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ON THE SLOPE OF RATIONAL FIBERED SURFACES

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Abstract

Given a rational fibered surface $f : X \to \mathbb{P}^1$ of genus g we prove the inequality $\frac{6n+5}{n+1} - \frac{9n+12}{2g} \le \lambda_f$, provided that the genus g is sufficiently high with respect to the gonality $2n+3$ of the general fibre.

1. Introduction

We work over the complex field $\mathbb{C}$. Let $f : X \to B$ be a surjective morphism between a projective nonsingular surface X and a projective, nonsingular curve B, with connected fibres. We say that f is a fibration and X a fibered surface. In what follows we assume that the fibration is relatively minimal (i.e. the fibers do not contain (-1)–curves) and the genus of the general fiber F is $g \ge 2$.

We denote by K the canonical divisor of X and for a surface T we write K_T only if it is necessary to distinguish what surface we are referring to. The basic numerical invariants associated with a fibration are: if $K_f := K - f^*K_B$ is the relative canonical divisor, define:

$$\chi_f := \deg f_*K_f = \chi(\mathcal{O}_X) - (g-1)(b-1), \text{ with } b = \text{the genus of } B,$$

the slope:

$$\lambda_f = \frac{K_f^2}{\chi_f},$$

and the relative Euler characteristic:

$$e_f := e(X) - 4(g-1)(b-1).$$

These numbers are related to the relative Noether formula:

$$12\chi_f = K_f^2 + e_f$$

(for these relationships see, for instance, [12]). The numbers K_f^2, χ_f and λ_f are positive if and only if f is non-isotrivial ([2], [10], [12]). Thus, the problem of the "geography"of fibered surfaces naturally arises.

The most important result in this direction is the slope inequality obtained by Xiao ([13]):

$$\frac{4(g-1)}{g} \le \lambda_f \le 12.$$

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The search of sharper inequalities is a natural issue when imposing conditions on the fibration. According to [9]: "Ashikaga conjectured that the lower bound of the slope of "general"fibrations of genus g is given as a function in g that approaches 6 from below as g grows ". Probably the most general result in this direction is the inequality:

$$\frac{6(g-1)}{g+1} \leq \lambda_f,$$

if the general fiber is of maximal Clifford index ([9]). See also [3] and the references therein.

Consider, for instance, one of the simplest examples of a fibration: let X be the (rational) surface obtained after blowing up the base locus of a pencil of nonsingular plane curves of degree d transversely intersecting. Remember that if X is rational, then B is necessarily rational and $\chi_f = g$, meaning in particular that if $g > 0$ the fibration is not isotrivial. Therefore, in our particular case:

$$3d(d-3) = 6(g-1) \text{ and } K_f^2 = 3d^2 - 12d + 9.$$

Thus, we obtain:

$$6 - \frac{3(d-1)}{g} = \lambda_f.$$

The general fibre is far from being of maximal Clifford index, nevertheless the slope is not much less than 6 (d being of the order of $\sqrt{g}$). Note also that the number d is just the gonality of the general fibre F plus one (see [7]).

The main goal of this paper is to prove that a similar bound can be obtained for any fibration of sufficiently high genus defined on a rational surface:

Theorem 1.1. *Let $f : X \to \mathbb{P}^1$ be a relatively minimal fibration of genus g on the rational surface X, and let $n \in \mathbb{N}$ be a given natural number. Assume $(3n^3+20n^2+39n-2) \leq 6(g-1)$ and the gonality $gon(F)$ of the general fibre F is at least $2n+3$. Then,*

$$\frac{6n+5}{n+1}g - \frac{9n+12}{2} \leq K_f^2.$$

In particular,

$$\frac{6n+5}{n+1} - \frac{9n+12}{2g} \leq \lambda_f.$$

The proof of Theorem 1.1 splits out into two parts. For any effective divisor D we denote by D_+ its positive or nef part (see the following section). First, we obtain the bound for K_f^2 assuming that $|(lK+F)_+ + K|$ defines a birational map for all $1 \leq l \leq n$:

Proposition 1.2. *If for all $1 \leq l \leq n-1$, the linear systems $|(lK+F)_+ + K|$ are nonempty and $|((n-1)K+F)_+ + K|$ defines a birational map, then*

$$\frac{6n-1}{n}g - \frac{9n+3}{2} \leq K_f^2.$$

Next, we give sufficient conditions for $|(lK+F)_+ + K|$ defining a birational map:

