3 + m\beta, \tag{1.8}$$

where $\{\alpha_j\}$ and β are 4×4 Hermitian matrices such that

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{1.9}$$

We can easily see that these matrices also satisfy the relation (1.2). It is known that $H_0(\lambda)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$. (See Theorem 4.3 in [10].) Now we consider the spectrum of $H_0(\lambda)$. At first we rewrite $H_0(\lambda)$ as follows.

$$H_0(\lambda) = Q_0 + m\beta = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, \tag{1.10}$$

with $D_0 = \sigma \cdot (P - \mathbf{a}_0)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$.

By using Foldy-Wouthuysen transform, explained in detail in the following section, $H_0(\lambda)$ can be diagonalized by a unitary operator U_{FW}.

$$U_{FW} H_0(\lambda) U_{FW}^{-1} = \begin{pmatrix} \sqrt{D_0^2 + m^2} & 0 \\ 0 & -\sqrt{D_0^2 + m^2} \end{pmatrix}. \tag{1.11}$$

From the commutation relation (1.2) we have

$$D_0^2 = \left(P_1 + \frac{\lambda x_2}{2}\right)^2 + \left(P_2 - \frac{\lambda x_1}{2}\right)^2 + P_3^2 - \lambda\beta. \tag{1.12}$$

We can easily see that $\sigma(D_0^2) = [0, \infty)$. So we have

$$\sigma(H_0(\lambda)) = (-\infty, -m] \cup [m, \infty).$$

Therefore in the 3 dimensional case, the spectrum of $H_0(\lambda)$ is absolutely continuous.

Let us consider the perturbation of $H_0(\lambda)$: We put

$$H(\lambda) = H_0(\lambda) + V. \tag{1.13}$$

Our aim is to show the so-called limiting absorption principle, namely the existence of the boundary value of the resolvent $(z - H(\lambda))^{-1}$ on the real axis. The precise assumption on V will be given in Section 3. It is closely related to the absence of singular continuous spectrum of the operator and the asymptotic completeness of the wave operator associated with $H_0(\lambda)$ and $H(\lambda)$. To prove the limiting absorption principle, we use Mourre's commutator method, which makes great progress for various Schrödinger operators. (For example, see [8].)

Suppose we are given a self-adjoint operator H on a separable Hilbert space. For a closed interval $I \subset \mathbb{R}$ we denote the spectral measure, corresponding to the interval as $E_I(H)$. Once we find some self-adjoint operator A satisfying the following inequality, we obtain many informations about H :

$$E_I(H)i[H, A]E_I(H) \geq \alpha E_I(H) + K, \tag{1.14}$$

where α is a positive number and K is a compact operator. To be accurate, we can see the following properties hold.

(i) $\sigma_{pp}(H) \cap I$, the eigenvalues of H in I, are discrete.

(ii) The boundary value of the resolvent on $I \setminus \sigma_{pp}(H)$ exists in some weighted Hilbert space. (**limiting absorption principle**)

For example, we consider a usual Schrödinger operator

$$H = -\Delta + V(x).$$

Here $V(x)$ is a real valued function, which is decaying as $|x| \to \infty$. In this case we choose $A = 1/(2i)\{x \cdot \nabla_x + \nabla_x \cdot x\}$ as the conjugate operator. Then the Mourre's inequality holds for any compact interval $I \subset \mathbb{R} \setminus \{0\}$. As a result one can show that $\sigma_{pp}(H)$ is discrete with no accumulation point except $\{0\}$. We denote $\langle \cdot \rangle = (| \cdot |^2 + 1)^{1/2}$. Then we can also see the boundary values

$$\langle x \rangle^{-s}(H - \mu \mp i0)^{-1}\langle x \rangle^{-s}$$

exist in the operator norm for $s > 1/2$ and $\mu \in \mathbb{R} \setminus (\{0\} \cup \sigma_{pp}(H))$.

As for the Schrödinger operator with constant magnetic field, Iwashita [6] shows the limiting absorption principle for long-range potential by using commutator method. In [6] the following self-adjoint operator is considered.

$$\tilde{H} = \left(P_1 + \frac{\lambda x_2}{2}\right)^2 + \left(P_2 - \frac{\lambda x_1}{2}\right)^2 + P_3^2 + V(x). \tag{1.15}$$

A self-adjoint operator $A = 1/2(P_3 \cdot x_3 + x_3 \cdot P_3)$ is used as the conjugate operator. As a result, the existence of the boundary values

$$\langle x_3 \rangle^{-s}(\tilde{H} - \mu \mp i0)^{-1}\langle x_3 \rangle^{-s}$$

is proved for $s > 1/2$ and $\mu \in \mathbb{R} \setminus (\{\lambda(2n+1) | n = 0, 1, 2, \ldots\} \cup \sigma_{pp}(\tilde{H}))$.

Commutator method is also used for the free Dirac Hamiltonian and that with a scalar potential, which is decaying as $|x| \to \infty$. (See [2].) As for the electromagnetic Dirac Hamiltonian, asymptotic behavior of the solution of the Dirac equation is investigated in [3]. In their paper, the time dependent electromaginetic field $a_j(x, t)$ is required to satisfy the following properties.

