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2-STRATIFOLD SPINES OF CLOSED 3-MANIFOLDS

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Abstract
2-stratifolds are a generalization of 2-manifolds in that there are disjoint simple closed branch curves. We obtain a list of all closed 3-manifolds that have a 2-stratifold as a spine.

1. Introduction

2-*stratifolds* form a special class of 2-dimensional stratified spaces. A (closed with empty 0-stratum) 2-stratifold is a compact connected 2-dimensional cell complex X that contains a 1-dimensional subcomplex $X^{(1)}$, consisting of branch curves, such that $X - X^{(1)}$ is a (not necessarily connected) 2-manifold. The exact definition is given in section 2. X can be constructed from a disjoint union $X^{(1)}$ of circles and compact 2-manifolds W^2 by attaching each component of ∂W^2 to $X^{(1)}$ via a covering map $\psi : \partial W^2 \to X^{(1)}$, with $\psi^{-1}(x) > 2$ for $x \in X^{(1)}$. A slightly more general class of 2-dimensional stratified spaces, called *multibranched surfaces* and which have been defined and studied in [13], is obtained by allowing boundary curves, i.e. considering a covering map $\psi : \partial W' \to X^{(1)}$, where $\partial W'$ is a sub collection of the components of ∂W^2.

2-stratifolds arise as the nerve of certain decompositions of 3-manifolds into pieces where they determine whether the $\mathcal{G}$-category of the 3-manifold is 2 or 3 ([6]). They are related to *foams*, which include special spines of 3-dimensional manifolds and which have been studied by Khovanov [10] and Carter [4]. Simple 2-dimensional stratified spaces arise in Topological Data Analysis [2], [11].

Matsuzaki and Ozawa [13] show that 2-stratifolds can be embedded in $\mathbb{R}^4$. Furthermore they show that they can be embedded into some orientable closed 3-manifold if and only if their branch curves satisfy a certain regularity condition. However, the embeddings are not π_1-injective, i.e. the induced homomorphism of fundamental groups is not injective. In fact, there are many 2-stratifolds whose fundamental group is not isomorphic to a 3-manifold group; for example there are infinitely many 2-stratifolds with (Baumslag-Solitar) non-Hopfian fundamental groups. These can not be embedded as π_1-injective subcomplexes into 3-manifolds since 3-manifold groups are residually finite.

Further comparing properties of 2-stratifold groups with 3-manifold groups we note that a 2-stratifold group G is the fundamental group of a graph of groups where each edge group is cyclic and each vertex group is an F-group or a free product of cyclic groups. (This is described in detail in section 2). If G is torsion free then the vertex groups are surface groups. Since the latter (except for Z_2) are left-orderable it follows from Corollary 3.6 of [5] that G

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is left-orderable. On the other hand, some torsion free 3-manifold groups are left orderable and some are not. For example it is shown in [3] that groups of compact P^2-irreducible 3-manifolds M with first Betti number > 0 are left-orderable, but not all Haken manifolds have left orderable groups. Thus the following question arises:

Question 1. Which 3-manifolds M have fundamental groups isomorphic to the fundamental group of a 2-stratifold?

The fundamental group of a closed 2-manifold S is isomorphic to the fundamental group of a closed 3-manifold M if and only if S is the 2-sphere or projective plane and M is S^3 or P^3, respectively. Since S^2 is not a spine of S^3, the only closed 3-manifold with a (closed) 2-manifold spine is P^3. This motivates the next question:

Question 2. Which closed 3-manifolds M have spines that are 2-stratifolds?

The main results of this paper are Theorem 1 which answers question 1 for closed 3-manifolds and Theorem 2, which answers question 2 by showing that a closed 3-manifold M has a 2-stratifold spine if and only if M is a connected sum of lens spaces, S^2-bundles over S^1, and $P^2 \times S^1$'s.

2. 2-stratifolds and their graphs.

In this section we review the definitions of a 2-stratifold X and its associated graph G_X given in [7].