Proposition 1.3. *If $(3n^3+20n^2+39n-2) \leq 6(g-1)$ and the gonality $gon(F)$ of the general fibre F is at least $2n+3$, then for $1 \leq l \leq n$, $|(lK+F)_+ + K| \neq \emptyset$ and $|(nK+F)_+ + K|$ defines a birational map.*

Theorem 1.1 follows at once from these two propositions and the fact that $\chi_f = g$. Theorem 1.1 generalizes the main results of [1], that in some sense can be considered as the first step of an inductive procedure.

In Section 2 we present some preliminaries and fix the notation. Section 3 is devoted to the proof of Proposition 1.2 and section 4 to the proof of Proposition 1.3.

2. Preliminaries

Let $f : X \to \mathbb{P}^1$ be a relatively minimal fibration, with X a rational surface. We assume that the genus g of the general fibre F is at least 2.

Remember that in general, if $f : X \to B$ is a fibered surface, the relative canonical divisor of f, denoted by K_f is $K_f := K - f^*K_B$. In our case $K_f = K + 2F$ and we have:

$$K_f^2 = K^2 + 8(g-1).$$

We will use systematically the Zariski decomposition of an effective divisor D ([5], [14]). Recall that given an effective divisor D (or more generally a pseudo-effective divisor) there are effective $\mathbb{Q}-$divisors D_+ and D_- such that:

. $D = D_+ + D_-$,
. D_+ is nef,
. for all irreducible components Γ of D_-, $D_+.\Gamma = 0$,
. the matrix of intersection of the irreducible components of D_- is negative definite.

This is the Zariski (or Zariski-Fujita) decomposition of the divisor D. It is unique as a sum of Néron-Severi classes.

In what follows when writing $D = D_+ + D_-$ without any further reference to D, we are implicity assuming that D is effective.

Remark 2.1. If D is effective, then for all $n \in \mathbb{N}$ we have ([5], Lemma 14.17):

$$H^0(X, nD) = H^0(X, n(D_+)).$$

Now, we follow the exposition in [6]. Let C be a nef divisor on the rational surface X. Assume $C + nK$ is effective and Δ is a curve such that $\Delta.(C + nK) < 0$. Then $\Delta^2 < 0$ and $\Delta.K < 0$. Therefore Δ is a $(-1)-$curve. This procedure can be repeated, after contracting Δ, in order to obtain the Zariski decomposition of $nK + C$. The description of $(C + nK)_-$ is given in terms of the so called $(-1)-$cycles, we say that an effective divisor Θ in X is a $(-1)-$cycle if a birational morphism exists:

$$\pi : X \to T,$$

such that T is nonsingular, a point $p \in T$ exists such that $\Theta = \pi^*(\mathcal{I}_p)$ (($\mathcal{I}_p$) denoting the ideal sheaf of p) and $\pi : X - \Theta \to T - \{p\}$ is an isomorphism. In this way a $(-1)-$cycle is a non necessarily irreducible generalization of a $(-1)-$curve.

The structure of $(-1)-$cycles have been investigated in [4], [8], and [14]. One of its most important properties, generalizing those of $(-1)-$curves, is that $\Theta^2 = \Theta.K = -1$.

The result on the Zariski decomposition of $C + nK$ is:

Proposition 2.2 ([6, Proposition 4.2]). *Let C be a nef divisor on a rational surface X. Let n be a positive integer and assume that $|C + nK_X| \neq \emptyset$. Then:*

$$(C + nK_X)_- = \sum_{i=0}^{n-1} \sum_{j=1}^{h_i} (n-i)\Theta_{i,j},$$

where $\Theta_{i,j}$ are the $(-1)-$cycles on X satisfying:

(1) $\Theta_{i,j}.\Theta_{h,k} = 0$ *if* $(h,k) \neq (i,j)$ *and*

(2) $C.\Theta_{i,j} = i$.