(i) Each $a_j(x, t)$ satisfies the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) a_j(x, t) = 0.$$

(ii) The initial data $a_j(x,0)$ and $\partial_t a_j(x,0)$ are compactly supported in $\mathbb{R}^3$.
Hachem [5] showed the limiting absorption principle for the following electromagnetic Dirac Hamiltonian with a short-range potential $V(x)$.

$$H = \alpha_1 P_1 + \alpha_2(P_2 + \lambda x_1) + \alpha_3 P_3 + m\beta + V(x).$$

His idea is roughly as follows. First let us consider the case $V \equiv 0$. By passing to the Fourier transformation with respect to x_2, x_3-variables we denote $F_0 H F_0^*$ as $D(p)$ ($p = (p_2, p_3)$). Then we have

$$D(p)^2 = A_+(p) \oplus A_-(p) \oplus A_+(p) \oplus A_-(p), \tag{1.16}$$

where $A_\pm(p)$ are harmonic oscillators defined as follows.

$$A_\pm(p) = -\frac{d^2}{dx_1^2} + (\lambda x_1 + p_2)^2 \pm \lambda + p_3^2 + m^2 \tag{1.17}$$

He then switched on the short-range potential $V(x)$ by perturbative argument. Roughly speaking, his assumption means that the absolute value of each component of V is dominated from above by $C\langle x'\rangle^{-1-\epsilon}\langle x\rangle^{-\epsilon}$ ($x' = (x_2, x_3)$) for sufficiently large x. We remark that $\epsilon > 0$ is used as a sufficiently small parameter throughout this paper. To be accurate, $\langle x'\rangle^{1+\epsilon} V(x)$ is required to be a $H_0(\lambda)$-compact operator.

In this paper we treat directly the following operator

$$H(\lambda) = \alpha_1\left(P_1 + \frac{\lambda}{2}x_2\right) + \alpha_2\left(P_2 - \frac{\lambda}{2}x_1\right) + \alpha_3 P_3 + m\beta + V(x), \tag{1.18}$$

where $V(x)$ is a matrix potential. Our strategy is to apply Mourre's commutator method directly to this operator, which enables us to include the long-range diagonal components for $V(x)$. In this case it seems that an appropriate choice of the conjugate operator is

$$\frac{P_3}{\langle P_3\rangle} \cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3\rangle},$$

which is inspired by [11], when we proved the limiting absorption principle for time-periodic Schrödinger operator. In fact the method of the proof shares many ideas in common with [11]. Namely we rewrite $H_0(\lambda)$ by a direct integral and the conjugate operator A acts on each space of fiber. Our main results are Theorem 3.4 and Corollary 3.7.

2. Conjugate operator

Let us recall

$$Q_0 = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix}, \quad D_0 = \sigma(P - \mathbf{a}_0), \tag{2.1}$$

with

$$\mathbf{a}_0(x) = \left(-\frac{\lambda x_2}{2}, \frac{\lambda x_1}{2}, 0\right). \tag{2.2}$$

The Dirac Hamiltonian $Q_0 + m\beta$ can be diagonalized by sandwiching it between a unitary operator U and $U^* = U^{-1}$. In the beginning of this section we introduce a unitary operator which diagonalizes the self-adjoint operator $H_0(\lambda)$. Secondly we give a conjugate operator associated with the diagonalized Dirac Hamiltonian. Finally we show Mourre's inequality for original Hamiltonians $H_0(\lambda)$ and $H(\lambda)$.

Let Q_0 be the self-adjoint operator as in (2.1) and $|Q_0| = \sqrt{Q_0^2}$, $|H_0(\lambda)| = \sqrt{H_0(\lambda)^2}$. We define a unitary operator U_{FW}, which diagonalizes $H_0(\lambda)$, in the following way.

DEFINITION 2.1. (i) At first we define a signature function associated with Q_0 by

$$\operatorname{sgn} Q_0 = \begin{cases} \dfrac{Q_0}{|Q_0|}, & \text{on } (\ker Q_0)^{\perp} \\ 0, & \text{on } (\ker Q_0) \end{cases} \tag{2.3}$$

We note that $\operatorname{sgn} Q_0$ is isometory on $(\ker Q_0)^{\perp}$.
(ii) We can easily see that $m/|H_0(\lambda)| \leq 1$. So we denote the square root of $1/2(1 \pm m/|H_0(\lambda)|)$ as $a_{\pm}$. i.e.

$$a_{\pm} = \frac{1}{\sqrt{2}}\sqrt{1 \pm \frac{m}{|H_0(\lambda)|}}. \tag{2.4}$$

(iii) Combining these operators we define the operator U_{FW} as

$$U_{FW} = a_+ + \beta(\operatorname{sgn} Q_0)a_-. \tag{2.5}$$

Lemma 2.2. (i) *U_{FW} is a unitary operator on $L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$. Further,*

$$U_{FW}^* = U_{FW}^{-1} = a_+ - \beta(\operatorname{sgn} Q_0)a_-. \tag{2.6}$$

(ii) *$H_0(\lambda)$ can be diagonalized by U_{FW} as follows.*

$$U_{FW} H_0(\lambda) U_{FW}^{-1} = |H_0(\lambda)|\beta = \begin{pmatrix} \sqrt{D_0^2 + m^2} & 0 \\ 0 & -\sqrt{D_0^2 + m^2} \end{pmatrix}. \tag{2.7}$$

Proof. See 5.6.1 in [10]. □

We denote the diagonalized Dirac Hamiltonian as $\hat{H}_0(\lambda)$. i.e.

$$\hat{H}_0(\lambda) = U_{FW} H_0(\lambda) U_{FW}^{-1}.$$

We rewrite (1.12) as follows.

$$D_0^2 = \begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix}.$$

Here $D_\pm$ are the operators acting on $L^2(\mathbb{R}^3)$ such that

$$D_\pm = \left(P_1 + \frac{\lambda}{2}x_2\right)^2 + \left(P_2 - \frac{\lambda}{2}x_1\right)^2 + P_3^2 \pm \lambda.$$

It is well-known that $(P_1 + \lambda/2x_2)^2 + (P_2 - \lambda/2x_1)^2$ has eigenvalues

$$\{\lambda(2n+1) \mid n = 0, 1, 2, \ldots\}.$$

We denote the eigenprojection on each eigenspace as Π_n. With these projections, $\sqrt{D_0^2 + m^2}$ can be rewritten as follows.

$$\sqrt{D_0^2 + m^2} = \sum_{n=0}^{\infty} \begin{pmatrix} d_n \otimes \Pi_n & 0 \\ 0 & d_{n+1} \otimes \Pi_n \end{pmatrix}, \tag{2.8}$$

with $d_n = d_n(P_3) = \sqrt{2\lambda n + P_3^2 + m^2}$.