A (closed) 2-stratifold is a compact 2-dimensional cell complex X that contains a 1-dimensional subcomplex $X^{(1)}$, such that $X - X^{(1)}$ is a 2-manifold ($X^{(1)}$ and $X - X^{(1)}$ need not be connected). A component $C \approx S^1$ of X^1 has a regular neighborhood $N(C) = N_\pi(C)$ that is homeomorphic to $(Y\times[0,1])/(y,1) \sim (h(y),0)$, where Y is the closed cone on the discrete space $\{1,2,...,d\}$ (for $d \geq 3$) and $h : Y \to Y$ is a homeomorphism whose restriction to $\{1,2,...,d\}$ is the permutation $\pi : \{1,2,...,d\} \to \{1,2,...,d\}$. The space $N_\pi(C)$ depends only on the conjugacy class of $\pi \in S_d$ and therefore is determined by a partition of d. A component of $\partial N_\pi(C)$ corresponds then to a summand of the partition determined by π. Here the neighborhoods $N(C)$ are chosen sufficiently small so that for disjoint components C and C' of X_1, $N(C)$ is disjoint from $N(C')$.

Note that X may also be described as a quotient space $W \cup_\psi X^{(1)}$, where $\psi : \partial W \to X^{(1)}$ is a covering map (and $|\psi^{-1}(x)| > 2$ for every $x \in X^{(1)}$).

We construct an associated bicolored graph $G = G_X$ of $X = X_G$ by letting the white vertices w of G_X be the components W of $M := \overline{X - \cup_j N(C_j)}$ where C_j runs over the components of X^1; the black vertices b_j are the C_j's. An edge e is a component S of ∂M; it joins a white vertex w corresponding to W with a black vertex b corresponding to C_j if $S = W \cap N(C_j)$. The number of boundary components of W is the number of adjacent edges of W. G_X embeds naturally as a retract into X_G.

We label the white vertices w with the genus g of W; here we use Neumann's [16] convention of assigning negative genus g to nonorientable surfaces; for example the genus g of the projective plane or the Moebius band is -1, the genus of the Klein bottle is -2. We orient all components C_j and S of $X^{(1)}$ and ∂W, resp., and assign a label m to an edge e, where $|m|$

is the summand of the partition π corresponding to the component $S \subset \partial N_\pi(C)$; the sign of m is determined by the orientation of C_j and S. In terms of attaching maps, m is the degree of the covering map $\psi : S \to C_j$ for the corresponding components of ∂W and $X^{(1)}$.

(Note that the partition π of a black vertex is determined by the labels of its adjacent edges).

3. Structure of $\pi_1(X_G)$

In this section we obtain a natural presentation for the fundamental group of a 2-stratifold X_G with associated bicolored graph $G = G_X$ and describe $\pi_1(X_G)$ as the fundamental group of a graph of groups $\mathcal{G}$ with the same underlying graph G.

For a given white vertex w, the compact 2-manifold W has conveniently oriented boundary curves $s_1, \dots, s_p$ such that

$$(*) \qquad \pi_1(W) = \langle s_1, \dots, s_p, y_1, \dots, y_n : s_1 \cdots s_p \cdot q = 1 \rangle$$

where $q = [y_1, y_2] \dots [y_{2g-1}, y_{2g}]$, if W is orientable of genus g and $n = 2g$, $q = y_1^2 \dots y_n^2$, if W is non-orientable of genus $-n$.

Let $\mathcal{B}$ be the set of black vertices, $\mathcal{W}$ the set of white vertices and choose a fixed maximal tree T of G. Choose orientations of the black vertices and of all boundary components of M such that all labels of edges in T are positive.

