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ASYMPTOTICS OF EIGENVALUES OF THE LAPLACIAN WITH SMALL SPHERICAL ROBIN BOUNDARY

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1. Introduction

Let Ω be a bounded domain in $\boldsymbol{R}^N$ with C^∞ boundary $\partial\Omega$. Let $\tilde{w}$ be a fixed point in Ω and $B(\varepsilon, \tilde{w})$ be the ball of radius ε with the center $\tilde{w}$. We put $\Omega_\varepsilon=\Omega\backslash\overline{B(\varepsilon, \tilde{w})}$. Consider the following eigenvalue problem

$$(1.1) \qquad \begin{aligned} -\Delta u(x) &= \lambda u(x) && x\in\Omega_\varepsilon \\ u(x) &= 0 && x\in\partial\Omega \\ u(x)+k\varepsilon^\sigma\frac{\partial u}{\partial\nu_x}(x) &= 0 && x\in\partial B(\varepsilon, \tilde{w}) . \end{aligned}$$

Here k denotes a positive constant. And σ is a real number. Here $\partial/\partial\nu_x$ denotes the derivative along the exterior normal direction with respect to Ω_ε.

Let $\mu_j(\varepsilon)>0$ be the j-th eigenvalue of (1.1). Let μ_j be the j-th eigenvalue of the problem

$$(1.2) \qquad \begin{aligned} -\Delta u(x) &= \lambda u(x) && x\in\Omega \\ u(x) &= 0 && x\in\partial\Omega . \end{aligned}$$

Let $G(x,y)$ (resp, $G_\varepsilon(x, y)$) be the Green function of the Laplacian in Ω (resp. Ω_ε) associated with the boundary condition (1.2) (resp. (1.1)).

Main aim of this paper is to show the following Theorems. Let $\varphi_i(x)$ be the L^2-normalized eigenfunction associated with μ_j. We have the following.

Theorem 1. *Assume $N=3$. We fix j and $\sigma\geqq 1$. Suppose that μ_j is simple. Then, for any fixed $s\in(0, 1)$,*

$$(1.3) \qquad \begin{aligned} \mu_j(\varepsilon) &= \mu_j+P_j\varepsilon+O(\varepsilon^{2-s}) && (\quad \sigma\geqq 2) \\ \mu_j(\varepsilon) &= \mu_j+P_j\varepsilon+O(\varepsilon^\sigma) && (1<\sigma<2) \\ \mu_j(\varepsilon) &= \mu_j+(1+k)^{-1}P_j\varepsilon+O(\varepsilon^{2-s}) && (\quad \sigma=1) , \end{aligned}$$

where

$$P_j = 4\pi\varphi_j(\tilde{w})^2 .$$

Theorem 2. *Assume $N=3$. We fix j and $\sigma<1$. Suppose that μ_j is simple. Then,*

$$
\begin{aligned}
(1.4)\qquad & \mu_j(\varepsilon) = \mu_j+Q_j\varepsilon^{2-\sigma}+O(\varepsilon^{3-2\sigma}) && (0\leqq\sigma<1)\\
& \mu_j(\varepsilon) = \mu_j+Q_j\varepsilon^{2-\sigma}+R_j\varepsilon^3+O(\varepsilon^{3-2\sigma}) && (-1/2<\sigma<0)\\
& \mu_j(\varepsilon) = \mu_j+Q_j\varepsilon^{2-\sigma}+R_j\varepsilon^3+O(\varepsilon^4) && (-2<\sigma\leqq-1/2)\\
& \mu_j(\varepsilon) = \mu_j+R_j\varepsilon^3+O(\varepsilon^4) && (\sigma\leqq-2)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
Q_j &= (4\pi/k)\,\varphi_j(\tilde{w})^2\\
R_j &= -\pi(2\,|\mathrm{grad}\,\varphi_j(\tilde{w})|^2-(4/3)\mu_j\varphi_j(\tilde{w})^2)\,.
\end{aligned}
$$

REMARK. The case $N=2$ is treated in Ozawa [10] and [11]. The singularity of $G(x,y)$ near $x=y$ in the case $N=3$ is stronger than that of the case $N=2$. When we use the Sobolev embedding; $W^{2,p}(\Omega)\hookrightarrow C^{2-N/p}(\bar{\Omega})$, we must take p larger as N increases. Therefore we may need some change of the mehtod devloped in the above papers.

When $N\geqq4$, we do not know whether the method we have used can be applied or not.

For the related papers we have Besson [2], Chavel and Feldman [3], Ozawa [8], [9], Rauch and Taylor [12] and the references in the above papers.

For other related problems on singular variation of domains the readers may refer to Arrieta, Hale and Han [1], Jimbo [4], Jimbo and Morita [5]. The Poisson equation with many small Robin holes is discussed in Kaizu [6], [7].

2. Outline of proof of Theorem 1 and Theorem 2

Hereafter we assume $N=3$.

We introduce the following kernel $p_\varepsilon(x,y)$.

$$
\begin{aligned}
(2.1)\qquad p_\varepsilon(x,y) = G(x,y)&+g(\varepsilon)G(x,\tilde{w})G(\tilde{w},y)\\
&+h(\varepsilon)\langle\nabla_w G(x,\tilde{w}),\,\nabla_w G(\tilde{w},y)\rangle\\
&+i(\varepsilon)\langle H_w G(x,\tilde{w}),\,H_w G(\tilde{w},y)\rangle\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\langle\nabla_w u(\tilde{w}),\,\nabla_w v(\tilde{w})\rangle &= \sum_{n=1}^{3}\frac{\partial u}{\partial w_n}\frac{\partial v}{\partial w_n}\Big|_{w=\tilde{w}}\\
\langle H_w u(\tilde{w}),\,H_w v(\tilde{w})\rangle &= \sum_{m,n=1}^{3}\frac{\partial^2 u}{\partial w_m\partial w_n}\frac{\partial^2 v}{\partial w_m\partial w_n}\Big|_{w=\tilde{w}}
\end{aligned}
$$

when $w=(w_1,w_2,w_3)$ is an orthonormal frame of $\boldsymbol{R}^3$. Here $g(\varepsilon)$, $h(\varepsilon)$, $i(\varepsilon)$ are determined so that

$$
(2.2)\qquad p_\varepsilon(x,y)+k\varepsilon^\sigma\frac{\partial}{\partial\nu_x}p_\varepsilon(x,y)\qquad x\in\partial B(\varepsilon,\tilde{w})
$$

is small in some sense.

If we put

$$g(\varepsilon) = -(\gamma+(4\pi\varepsilon)^{-1}+k(4\pi)^{-1}\varepsilon^{\sigma-2})^{-1} \tag{2.3}$$

$$\begin{aligned} h(\varepsilon) &= (k\varepsilon^{\sigma}-\varepsilon)/((4\pi)^{-1}\varepsilon^{-2}+k(2\pi)^{-1}\varepsilon^{\sigma-3}) && (\sigma<1) \\ &= 0 && (\sigma\geqq 1) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} i(\varepsilon) &= k\varepsilon^{\sigma+1}/(3(4\pi)^{-1}\varepsilon^{-3}+9k(4\pi)^{-1}\varepsilon^{\sigma-4}) && (\sigma<1) \\ &= 0 && (\sigma\leqq 1)\,, \end{aligned} \tag{2.5}$$

the above aim for (2.2) to be small is attained.
Here

$$\gamma = \lim_{x\to\tilde{w}} (G(x,\tilde{w})-(4\pi)^{-1}\,|x-\tilde{w}|^{-1})\,.$$

We put

$$(\boldsymbol{G}f)(x) = \int_{\Omega} G(x,y)f(y)dy$$

$$(\boldsymbol{G}_{\varepsilon}f)(x) = \int_{\Omega_\varepsilon} G_{\varepsilon}(x,y)f(y)dy$$

and

$$(\boldsymbol{P}_{\varepsilon}f)(x) = \int_{\Omega_\varepsilon} p_{\varepsilon}(x,y)f(y)dy$$

Let T and T_ε be operators on Ω and Ω_ε, respectively. Then, $||T||_p$, $||T_\varepsilon||_{p,\varepsilon}$ denotes the operator norm on $L^p(\Omega)$, $L^p(\Omega_\varepsilon)$, respectively. Let f and f_ε be functions on Ω and Ω_ε, respectively. Then, $||f||_p$, $||f_\varepsilon||_{p,\varepsilon}$ denotes the norm on $L^p(\Omega)$, $L^p(\Omega_\varepsilon)$, respectively.

At first we outline the proof of Theorem 1. A crucial part of our proof of Theorem 1 is the following.

Theorem 3. *Fix $\sigma\geqq 1$ and $s\in(0,1)$. Then there exists a constant C_s independent of ε such that*

$$||(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)f||_{2,\varepsilon}\leqq C_s\varepsilon^{2-s}\,||f||_{p,\varepsilon} \tag{2.6}$$

holds for any $f\in L^p(\Omega_\varepsilon)$ ($p>3$).

We put

$$\begin{aligned} \tilde{p}_\varepsilon(x,y) = G(x,y)&+g(\varepsilon)G(x,\tilde{w})G(\tilde{w},y)\chi_\varepsilon(x)\chi_\varepsilon(y) \\ &+h(\varepsilon)\langle\nabla_w G(x,\tilde{w}),\,\nabla_w G(\tilde{w},y)\rangle\,\chi_\varepsilon(x)\chi_\varepsilon(y) \\ &+i(\varepsilon)\langle H_w G(x,\tilde{w}),\,H_w G(\tilde{w},y)\rangle\,\chi_\varepsilon(x)\chi_\varepsilon(y) \end{aligned} \tag{2.7}$$

for the characteristic function $\chi_\varepsilon(x)$ of $\overline{\Omega}_\varepsilon$.

And we put

$$(\tilde{\boldsymbol{P}}_\varepsilon f)(x)=\int_\Omega \tilde{p}_\varepsilon(x,y)f(y)dy\,.$$

Since $\boldsymbol{G}_\varepsilon$ is approximated by $\boldsymbol{P}_\varepsilon$ and the difference between $\boldsymbol{P}_\varepsilon$ and $\tilde{\boldsymbol{P}}_\varepsilon$ is small in some sense, we know that everything reduces to our investigation of the perturbative analysis of $\boldsymbol{G}\rightarrow\tilde{\boldsymbol{P}}_\varepsilon$.