Combining (2.7) and (2.8), we have

$$f(\hat{H}_0(\lambda)) = \sum_{n=0}^{\infty} \begin{pmatrix} f(d_n) \otimes \Pi_n & & & \\ & f(d_{n+1}) \otimes \Pi_n & & \\ & & f(-d_n) \otimes \Pi_n & \\ & & & f(-d_{n+1}) \otimes \Pi_n \end{pmatrix},$$

for any Borel function f.

Now we define the conjugate operator. At first we define

$$\hat{A} = \frac{1}{2}\left\{\frac{P_3}{\langle P_3 \rangle} \cdot x_3 + x_3 \cdot \frac{P_3}{\langle P_3 \rangle}\right\}. \tag{2.9}$$

We note that $\hat{A}$ is essentially self-adjoint operator on $D(|x_3|)$. (It is obtained by use of Nelson's commutator theorem [9].) The conjugate operator for the Dirac Hamiltonian associated with constant magnetic field is defined by sandwiching $\hat{A}\beta$ between U_{FW}^{-1} and U_{FW} :

$$A = U_{FW}^{-1}(\hat{A}\beta)U_{FW}. \tag{2.10}$$

Letting F be a Fourier transformation with respect to x_3-variable. We define the self-adjoint operator A_F by $F\hat{A}F^{-1}$. Then we have

$$(e^{itA_F}\phi)(x_1, x_2, p_3) = \left|\frac{\partial\Gamma_t}{\partial p_3}(p_3)\right|^{1/2}\phi(x_1, x_2, \Gamma_t(p_3)), \tag{2.11}$$

for $\phi \in L^2(\mathbb{R}_x^2 \times \mathbb{R}_p)$. Here Γ_t is a solution of the following equation.

$$\begin{cases} \dfrac{d}{dt}\Gamma_t(p_3) = \langle\Gamma_t(p_3)\rangle^{-1}\Gamma_t(p_3) \\ \Gamma_0(p_3) = p_3 \end{cases} \tag{2.12}$$

For the proof, see Appendix 1 in [8]. Therefore the unitary group $e^{it\hat{A}\beta}$ is rewritten

$$(Fe^{it\hat{A}\beta}F^{-1}\phi)(x_1, x_2, p_3) = \begin{pmatrix} \left|\dfrac{\partial\Gamma_t}{\partial p_3}(p_3)\right|^{1/2}\phi_1(x_1, x_2, \Gamma_t(p_3)) \\ \left|\dfrac{\partial\Gamma_t}{\partial p_3}(p_3)\right|^{1/2}\phi_2(x_1, x_2, \Gamma_t(p_3)) \\ \left|\dfrac{\partial\Gamma_{-t}}{\partial p_3}(p_3)\right|^{1/2}\phi_3(x_1, x_2, \Gamma_{-t}(p_3)) \\ \left|\dfrac{\partial\Gamma_{-t}}{\partial p_3}(p_3)\right|^{1/2}\phi_4(x_1, x_2, \Gamma_{-t}(p_3)) \end{pmatrix}. \tag{2.13}$$

Before we compute the commutator $i[H(\lambda), A]$, we have to care that the following matters hold.

Lemma 2.3. (i) *A is a self-adjoint operator on $\mathbb{H}$.*
(ii) *e^{-itA} leaves $D(H_0(\lambda))$ invariant, i.e.*

$$\sup_{|t|\leq 1} \|H_0(\lambda)e^{itA}(H_0(\lambda) + i)^{-1}\phi\|_{\mathbb{H}} < \infty \quad \textit{for} \quad \phi \in \mathbb{H}, \tag{2.14}$$

where $\|\cdot\|_{\mathbb{H}}$ denotes the operator norm on $\mathbb{H}$.

Proof. The self-adjointness of A is easily obtained from that of $\hat{A}\beta$. To see the invariance of $D(H_0(\lambda))$, it is sufficient to show the following

$$\sup_{|t|\leq 1} \|\hat{H}_0(\lambda)e^{it\hat{A}\beta}(\hat{H}_0(\lambda) + i)^{-1}\phi\|_{\mathbb{H}} < \infty \quad \text{for} \quad \phi \in \mathbb{H}. \tag{2.15}$$

From the arguements (2.11) and (2.13), we have

$$F\hat{H}_0(\lambda)e^{it\hat{A}\beta}(\hat{H}_0(\lambda)+i)^{-1}F^{-1}\phi$$
$$=\sum_{n=0}^{\infty}\begin{pmatrix} D^{+}_{n,t}\left|\dfrac{\partial\Gamma_t}{\partial p_3}(p_3)\right|^{1/2}\phi_1(x_1,x_2,\Gamma_t(p_3)) \\ D^{+}_{n+1,t}\left|\dfrac{\partial\Gamma_t}{\partial p_3}(p_3)\right|^{1/2}\phi_2(x_1,x_2,\Gamma_t(p_3)) \\ D^{-}_{n,-t}\left|\dfrac{\partial\Gamma_{-t}}{\partial p_3}(p_3)\right|^{1/2}\phi_3(x_1,x_2,\Gamma_{-t}(p_3)) \\ D^{-}_{n+1,-t}\left|\dfrac{\partial\Gamma_{-t}}{\partial p_3}(p_3)\right|^{1/2}\phi_4(x_1,x_2,\Gamma_{-t}(p_3)) \end{pmatrix}$$

where $D^{\pm}_{n,\alpha}=d_n(p_3)(d_n(\Gamma_\alpha(p_3))\pm i)^{-1}\otimes\Pi_n$. By integrating (2.12), we have

$$|\Gamma_\alpha(p_3)-p_3|\le 1 \quad (|\alpha|\le 1). \tag{2.16}$$

So (2.14) is obtained from the fact that $D^{\pm}_{n,\alpha}$ is bounded uniformly for $n\in\mathbb{N}$ and $|\alpha|\le 1$. □

Before we show Mourre's inequality, we introduce the usual functional calculus, started by Helffer and Sjöstrand.