Then $\pi_1(X_G)$ has a natural presentation with
generators:
$\{b\}_{b \in \mathcal{B}}$
$\{s_1, \dots, s_p, y_1, \dots, y_n\}$, one set for each $w \in \mathcal{W}$, as in $(*)$
$\{t_i\}$, one t_i for each edge $c_i \in G{-}T$ between w and b
and *relations*:
$s_1 \cdots s_p \cdot q = 1$, one for each $w \in \mathcal{W}$, as in $(*)$
$b^m = s_i$, for each edge $s_i \in T$ between w and b with label $m \geq 1$
$t_i^{-1} s_i t_i = b^{m_i}$, for each edge $s_i \in G{-}T$ between w and b with label $m_i \in \mathbb{Z}$.

As an example we show in Figure 1 (the graph of) a 2-stratifold X_G with $\pi_1(X_G) = \mathcal{F}$, an F-group as in Proposition (III)5.3 of [12], with presentation

$$(\mathcal{F}) \qquad \mathcal{F} = \langle c_1, \dots, c_p, y_1, \dots, y_n : c_1^{m_1}, \dots, c_p^{m_p},\ c_1 \cdots c_p \cdot q = 1 \rangle$$

where $p, n \geq 0$, all $m_i > 1$ and $q = [y_1, y_2] \dots [y_{2g-1}, y_{2g}]$ or $q = y_1^2 \dots y_n^2$.

Here we have denoted the generators corresponding to the black vertices by c_i, rather than b_i, to indicate that the finite order elements correspond to attaching disks along the boundary curves of W.

The fundamental group of X_G is best described as the fundamental group of a graph of

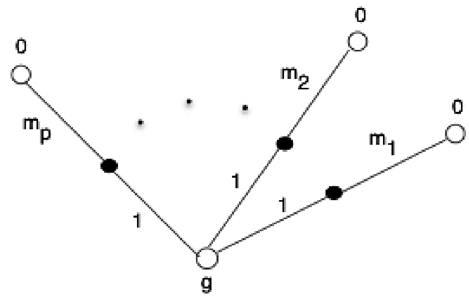

Fig. 1. F-group

groups [8].

If $\pi_1(X_G)$ has no elements of finite order, then $\pi_1(X_G)$ is the fundamental group of a graph of groups $\mathcal{G}$, with underlying graph G, the groups of white vertices are the fundamental groups of the $W's$, the groups of the black vertices and edges are (infinite) cyclic.

Elements of finite order occur when a generator b of a black vertex has finite order $o(b) \geq 1$. In this case we attach 2-cells d_b and d_e to C_b, the circle corresponding to b, as follows: d_b is attached by a map of degree $o(b)$. If e is an edge joining b to w with label m, attach d_e with degree $o(c) = o(b)/(o(b), m)$. Letting $\hat{X}_b = N(C_b) \cup d_b \cup (\cup d_e)$, where e runs over the edges having b as an endpoint, $\hat{X}_w = W \cup (\cup d_e)$, where e runs over the edges incident to w, and $\hat{X}_e = (\hat{X}_b \cap \hat{X}_w)$, for an edge e joining b to w, we obtain a graph of CW-complexes that determines a graph of groups $\mathcal{G}$ with the same underlying graph as G_X.

The vertex groups are $G_b = \pi_1(\hat{X}_b)$ and $G_w = \pi_1(\hat{X}_w)$, the edge groups are $G_e = \pi_1(\hat{X}_e)$, the monomorphisms $\delta : G_e \to G_b$ (resp. $G_e \to G_w$) are induced by inclusion. Then (see for example [17],[18]) $\pi_1\mathcal{G} \cong \pi_1(\hat{X})$.

Note that the groups G_b of the black vertices and the groups G_e of the edges are cyclic. For a white vertex w with edges $e_1, \dots e_p$ labelled $m_1, \dots m_p$ with associated vertex space $X_w = W \cup_{i=1}^{r} d_{e_i}$ we obtain

$$G_w = \langle c_1, \dots, c_p, y_1, \dots, y_n : c_1 \cdots c_p \cdot q = 1 \, , \, c_1^{m_1} = \cdots = c_r^{m_r} = 1 \rangle$$

where q is as in ($\mathcal{F}$), $1 \leq r \leq p$ and $k_i \geq 1$.