Next we outline the proof of Theorem 2. One important part of our proof of Theorem 2 is the following.

Theorem 4. *Fix $\sigma<1$. Then, there exists a constant C such that*

$$\begin{aligned}(2.8)\qquad ||(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)(\chi_\varepsilon\varphi_j)||_{2,\varepsilon}&\leqq C\varepsilon^{4-\sigma}\qquad (0\leqq\sigma<1)\\ &\leqq C\varepsilon^4\qquad\quad (\quad \sigma<0)\end{aligned}$$

hold.

We fix j and put

$$\begin{aligned}(2.9)\qquad \bar{p}_\varepsilon(x,y)=G(x,y)&-(4\pi/3)\mu_j\varepsilon^3G(x,\tilde{w})G(\tilde{w},y)\xi_\varepsilon(x)\xi_\varepsilon(y)\\ &+g(\xi)G(x,\tilde{w})G(\tilde{w},y)\xi_\varepsilon(x)\xi_\varepsilon(y)\\ &+h(\varepsilon)\langle\nabla_wG(x,\tilde{w}),\nabla_wG(\tilde{w},y)\rangle\xi_\varepsilon(x)\xi_\varepsilon(y)\\ &+i(\varepsilon)\langle H_wG(x,\tilde{w}),H_wG(\tilde{w},y)\rangle\xi_\varepsilon(x)\xi_\varepsilon(y)\,,\end{aligned}$$

where $\xi_\varepsilon(x)\in C^\infty(\boldsymbol{R}^3)$ satisfies $|\xi_\varepsilon(x)|\leqq1$, $\xi_\varepsilon(x)=1$ for $x\in\boldsymbol{R}^3\backslash\overline{B(\varepsilon,\tilde{w})}$, $\xi_\varepsilon(x)=0$ for $x\in B(\varepsilon/2,\tilde{w})$ and $\xi_\varepsilon(x-\tilde{w})$ is rotationary invariant.
Furthermore we put

$$(\bar{\boldsymbol{P}}_\varepsilon f)(x)=\int_\Omega \bar{p}_\varepsilon(x,y)f(y)dy\,.$$

The other important part of our proof of Theorem 2 is the following.

Theorem 5. *Fix $\sigma<1$. Then, there exists a constant C such that*

$$\begin{aligned}(2.10)\qquad ||(\chi_\varepsilon\bar{\boldsymbol{P}}_\varepsilon-\boldsymbol{P}_\varepsilon\chi_\varepsilon)\varphi_j||_{2,\varepsilon}&\leqq C\varepsilon^{4-\sigma}\qquad (0<\sigma<1)\\ &\leqq C\varepsilon^4\qquad\quad (\quad \sigma\leqq0)\end{aligned}$$

hold.

Since (2.8) and (2.10) are both $o(\varepsilon^3+\varepsilon^{2-\sigma})$, we know that everything reduces to our investigation of the perturbative analysis of $\boldsymbol{G}\rightarrow\bar{\boldsymbol{P}}_\varepsilon$.

3. Estimation of L^p-norm

We write $B(\varepsilon,\tilde{w})=B_\varepsilon$. In this section we show the following propositions.

Proposition 3.1. *Fix $\sigma \geqq 1$. Assume that $u_\varepsilon(x) \in C^\infty(\bar{\Omega}_\varepsilon)$ satisfies*

$$
\begin{aligned}
(3.1) \qquad & \Delta u_\varepsilon(x) = 0 \qquad x \in \Omega_\varepsilon \\
& u_\varepsilon(x) = 0 \qquad x \in \partial\Omega \\
& u_\varepsilon(x) + k\varepsilon^\sigma \frac{\partial u_\varepsilon}{\partial \nu_x}(x) = M(\omega) \quad x = \tilde{w} + \varepsilon\omega \in \partial B_\varepsilon \, (\omega \in S^2) .
\end{aligned}
$$

We fix $s \in (0, 1)$. Then,

$$
(3.2) \qquad ||u_\varepsilon||_{2,\varepsilon} \leqq C_s \varepsilon^{1-s} \operatorname*{Max}_\omega |M(\omega)|
$$

holds for a constant C_s independent of ε.

Proposition 3.2. *Fix $\sigma < 2$. Under the same assumptions of u_ε in Proposition 3.1,*

$$
(3.3) \qquad ||u_\varepsilon||_{2,\varepsilon} \leqq C\varepsilon^{2-\sigma} \operatorname*{Max}_\omega |M(\omega)|
$$

holds for a constant C independent of ε.

We take the same procedure as in Ozawa [9, section 1, pp. 260–262] to prove the above Propositions. But we need some change of the method developed in the above paper, since we put the Robin condition on ∂B_ε and we assume that $N=3$.

At first we prepare two Lemmas.

Lemma 3.3. *Fix $\alpha \in C^\infty(S^2)$ and $q > 1$. Then there exists at least one solution of*

$$
(3.4) \qquad \Delta v_\varepsilon(x) = 0 \qquad x \in \boldsymbol{R}^3 \backslash \bar{B}_\varepsilon
$$

$$
(3.5) \qquad v_\varepsilon(x) + k\varepsilon^\sigma \frac{\partial v_\varepsilon}{\partial \nu_x}(x) = \alpha(\omega) \quad x = \tilde{w} + \varepsilon\omega \in \partial B_\varepsilon (\omega \in S^2)
$$

satisfying

$$
(3.6) \qquad |v_\varepsilon(x)| \leqq C\varepsilon^{2-\sigma} \operatorname*{Max}_\omega |\alpha(\omega)| \, r^{-1}(\log(r/(r-\varepsilon)))^{1/2}
$$

$$
(3.7) \qquad |v_\varepsilon(x)| \leqq C_q \varepsilon^{1-\sigma/q} \operatorname*{Max}_\omega |\alpha(\omega)| \, (r-\varepsilon)^{-1/q'}
$$

for $r = |x - \tilde{w}| > \varepsilon$ and

$$
(3.8) \qquad ||v_\varepsilon||_{2,\varepsilon} \leqq C'_q \varepsilon^{1-(\sigma-1)/(2q)} \operatorname*{Max}_\omega |\alpha(\omega)| ,
$$

where q' satisfies $(1/q) + (1/q') = 1$.

Proof. We put $x = \tilde{w} + r\omega$ $(\omega \in S^2)$ and

$$
\omega = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta) \quad (0 \leqq \theta < \pi, \ 0 \leqq \varphi < 2\pi).
$$

Let $P_n(z)$ be the Legendre polynomial and $P_n^m(z)$ be the associated Legendre function, that is,

$$P_n^m(z) = (1-z^2)^{m/2} \cdot d^m P_n(z)/dz^m \qquad (|z|<1,\ m \in \boldsymbol{Z}_+)\,.$$

It is well-known that $\{P_n^m(\cos\theta)\cos m\varphi,\ P_n^m(\cos\theta)\sin m\varphi;\ 0\leqq m\leqq n\}_{n=0}^{\infty}$ is a complete orthogonal system of $L^2(S^2)$ consisting of eigenfunction of the Laplace-Beltrami operator Δ_{S^2} whose eigenvalue is $-n(n+1)$.

Therefore we have the Fourier expansion

$$\alpha(\omega) = \sum_{n=0}^{\infty} Y_n(\theta, \varphi)\,, \tag{3.9}$$

where

$$Y_n(\theta, \varphi) = \sum_{m=0}^{n} (a_{n,m}\cos m\varphi + b_{n,m}\sin m\varphi)\, P_n^m(\cos\theta)\,. \tag{3.10}$$

By the Parseval relation, we see

$$\begin{aligned}&\sum_{n=0}^{\infty}(2n+1)^{-1}(a_{n,0}^2+\sum_{m=1}^{n}((n+m)!/2\cdot(n-m)!)(a_{n,m}^2+b_{n,m}^2)) \\ &= C\|\alpha\|_{L^2(S^2)}^2 \leqq C'(\operatorname*{Max}_{\omega}|\alpha(\omega)|)^2\,.\end{aligned} \tag{3.11}$$

We put

$$v_\varepsilon(x) = \sum_{n=0}^{\infty}(\sum_{m=0}^{n}(s_{n,m}\cos m\varphi + t_{n,m}\sin m\varphi)P_n^m(\cos\theta))r^{-(n+1)}\,.$$

Then, it satisfies $\Delta v_\varepsilon(x)=0$ for $x \in \boldsymbol{R}^3\backslash\bar{B}_\varepsilon$.
We see that

$$\begin{aligned}v_\varepsilon(x)+k\varepsilon^{\sigma}\frac{\partial v_\varepsilon}{\partial v_x}(x)_{|x\in\partial B_\varepsilon} &= \alpha(\omega) \\ &= \sum_{n=0}^{\infty}(\sum_{m=0}^{n}(a_{n,m}\cos m\varphi + b_{n,m}\sin m\varphi)P_n^m(\cos\theta))\end{aligned}$$

implies

$$\begin{aligned}a_{n,m} &= \varepsilon^{-(n+1)}(1+(n+1)k\varepsilon^{\sigma-1})s_{n,m} \\ b_{n,m} &= \varepsilon^{-(n+1)}(1+(n+1)k\varepsilon^{\sigma-1})t_{n,m}\end{aligned}$$

for $0\leqq m\leqq n,\ n\geqq 0$.
Thus we have

$$v_\varepsilon(x) = \sum_{n=0}^{\infty} Y_n(\theta, \varphi)\,(\varepsilon/r)^{n+1}(1+(n+1)k\varepsilon^{\sigma-1})^{-1}\,, \tag{3.12}$$

and

$$|v_\varepsilon(x)|^2 \leq (\sum_{n=0}^{\infty} Y_n(\theta, \varphi)^2)\sum_{n=0}^{\infty}(\varepsilon/r)^{2n+2}(1+(n+1)k\varepsilon^{\sigma-1})^{-2}\,. \tag{3.13}$$

Since (3.9) holds in $L^2(S^2)$, we see that

$$\int_{\omega\in S^2} |v_\varepsilon(x)|^2\, d\omega$$
$$\leqq \|\alpha\|^2_{L^2(S^2)} \sum_{n=0}^{\infty} (\varepsilon/r)^{2n+2}(1+(n+1)k\varepsilon^{\sigma-1})^{-2}$$
$$\leqq C(\operatorname{Max}_{\omega}|\alpha(\omega)|)^2 (\sum_{n=0}^{\infty} (\varepsilon/r)^{2(n+1)q'})^{1/q'}$$
$$\times (\sum_{n=0}^{\infty} (1+(n+1)k\varepsilon^{\sigma-1})^{-2q})^{1/q}$$
$$\leqq C(\operatorname{Max}_{\omega}|\alpha(\omega)|)^2 (\varepsilon/r)^2 (\sum_{n=0}^{\infty} (\varepsilon/r)^{n})^{1/q'}$$
$$\times (\int_0^{\infty} (1+k\varepsilon^{\sigma-1}t)^{-2q} dt)^{1/q}$$
$$= C(\operatorname{Max}_{\omega}|\alpha(\omega)|)^2 (\varepsilon/r)^2 (r/(r-\varepsilon))^{1/q'}((2q-1)k\varepsilon^{\sigma-1})^{-1/q}.$$

Therefore, we have

$$\|v_\varepsilon\|^2_{2,\varepsilon} \leqq \int_\varepsilon^R (\int_{\omega\in S^2} |v_\varepsilon(x)|^2 d\omega) r^2 dr$$
$$\leqq C_q(\operatorname{Max}_{\omega}\ |\alpha(\omega)|)^2\ \varepsilon^{2+(1-\sigma)/q}.$$

Thus we get (3.8).