Suppose that $f\in C^\infty(\mathbb{R})$ satisfies the following condition for some $m_0\in\mathbb{R}$.

$$|f^{(k)}(t)|\le C_k(1+|t|)^{m_0-k}, \quad {}^\forall k\in\mathbb{N}\cup\{0\}. \tag{2.17}$$

Then we can construct an almost analytic extension $\tilde{f}(z)$ of $f(t)$ having the following properties

$$\tilde{f}(t)=f(t), \quad t\in\mathbb{R},$$
$$\operatorname{supp}\tilde{f}\subset\{z;|\operatorname{Im}z|\le 1+|\operatorname{Re}z|\},$$

$$|\partial_{\bar{z}}\tilde{f}(z)|\le C_N|\operatorname{Im}z|^N\langle z\rangle^{m_0-1-N}, \quad {}^\forall N\in\mathbb{N}. \tag{2.18}$$

Then for all f, satisfying (2.17) for $m_0<0$ and a self-adjoint operator H, we have

$$f(H)=\frac{1}{2\pi i}\int_{\mathbb{C}}\frac{\partial\tilde{f}}{\partial\bar{z}}(z)(z-H)^{-1}dz\wedge d\bar{z}. \tag{2.19}$$

With this form, we can compute the commutator of an operator P and $g(A)$ in the following way.

For operators P and Q, we define $ad^0_Q(P)=P$ and inductively $ad^m_Q(P)=[ad^{m-1}_Q(P),Q]$ for $m\in\mathbb{N}$.

Lemma 2.4. *Let A and P be self-adjoint operators on $\mathbb{H}$. Suppose that $ad_A^m(P)(A+i)^{-m}$ extends to a bounded operator for $1 \leq m \leq n$. Then for any $g \in C^\infty(\mathbb{R})$ satisfying* (2.17) *with $m_0 < 0$, we have*

$$Pg(A) = \sum_{m=0}^{n-1} \frac{g^{(m)}(A)}{m!} ad_A^m(P) + \frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{g}(z) R^r_{n,A,P}(z) dz \wedge d\bar{z} \tag{2.20}$$

where $R^r_{n,A,P}(z) = (z-A)^{-n} ad_A^n(P)(z-A)^{-1}$, and

$$g(A)P = \sum_{m=0}^{n-1} ad_A^m(P)\frac{(-1)^m}{m!} g^{(m)}(A) + \frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{g}(z) R^l_{n,A,P}(z) dz \wedge d\bar{z} \tag{2.21}$$

where $R^l_{n,A,P}(z) = (z-A)^{-1} ad_A^n(P)(z-A)^{-n}$ and $\tilde{g}(z)$ denotes an almost analytic extension of $g(t)$.

For the proof of above results, see [4].

3. Limiting absorption principle for long-range potentials

Now we show the Mourre's inequality for the Dirac Hamiltonian by choosing A defined in the previous section as the conjugate operator.

Lemma 3.1. *Let $\mathbb{R}_{\mathbb{N}}$ be the following discrete subset of $\mathbb{R}$*

$$\mathbb{R}_{\mathbb{N}} = \{\pm\sqrt{2\lambda n + m^2} \quad | \; n = 0, 1, 2, \ldots\} \subset \mathbb{R}.$$

We take a compact interval $I \subset \mathbb{R} \setminus \mathbb{R}_{\mathbb{N}}$ arbitrarily. Then there exists $\alpha > 0$ such that the following inequality holds for any real valued $f \in C_0^\infty(I)$

$$f(H_0(\lambda)) i [H_0(\lambda), A] f(H_0(\lambda)) \geq \alpha f(H_0(\lambda))^2. \tag{3.1}$$

Proof. By the relations (2.7) and (2.10), it is sufficient to show the inequality

$$f(\hat{H}_0(\lambda)) i [\hat{H}_0(\lambda), \hat{A}\beta] f(\hat{H}_0(\lambda)) \geq \alpha f(\hat{H}_0(\lambda))^2. \tag{3.2}$$

We rewrite the commutator as follow.

$$i\left[\hat{H}_0(\lambda), \hat{A}\beta\right] = \begin{pmatrix} i\left[\sqrt{D_0^2+m^2}, \hat{A}\right] & \\ & i\left[\sqrt{D_0^2+m^2}, \hat{A}\right] \end{pmatrix}. \tag{3.3}$$