If all $k_i \geq 2$ and $r = p$ then G_w is an F-group ([12] p. 126-127). If $r < p$ it is a free product of cyclic groups.

4. Necessary Conditions

In this section we show that a 2-stratifold group that is a closed 3-manifold group is a free product of cyclic or $\mathbb{Z}\times\mathbb{Z}_2$ groups.

First consider an F-group $\mathcal{F}$ as in ($\mathcal{F}$).

Proposition 1 ([12] Proposition (III)7.4). *Let H be a subgroup of an F-group. If H has finite index then H is an F-group. If H has infinite index then H is a free product of cyclic groups.*

Proposition 2 ([12] p.132). (a) *$\mathcal{F}$ is finite non-cyclic if and only if $n = 0$, $p = 3$ and $(m_1, m_2, m_3) = (2, 2, m)$ ($m \geq 2$) (the dihedral group of order $2m$) or $(m_1, m_2, m_3) = (2, 3, k)$*

for $k = 3, 4$ or 5 (the tetrahedral, octahedral, dodecahedral groups). In each case, c_1 is a non-central element of order 2.

(b) *$\mathcal{F}$ is finite cyclic if and only if $n = 0$, $p \leq 2$ (the 2-sphere orbifold with at most two cone points) or $n = 1$, $p \leq 1$ (the projective plane orbifold with at most one cone point).*

Lemma 1. *$\mathcal{F}$ is not a non-trivial free product.*

Proof. If $\mathcal{F} = A * B$ with A, B non-trivial, then A and B have infinite index and so, by Proposition 1, A, B and $\mathcal{F}$ are free products of cyclic groups. However, $\mathcal{F}$ is not such a group since it contains a subgroup isomorphic to the fundamental group of an orientable closed surface of genus ≥ 1 (see the remark after Proposition (III)7.12 in [12]). □

The following remark is easy to see.

Remark 1. If $\mathcal{F} \neq \mathbb{Z}_2$ then $\mathcal{F}$ has no elements of finite order if and only if $\mathcal{F}$ is a surface group.

Lemma 2. *If M is an orientable (not necessarily closed or compact) 3-manifold with $\pi_1(M) \cong \mathcal{F}$ then $\pi_1(M)$ is cyclic or a surface group.*

Proof. We may assume that ∂M contains no 2-spheres. By Scott's Core Theorem we may assume that M is compact and by Lemma 1 that M is irreducible.

If $\pi_1(M)$ is infinite then M is aspherical (see e.g. [1]). It follows that $\pi_1(M)$ is torsion-free and from Remark 1 that $\pi_1(M)$ is a surface group.

If $\pi_1(M)$ is finite then M is closed. If $\pi_1(M)$ is also non-cyclic then by Proposition 2, $\pi_1(M)$ contains a non-central element of order 2. This can not happen by Milnor [15]. □

We now consider a 2-stratifold X_G with $\pi_1(X_G) = \pi(\mathcal{G})$ as in section 3.

Up to conjugacy, the only elements of finite order of $\pi_1(X_G)$ are contained in the vertex groups; they correspond to black vertices of finite order and elements of white vertices w whose corresponding group in $\mathcal{G}$ is finite. The latter are described in Proposition 2. It is also shown in [12] (proof of Proposition (III)7.12) that in an infinite F-group the only elements of finite order are the obvious ones, namely conjugates of powers of $c_1, \ldots, c_p$.

For a group H, denote by QH be the quotient group of H modulo the smallest subgroup of H containing all elements of finite order of H.

Let w be a white vertex in G_X. We say that w is a *white hole*, if w has label -1, all of its (black) neighbors have finite order and at most one of its neighbors has order > 1.

If G_X has more than one vertex, note that $Q\pi_1(X_G)$ is obtained from $\pi_1(X_G)$ by killing the open stars of all the black vertices representing elements of finite order ≥ 1 of $\pi_1(X_G)$ and deleting the white holes. In the example of Figure 1, when genus $g = -1$ (and so $n = 1$), $Q\pi_1(X_G) = \mathbb{Z}_2$. (Note that the white vertex of genus -1 is not a white hole if $p \geq 2, m_i > 1$).