Using the Schwarz inequality and the relation

$$P_n(\cos\theta)^2 + \sum_{m=1}^{n} (2\cdot(n-m)!/(n+m)!) P_n^m(\cos\theta)^2 = 1\,,$$

we see

$$(3.13)\qquad |Y_n(\theta,\varphi)|^2 \leqq a^2_{n,0} + \sum_{m=1}^{n} ((n+m)/!2\cdot(n-m)!)\,(a^2_{n,m}+b^2_{n,m})\,.$$

From (3.11), (3.12) and (3.13), we have

$$|v_\varepsilon(x)| \leqq C\ \operatorname{Max}_{\omega}|\alpha(\omega)|\, R(\varepsilon,\sigma,r)^{1/2}(\varepsilon/r)\,,$$

where

$$R(\varepsilon,\sigma,r) = \sum_{n=0}^{\infty} (\varepsilon/r)^{2n}(n+1)\,(1+(n+1)k\varepsilon^{\sigma-1})^{-2}\,.$$

Since

$$R(\varepsilon,\sigma,r) \leqq C\ \varepsilon^{2(1-\sigma)} \sum_{n=0}^{\infty} (n+1)^{-1}(\varepsilon/r)^{n+1}$$
$$\leqq C\ \varepsilon^{2(1-\sigma)} \log(r/(r-\varepsilon))$$

and

$$R(\varepsilon,\sigma,r) \leq (\sum_{n=0}^{\infty} (n+1)\,(\varepsilon/r)^{2nq'})^{1/q'}$$
$$\times (\sum_{n=0}^{\infty} (n+1)\,(1+(n+1)k\varepsilon^{\sigma-1})^{-2q})^{1/q}$$

$$\leqq \left(\sum_{n=0}^{\infty}(n+1)(\varepsilon/r)^n\right)^{1/q'}$$

$$\times\left(k^{-1}\varepsilon^{1-\sigma}\sum_{n=0}^{\infty}(1+(n+1)k\varepsilon^{\sigma-1})^{1-2q}\right)^{1/q}$$

$$\leqq (r/(r-\varepsilon))^{2/q'}\left(k^{-1}\varepsilon^{1-\sigma}\int_0^\infty (1+k\varepsilon^{\sigma-1}t)^{1-2q}dt\right)^{1/q}$$

$$=(r/(r-\varepsilon))^{2/q'}(2(q-1)k^2\varepsilon^{2(\sigma-1)})^{-1/q}$$

hold, we get (3.6) and (3.7). q.e.d.

Lemma 3.4. *Fix $\beta\in C^\infty(\partial B_\varepsilon)$. Assume that $g\in H^2(\Omega_\varepsilon)$ satisfies*

$$\begin{aligned} \Delta g(x) &= 0 \qquad x\in\Omega_\varepsilon \\ g(x) &= 0 \qquad x\in\partial\Omega \\ g(x)+k\varepsilon^\sigma\frac{\partial g}{\partial\nu_x}(x) &= \beta(x) \qquad x\in\partial B_\varepsilon\,. \end{aligned} \tag{3.14}$$

Then,

$$\int_{\Omega_\varepsilon}|\nabla g(x)|^2\,dx\leqq 4\pi k^{-1}\varepsilon^{2-\sigma}(\operatorname*{Max}_{\partial B_\varepsilon}|\beta(x)|)^2\,.$$

Proof. Since $g\in H^2(\Omega_\varepsilon)$, we have Green's formula:

$$\int_{\Omega_\varepsilon}(g\cdot\Delta g+|\nabla g|^2)\,dx=\int_{\partial\Omega_\varepsilon}g\,\frac{\partial g}{\partial\nu_x}\,d\sigma_x\,.$$

By (3.14), we can see that

$$\int_{\Omega_\varepsilon}|\nabla g|^2\,dx=\int_{\partial B_\varepsilon}g\,\frac{\partial g}{\partial\nu_x}\,d\sigma_x=\int_{\partial B_\varepsilon}(\beta(x)-k\varepsilon^\sigma\frac{\partial g}{\partial\nu_x}(x))\frac{\partial g}{\partial\nu_x}(x)d\sigma_x\,.$$

Therefore, we have

$$\int_{\Omega_\varepsilon}|\nabla g|^2\,dx+k\varepsilon^\sigma\int_{\partial B_\varepsilon}\left|\frac{\partial g}{\partial\nu_x}\right|^2d\sigma_x=\int_{\partial B_\varepsilon}\beta(x)\,\frac{\partial g}{\partial\nu_x}(x)d\sigma_x\,. \tag{3.15}$$

Using the Schwarz inequality, we have

$$\begin{aligned} k\varepsilon^\sigma\int_{\partial B_\varepsilon}\left|\frac{\partial g}{\partial\nu_x}\right|^2d\sigma_x &\leqq \int_{\partial B_\varepsilon}\beta(x)\,\frac{\partial g}{\partial\nu_x}(x)d\sigma_x \\ &\leqq \left(\int_{\partial B_\varepsilon}\beta(x)^2\,d\sigma_x\right)^{1/2}\left(\int_{\partial B_\varepsilon}\left|\frac{\partial g}{\partial\nu_x}\right|^2d\sigma_x\right)^{1/2}. \end{aligned}$$

Thus,

$$\left(\int_{\partial B_\varepsilon}\left|\frac{\partial g}{\partial\nu_x}\right|^2d\sigma_x\right)^{1/2}\leqq k^{-1}\varepsilon^{-\sigma}\left(\int_{\partial B_\varepsilon}\beta(x)^2d\sigma_x\right)^{1/2}. \tag{3.16}$$

From (3.15) and (3.16), we get

$$\begin{aligned}\int_{\Omega_\varepsilon} |\nabla g|^2 \, dx &\leqq \int_{\partial B_\varepsilon} \beta(x) \frac{\partial g}{\partial \nu_x}(x) \, d\sigma_x \\ &\leqq \Big(\int_{\partial B_\varepsilon} \beta(x)^2 \, d\sigma_x\Big)^{1/2} \Big(\int_{\partial B_\varepsilon} \Big|\frac{\partial g}{\partial \nu_x}\Big|^2 d\sigma_x\Big)^{1/2} \\ &\leqq k^{-1}\varepsilon^{-\sigma} \int_{\partial B_\varepsilon} \beta(x)^2 \, d\sigma_x \\ &\leqq 4\pi k^{-1} \varepsilon^{2-\sigma} (\underset{\partial B_\varepsilon}{\mathrm{Max}} |\beta(x)|)^2 .\end{aligned}$$

q.e.d.

Now we are in a position to prove Propositions 3.1 and 3.2.

Proof of Proposition 3.1. Let $u_\varepsilon(x)$ be as in Proposition 3.1. We take an arbitrary $q>\sigma$. Firstly we put $\alpha(\omega)=M(\omega)$ and we take $v_\varepsilon^{(0)}$ so that it satisfies (3.4), (3.5), (3.7) and (3.8). Then $v_\varepsilon^{(0)}$ may not satisfy $v_\varepsilon^{(0)}(x)=0$ for $x\in\partial\Omega$. Let $v_\varepsilon^{(1)}$ be the harmonic function in Ω satisfying $v_\varepsilon^{(1)}(x)=v_\varepsilon^{(0)}(x)$ for $x\in\partial\Omega$. Put

$$M_\varepsilon = \underset{\omega}{\mathrm{Max}}\ |M(\omega)| .$$

Then from (3.7) we see that $\mathrm{Max}\{|v_\varepsilon^{(1)}(x)|\,;\, x\in\bar\Omega\} \leqq \hat{C}_q \varepsilon^{1-\sigma/q} M_\varepsilon$ and $\mathrm{Max}\{|v_\varepsilon^{(1)}(x) + k\varepsilon^\sigma \partial v_\varepsilon^{(1)}(x)/\partial\nu_x|\,;\, x\in\partial B_\varepsilon\} \leqq \hat{C}_q \varepsilon^{1-\sigma/q} M_\varepsilon$, where $\hat{C}_q$ is a constant independent of ε. Secondly, we put $\alpha(\omega)=v_\varepsilon^{(1)}(x)+k\varepsilon^\sigma \partial v_\varepsilon^{(1)}(x)/\partial\nu_x$ for $x=\tilde{w}+\varepsilon\omega\in\partial B_\varepsilon$ and we take $v_\varepsilon^{(2)}$ so that it satisfies (3.4), (3.5), (3.7) and (3.8). Let $v_\varepsilon^{(3)}$ be the harmonic function in Ω satisfying $v_\varepsilon^{(3)}(x)=v_\varepsilon^{(2)}(x)$ for $x\in\partial\Omega$. Then, $\mathrm{Max}\{|v_\varepsilon^{(3)}(x)|\,;\, x\in\bar\Omega\} \leqq (\hat{C}_q\varepsilon^{1-\sigma/q})^2 M_\varepsilon$ and $\mathrm{Max}\{|v_\varepsilon^{(3)}(x)+k\varepsilon^\sigma\partial v_\varepsilon^{(3)}(x)/\partial\nu_x|\,;\, x\in\partial B_\varepsilon\} \leqq (\hat{C}_q\varepsilon^{1-\sigma/q})^2 M_\varepsilon$.