Remark 2.3. If $n = 1$ we have:

$$(C + K)_- = \sum \Theta_{ij}, \text{ with } \Theta_{ij}.C = 0, \quad \Theta_{ij}.\Theta_{lk} = 0$$

and from this it follows that:

$$K.(C + K)_- = (C + K)_-^2.$$

The other main ingredient in the proof of Theorem 1.1 is Reider's method ([11]), in particular its following consequence:

Lemma 2.4. *Let C be a nef divisor in X.*

(1) *If $C^2 \geq 5$ and $|C + K| = \emptyset$, then X admits a base point free pencil $|E|$ with $E.C = 1$.*

(2) *If $C^2 \geq 10$ and $|C + K|$ does not define a birational map, then X admits a base point free pencil E with $E.C = 1$ or 2.*

3. Proof of Proposition 1.2

Before proceeding with the proof of Proposition 1.2 we need the following Lemma and its consequences, which are on the base of all our computations. Its provides a recursive method for computing Zariski's decomposition of the $n-$adjoint linear system $|nK + F|$.

Lemma 3.1. *Let $f : X \to \mathbb{P}^1$ be a relatively minimal fibration on the rational surface X. If $(n-1)K + F$ and $((n-1)K + F)_+ + K$ are effective, then:*

(1) *$nK + F$ is effective,*

(2) $(nK + F)_+ = (((n-1)K + F)_+ + K)_+$ *and*

$$(nK + F)_- = ((n-1)K + F)_- + (((n-1)K + F)_+ + K)_-,$$

(3) $((n-1)K + F)_-.(((n-1)K + F)_+ + K)_- = ((n-1)K + F)_-.K.$

Proof. 1) The effectiveness of $nK+F$ follows from the effectiveness of $((n-1)K+F)_+ + K$ and $((n-1)K + F)_-$.

2) Write:

$$nK + F = (((n-1)K + F)_+ + K)_+ + (((n-1)K + F)_+ + K)_- + ((n-1)K + F)_-.$$

We want to prove that this is the Zariski decomposition of $nK + F$. By the uniqueness of such a decomposition and taking into account that $(((n-1)K + F)_+ + K)_+$ is nef, it is sufficient to prove the following claims:

a) $(((n-1)K+F)_+ + K)_+.((n-1)K+F)_- = 0$.

b) The intersection matrix of the irreducible components of

$$\mathrm{Supp}\,((((n-1)K+F)_+ + K)_- + ((n-1)K+F)_-)$$

is negative definite.

From Proposition 2.2 applied to $(n-1)K+F$ we have:

$$((n-1)K+F)_- = \sum_{i=1}^{n-2}\sum_{j=1}^{h_i}(n-1-i)\Theta_{ij},$$

where Θ_{ij} are (-1)−cycles.

Using that $\Theta_{ij}.((n-1)K+F)_+ = 0$ and applying Remark 2.3 we see that $\mathrm{Supp}(((n-1)K+F)_-) \leq \mathrm{Supp}((((n-1)K+F)_+ + K)_-)$ and from this a) follows. Moreover, it also follows that:

$$\mathrm{Supp}\,((((n-1)K+F)_+ + K)_- + ((n-1)K+F)_-) = \mathrm{Supp}((((n-1)K+F)_+ + K)_-),$$

therefore b) is true, and we conclude (2).

Finally, writing:

$$(((n-1)K+F)_+ + K)_- = -(((n-1)K+F)_+ + K)_+ + ((n-1)K+F)_+ + K,$$

we see that (3) is a direct consequence of claim a). □

Corollary 3.2. *Let $f : X \to \mathbb{P}^1$ be a relatively minimal fibration on a rational surface X. If for all $1 \leq l \leq n-1$, $(lK+F)_+ + K$ is effective, then:*

(1) $(nK+F)_- = \sum_{l=1}^{n-1}((lK+F)_+ + K)_-$.

(2) $(nK+F)_-.K = \sum_{l=1}^{n-1}((lK+F)_+ + K)_-^2$.

(3) $(nK+F)_-^2 = \sum_{l=1}^{n-1}(2(n-l)-1)((lK+F)_+ + K)_-^2$.

(4) $p_a((nK+F)_-) - 1 = \sum_{l=1}^{n-1}(n-l)(p_a(((lK+F)_+ + K)_-) - 1)$.

(5) $\dfrac{n}{n+1}2p_a((nK+F)_- - 1) - (nK+F)_-^2 = \sum_{l=1}^{n-1}\left(\dfrac{-2(n-l)}{n+1} + 1\right)(p_a((lK+F)_+ + K)_-) - 1)$.