Proposition 3. *If $Q(\pi_1(X_G))$ has no elements of order 2, then $H_3(Q\pi_1(X_G)) = 0$.*

Proof. Let G' be the labelled subgraph of G_X obtained by deleting the open stars of all black vertices representing elements of finite order of $\pi_1(X_G)$ and all white holes. ($\pi_1(X_\emptyset) = 1$ by definition). Let C be a component of G'. Then $Q\pi(X_G) = L * (*_C(\pi(X_C))$, the free product of a free group L with the free product of the $\pi(X_C)$ where C runs over the components of G'.

If C consists of only one (white) vertex, then X_C is a closed 2-manifold, different from P^2, since by assumption $Q(\pi_1(X_G))$ has no elements of order 2. We may ignore the C's consisting of spheres, since they do not contribute to $Q\pi(X_G)$. (A nonseparating 2-sphere only changes the rank of L). In all other cases X_C is the total space of a bicolored graph of spaces with white vertex spaces 2-manifolds with boundary, edge spaces circles, and black vertex spaces homotopy equivalent to circles.

Thus every vertex and edge space of X_C is aspherical (with free fundamental group) of dimension ≤ 2. By Proposition 3.6 (ii) of [17], X_C is aspherical. It follows that $Q\pi_1(X_G)$ has (co)homological dimension ≤ 2 and so $H_3(Q\pi_1(X_G)) = 0$. □

The assumption that $Q(\pi_1(X_G))$ has no elements of finite order is satisfied if $\pi_1(X_G)$ is a 3-manifold group: We claim that $Q\pi_1(M)$ is torsion free if M is a closed orientable 3-manifold.

For let $M = M_1\#\ldots\#M_k$ be its prime decomposition. If M_i is irreducible with infinite fundamental group, then M_i is aspherical and so $\pi_1(M_i)$ is torsion free; if M_i has finite fundamental group, then $Q\pi_1(M_1) = 1$. Now the claim follows since $Q\pi_1(M) = Q\pi_1(M_1) * \cdots * Q\pi_1(M_k)$.

Lemma 3. *Let M be a closed orientable 3-manifold with prime decomposition $M = M_1\#\ldots\#M_k$. If $\pi_1(M) \cong \pi_1(X_G)$, then each $\pi_1(M_i)$ is infinite cyclic or finite.*

Proof. If there is some M_i with $\pi_1(M_i) \neq \mathbb{Z}$, then M_i is irreducible. If $\pi_1(M_i)$ is infinite then M_i is aspherical and hence $H_3(Q\pi_1(M_i)) = H_3(\pi_1(M_i)) = H_3(M_i) \neq 0$. Since $Q\pi_1(M) = Q\pi_1(M_1) * \cdots * Q\pi_1(M_k)$ it follows that $H_3(Q\pi_1(M)) \neq 0$, which contradicts Proposition 3. □

REMARK 2. One can give an alternate proof of Lemma 3 by using the fact that 3-manifold groups are virtually torsion free. Using this fact one can show (by looking at torsion-free subgroups instead of torsion-free quotients) that if G is both a 2-stratifold group and a 3-manifold group, then the virtual cohomological dimension of G is at most 2. By a slight modification of the above proof of Lemma 3 this implies that no connected sum summand of a closed orientable 3-manifold M is aspherical.