By repeating this procedure we have

$$\begin{aligned} \Delta v_\varepsilon^{(2n+1)}(x) &= 0 \qquad x\in\Omega \\ v_\varepsilon^{(2n+1)}(x) &= v_\varepsilon^{(2n)}(x) \qquad x\in\partial\Omega \end{aligned}$$

and

$$\Delta v_\varepsilon^{(2n+2)}(x) = 0 \qquad x\in \boldsymbol{R}^3\backslash\bar{B}_\varepsilon$$

$$v_\varepsilon^{(2n+2)}(x)+k\varepsilon^\sigma \frac{\partial v_\varepsilon^{(2n+2)}}{\partial\nu_x}(x) = v_\varepsilon^{(2n+1)}(x)+k\varepsilon^\sigma \frac{\partial v_\varepsilon^{(2n+1)}}{\partial\nu_x}(x) \qquad x\in\partial B_\varepsilon$$

for $n=0, 1, 2, \cdots$.

Then, by induction,

$$\underset{\bar\Omega}{\mathrm{Max}} |v_\varepsilon^{(2n+1)}(x)| \leqq (\hat{C}_q \varepsilon^{1-\sigma/q})^{n+1} M_\varepsilon \tag{3.17}$$

$$\underset{\partial B_\varepsilon}{\mathrm{Max}} \Big|v_\varepsilon^{(2n+1)}(x)+k\varepsilon^\sigma \frac{\partial v_\varepsilon^{(2n+1)}}{\partial\nu_x}(x)\Big| \leqq (\hat{C}_q \varepsilon^{1-\sigma/q})^{n+1} M_\varepsilon \tag{3.18}$$

$$|v_\varepsilon^{(2n)}(x)| \leqq (\hat{C}_q \varepsilon^{1-\sigma/q})^{n+1} M_\varepsilon (r-\varepsilon)^{-1/q'} \qquad (r>\varepsilon) \tag{3.19}$$

$$\|v_\varepsilon^{(2n)}\|_{2,\varepsilon} \leqq C_q' \varepsilon^{1-(\sigma-1)/(2q)} (\hat{C}_q \varepsilon^{1-\sigma/q})^n M_\varepsilon \tag{3.20}$$

hold for $n \geqq 0$.

Since $q>\sigma$, we can take ε so that $\hat{C}_q \varepsilon^{1-\sigma/q}<1/2$. We put

$$w_\varepsilon(x) = \sum_{n=0}^{\infty} (-1)^n v_\varepsilon^{(n)}(x) . \tag{3.21}$$

From (3.17) and (3.19), we can see that the right hand side of (3.21) is uniformly convergent on $\bar{\Omega} \backslash B_\eta$ for any $\eta>\varepsilon$. Since $v_\varepsilon^{(n)}$ is harmonic in Ω_ε, we see that $w_\varepsilon(x)$ is harmonic in Ω_ε, $w_\varepsilon(x)=0$ for $x \in \partial\Omega$ and

$$\frac{\partial w_\varepsilon}{\partial x_j}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\partial v_\varepsilon^{(n)}}{\partial x_j}(x) \qquad x \in \Omega_\varepsilon , \qquad j=1,2,3 .$$

We put

$$g_\varepsilon^{(n)}(x) = u_\varepsilon(x) - \sum_{i=0}^{2n+1} (-1)^i v_\varepsilon^{(i)}(x) .$$

Then,

$$\nabla g_\varepsilon^{(n)}(x) \to \nabla(u_\varepsilon - w_\varepsilon)(x) \qquad (n \to \infty) \qquad \text{for} \quad x \in \Omega_\varepsilon . \tag{3.22}$$

It is easy to see that $g_\varepsilon^{(n)}$ is harmonic in Ω_ε, $g_\varepsilon^{(n)}(x)=0$ for $x \in \partial\Omega$ and

$$g_\varepsilon^{(n)}(x) + k\varepsilon^\sigma \frac{\partial g_\varepsilon^{(n)}}{\partial \nu_x}(x) = v_\varepsilon^{(2n+1)}(x) + k\varepsilon^\sigma \frac{\partial v_\varepsilon^{(2n+1)}}{\partial \nu_x}(x) \qquad x \in \partial B_\varepsilon .$$

Therefore, by Lemma 3.4 and (3.18), we have

$$\begin{aligned}\int_{\Omega_\varepsilon} |\nabla g_\varepsilon^{(n)}|^2 \, dx &\leqq 4\pi k^{-1} \varepsilon^{2-\sigma} \operatorname*{Max}_{\partial B_\varepsilon} \left| v_\varepsilon^{(2n+1)}(x) + k\varepsilon^\sigma \frac{\partial v_\varepsilon^{(2n+1)}}{\partial \nu_x}(x) \right| \\ &\leqq 4\pi k^{-1} \varepsilon^{2-\sigma} (\hat{C}_q \varepsilon^{1-\sigma/q})^n M_\varepsilon .\end{aligned}$$

Using Fatou's Lemma and (3.22), we see that

$$\int_{\Omega_\varepsilon} |\nabla(u_\varepsilon - w_\varepsilon)|^2 \, dx \leqq \liminf_{n \to \infty} \int_{\Omega_\varepsilon} |\nabla g^{(n)}|^2 \, dx \leqq 0 .$$

Thus, $u_\varepsilon - w_\varepsilon = \text{constant}$ a.e. Ω_ε. Since $u_\varepsilon(x)=w_\varepsilon(x)=0$ for $x \in \partial\Omega$, $u_\varepsilon = w_\varepsilon$ a.e. Ω_ε. Therefore,

$$u_\varepsilon(x) = \sum_{n=0}^{\infty} (-1)^n v_\varepsilon^{(n)}(x) \qquad x \in \Omega_\varepsilon . \tag{3.23}$$

From (3.17) and (3.20), we have

$$\begin{aligned}\|\sum_{n=0}^{2n'+1}(-1)^n v_\varepsilon^{(n)}\|_{2,\varepsilon} &\leqq \sum_{n=0}^{n'}(\|v_\varepsilon^{(2n)}\|_{2,\varepsilon}+\|v_\varepsilon^{(2n+1)}\|_{2,\varepsilon}) \\ &\leqq \sum_{n=0}^{n'}(C'_q\varepsilon^{1-(\sigma-1)/(2q)}+\hat{C}_q\varepsilon^{1-\sigma/q})(1/2)^n M_\varepsilon \\ &\leqq C_q\varepsilon^{1-\sigma/q}M_\varepsilon\,.\end{aligned}$$

Using Fatou's Lemma and (3.23), we see that

$$\begin{aligned}\int_{\Omega_\varepsilon}|u_\varepsilon(x)|^2\,dx &\leqq \liminf_{n'\to\infty}\int_{\Omega_\varepsilon}|\sum_{n=0}^{2n'+1}(-1)^n v_\varepsilon^{(n)}(x)|^2\,dx \\ &\leqq (C_q\varepsilon^{1-\sigma/q}M_\varepsilon)^2\,.\end{aligned}$$

Thus we get (3.2). q.e.d.

Proof of Proposition 3.2. Let $\{v_\varepsilon^{(n)}(x)\}_{n=0}^{\infty}$ be the sequence of functions as in the proof of Proposition 3.1. Then, by using (3.6), we can get

$$\text{(3.24)}\qquad \begin{aligned}&\operatorname*{Max}_{\bar{\Omega}}|v_\varepsilon^{(2n+1)}(x)|\leqq(\hat{C}\varepsilon^{2-\sigma})^{n+1}M_\varepsilon \\ &\operatorname*{Max}_{\partial B_\varepsilon}|v_\varepsilon^{(2n+1)}(x)+k\varepsilon^\sigma\frac{\partial v_\varepsilon^{(2n+1)}}{\partial\nu_x}(x)|\leqq(\hat{C}\varepsilon^{2-\sigma})^{n+1}M_\varepsilon\end{aligned}$$

$$\text{(3.25)}\qquad |v_\varepsilon^{(2n)}(x)|\leqq(\hat{C}\varepsilon^{2-\sigma})^{n+1}M_\varepsilon r^{-1}(\log(r/(r-\varepsilon)))^{1/2}$$

for $n\geqq 0$. Here $\hat{C}$ is a constant independent of ε and $M_\varepsilon=\operatorname*{Max}_{\omega}|M(\omega)|$.

Since $\sigma<2$, we can take ε so that $\hat{C}\varepsilon^{2-\sigma}<1/2$. Then, by the same argument as in the proof of Proposition 3.1, we can see that

$$\text{(3.26)}\qquad u_\varepsilon(x)=\sum_{n=0}^{\infty}(-1)^n v_\varepsilon^{(n)}(x)\qquad x\in\Omega_\varepsilon\,.$$

From (3.24), (3.25) and (3.26), we have

$$\text{(3.27)}\qquad \begin{aligned}|u_\varepsilon(x)| &\leqq \sum_{n=0}^{\infty}(|v_\varepsilon^{(2n)}(x)|+|v_\varepsilon^{(2n+1)}(x)|) \\ &\leqq C\varepsilon^{2-\sigma}M_\varepsilon r^{-1}(\log(r/(r-\varepsilon)))^{1/2}\qquad (r>\varepsilon)\,.\end{aligned}$$

Now (3.3) easily follows from (3.27). q.e.d.

4. Proof of Theorem 3

From this section to section 7, we assume $\sigma\geqq 1$. By (2.3) we see that

$$\text{(4.1)}\qquad \begin{aligned}g(\varepsilon) &= -4\pi\varepsilon+O(\varepsilon^2+\varepsilon^\sigma)\qquad (\sigma>1) \\ &= -4\pi(1+k)^{-1}\varepsilon+O(\varepsilon^2)\qquad (\sigma=1)\,.\end{aligned}$$

We take an arbitrary fixed point $x\in\partial B_\varepsilon$. Without loss of generality we may assume $\tilde{w}=0$ and $x=\varepsilon e_1$. Here we put $e_1=(1,0,0)$. We put $p_\varepsilon(x,y)$ as before and

$$S(x,y)=G(x,y)-(4\pi)^{-1}|x-y|^{-1}.$$

Then, $S(x,y)\in C^\infty(\Omega\times\Omega)$ and

$$\begin{aligned}&p_\varepsilon(x,y)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}p_\varepsilon(x,y)_{|x=\varepsilon e_1}\\ &=G(x,y)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}G(x,y)-g(\varepsilon)k\varepsilon^\sigma\frac{\partial}{\partial x_1}S(x,\tilde{w})G(\tilde{w},y)\\ &\quad+g(\varepsilon)\left((4\pi)^{-1}\varepsilon^{-1}+S(x,\tilde{w})+k(4\pi)^{-1}\varepsilon^{\sigma-2}\right)G(\tilde{w},y)\end{aligned}$$

for $\tilde{w}=0$, $x=\varepsilon e_1$.