We proceed the calculus more precisely to see that

$$(3.4)\qquad i\left[\sqrt{D_0^2+m^2},\hat{A}\right]=\sum_{n=0}^{\infty}\begin{pmatrix} i\left[d_n,\hat{A}\right]\otimes\Pi_n & \\ & i\left[d_{n+1},\hat{A}\right]\otimes\Pi_n\end{pmatrix}$$

by (2.8). From (3.3) and (3.4) the left hand side of (3.2) is rewritten as

$$(3.5)\qquad f(\hat{H}_0(\lambda))i\left[\hat{H}_0(\lambda),\hat{A}\beta\right]f(\hat{H}_0(\lambda))=\begin{pmatrix} I_1 & & & \\ & I_2 & & \\ & & I_3 & \\ & & & I_4\end{pmatrix}$$

where

$$I_1=\sum_{n=0}^{\infty}f(d_n)i[d_n,\hat{A}]f(d_n)\otimes\Pi_n,$$
$$I_2=\sum_{n=0}^{\infty}f(d_{n+1})i[d_{n+1},\hat{A}]f(d_{n+1})\otimes\Pi_n,$$
$$I_3=\sum_{n=0}^{\infty}f(-d_n)i[d_n,\hat{A}]f(-d_n)\otimes\Pi_n,$$
$$I_4=\sum_{n=0}^{\infty}f(-d_{n+1})i[d_{n+1},\hat{A}]f(-d_{n+1})\otimes\Pi_n.$$

We note that all the sum in $I_1,\dots,I_4$ are finite since f is a compactly supported function. By an elementary caluculus, we have

$$(3.6)\qquad i[d_l,\hat{A}]=\frac{P_3^2}{\sqrt{2\lambda l+P_3^2+m^2}\langle P_3\rangle}\qquad (l\in\mathbb{N}\cup\{0\}).$$

Since $\operatorname{supp} f\subset I\subset\mathbb{R}\setminus\mathbb{R}_{\mathbb{N}}$, P_3 is away from zero when $P_3\in\operatorname{supp} f(d_l(P_3))$ or $P_3\in\operatorname{supp} f(-d_l(P_3))$. So there exist $C_l>0$ such that

$$f(d_l)i[d_l,\hat{A}]f(d_l)\otimes\Pi_l\geq C_l f(d_l)^2\otimes\Pi_l,$$
$$f(-d_l)i[d_l,\hat{A}]f(-d_l)\otimes\Pi_l\geq C_l f(-d_l)^2\otimes\Pi_l.$$

Since only a finite number of $l=l_j\quad(j=1,\dots,N)$ is concerned, we have (3.2) with $\alpha=\inf_{j=1\dots,N}C_{l_j}$. □

Now we give the assumption for the potential, which is necessary to prove Mourre's inequality associated to $H(\lambda)$. After that we give an example of V satisfying this assumption. The potential V consists of a sum of long-range part and short-range part. In our case short-range potential means $V(x) = O(\langle x\rangle^{-\epsilon}\langle x_3\rangle^{-1-\epsilon})$ as $|x| \to \infty$. And long-range part is a multiplication of a real valued function $\varphi(x)$ such that $\varphi(x) = O(\langle x\rangle^{-\epsilon})$ as $|x| \to \infty$. More precisely we assume that V satisfies the following.

ASSUMPTION 3.2. $V = V(x)$ is a multiplicative operator of a 4×4 Hermitian matrix satisfying the following properties.
(i) V is a $H_0(\lambda)$-compact operator.
(ii) The form $[V, A]$ can be extended to a $H_0(\lambda)$-compact operator.

For example a 4×4 matrix $V(x)$ satisfying the following inequality is $H_0(\lambda)$-compact.

$$|V(x)| \le C\langle x\rangle^{-\epsilon} \quad (x \in \mathbb{R}^3). \tag{3.7}$$

It is owing to the fact that $V(x)(-\Delta_x + 1)^{-1}$ is compact. (It is due to Theorem 2.6 in [1].) Under this assumption we show Mourre's inequality for $H(\lambda)$.

Lemma 3.3. *Suppose V satisfies* Assumption 3.2.
(i) *We take $\mu \in \mathbb{R}\setminus\mathbb{R}_{\mathbb{N}}$ and $\delta > 0$ so that the closed interval $I \equiv [\mu-\delta, \mu+\delta] \subset \mathbb{R}\setminus\mathbb{R}_{\mathbb{N}}$. There exist $\alpha > 0$ and a compact operator K such that the following inequality holds for all $f \in C_0^\infty(I)$.*

$$f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \ge \alpha f(H(\lambda))^2 + K. \tag{3.8}$$

(ii) *There is no accumulation point of $\sigma_{pp}(H(\lambda))$ in $\mathbb{R}\setminus\mathbb{R}_{\mathbb{N}}$. For $\mu \in \mathbb{R}\setminus(\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$, there exist $\delta_0 > 0$ and $\alpha_0 > 0$ such that the following inequality holds for all $f \in C_0^\infty([\mu-\delta_0, \mu+\delta_0])$.*

$$f(H(\lambda))i[H(\lambda), A]f(H(\lambda)) \ge \alpha_0 f(H(\lambda))^2. \tag{3.9}$$

Proof. From (2.19) we have

$$f(H(\lambda)) - f(H_0(\lambda)) = \frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar z} f(z)(z-H(\lambda))^{-1}V(z-H_0(\lambda))^{-1}dz\wedge d\bar z$$

for $f \in C_0^\infty(\mathbb{R})$. We can easily see that $f(H(\lambda)) - f(H_0(\lambda))$ is a compact operator since $V(H_0(\lambda)+i)^{-1}$ is compact. Combing this fact and (3.1), we have (3.8) by replacing $f(H_0(\lambda))$ in (3.1) by $f(H(\lambda))$. As for the non-existence of the accumulation point of $\sigma_{pp}(H(\lambda))$, see Theorem 2.2 in [7]. (3.9) follows from the arguement in [8] □

With this inequality we have the limiting absorption principle for the Dirac Hamiltonian.