Lemma 4. *Let M be a closed 3-manifold and suppose $\pi_1(M) = \pi_1(X_G)$. Then any finite subgroup H of $\pi_1(M)$ is cyclic.*

Proof. $\pi_1(X_G) \cong \pi_1(\mathcal{G})$ where $\mathcal{G}$ is a graph of groups in which the groups of black vertices are cyclic and the groups of white vertices are F-groups or free products of finitely many cyclic groups. The finite group H is non-splittable (i.e. not a non-trivial HHN extension or free product with amalgamation). By Corollary 3.8 and the Remark after Theorem 3.7 of [17], H is a cyclic group or isomorphic to a subgroup of an F-group. If H is not cyclic, then (since H is not a non-trivial free product of cyclic groups), H is itself an F-group by

Proposition 1. Since H is a 3-manifold group it follows from Lemma 2 that this case can not occur. □

Corollary 1. *Let M be a closed orientable 3-manifold. If $\pi_1(M) \cong \pi_1(X_G)$, then $\pi_1(M)$ is a free product of cyclic groups.*

Proof. This follows from Lemmas 3 and 4. □

Theorem 1. *Let M be a closed 3-manifold. If $\pi_1(M) \cong \pi_1(X_G)$, then $\pi_1(M)$ is a free product of groups, where each factor is cyclic or $\mathbb{Z}\times\mathbb{Z}_2$.*

Proof. If M is orientable this is Corollary 1 (with each factor cyclic). Thus assume M is non-orientable and let $p : \tilde{M} \to M$ be the 2-fold orientable cover of M. Then $\pi_1(\tilde{M}) = \pi_1(X_{\tilde{G}})$ for the 2-stratifold $X_{\tilde{G}}$, which is the 2-fold cover of X_G corresponding to the orientation subgroup of $\pi_1(M)$. Hence $\pi_1(\tilde{M})$ is a free product of cyclic groups.

Let $M = \hat{M}_1\# \ldots \#\hat{M}_k$ be a prime decomposition of $M = M_1\cup\cdots\cup M_k$. If M_i is orientable, then M_i lifts to two homeomorphic copies $\tilde{M}_{i1}$, $\tilde{M}_{i2}$ of M_i, with each $\hat{\tilde{M}}_{ij}$ a factor of the prime decomposition of $\tilde{M}$ and it follows that $\pi_1(M_i)$ is cyclic.

If $\hat{M}_i$ is non-orientable and P^2-irreducible, then M_i lifts to $\tilde{M}_i$, where $\hat{\tilde{M}}_i$ is irreducible. Then $\pi_1(\hat{\tilde{M}}_i)$, being a factor of the free product decomposition of $\pi_1(\tilde{M})$, is finite cyclic, which can not occur since $\pi_1(M_i)$ is infinite.

If $\hat{M}_i$ is non-orientable irreducible, contains P^2's, but is not $P^2\times S^1$, then by Proposition (2.2) of [19], M_i splits along two-sided P^2's into 3-manifolds $N_1, \ldots, N_m$ such that the fundamental group of the lifts $\tilde{N}_i$ is indecomposable, torsion free and not isomorphic to $\mathbb{Z}$. Since $\pi_1(\tilde{N}_i)$ is a factor of the free product decomposition of $\pi_1(\tilde{M})$, this can not happen.

Therefore each non-orientable M_i is either the S^2-bundle over S^1 or $P^2\times S^1$, which proves the Theorem. □

5. Realizations of spines.

Recall that a subpolyhedron P of a 3-manifold M is a *spine* of M, if $M - Int(B^3)$ collapses to P, where B^3 is a 3-ball in M.

An equivalent definition is that $M - P$ is homeomorphic to an open 3-ball (Theorem 1.1.7 of [14]).

We first construct 2-stratifold spines of lens spaces (different from S^3), the non-orientable S^2-bundle over S^1, and $P^2\times S^1$.

Example 1. Lens space $L(0, 1) = S^3$.

S^3 does not have a 2-stratifold spine. Otherwise such a spine X would be a deformation retract of the 3-ball and therefore contractible. However there are no contractible 2-stratifolds [7].

Example 2. Lens spaces $L(p, q)$ with $q \neq 0, 1$.

Let r be the rotation of the disk D^2 about its center c with angle $2\pi p/q$, let $1 \in S^1 \subset D^2$ and let $x_i = r^{i-1}(1)$, $i = 1, \ldots, q$. Let $Y \subset D^2$ be the cone of $\{x_1, \ldots, x_q\}$ with cone point c. Embed $Y\times I/(x_i, 0) \sim (x_{i+1}, 1)$ into the solid torus $V = D^2\times I/(x, 0) \sim (r(x), 1)$.