Since $\gamma=S(\tilde{w},\tilde{w})$, $S(x,\tilde{w})=\gamma+O(\varepsilon)$ as $\varepsilon\to 0$. By (4.1),

$$\begin{aligned}&p_\varepsilon(x,y)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}p_\varepsilon(x,y)_{|x=\varepsilon e_1}\\ &=G(x,y)-G(\tilde{w},y)+O(\varepsilon^2)G(\tilde{w},y)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}G(x,y)\end{aligned}\tag{4.2}$$

for $\tilde{w}=0$, $x=\varepsilon e_1$.

We take an arbitrary $f\in L^p(\Omega_\varepsilon)$ and let $\tilde{f}$ be the extension of f to Ω defined by 0 on B_ε. Then we have

$$\begin{aligned}&(\boldsymbol{P}_\varepsilon f)(x)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}(\boldsymbol{P}_\varepsilon f)(x)_{|x=\varepsilon e_1}\\ &=(\boldsymbol{G}\tilde{f})(x)-(\boldsymbol{G}\tilde{f})(\tilde{w})+O(\varepsilon^2)(\boldsymbol{G}\tilde{f})(\tilde{w})-k\varepsilon^\sigma\frac{\partial}{\partial x_1}(\boldsymbol{G}\tilde{f})(x)\end{aligned}\tag{4.3}$$

for $\tilde{w}=0$, $x=\varepsilon e_1$.

By the Sobolev embedding theorem and a priori estimate

$$\|\boldsymbol{G}\tilde{f}\|_{C^{1+\tau}(\bar{\Omega})}\leq C\|\tilde{f}\|_p\leqq C\|f\|_{p,\varepsilon}\tag{4.4}$$

hold for $\tau=1-3/p$ $(p>3)$. Therefore we have

$$\left|(\boldsymbol{P}_\varepsilon f)(x)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}(\boldsymbol{P}_\varepsilon f)(x)\right|_{x=\varepsilon e_1}\leqq C\varepsilon\|f\|_{p,\varepsilon}.$$

We put $u_\varepsilon=(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)f$. Then u_ε satisfies (3.1) because $\boldsymbol{G}_\varepsilon f$ satisfies the given Robin condition on ∂B_ε. By Proposition 3.1, we have (2.6).

5. Convergence of eigenvalues for $\sigma\geqq 1$

We put $\tilde{p}_\varepsilon(x,y)$, $\tilde{\boldsymbol{P}}_\varepsilon$ as before. Then,

$$\tilde{\boldsymbol{P}}_\varepsilon = A_0 + g(\varepsilon)A_1\,, \tag{5.1}$$

where $A_0 = \boldsymbol{G}$ and

$$(A_1 f)(x) = G(x, \tilde{w})\chi_\varepsilon(x)(\boldsymbol{G}\chi_\varepsilon f)(\tilde{w})\,. \tag{5.2}$$

Since

$$|(A_1 f)(x)| \leqq C|x-\tilde{w}|^{-1}\chi_\varepsilon(x)\|f\|_p \qquad (p>3/2)\,,$$

we have

$$\begin{aligned} \|A_1 f\|_p &\leqq C\|f\|_p \qquad (3/2<p<3) \\ &\leqq C\varepsilon^{3/p-1}\|f\|_p \qquad (p>3)\,. \end{aligned} \tag{5.3}$$

From (4.1), (5.1) and (5.3) we have

$$\|(\tilde{\boldsymbol{P}}_\varepsilon - \boldsymbol{G})f\|_2 \leqq |g(\varepsilon)|\ \|A_1 f\|_2 \leqq C\varepsilon\|f\|_2$$

for any $f \in L^2(\Omega)$. Therefore we get the following.

Lemma 5.1. *There exists a constant C independent of ε such that*

$$\|\tilde{\boldsymbol{P}}_\varepsilon - \boldsymbol{G}\|_2 \leqq C\varepsilon \tag{5.4}$$

holds.

Next we want to estimate $\|\chi_\varepsilon\tilde{\boldsymbol{P}}_\varepsilon\chi_\varepsilon - \tilde{\boldsymbol{P}}_\varepsilon\|_2$. It does not exceed

$$\|(1-\chi_\varepsilon)\tilde{\boldsymbol{P}}_\varepsilon\chi_\varepsilon\|_2 + \|\tilde{\boldsymbol{P}}_\varepsilon(1-\chi_\varepsilon)\|_2\,. \tag{5.5}$$

Notice that $(1-\chi_\varepsilon)\chi_\varepsilon = 0$ in $g(\varepsilon)$ term. By (5.1),

$$\|(1-\chi_\varepsilon)\tilde{\boldsymbol{P}}_\varepsilon v\|_2 \leqq C|B_\varepsilon|^{1/2}\|\boldsymbol{G}v\|_2 \leqq C\varepsilon^{3/2}\|v\|_2$$

hold for any $v \in L^2(\Omega)$. Therefore we get

$$\begin{aligned} &\|(1-\chi_\varepsilon)\tilde{\boldsymbol{P}}_\varepsilon\|_2 \leqq C\varepsilon^{3/2} \\ &\|(1-\chi_\varepsilon)\tilde{\boldsymbol{P}}_\varepsilon\chi_\varepsilon\|_2 \leqq C\varepsilon^{3/2}\,. \end{aligned} \tag{5.6}$$

Since we have the duality

$$((1-\chi_\varepsilon)\tilde{\boldsymbol{P}}_\varepsilon)^* = \tilde{\boldsymbol{P}}_\varepsilon(1-\chi_\varepsilon)\,,$$

we get

$$\|\tilde{\boldsymbol{P}}_\varepsilon(1-\chi_\varepsilon)\|_2 \leqq C\varepsilon^{3/2}\,. \tag{5.7}$$

Summing up these facts, we get the following.

Lemma 5.2. *There exists a constant C independent of ε such that*

$$\|\chi_\varepsilon \tilde{P}_\varepsilon \chi_\varepsilon - \tilde{P}_\varepsilon\|_2 \leqq C\varepsilon^{3/2}$$

holds.

Notice that the j-the eigenvalue of P_ε is equal to the j-th eigenvalue of $\chi_\varepsilon \tilde{P}_\varepsilon \chi_\varepsilon$. By virtue of Theorem 3, Lemmas 5.1 and 5.2, we see that there exists a constant C independent of ε such that

$$(5.8) \qquad |\mu_j(\varepsilon)^{-1} - \mu_j^{-1}| \leqq C(\varepsilon^{2-s} + \varepsilon + \varepsilon^{3/2}) \leqq C \cdot \varepsilon$$

hold.

For the later convenience we estimate $\|(\chi_\varepsilon \tilde{P}_\varepsilon - P_\varepsilon \chi_\varepsilon) f\|_{2,\varepsilon}$. We put $v_\varepsilon = (\chi_\varepsilon \tilde{P}_\varepsilon - P_\varepsilon \chi_\varepsilon) f$. Then, $v_\varepsilon = (G\hat{\chi}_\varepsilon f)$ on Ω_ε. Here $\hat{\chi}_\varepsilon$ is the characteristic function on B_ε. Thus v_ε satisfies $\Delta v_\varepsilon(x) = 0$ for $x \in \Omega_\varepsilon$, $v_\varepsilon(x) = 0$ for $x \in \partial\Omega$ and

$$(5.9) \qquad |v_\varepsilon(x)| \leqq C\left(\int_{B_\varepsilon} |x-y|^{-p'} dy\right)^{1/p'} \|f\|_p \leqq \begin{cases} C\varepsilon^{2-3/p} \|f\|_p & (3/2 < p < \infty) \\ C\varepsilon^2 \|f\|_\infty & (p = \infty) \end{cases}$$

for $x \in \partial B_\varepsilon$.

By the maximum principle, we get the following.

Lemma 5.3. *There exists a constant C independent of ε such that*

$$(5.10) \qquad \begin{aligned} \|(\chi_\varepsilon \tilde{P}_\varepsilon - P_\varepsilon \chi_\varepsilon) f\|_{2,\varepsilon} &\leqq C\varepsilon^{2-3/p} \|f\|_p \qquad (3/2 < p < \infty) \\ &\leqq C\varepsilon^2 \|f\|_\infty \qquad (p = \infty) \end{aligned}$$

hold for any $f \in L^p(\Omega)$.

6. Perturbational calculus for $\tilde{P}_\varepsilon$

In this section we consider the behaviour of eigenvalues of $\tilde{P}_\varepsilon$ as ε tends to 0. We put

$$\lambda(\varepsilon) = \lambda_0 + g(\varepsilon)\lambda_1$$
$$\psi(\varepsilon) = \psi_0 + g(\varepsilon)\psi_1$$

so that $\lambda(\varepsilon)$ and $\psi(\varepsilon)$ is an approximate eigenvalue of $\tilde{P}_\varepsilon$ and an approximate eigenfunction of $\tilde{P}_\varepsilon$, respectively.

Let λ_0 be a simple eigenvalue of A_0 and ψ_0 be a solution of

$$(6.1) \qquad (A_0 - \lambda_0)\psi_0 = 0\,, \qquad \|\psi_0\|_2 = 1\,.$$

Next we solve the following equations:

$$(6.2) \qquad (A_0 - \lambda_0)\psi_1 = (\lambda_1 - A_1)\psi_0$$

$$(\psi_0, \psi_1)_2 = 0\,, \tag{6.3}$$

where $(\ ,\ )_2$ denotes the inner product on $L^2(\Omega)$.

By the Fredholm alternative theory, we see that

$$\lambda_1 = (A_1\psi_0, \psi_0)_2 \tag{6.4}$$

is the condition such that the unique solution of ψ_1 of (6.2) and (6.3) exists.