Proof. First of all, observe that from Lemma 3.1 (1) we have that $nK+F$ is also effective. So, it makes sense to speak about its positive and negative parts.

1) The proof is by induction on n. If $n = 2$, considering that $K+F$ is nef, we can write $2K+F = (K+F)_+ + K$ and we obtain the statement in this case. If $n > 2$, assume that:

$$((n-1)K+F)_- = \sum_{l=1}^{n-2}((lK+F)_+ + K)_-.$$

From Lemma 3.1 (2) we have:

$$(nK+F)_- = ((n-1)K+F)_- + (((n-1)K+F)_+ + K)_-,$$

substituting in the previous expression we obtain part (1). of the Corollary.

2) follows from part (1) and Remark 2.3.

3) We use induction on n, the case $n=2$, being clear from part (1). Assume $n>2$ and :

$$((n-1)K+F)_-^2 = \sum_{l=1}^{n-2}(2(n-l)-3)((lK+F)_+ + K)_-^2.$$

From parts (2) and (3) of Lemma 3.1 we obtain:

$$(nK+F)_-^2 = ((n-1)K+F)_-^2 + 2((n-1)K+F)_-.K + (((n-1)K+F)_+ + K)_-^2$$

Thus, substituting and using the previous part (2), we have:

$$\begin{aligned}(nK+F)_-^2 &= \sum_{l=1}^{n-2}(2(n-l)-1)((lK+F)_+ + K)_-^2 + (((n-1)K+F)_+ + K)_-^2\\ &= \sum_{l=1}^{n-1}(2(n-l)-1)((lK+F)_+ + K)_-^2.\end{aligned}$$

4) This follows at once from (2) and (3)

5) Follows from (3) and (4). □

Now, we can proceed with the proof of Proposition 1.2.

Proof of Proposition 1.2. First, we need the following:

Lemma 3.3. *Let $f: X \to \mathbb{P}^1$ be a relatively minimal fibration on the rational surface X. If $|((n-1)K+F)_+ + K|$ defines a birational map, then $(nK+F)_+$ is big and*

$$0 \le h^0((nK+F)_+ + K) = \frac{1}{2}n(n+1)K_f^2 - (4n^2+2n-1)(g-1) - (p_a((nK+F)_-) - 1) + 1.$$

Proof. Since $((n-1)K+F)_+ + K$ is effective, from Lemma 3.1 parts (1) and (2) we know that $nK+F$ is effective and:

$$(nK+F)_+ = (((n-1)K+F)_+ + K)_+. \tag{1}$$

Now, from

$$H^0(((n-1)K+F)_+ + K) \cong H^0((((n-1)K+F)_+ + K)_+)$$

(Remark 2.1), we deduce that $|((n-1)K+F)_+ + K)_+|$ defines a birational map and in consequence is big. It follows that $(nK+F)_+$ is big. Thus, $(nK+F)_+$ is big and nef, and applying Mumford's Vanishing Theorem:

$$0 \le h^0((nK+F)_+ + K) = \frac{1}{2}(nK+F)_+.((nK+F)_+ + K) + 1.$$

Substituting $(nK+F)_+ = nK+F-(nK+F)_-$ we get:

$$\begin{aligned}0 \le h^0((nK+F)_+ + K) &= \frac{1}{2}(nK+F-(nK+F)_-).(nK+F-(nK+F)_- + K) + 1\\ &= \frac{1}{2}(nK+F).((n+1)K+F)\end{aligned}$$

$$+\frac{1}{2}(nK+F)_-.((nK+F)_- - 2(nK+F) - K) + 1.$$

Moreover, from $(nK+F).(nK+F)_- = (nK+F)_-^2$:

$$0 \le h^0((nK+F)_+ + K) = \frac{1}{2}(nK+F).[(n+1)K+F] - \frac{1}{2}(nK+F)_-.((nK+F)_- + K) + 1.$$

The last inequality follows from $K^2 = K_f^2 - 8(g-1)$:

$$\begin{aligned}(nK+F).((n+1)K+F) &= n(n+1)K^2 + (2n+1)K.F\\ &= n(n+1)(K_f^2 - 8(g-1)) + 2(2n+1)(g-1)\\ &= n(n+1)K_f^2 - 2(4n^2+2n-1)(g-1).\end{aligned}$$