The punctured lens space $L(p, q)$ is obtained from V by attaching a 2-handle $D{\times}I$ with ∂D attached to the boundary curve of $(Y{\times}I)/_\sim$. Then $L(p, q)$ deformation retracts to $(Y{\times}I)/_\sim \cup D$, which is the 2-stratifold with one white vertex of genus 0, one black vertex, and one edge with label q.

Example 3. Lens space $L(1, 0) = S^2{\times}S^1$ and non-orientable S^2-bundle over S^1.

Consider $S^2\tilde{\times}S^1$, the non-orientable S^2-bundle over S^1, as as the quotient space $q(S^2{\times}I)$ under the quotient map $q : S^2{\times}I \to S^2\tilde{\times}S^1$ that identifies $(x, 0)$ with $(\alpha(x), 1)$, $x \in S^2$, where α is the antipodal map.

Let $D_0 \subset S^2{\times}\{0\}$ be a disk and B_1 be the 3-ball $D_0{\times}I \subset S^2{\times}I$, let D_1 be the disk $B_1 \cap S^2{\times}\{1\}$, let A be the annulus $\partial B_1 - (Int(D_0) \cup Int(D_1))$, and let B_2 be the ball $S^2{\times}I - Int(B_1)$, see Figure 2. Then $S^2{\times}I - Int(B_2) = S^2{\times}\{0\} \cup B_1 \cup S^2{\times}\{1\}$ and $S^2\tilde{\times}S^1 - Int(B_2) = q(S^2{\times}I - Int(B_2)) = S^2 \cup q(B_1)$, where $S^2 = q(S^2{\times}\{0\}) = q(S^2{\times}\{1\})$. Collapsing the ball $q(B_1)$ across the free face $q(D_1)$ onto $q(A) \cup q(D_0)$ we obtain a collapse of $S^2\tilde{\times}S^1 - Int(B_2)$ onto $(S^2 - Int(q(D_1))) \cup q(A)$, which is a torus with a disk attached. This is a 2-stratifold X_G with graph G_X in Figure 3(b). (The white vertices have genus 0).

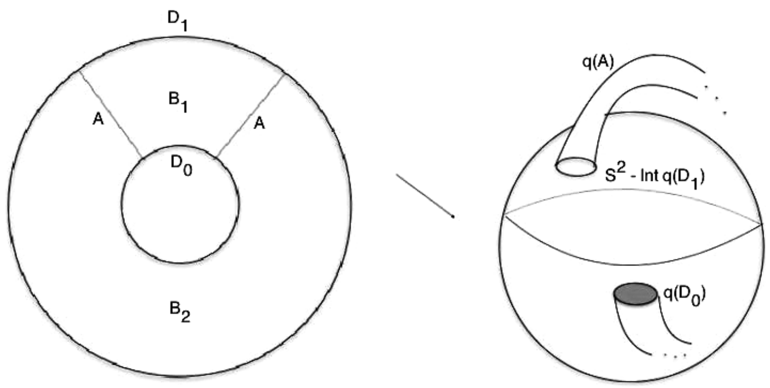

Fig. 2. $S^2\tilde{\times}S^1 - Int(B_2) \searrow S^1\tilde{\times}S^1 \cup D^2$

A similar construction, considering $S^2{\times}S^1$ as the obvious quotient space of $q : S^2{\times}I \to S^2{\times}S^1$ and first isotoping the ball B_1 such that $D_0 \cap D_1 = \emptyset$, we obtain a collapse of $S^2{\times}S^1 - Int(B_2)$ onto a Kleinbottle with a disk attached. This is a 2-stratifold X_G with graph G_X in Figure 3(a).