Hereafter we put $\lambda_0=\mu_j^{-1}$. Then $\psi_0=\varphi_j$. It is easy to see

$$\lambda_1 = |(G\chi_\varepsilon\varphi_j)(\tilde{w})|^2 = \mu_j^{-2}\varphi_j(\tilde{w})^2+O(\varepsilon^2) \tag{6.5}$$

$$(\tilde{P}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon) = g(\varepsilon)^2(A_1-\lambda_1)\psi_1\,. \tag{6.6}$$

From (5.3), (6.2), (6.4) and (6.6), we have the following.

Lemma 6.1. *There exists a constant C independent of ε such that*

$$\|(\tilde{P}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon)\|_{2,\varepsilon}\leqq Cg(\varepsilon)^2\leqq C\varepsilon^2 \tag{6.7}$$

hold.

By (5.3), (6.2) and (6.4), we have

$$\|\psi_1\|_p,\ \|A_1\|_p\leqq C\varepsilon^{3/p-1} \qquad (p>3)\,. \tag{6.8}$$

Now we have the following.

Lemma 6.2. *Fix $s\in(0, 1)$. Then, there exist constants C, C_s independent of ε such that*

$$\|(P_\varepsilon-G_\varepsilon)(\chi_\varepsilon\psi(\varepsilon))\|_{2,\varepsilon}\leqq C_s\varepsilon^{2-s} \tag{6.9}$$

$$\|(\chi_\varepsilon\tilde{P}_\varepsilon-P_\varepsilon\chi_\varepsilon)\psi(\varepsilon)\|_{2,\varepsilon}\leqq C\varepsilon^2 \tag{6.10}$$

hold.

Proof. By (6.8), Theorem 3 and Lemma 5.3, we have

$$\begin{aligned}&\|(P_\varepsilon-G_\varepsilon)\,(\chi_\varepsilon\psi(\varepsilon))\|_{2,\varepsilon}\\&\leqq C_s\varepsilon^{2-s}(1+|g(\varepsilon)|\,\varepsilon^{3/p-1})\leqq C_s\varepsilon^{2-s} \qquad (p>3)\end{aligned}$$

and

$$\begin{aligned}&\|(\chi_\varepsilon\tilde{P}_\varepsilon-P_\varepsilon\chi_\varepsilon)\psi(\varepsilon)\|_{2,\varepsilon}\\&\leqq C(\varepsilon^2+|g(\varepsilon)|\,\varepsilon^{2-3/p}\varepsilon^{3/p-1})\leqq C\varepsilon^2 \qquad (p>3)\,.\end{aligned}$$

q.e.d.

7. Proof of Theorem 1

Now we are in a position to prove Theorem 1. We fix $s\in(0, 1)$. Then, by Lemmas 6.1 and 6.2, we have

$$\begin{aligned}&||(\boldsymbol{G}_\varepsilon-\lambda(\varepsilon))\,(\chi_\varepsilon\psi(\varepsilon))||_{2,\varepsilon}\\ &\leqq||(\boldsymbol{G}_\varepsilon-\boldsymbol{P}_\varepsilon)\,(\chi_\varepsilon\psi(\varepsilon))||_{2,\varepsilon}+||(\boldsymbol{P}_\varepsilon\chi_\varepsilon-\chi_\varepsilon\tilde{\boldsymbol{P}}_\varepsilon)\psi(\varepsilon)||_{2,\varepsilon}\\ &\qquad+||\chi_\varepsilon(\tilde{\boldsymbol{P}}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon)||_{2,\varepsilon}\\ &\leqq C_s\varepsilon^{2-s}\,.\end{aligned}$$

Since $||\psi(\varepsilon)||_{2,\varepsilon}\in(1/2, 2)$ for small ε, there exists at least one eigenvalue $\lambda^*(\varepsilon)$ of $\boldsymbol{G}_\varepsilon$ satisfying

$$|\lambda^*(\varepsilon)-\lambda(\varepsilon)|\leqq C_s\varepsilon^{2-s}\,. \tag{7.1}$$

We here represent $\lambda(\varepsilon)$ explicitly as follows:

$$\begin{aligned}\lambda(\varepsilon) &= \mu_j^{-1}+g(\varepsilon)\,(\mu_j^{-2}\varphi_j(\tilde{w})^2+0(\varepsilon^2))\\ &=\begin{cases}\mu_j^{-1}-4\pi\mu_j^{-2}\varphi_j(\tilde{w})^2\varepsilon+0(\varepsilon^2+\varepsilon^\sigma) & (\sigma>1)\\ \mu_j^{-1}-4\pi(1+k)^{-1}\mu_j^{-2}\varphi_j(\tilde{w})^2\varepsilon+0(\varepsilon^2) & (\sigma=1)\end{cases}\end{aligned} \tag{7.2}$$

By (7.1), (7.2) and the fact (5.8), we see that $\lambda^*(\varepsilon)$ must be $\mu_j(\varepsilon)^{-1}$. Then, (1.3) easily follows from (7.1) and (7.2). Therefore we get the desired Theorem 1.

8. Proof of Theorem 4

From this section we assume $\sigma<1$. By (2.3), (2.4) and (2.5), we see that

$$\begin{aligned}g(\varepsilon) &= -(4\pi/k)\varepsilon^{2-\sigma}+O(\varepsilon^{3-2\sigma})\\ h(\varepsilon) &= 2\pi\varepsilon^3+O(\varepsilon^{4-\sigma})\\ i(\varepsilon) &= (4\pi/9)\varepsilon^5+O(\varepsilon^{6-\sigma})\,.\end{aligned} \tag{8.1}$$

We take an arbitrary $x\in\partial B_\varepsilon$. Without loss of generality we may assume that $\tilde{w}=0$ and $x=\varepsilon e_1$. We put $S(x, y)$ as before. Then, the same calculation as in p. 263 of Ozawa [9] yields

$$\begin{aligned}&\langle\nabla_w G(x, \tilde{w}),\ \nabla_w G(\tilde{w}, y)\rangle\\ &= (4\pi)^{-1}\varepsilon^{-2}\frac{\partial}{\partial w_1}G(\tilde{w}, y)+\langle\nabla_w S(x, \tilde{w}),\ \nabla_w G(\tilde{w}, y)\rangle\end{aligned} \tag{8.2}$$

$$\begin{aligned}&\frac{\partial}{\partial x_1}\langle\nabla_w G(x, \tilde{w}),\ \nabla_w G(\tilde{w}, y)\rangle\\ &= -(2\pi)^{-1}\varepsilon^{-3}\frac{\partial}{\partial w_1}G(\tilde{w}, y)+\frac{\partial}{\partial x_1}\langle\nabla_w S(x, \tilde{w}),\ \nabla_w G(\tilde{w}, y)\rangle\end{aligned} \tag{8.3}$$

$$\begin{aligned}&\langle H_w G(x, \tilde{w}),\ H_w G(\tilde{w}, y)\rangle-\langle H_w S(x, \tilde{w}),\ H_w G(\tilde{w}, y)\rangle\\ &= 3(4\pi)^{-1}\varepsilon^{-3}\frac{\partial^2}{\partial w_1^2}G(\tilde{w}, y)-(4\pi)^{-1}\varepsilon^{-3}\Delta_w G(\tilde{w}, y)\end{aligned} \tag{8.4}$$

(8.5)
$$\frac{\partial}{\partial x_1}\langle H_w G(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle - \frac{\partial}{\partial x_1}\langle H_w S(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle$$
$$= -9(4\pi)^{-1}\varepsilon^{-4}\frac{\partial^2}{\partial w_1^2}G(\tilde{w}, y) + 3(4\pi)^{-1}\varepsilon^{-4}\Delta_w G(\tilde{w}, y)$$

for $x=\varepsilon e_1$, $\tilde{w}=0$. We recall that

(8.6)
$$\Delta_w G(\tilde{w}, y) = 0 \qquad \text{for} \quad y\in\Omega_\varepsilon .$$

We put $p_\varepsilon(x, y)$ as before. By (8.2), (8.3), (8.4), (8.5) and (8.6), we have

(8.7)
$$p_\varepsilon(x, y) - k\varepsilon^\sigma \frac{\partial}{\partial x_1} p_\varepsilon(x, y)_{|x=\varepsilon e_1} = \sum_{j=1}^{7} L_j ,$$

where

$$L_1 = G(x, y)$$
$$L_2 = g(\varepsilon)\,((4\pi\varepsilon)^{-1} + \gamma + (4\pi)^{-1}k\varepsilon^{\sigma-2})G(\tilde{w}, y)$$
$$L_3 = g(\varepsilon)O(\varepsilon^\sigma)G(\tilde{w}, y)$$
$$L_4 = (4\pi)^{-1}(\varepsilon^{-2} + 2k\varepsilon^{\sigma-3})h(\varepsilon)\frac{\partial}{\partial w_1}G(\tilde{w}, y) - k\varepsilon^\sigma\frac{\partial}{\partial x_1}G(x, y)$$
$$L_5 = 3(4\pi)^{-1}(\varepsilon^{-3} + 3k\varepsilon^{\sigma-4})i(\varepsilon)\frac{\partial^2}{\partial w_1^2}G(\tilde{w}, y)$$
$$L_6 = h(\varepsilon)\,\langle\nabla_w S(x, \tilde{w}),\ \nabla_w G(\tilde{w}, y)\rangle$$
$$\qquad - k\varepsilon^\sigma h(\varepsilon)\frac{\partial}{\partial x_1}\langle\nabla_w S(x, \tilde{w}),\ \nabla_w G(\tilde{w}, y)\rangle$$
$$L_7 = i(\varepsilon)\,\langle H_w S(x, \tilde{w}),\ H_w G(\tilde{w}, y)\rangle$$
$$\qquad - k\varepsilon^\sigma i(\varepsilon)\frac{\partial}{\partial x_1}\langle H_w S(x, \tilde{w}),\ H_w G(\tilde{w}, y)\rangle$$

for $\tilde{w}=0$, $x=\varepsilon e_1$.
Here we used the fact that

$$S(x, \tilde{w}) = \gamma + O(\varepsilon) \qquad \text{as} \quad \varepsilon\to 0 .$$

By (2.3), (2.4), (2.5) and (8.6), we get the following.