□

Now, note that being $((lK+F)_+ + K)_-$ a sum of (-1)−cycles we can contract its support obtaining a non-singular surface T with $K = \pi^*K_T + ((lK+F)_+ + K)_-$, where $\pi : X \to T$ is the aforementioned contraction. Thus,

$$K^2 = (\pi^*K_T)^2 + (((lK+F)_+ + K)_-)^2 = (\pi^*K_T)^2 + p_a(((lK+F)_+ + K)_-) - 1.$$

Substituting $K_f^2 = K^2 + 8(g-1)$, and observing that, since T is rational, $K_T^2 \le 9$ we obtain:

$$K_f^2 - p_a(((lK+F)_+ + K)_-) + 1 \le 8(g-1) + 9. \tag{2}$$

Thus, combining Corollary 3.2 (4), Lemma 3.3 and (2) we have:

$$\begin{aligned}&\frac{n(n-1)}{2}(K_f^2 - 8(g-1) - 9) + (4n^2+2n-1)(g-1)\\ &= (\sum_{l=1}^{n-1}(n-l))(K_f^2 - 8(g-1) - 9) + (4n^2+2n-1)(g-1)\\ &\le \sum_{l=1}^{n-1}(n-l)[p_a((lK+F)_+ - 1] + (4n^2+2n-1)(g-1) \le \frac{n(n+1)}{2}K_f^2 + 1.\end{aligned}$$

From this the Proposition follows. □

4. Proof of Proposition 1.3

The proof is based on Reider's method applied to the linear system $(nK+F)_+ + K$. The next Lemma will be useful for giving an estimate of the self-intersection of $(nK+F)_+$:

Lemma 4.1. *If* $|(lK+F)_+ + K| \ne \emptyset$ *for all* $1 \le l \le n-1$*and* $|(n-1)K+F|$ *defines a birational map, then:*

(1) $-2(2n+1)(g-1) - 9n(n+1) - 2 + 2\sum_{l=1}^{n-2}(n-l)(p_a(((lK+F)_+ + K)_-) - 1)$
$\le (n-1)(n+2)(p_a(((n-1)K+F)_+ + K)_-) - 1).$

(2) *If, moreover,* $|(lK+F)_+ + K|$ *defines a birational map for* $1 \le l \le n-1$*, then*

$$n(nK+F)_-.K - (nK+F)_-^2 \ge -\frac{(n-1)}{(n+2)}\Big(2(n+3)(g-1) + 3n^2 + 12n + 20\Big).$$

Proof. 1) By Lemma 3.3 and Corollary 3.2 (4) we have:

$$h^0((nK+F)_+ + K) = \frac{1}{2}n(n+1)K_f^2 - (4n^2+2n-1)(g-1) - \sum_{l=1}^{n-1}(n-l)(p_a((lK+F)_+ + K)_-) - 1) + 1.$$

Using equation (2), we can write:

$$\begin{aligned} 0 \le\ & \frac{1}{2}n(n+1)(8(g-1)+9) - (4n^2+2n-1)(g-1) \\ & - \sum_{l=1}^{n-2}(n-l)(p_a((lK+F)_+ + K)_-) - 1) \\ & + \left(\frac{1}{2}n(n+1) - 1\right)(p_a(((n-1)K+F)_+ + K)_-) - 1) + 1. \end{aligned}$$

The inequality follows after regrouping.

2) The proof is by induction on n. If $n = 2$, using that $K + F$ is nef we can, using Lemma 3.1 (2), write:

$$\begin{aligned} 2(2K+F)_-.K - (2K+F)_-^2 &= 2((K+F)_+ + K)_-.K - ((K+F)_+ + K)_-^2 \\ &= p_a(((K+F)_+ + K)_-) - 1. \end{aligned}$$

Moreover, since $|(K+F)_+ + K|$ defines a birational map it follows from the previous part that part 2) is true for $n = 2$.