Theorem 3.4. *Let $s > 1/2$. Suppose V satisfies* Assumption 3.2. *Then for $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$, the following limits*

$$R^{\pm}(\mu) = \lim_{\epsilon \downarrow 0} \langle x_3 \rangle^{-s} (H(\lambda) - \mu \mp i\epsilon)^{-1} \langle x_3 \rangle^{-s} \tag{3.10}$$

exist and $R^{\pm}(\mu)$ are continuous with respect to $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$.

Sketch of proof
From (3.9) and Theorem 2.2 in [7], we can see that the boundary value $\langle A \rangle^{-s}(H(\lambda) - \mu \mp i0)^{-1}\langle A \rangle^{-s}$ exist for $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$. To see the existence of (3.10), it is sufficient to show the boundedness of $\langle A \rangle^{s} \langle x_3 \rangle^{-s}$. Since $\langle \hat{A} \rangle^{s} \langle x_3 \rangle^{-s}$ is bounded, it is sufficient to show $\langle x_3 \rangle^{s} U_{FW} \langle x_3 \rangle^{-s}$ is bounded. We prove it in the following Lemma. Before that we introduce smooth functions. Let $\chi(t) \in C^{\infty}(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} \dfrac{1}{\sqrt{2}} & \left(t > -\dfrac{m^2}{3} \right) \\ 0 & \left(t < -\dfrac{2m^2}{3} \right). \end{cases} \tag{3.11}$$

With this function we define $F_{\pm}(t)$ and $F_{\chi,\pm}$ as follows.

$$\begin{aligned} F_{+}(t) &= \chi(t) \sqrt{1 + \frac{m}{\sqrt{t + m^2}}} \\ F_{-}(t) &= \chi(t) \left(\sqrt{1 + \frac{m}{\sqrt{t + m^2}}} \right)^{-1} \frac{1}{\sqrt{t + m^2}} \\ F_{\chi,+}(t) &= F_{+}(t) - \chi(t) \\ F_{\chi,-}(t) &= \sqrt{t + m^2} F_{-}(t) - \chi(t) \end{aligned}$$

Then we can easily verify that

$$\begin{aligned} a_{+} &= F_{+}(Q_0^2), \\ a_{-} \operatorname{sgn} Q_0 &= F_{-}(Q_0^2) Q_0 = Q_0 F_{-}(Q_0^2). \end{aligned}$$

Obviously $[Q_0, F_{-}(Q_0^2)] = 0$. By the construction of these functions, we can also see that $F_{\chi,\pm}(t)$ satisfy (2.17) with $m_0 < 0$. So we apply the functional calculus in Section 2 to $F_{\chi,\pm}(t)$ and see the following properties hold.

Lemma 3.5. *Suppose $0 \le s \le 2$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Then*
(i) *For $0 < s \le 1$, there exists $C_s > 0$ such that*

$$\| \langle x \rangle^{s} (z - Q_0^2)^{-1} \langle x \rangle^{-s} \|_{\mathbb{H}} \le C_s (|\operatorname{Im} z|^{-1} + |\operatorname{Im} z|^{-2} \langle z \rangle). \tag{3.12}$$

(ii) *For* $1 < s \leq 2$, *there exists* $C'_s > 0$ *such that*

$$\|\langle x\rangle^s (z - Q_0^2)^{-1}\langle x\rangle^{-s}\|_{\mathbb{H}} \leq C'_s(|\operatorname{Im} z|^{-1} + |\operatorname{Im} z|^{-2}\langle z\rangle + |\operatorname{Im} z|^{-3}\langle z\rangle^2). \tag{3.13}$$

(iii) $\langle x\rangle^s F_+(Q_0^2)\langle x\rangle^{-s}$ *and* $\langle x\rangle^s F_-(Q_0^2)Q_0\langle x\rangle^{-s}$ *are bounded operators.*

Proof. For the proof of (i) and (ii), we use the resolvent equation. Suppose $0 < s \leq 1$. Then

$$\langle x\rangle^s (z - Q_0^2)^{-1}\langle x\rangle^{-s} = (z - Q_0^2)^{-1} + (z - Q_0^2)^{-1}(Q_0^2 + 1) \tag{3.14}$$

$$\times (Q_0^2 + 1)^{-1}[\langle x\rangle^s, Q_0^2](z - Q_0^2)^{-1}\langle x\rangle^{-s}. \tag{3.15}$$

From the boundedness of $(Q_0^2 + 1)^{-1}[\langle x\rangle^s, Q_0^2]$ and the following estimate

$$\|(z - Q_0^2)^{-1}(Q_0^2 + 1)\|_{\mathbb{H}} \leq C(|\operatorname{Im} z|^{-1}\langle z\rangle + 1), \tag{3.16}$$

we obtain (i). As for the case $1 < s \leq 2$, we rewrite the last term $(z - Q_0^2)^{-1}[\langle x\rangle^s, Q_0^2](z - Q_0^2)^{-1}\langle x\rangle^{-s}$ as

$$(z - Q_0^2)^{-1}(Q_0^2 + 1)(Q_0^2 + 1)^{-1}[\langle x\rangle^s, Q_0^2]\langle x\rangle^{-s+1} \tag{3.17}$$

$$\times \langle x\rangle^{s-1}(z - Q_0^2)^{-1}\langle x\rangle^{-s+1}\langle x\rangle^{-1}. \tag{3.18}$$

By using the result for $0 < s \leq 1$, we have the inequality for $1 < s \leq 2$.