(8.8)
$$p_\varepsilon(x, y) - k\varepsilon^\sigma\frac{\partial}{\partial x_1}p_\varepsilon(x, y)_{|x=\varepsilon e_1}$$
$$= G(x, y) - G(\tilde{w}, y) - \varepsilon\frac{\partial}{\partial w_1}G(\tilde{w}, y) + L_3 + L_6 + L_7$$
$$- k\varepsilon^\sigma\Big(\frac{\partial}{\partial x_1}G(x, y) - \frac{\partial}{\partial w_1}G(\tilde{w}, y) - \varepsilon\frac{\partial^2}{\partial w_1^2}G(\tilde{w}, y)\Big)$$

for $\tilde{w}=0,\ x=\varepsilon e_1$.

We take an arbitrary $f\in L^p(\Omega_\varepsilon)$ and let $\tilde{f}$ be the extension of f to Ω defined by 0 on B_ε. By (8.8),

$$
\begin{aligned}
(8.9)\qquad & (\boldsymbol{P}_\varepsilon f)(x)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}(\boldsymbol{P}_\varepsilon f)(x)_{|x=\varepsilon e_1} \\
&= (\boldsymbol{G}\tilde{f})(x)-(\boldsymbol{G}\tilde{f})(\tilde{w})-\varepsilon\frac{\partial}{\partial w_1}(\boldsymbol{G}\tilde{f})(\tilde{w})+I_0(\varepsilon,\tilde{f}) \\
&\quad -k\varepsilon^\sigma\Big(\frac{\partial}{\partial x_1}(\boldsymbol{G}\tilde{f})(x)-\frac{\partial}{\partial w_1}(\boldsymbol{G}\tilde{f})(\tilde{w})-\varepsilon\frac{\partial^2}{\partial w_1^2}(\boldsymbol{G}\tilde{f})(\tilde{w})\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
I_0(\varepsilon,\tilde{f}) &= g(\varepsilon)O(\varepsilon^\sigma)(\boldsymbol{G}\tilde{f})(\tilde{w}) \\
&\quad +h(\varepsilon)\langle\nabla_w S(x,\tilde{w}),\ \nabla_w(\boldsymbol{G}\tilde{f})(\tilde{w})\rangle \\
&\quad -k\varepsilon^\sigma h(\varepsilon)\frac{\partial}{\partial x_1}\langle\nabla_w S(x,\tilde{w}),\ \nabla_w(\boldsymbol{G}\tilde{f})(\tilde{w})\rangle \\
&\quad +i(\varepsilon)\langle H_w S(x,\tilde{w}),\ H_w(\boldsymbol{G}\tilde{f})(\tilde{w})\rangle \\
&\quad -k\varepsilon^\sigma i(\varepsilon)\frac{\partial}{\partial x_1}\langle H_w S(x,\tilde{w}),\ H_w(\boldsymbol{G}\tilde{f})(\tilde{w})\rangle
\end{aligned}
$$

for $\tilde{w}=0,\ x=\varepsilon e_1$.

By (4.4), we have

$$
\begin{aligned}
(8.10)\qquad & |(\boldsymbol{G}\tilde{f})(\tilde{w})|\leqq C\|f\|_{p,\varepsilon} \\
& \Big|(\boldsymbol{G}\tilde{f})(x)-(\boldsymbol{G}\tilde{f})(\tilde{w})-\varepsilon\frac{\partial}{\partial w_1}(\boldsymbol{G}\tilde{f})(\tilde{w})\Big|\leqq C\varepsilon^{2-3/p}\|f\|_{p,\varepsilon} \\
& \left|\frac{\partial}{\partial x_1}(\boldsymbol{G}\tilde{f})(x)-\frac{\partial}{\partial w_1}(\boldsymbol{G}\tilde{f})(\tilde{w})\right|\leqq C\varepsilon^{1-3/p}\|f\|_{p,\varepsilon}
\end{aligned}
$$

for $\tilde{w}=0,\ x=\varepsilon e_1,\ p>3$.

Furthermore,

$$
\begin{aligned}
(8.11)\qquad \left|\frac{\partial}{\partial w_n}(\boldsymbol{G}\tilde{f})(\tilde{w})\right| &\leqq C\Big(\int_{\Omega_\varepsilon}|y-\tilde{w}|^{-2p'}dy\Big)^{1/p'}\|\tilde{f}\|_p \\
&\leqq\begin{cases} C\varepsilon^{1-3/p}\|f\|_{p,\varepsilon} & (1<p<3) \\ C\|f\|_{p,\varepsilon} & (p>3)\end{cases}
\end{aligned}
$$

for $1\leqq n\leqq 3$, where p' satisfies $(1/p)+(1/p')=1$. Also,

$$
\begin{aligned}
(8.12)\qquad \left|\frac{\partial^2}{\partial w_m\partial w_n}(\boldsymbol{G}\tilde{f})(\tilde{w})\right| &\leqq C\Big(\int_{\Omega_\varepsilon}|y-\tilde{w}|^{-3p'}dy\Big)^{1/p'}\|\tilde{f}\|_p \\
&\leqq C\varepsilon^{-3/p}\|f\|_{p,\varepsilon}\qquad (p>1)
\end{aligned}
$$

for $1 \leqq m, n \leqq 3$.

Summing up these facts, we get

$$|(\boldsymbol{P}_\varepsilon f)(x) - k\varepsilon^\sigma \frac{\partial}{\partial x_1}(\boldsymbol{P}_\varepsilon f)(x)|_{x=\varepsilon e_1} \leqq C\varepsilon^{1+\sigma-3/p}||f||_{p,\varepsilon}$$

for $p>3$.

Therefore we have the following by Proposition 3.2.

Lemma 8.1. *For a constant C independent of ε,*

$$||(\boldsymbol{P}_\varepsilon - \boldsymbol{G}_\varepsilon)f||_{2,\varepsilon} \leqq C\varepsilon^{3-3/p}||f||_{p,\varepsilon} \tag{8.13}$$

holds for any $f \in L^p(\Omega_\varepsilon)$ $(p>3)$.

The right hand side of (8.13) is not $O(\varepsilon^3)$. On the other hand, the right hand side of (2.8) is $o(\varepsilon^3)$. Therefore we need some sharper estimate to get Theorem 4.

We put $v_\varepsilon(x) = ((\boldsymbol{P}_\varepsilon - \boldsymbol{G}_\varepsilon)(\chi_\varepsilon \varphi_j))(x)$. As we get (8.9),

$$\begin{aligned} & v_\varepsilon(x) - k\varepsilon^\sigma \frac{\partial}{\partial x_1} v_\varepsilon(x)|_{x=\varepsilon e_1} \\ & = I_1(\varepsilon) - I_2(\varepsilon) - k\varepsilon^\sigma (I_3(\varepsilon) - I_4(\varepsilon)) + I_5(\varepsilon)\,, \end{aligned} \tag{8.14}$$

where

$$I_1(\varepsilon) = (\boldsymbol{G}\varphi_j)(x) - (\boldsymbol{G}\varphi_j)(\tilde{w}) - \varepsilon \frac{\partial}{\partial w_1}(\boldsymbol{G}\varphi_j)(\tilde{w})$$

$$I_2(\varepsilon) = (\boldsymbol{G}\hat{\chi}_\varepsilon \varphi_j)(x) - (\boldsymbol{G}\hat{\chi}_\varepsilon \varphi_j)(\tilde{w}) - \varepsilon \frac{\partial}{\partial w_1}(\boldsymbol{G}\hat{\chi}_\varepsilon \varphi_j)(\tilde{w})$$

$$I_3(\varepsilon) = \frac{\partial}{\partial x_1}(\boldsymbol{G}\varphi_j)(x) - \Big(\frac{\partial}{\partial w_1} + \varepsilon \frac{\partial^2}{\partial w_1^2}\Big)(\boldsymbol{G}\varphi_j)(\tilde{w})$$

$$I_4(\varepsilon) = \frac{\partial}{\partial x_1}(\boldsymbol{G}\hat{\chi}_\varepsilon \varphi_j)(x) - \Big(\frac{\partial}{\partial w_1} + \varepsilon \frac{\partial^2}{\partial w_1^2}\Big)(\boldsymbol{G}\hat{\chi}_\varepsilon \varphi_j)(\tilde{w})$$

for $\tilde{w}=0$, $x=\varepsilon e_1$, and $I_5(\varepsilon)$ is given by replacing f with $\chi_\varepsilon \varphi_j$ in the term $I_0(\varepsilon, \tilde{f})$ of (8.9).

Since $\boldsymbol{G}\varphi_j = \mu_j^{-1}\varphi_j$,

$$|I_1(\varepsilon)| \leqq C\varepsilon^2,\ |I_3(\varepsilon)| \leqq C\varepsilon^2\,. \tag{8.15}$$

Using (8.11), (8.12) with $f=\chi_\varepsilon \varphi_j$, we have

$$|I_5(\varepsilon)| \leqq C(\varepsilon^2 + \varepsilon^{3+\sigma})\,. \tag{8.16}$$

Furthermore,

$$(8.17)\qquad \begin{aligned}|I_2(\varepsilon)| &\leqq C\varepsilon^{2-3/p}\|\hat{\chi}_\varepsilon\varphi_j\|_p \qquad (p>3)\\ &\leqq C\varepsilon^{2-3/p}|B_\varepsilon|^{1/p}\leqq C\varepsilon^2\,.\end{aligned}$$