Assume that $n > 2$. Since $((n-1)K+F)_+ + K$ is effective it follows from Lemma 3.1 (1) that $nK + F$ is effective and from Lemma 3.1 (2):

$$\begin{aligned} n(nK+F)_-.K - (nK+F)_-^2 &= n[((n-1)K+F)_- + (((n-1)K+F)_+ + K)_-].K \\ &\quad - [((n-1)K+F)_- + (((n-1)K+F)_+ + K)_-]^2 \\ &= (n-1)((n-1)K+F)_-.K - ((n-1)K+F)_-^2 \\ &\quad - ((n-1)K+F)_-.K + (n-1)(((n-1)K+F)_+ + K)_-^2 \\ &= (n-1)((n-1)K+F)_-.K - ((n-1)K+F)_-^2 \\ &\quad - \sum_{l=1}^{n-2}(p_a((lK+F)_+ + K)_-) - 1) \\ &\quad + (n-1)(p_a(((n-1)K+F)_+ + K)_-) - 1) \\ &\text{(by Corollary 3.2 (2) applied to } ((n-1)K+F)_-) \\ &\ge (n-1)((n-1)K+F)_-.K - ((n-1)K+F)_-^2 \\ &\quad + \frac{1}{n+2}(-2(2n+1)(g-1) - 9n(n+1) - 2) \\ &\quad + \frac{1}{n+2}\sum_{l=1}^{n-2}(-2(l+1)+n)(p_a((lK+F)_+ + K)_-) - 1) \\ &\text{(by part (1) applied to } ((n-1)K+F)_+ + K) \\ &\ge \left(1 - \frac{1}{n+2}\right)((n-1)((n-1)K+F)_-.K - ((n-1)K+F)_-^2) \end{aligned}$$

$$-\frac{1}{n+2}(2(2n+1)(g-1)+9n(n+1)+2)$$
$$\text{(by Corollary 3.2 (5) applied to } ((n-1)K+F)_-).$$

Finally, using induction, we obtain:

$$\begin{aligned} n(nK+F)_-.K-(nK+F)_-^2 &\geq -\frac{n-2}{n+2}(2(n+2)(g-1)+3(n-1)^2+12(n-1)+20) \\ &\quad -\frac{1}{n+2}(2(2n+1)(g-1)+9n(n+1)+2) \\ &= -\frac{1}{n+2}(S_1(g-1)+S_2) \end{aligned}$$

with $S_1 = 2(n^2-4)+2(2n+1)$ and $S_2 = (n-2)(3(n-1)^2+12(n-1)+20)+9n(n+1)+2$, from this the Lemma follows. □

Now, we reassume the proof of Proposition 1.3. The proof is by induction on n. If $n = 1$, we know from [1, Theorem 3.3 ii)] that $(K+F)_+ + K$ is effective and from the proof of the Theorem it follows that this linear system defines a birational map.

Assume Proposition 1.3 is true for all $1 \leq l \leq n-1$ and that $\frac{1}{6}(3n^3+20n^2+39n-2) \leq g-1$ and gon $F \geq 2n+3$. Since $\frac{1}{6}(3n^3+20n^2+39n-2)$ and $2n+3$ are increasing functions in n, we see, using induction, that $|(lK+F)_+ + K|$ defines a birational map for all $1 \leq l \leq n-1$. From Lemma 3.1 (1) it follows that $nK+F$ is effective. For what remains of this proof, we write

$$nK+F = P+N,$$

for the Zariski decomposition (i.e., $P = (nK+F)_+$ and $N = (nK+F)_-$).

Our first goal is to prove that $P^2 \geq 9$. From Lemma 3.3 we know that P is big and

$$\text{(3)} \qquad \frac{2}{n+1}\left[(4n^2+2n-1)(g-1)+(p_a(N)-1)-1\right] \leq nK_f^2.$$

Since $P^2 = (nK+F)^2 - N^2$ and $K = K_f - 2F$ we obtain from (3):

$$P^2 \geq \frac{2n}{n+1}\left[(4n^2+2n-1)(g-1)+\frac{1}{2}N.(N+K)-1\right]-4n(2n-1)(g-1)-N^2.$$

After regrouping we obtain:

$$P^2 \geq \frac{2n}{n+1}((4n^2+2n-1-2(2n-1)(n+1))(g-1)-1)+\left(\frac{n}{n+1}N.(N+K)-N^2\right).$$