With these estimates, we prove (iii). Since $\chi(Q_0^2) \equiv 1$, we can easily see that

$$\langle x\rangle^s F_+(Q_0^2)\langle x\rangle^{-s} = \langle x\rangle^s F_{\chi,+}(Q_0^2)\langle x\rangle^{-s} + I. \tag{3.19}$$

Since $F_{\chi,+}(t)$ satisfies (2.18) for $m_0 = -1/2$, $F_{\chi,+}(Q_0^2)$ can be rewritten as follows.

$$\frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar z}\tilde F_{\chi,+}(z)\langle x\rangle^s (z - Q_0^2)^{-1}\langle x\rangle^{-s} dz \wedge d\bar z. \tag{3.20}$$

From this formula and (i) (ii) we have

$$\begin{aligned}&\|\partial_{\bar z}\tilde F_{\chi,+}(z)\langle x\rangle^s (z - Q_0^2)^{-1}\langle x\rangle^{-s}\|_{\mathbb{H}}\\ &\qquad \leq C|\partial_{\bar z}\tilde F_{\chi,+}(z)|(|\operatorname{Im} z|^{-1} + |\operatorname{Im} z|^{-2}\langle z\rangle + |\operatorname{Im} z|^{-3}\langle z\rangle^2).\end{aligned}$$

From (2.18) we have

$$\|\partial_{\bar z}\tilde F_{\chi,+}(z)\langle x\rangle^s (z - Q_0^2)^{-1}\langle x\rangle^{-s}\|_{\mathbb{H}} \leq C\langle z\rangle^{-5/2}.$$

This implies the boundedness of $\langle x\rangle^s F_+(Q_0^2)\langle x\rangle^{-s}$.

In a similar way, we rewrite $\langle x\rangle^s F_-(Q_0^2)Q_0\langle x\rangle^{-s}$ as

$$\langle x\rangle^s F_{\chi,-}(Q_0^2)\langle x\rangle^{-s}\langle x\rangle^s \frac{Q_0}{\sqrt{Q_0^2+m^2}}\langle x\rangle^{-s} + \langle x\rangle^s \frac{Q_0}{\sqrt{Q_0^2+m^2}}\langle x\rangle^{-s}. \tag{3.21}$$

It is sufficient to show the boundedness of $\langle x\rangle^s Q_0/\sqrt{Q_0^2+m^2}\langle x\rangle^{-s}$. To see this, we denote $\chi(t)/\sqrt{t+m^2} \in C^\infty(\mathbb{R}^3)$ as $S(t)$ and its almost analytic extension as $\tilde{S}(z)$. We can easily see that $S(Q_0^2)\langle x\rangle^s Q_0\langle x\rangle^{-s}$ is bounded. So we obtain the boundedness of $\langle x\rangle^s F_-(Q_0^2)Q_0\langle x\rangle^{-s}$ if we show that $[\langle x\rangle^s, S(Q_0^2)]Q_0\langle x\rangle^{-s}$ is bounded. We rewrite it as follows.

$$\frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar z}\tilde{S}(z)\frac{Q_0^2+1}{z-Q_0^2}(Q_0^2+1)^{-1}[\langle x\rangle^s, Q_0^2]Q_0\langle x\rangle^{-s}\langle x\rangle^s(z-Q_0^2)^{-1}\langle x\rangle^{-s}dz\wedge d\bar{z}.$$

By an elementary calculus, we have $(Q_0^2+1)^{-1}[\langle x\rangle^s, Q_0^2]Q_0\langle x\rangle^{-s}$ bounded. Combining (i) and (ii), we have

$$\begin{aligned}\|[\langle x\rangle^s, S(Q_0^2)]Q_0\langle x\rangle^{-s}\|_{\mathbb{H}} &\le C\int_{\mathbb{C}} |\partial_{\bar z}\tilde{S}(z)|\{1+|\operatorname{Im} z|^{-1}\}\\ &\times\{|\operatorname{Im} z|^{-1}+|\operatorname{Im} z|^{-2}\langle z\rangle+|\operatorname{Im} z|^{-3}\langle z\rangle^2\}dz\wedge d\bar{z}<\infty.\end{aligned}$$

This implies the boundedness of $\langle x\rangle^s F_-(Q_0^2)Q_0\langle x\rangle^{-s}$. □

Next we give an example of V. It requires smoothness, but allows long-range part in its diagonal components.

Lemma 3.6. *Let V be a 4×4 Hermitian matrix of the form*

$$V(x) = (v_{ij}(x)) + \varphi(x)I_4 \equiv V_s(x) + V_l(x) \tag{3.22}$$

where $V_s(x) = (v_{ij}(x))$ is an Hermitian matrix and I_4 is an identity matrix. Suppose the following conditions hold. Then $V(x)$ satisfies Assumption 3.2.

There exist $\delta > 0$ such that the following inequalities hold for all multi-index α.

$$|\partial_x^\alpha v_{ij}(x)| \le C_\alpha\langle x\rangle^{-\delta-|\alpha|}\langle x_3\rangle^{-1} \quad (1\le i,j\le 4). \tag{3.23}$$

$\varphi(x)\in C^\infty(\mathbb{R}^3)$ is real valued and satisfies

$$|\partial_x^\alpha \varphi(x)| \le C'_\alpha\langle x\rangle^{-\delta-|\alpha|}. \tag{3.24}$$

The relatively compactness of $V(x)$ itself is clear since V satisfies (3.7). So we only have to show the relatively compactness of $[V, A]$. We prove the relatively compactness of $[V_s, A] = [V_s, U_{FW}^{-1}\hat{A}\beta U_{FW}]$ at first. From the boundedness of $\langle x_3\rangle^{-1}\hat{A}\beta$

and the relatively compactness of $V_s\langle x_3\rangle$, it is sufficient to show that $\langle x_3\rangle U_{FW}\langle x_3\rangle^{-1}$ and $\langle x_3\rangle U_{FW}^{-1}\langle x_3\rangle^{-1}$ are bounded operators in $\mathbb{H}$. We have already proved it in Lemma 3.5.