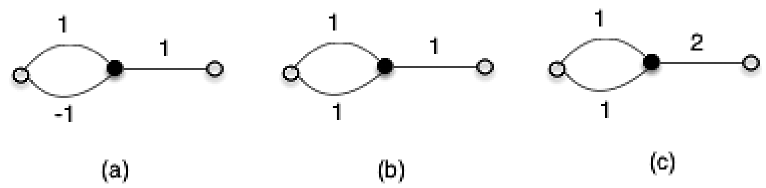

Fig. 3. 2-stratifold spines of punctured $S^2\tilde{\times}S^1$ and $P^2{\times}S^1$

We would like to thank Mario Eudave-Muñoz for pointing out that in this example the spine for the non-orientable (resp. orientable S^2-bundle over S^1) is a torus (resp. Kleinbottle) with a disk attached, rather than a Kleinbottle (resp. torus) with a disk attached.

Example 4. $P^2{\times}S^1$.

For a one-sided simple closed curve c in P^2 and a point t_0 in S^1 let $X = P^2{\times}\{t_0\} \cup c{\times}S^1 \subset P^2{\times}S^1$. Observe that the boundary of a regular neighborhood N of X in $P^2{\times}S^1$ is a 2-sphere. Since $P^2{\times}S^1$ is irreducible, ∂N bounds a 3-ball B^3 and therefore $P^2{\times}S^1 - Int(B^3) = N$, which collapses onto $X = X_G$, a 2-stratifold with graph in Figure 3(c).

Proposition 4. *If the closed* 3*-manifold* M_i $(i = 1, 2)$ *has a* 2*-stratifold spine and* M *is a connected sum of* M_1 *and* M_2*, then* M *has a* 2*-stratifold spine.*

Proof. Let K_i be a 2-stratifold spine of M_i. Let $K_1 \vee K_2$ be obtained by identifying, in the disjoint union of K_1 and K_2 a nonsingular point of K_1 with a nonsingular point of K_2. By Lemma 1 of [9], $K_1 \vee K_2$ is a spine of M. Though $K_1 \vee K_2$ is not a 2-stratifold, by performing the operation explained below (replacing the wedge point by a disk) we will change $K_1 \vee K_2$ to a 2-stratifold spine $K_1 \Delta K_2$.

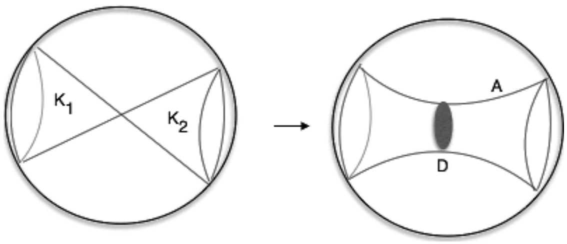

Fig.4. $K_1 \Delta K_2$

A 3-ball neighborhood B^3 of the wedge point of $K_1 \vee K_2$ intersects $K_1 \vee K_2$ in the double cone shown in Fig.4. Replace, in $K_1 \vee K_2$, $K_1 \vee K_2 \cap B^3$ by $A \cup D$, as shown in Fig. 4, where $A = S^1{\times}[0, 1]$ is an horizontal cylinder, $\partial A = (K_1 \vee K_2) \cap \partial B^3$, D is a vertical 2-disk with $A \cap D = \partial D = S^1{\times}(1/2)$. The result is a 2-stratifold $K_1 \Delta K_2$. There is a homeomorphism from $B^3 - (A \cup D)$ onto $B^3 - K_1 \vee K_2$ which is the identity on the boundary (roughly collapse D to a point) and so $M - K_1 \Delta K_2$ is homeomorphic to $M - K_1 \vee K_2$ which is homeomorphic to R^3.

Therefore $K_1 \Delta K_2$ is a 2-stratifold spine of M. □

Now Theorem 1 together with the examples and Proposition 4 yields our main Theorem. Here we do not consider S^3 to be a lens space.

Theorem 2. *A closed* 3*-manifold* M *has a* 2*-stratifold as a spine if and only if* M *is a connected sum of lens spaces,* S^2*-bundles over* S^1*, and* $P^2{\times}S^1$*'s.*

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