Using Lemma 4.1 (2) we get:

$$\begin{aligned} P^2 &\geq \frac{2n}{n+1}((g-1)-1)-\frac{2(n-1)}{(n+1)(n+2)}\left((n+3)(g-1)+\frac{3n^2+12n+20}{2}\right) \\ &= \frac{2}{(n+1)(n+2)}(P_1(g-1)-P_2), \end{aligned}$$

with $P_1 = n(n+2)-(n-1)(n+3) = 3$ and

$$P_2 = n(n+2)+\frac{1}{2}(n-1)(3n^2+12n+20) = \frac{1}{2}(3n^3+11n^2+12n-20).$$

Thus, since $\frac{1}{6}(3n^3 + 20n^2 + 39n - 2) \leq g - 1$:

$$P^2 \geq \frac{6}{(n+1)(n+2)}\left(\frac{1}{6}(3n^3 + 20n^2 + 39n - 2)\right) - \frac{1}{(n+1)(n+2)}(3n^3 + 11n^2 + 12n - 20) = 9. \tag{4}$$

In other words, the inequality $\frac{1}{6}(3n^3 + 20n^2 + 39n - 2) \leq g - 1$ implies that $P^2 \geq 9$. Now we use the hypothesis on the gonality of F in order to deduce that $|P+K|$ defines a birational map.

First of all, we note that $P+K$ is effective. Indeed, if $|P+K| = \emptyset$, then by Reider's method (Lemma 2.4 (1)) a base point free pencil $|E|$ exists on X such that $E.P = 1$. If E is not rational, then the map defined by $|P|$ must contract E. Now, from Lemma 3.1 (2) we have that $P = (((n-1)K+F)_+ + K)_+$ and since $H^0((((n-1)K+F)_+ + K)_+) = H^0(((n-1)K+F)_+ + K)$ (Remark 2.1), we conclude that E is contracted by $|((n-1)K + F)_+ + K|$, which contradicts the induction hypothesis.

Therefore, E must be rational and in consequence $E.K = -2$. Now, we claim that the support of N is $|E|$-vertical. Indeed, since P is big and nef and we are assuming that $|P+K| = \emptyset$, we obtain:

$$0 = h^0(P+K) = \frac{1}{2}P.(P+K) + 1,$$

$$P.(P+K) = -2.$$

From this and $P^2 \geq 9$ it is easy to deduce that $2P + K$ is effective. On the other hand $E.(2P + K) = 0$, therefore

$$2P + K = \sum E_i,$$

where E_i are $|E|$-vertical. It follows that $(2P + K)_+$ is the sum of a finite number of curves linearly equivalent to E and $(2P + K)_-$ is formed by a sum of $|E|$-vertical curves properly contained in fibres of $|E|$. Using the description of the negative part of $2P+K$ as the collection of $(-1)-$cycles Θ satisfying $P.\Theta = 2P.\Theta = 0$ (Proposition 2.2) it follows at once that N is $|E|$-vertical. Thus:

$$1 = P.E = (nK + F - N).E = -2n + E.F,$$

which contradicts the hypothesis on the gonality of F. We conclude that $|P + K|$ is effective.

The proof of the birationality of the map defined by $|P + K|$ follows the same lines. First, note that actually we have $P^2 \geq 10$. Indeed, since $|P + K|$ is effective, the inequality in (3) is strict and thus $P^2 \geq 10$.

Now assume that $P+K$ does not defines a birational map. Then, by Lemma 2.4 (2) a base point free pencil $|E|$ exists such that

$$E.P = 1 \quad \text{or} \quad 2.$$

Assume first that $E.P = 1$. If E is rational, then $E.(P + K) = -1$, which contradicts the effectiveness of $P + K$. If E is not rational we argue just as before in order to conclude that

E is contracted by $|((n-1)K + F)_+ + K|$.

Now, if $E.P = 2$ and E is not rational, then the system $|P|$ defines on E a hyperelliptic involution. Thus, $|P|$ does not separate points and by a similar argument as the used above we see that this implies that $|((n-1)K + F)_+ + K|$ does not define a birational map.

Finally, if $E.P = 2$ and E is rational, once again it can be proved that N is $|E|$-vertical, in this case using that $P + K$ is effective, $E.(P + K) = 0$ and thus $P + K$ is a sum of effective $|E|$-vertical divisors. We finally obtain:

$$2 = E.P = E.(nK + F - N) = -2n + E.F.$$

This conclude the proof of the Proposition 1.3 □

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