Next we treat the long-range term. The conjugate operator A can be decomosed into the sum of $J_1, \dots, J_4$ where

$$\begin{aligned} J_1 &= F_+(Q_0^2)\hat{A}\beta F_+(Q_0^2), \\ J_2 &= F_+(Q_0^2)\hat{A}\beta^2 F_-(Q_0^2)Q_0, \\ J_3 &= \beta F_-(Q_0^2)Q_0\hat{A}\beta F_+(Q_0^2), \\ J_4 &= \beta F_-(Q_0^2)Q_0\hat{A}\beta^2 F_-(Q_0^2)Q_0. \end{aligned}$$

We prove that the $H_0(\lambda)$ - compactness holds for each of $[V_l, J_1], \dots, [V_l, J_4]$. To see this we use the functional calculus again and rewrite J_1 as follows.

$$\begin{aligned} F_+(Q_0^2)\hat{A}\beta F_+(Q_0^2) &= \hat{A}\beta F_+(Q_0^2)^2 + [F_{\chi,+}(Q_0^2), \hat{A}\beta]F_+(Q_0^2) \\ &\equiv J_1' + J_1'' \end{aligned}$$

At first we prove the boundedness of J_1'' and conseqently the relatively compactness of $[V_l, J_1'']$. By using (2.19), we rewrite $[F_{\chi,+}(Q_0^2), \hat{A}\beta]$ as follows.

$$\frac{1}{2\pi i}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{F}_{\chi,+}(z)(z - Q_0^2)^{-1}[Q_0^2, \hat{A}\beta](z - Q_0^2)^{-1}dz \wedge d\bar{z}. \tag{3.25}$$

From (3.16) we have $[Q_0^2, \hat{A}\beta](z - Q_0^2)^{-1}$ is dominated from above by $C\{1 + |\operatorname{Im} z|\}$. So we have

$$\|[F_{\chi,+}(Q_0^2), \hat{A}\beta]\| \le C\int_{\mathbb{C}} |\partial_{\bar{z}}\tilde{F}_{\chi,+}(z)|\{|\operatorname{Im} z|^{-1} + |\operatorname{Im} z|^{-2}\langle z\rangle\}dz \wedge d\bar{z}. \tag{3.26}$$

Since the almost analytic extension $\tilde{F}_{\chi,+}(z)$ satisfies

$$|\partial_{\bar{z}}\tilde{F}_+(z)| \le C_N|\operatorname{Im} z|^N\langle z\rangle^{-3/2-N} \quad ({}^\forall N \in \mathbb{N}), \tag{3.27}$$

we have $[F_{\chi,+}(Q_0^2), \hat{A}\beta]$ is bounded and inductively $[V_l, J_1'']$ is $H_0(\lambda)$-compact. So we only have to show the relatively compactness of $[V_l, J_1']$.

$$[V_l, J_1'] = [V_l, \hat{A}\beta]F_+(Q_0^2)^2 + \hat{A}\beta[V_l, F_+(Q_0^2)^2]. \tag{3.28}$$

Clearly $[V_l, \hat{A}\beta]F_+(Q_0^2)$ is $H_0(\lambda)$-compact. Again we rewrite the commutator in the second term, by use of (2.19). Then we have $\langle x\rangle^{1+\delta}[V_l, F_+(Q_0^2)^2]$ is bounded. Combing these facts, we have the relatively compactness of $[V_l, J_1]$.

As for the commutator $[V_l, J_2], \dots, [V_l, J_4]$ we also replace $F_\pm$ by $F_{\chi,\pm}$ and use the functional calculus. The proof of relatively compactness of $[V_l, J_2]$ and $[V_l, J_3]$ are almost the same. We only give the proof for J_2. We also estimate the 'principle' part before we compute the commutator with V_l.

$$J_2 = \hat{A}F_+(Q_0^2)F_-(Q_0^2)Q_0 + [F_+(Q_0^2), \hat{A}]F_-(Q_0^2)Q_0 \tag{3.29}$$

It is sufficient to show that $[V_l, \hat{A}F_+(Q_0^2)F_-(Q_0^2)Q_0]$ is a $H_0(\lambda)$-compact operator. We decompose it into the following sum.

$$\begin{aligned}[V_l, \hat{A}]&F_+(Q_0^2)F_-(Q_0^2)Q_0 \\ &+\hat{A}[V_l, F_+(Q_0^2)F_-(Q_0^2)]Q_0 \\ &\qquad + \hat{A}F_+(Q_0^2)F_-(Q_0^2)[V_l, Q_0].\end{aligned}$$

We can easily see that the first and the third term is relatively compact since $\langle x\rangle^{1+\delta}[V_l, Q_0]$ is bounded. As for the second term, we can also see the relatively compactness in the same argument as we have done in the proof of Lemma 3.5 (iii).

As for J_4, the proof is similar. We rewrite it as

$$\hat{A}F_-(Q_0^2)^2Q_0^2 + [F_-(Q_0^2)Q_0, \hat{A}]F_-(Q_0^2)Q_0 \tag{3.30}$$

We can also obtain the relatively compactness by estimating the term $[V, \hat{A}F_-(Q_0^2)^2Q_0^2]$.

Corollary 3.7. *Let V be a 4×4 Hermitian matrix and $s > 1/2$. Suppose V satisfies the condition in* Lemma 3.6. *Then the following limits*

$$R^{\pm}(\mu) = \lim_{\epsilon\downarrow 0}\langle x_3\rangle^{-s}(H(\lambda) - \mu \mp i\epsilon)^{-1}\langle x_3\rangle^{-s} \tag{3.31}$$

exist for $\mu \in \mathbb{R} \setminus (\mathbb{R}_{\mathbb{N}} \cup \sigma_{pp}(H(\lambda)))$ and $R^{\pm}(\mu)$ are continuous with respect to μ.

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