Now we want to estimate $I_4(\varepsilon)$. We put $L(x,y)=(4\pi)^{-1}|x-y|^{-1}$. Then, we have

$$(8.18)\qquad I_4(\varepsilon)=I_6(\varepsilon)+I_7(\varepsilon)+I_8(\varepsilon)\,,$$

where

$$\begin{aligned}I_6(\varepsilon) &= \frac{\partial}{\partial x_1}\Big(\int_{B_\varepsilon}L(x,y)(\varphi_j(y)-\varphi_j(x))dy\Big)_{|x=\varepsilon e_1}\\ &\quad -\frac{\partial}{\partial w_1}\int_{B_\varepsilon}L(w,y)(\varphi_j(y)-\varphi_j(w))dy\\ &\quad -\varepsilon\frac{\partial^2}{\partial w_1^2}\int_{B_\varepsilon}L(w,y)\Big(\varphi_j(y)-\varphi_j(w)-\sum_{n=1}^{3}(y_n-w_n)\frac{\partial\varphi_j}{\partial w_n}(w)\Big)dy\\ I_7(\varepsilon) &= \frac{\partial}{\partial x_1}(\varphi_j(x)F(x))_{|x=\varepsilon e_1}-\Big(\frac{\partial}{\partial w_1}+\varepsilon\frac{\partial^2}{\partial w_1^2}\Big)(\varphi_j(w)F(w))\\ &\quad -\varepsilon\sum_{n=1}^{3}\frac{\partial^2}{\partial w_1^2}\Big(\frac{\partial\varphi_j}{\partial w_n}(w)K_n(w)\Big)\\ I_8(\varepsilon) &= \frac{\partial}{\partial x_1}(\boldsymbol{S}\hat{\chi}_\varepsilon\varphi_j)(x)_{|x=\varepsilon e_1}-\Big(\frac{\partial}{\partial w_1}+\varepsilon\frac{\partial^2}{\partial w_1^2}\Big)(\boldsymbol{S}\hat{\chi}_\varepsilon\varphi_j)(w)\end{aligned}$$

for $w=0$. Here we put operator $\boldsymbol{S}$ and functions F, K_n as follows:

$$(\boldsymbol{S}f)(x)=\int_\Omega S(x,y)f(y)dy$$

$$F(x)=\int_{B_\varepsilon}L(x,y)dy$$

and

$$K_n(w)=\int_{B_\varepsilon}L(w,y)(y_n-w_n)dy \qquad (n=1,2,3)\,.$$

It is easy to see that

$$(8.19)\qquad |I_8(\varepsilon)|\leqq C\varepsilon^2$$

$$(8.20)\qquad \begin{aligned}|I_6(\varepsilon)| &\leqq C\int_{B_\varepsilon}|x-y|^{-1}dy_{|x=\varepsilon e_1}+C\int_{B_\varepsilon}|\tilde{w}-y|^{-1}dy\\ &\quad +C\varepsilon\int_{B_\varepsilon}|\tilde{w}-y|^{-1}dy\\ &\leqq C\varepsilon^2\,.\end{aligned}$$

The simple calculation yields

$$(8.21)\qquad F(x) = (\varepsilon^3/3)|x|^{-1} \qquad \text{for}\quad x\in \boldsymbol{R}^3\backslash \bar{\boldsymbol{B}}_\varepsilon$$
$$= \varepsilon^2/2-|x|^2/6 \qquad \text{for}\quad x\in B_\varepsilon\,,$$

$$(8.22)\qquad K_n(w) = w_n(|w|^2/5-\varepsilon^2)/3 \qquad \text{for}\quad w\in B_\varepsilon\,.$$

Therefore, we see that

$$(8.23)\qquad F(x) = \varepsilon^2/3\,,\quad \partial F(x)/\partial x_1 = -\varepsilon/3$$

for $x=\varepsilon e_1$, and

$$(8.24)\qquad F(w) = \varepsilon^2/2\,,\quad \partial F(w)/\partial w_1 = 0\,,\quad \partial^2 F(w)/\partial w_1^2 = -1/3$$

$$(8.25)\qquad K_n(w) = \partial^2 K_n(w)/\partial w_1^2 = 0\,,\quad \partial K_n(w)/\partial w_1 = -\delta_{1,n}(\varepsilon^2/3)$$

for $w=0$, where $\delta_{1,n}$ is the Kronecker delta. Summing up these facts, we have

$$(8.26)\qquad I_7(\varepsilon) = -\varepsilon(\varphi_j(\varepsilon e_1)-\varphi_j(0))/3+O(\varepsilon^2) = O(\varepsilon^2)\,.$$

From (8.14), (8.15), (8.16), (8.17), (8.18), (8.19), (8.20) and (8.26), we see that

$$|v_\varepsilon(x)-k\varepsilon^\sigma \frac{\partial}{\partial x_1} v_\varepsilon(x)|_{x=\varepsilon e_1} \leqq C(\varepsilon^2+\varepsilon^{2+\sigma})\,.$$

By Proposition 3.2, we have

$$||v_\varepsilon||_{2,\varepsilon} \leqq C\varepsilon^4(1+\varepsilon^{-\sigma})\,.$$

Therefore, we get the desired Theorem 4.

9. Convergence of eigenvalues for $\sigma<1$

We put A_0, A_1 as before. Then,

$$(9.1)\qquad \tilde{\boldsymbol{P}}_\varepsilon = A_0+g(\varepsilon)A_1+h(\varepsilon)A_2+i(\varepsilon)A_3\,,$$

where

$$(9.2)\qquad (A_2 f)(x) = \langle \nabla_w G(x, \tilde{w}),\ \nabla_w(G\chi_\varepsilon f)(\tilde{w})\rangle\, \chi_\varepsilon(x)$$

$$(9.3)\qquad (A_3 f)(x) = \langle H_w G(x, \tilde{w}),\ H_w(G\chi_\varepsilon f)(\tilde{w})\rangle\, \chi_\varepsilon(x)\,.$$

Using (8.11) and (8.12), we have

$$(9.4)\qquad ||A_2 f||_p \leqq C\Big(\int_{\Omega_\varepsilon} |x-\tilde{w}|^{-2p}dx\Big)^{1/p}||\nabla_w(G\chi_\varepsilon f)||_\infty$$
$$\leqq \begin{cases} C\varepsilon^{-1}||f||_p & (3/2<p<3) \\ C\varepsilon^{3/p-2}||f|_p & (p>3)\,, \end{cases}$$

and

$$(9.5) \qquad \|A_3 f\|_p \leqq C\Big(\int_{\Omega_\varepsilon} |x-\tilde{w}|^{-3p} dx\Big)^{1/p} \|H_w(G\chi_\varepsilon f)\|_\infty$$
$$\leqq C\varepsilon^{-3}\|f\|_p \qquad (p>1)\,.$$

Here we put

$$(H_w v)(w) = \sum_{m,n=1}^{3} \frac{\partial^2 v}{\partial w_m \partial w_n}(w)\,.$$

From (5.3), (8.1), (9.1), (9.4) and (9.5),

$$\|(\tilde{\boldsymbol{P}}_\varepsilon - \boldsymbol{G})f\|_2 \leqq C(|g(\varepsilon)| + |h(\varepsilon)|\varepsilon^{-1} + |i(\varepsilon)|\varepsilon^{-3})\|f\|_2$$
$$\leqq C(\varepsilon^2 + \varepsilon^{2-\sigma})\|f\|_2$$

hold for any $f \in L^2(\Omega)$.

Therefore we get the following.

Lemma 9.1. *There exists a constant C independent of ε such that*

$$(9.6) \qquad \|\tilde{\boldsymbol{P}}_\varepsilon - \boldsymbol{G}\|_2 \leqq C(\varepsilon^2 + \varepsilon^{2-\sigma})$$

holds.

Notice that Lemma 5.2 is valid for $\sigma < 1$ because $(1-\chi_\varepsilon)\chi_\varepsilon = 0$. As we get (5.8),

$$(9.7) \qquad |\mu_j(\varepsilon)^{-1} - \mu_j^{-1}| \leqq C(\varepsilon^{3-3/p} + \varepsilon^2 + \varepsilon^{2-\sigma} + \varepsilon^{3/2}) \leqq C(\varepsilon^{3/2} + \varepsilon^{2-\sigma})$$

hold for a constant C independent of ε.

10. Perturbational calculus for $\bar{\boldsymbol{P}}_\varepsilon$

We recall (2.9). Then,

$$(10.1) \qquad \bar{\boldsymbol{P}}_\varepsilon = A_0 + \bar{g}(\varepsilon)\bar{A}_1 + h(\varepsilon)\bar{A}_2 + i(\varepsilon)\bar{A}_3\,,$$

where

$$(10.2) \qquad \bar{g}(\varepsilon) = g(\varepsilon) - (4\pi/3)\mu_j \varepsilon^3$$

and $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ is given by replacing χ_ε with ξ_ε in (5.2), (9.2), (9.3), respectively.

Furthermore we put $\lambda_0 = \mu_j^{-1}$, $\psi_0 = \varphi_j$ and

$$\lambda(\varepsilon) = \lambda_0 + \bar{g}(\varepsilon)\lambda_1 + h(\varepsilon)\lambda_2 + i(\varepsilon)\lambda_3$$
$$\psi(\varepsilon) = \psi_0 + \bar{g}(\varepsilon)\psi_1 + h(\varepsilon)\psi_2 + i(\varepsilon)\psi_3\,.$$

Then,

$$(A_0-\lambda_0)\psi_0=0\,,\quad \|\psi_0\|_2=1\,. \tag{10.3}$$

Next we consider the following equations:

$$(A_0-\lambda_0)\psi_n=(\lambda_n-\bar{A}_n)\psi_0,\quad (\psi_0,\psi_n)_2=0 \qquad (n=1,2,3)\,. \tag{10.4}$$

By the Fredholm alternative theory, we see that

$$\lambda_n=(\bar{A}_n\psi_0,\psi_0)_2 \qquad (n=1,2,3) \tag{10.5}$$

is the condition such that the unique solution ψ_n of (10.4) exists.

Since $\xi_\varepsilon=0$ on $B_{\varepsilon/2}$, $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ satisfies the same inequality as in (5.3), (9.4), (9.5), respectively. Then, by the Fredholm theory and the estimate of the $L^p(\Omega)$ norm of the right hand side of (10.4), we get the following.

Lemma 10.1. *For a constant C independent of ε,*

$$\begin{aligned}
\|\psi_1\|_p,\ \|\bar{A}_1\|_p &\leqq C && (3/2<p<3)\\
&\leqq C\varepsilon^{3/p-1} && (p>3)\\
\|\psi_2\|_p,\ \|\bar{A}_2\|_p &\leqq C\varepsilon^{-1} && (3/2<p<3)\\
&\leqq C\varepsilon^{3/p-2} && (p>3)\\
\|\psi_3\|_p,\ \|\bar{A}_3\|_p &\leqq C\varepsilon^{-3} && (p>1)
\end{aligned}$$

hold.

In view of (10.1), (10.3) and (10.4), we have

$$\begin{aligned}
&(\bar{\boldsymbol{P}}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon)\\
&=g(\varepsilon)^2(\bar{A}_1-\lambda_1)\psi_1+h(\varepsilon)^2(\bar{A}_2-\lambda_2)\psi_2+i(\varepsilon)^2(\bar{A}_3-\lambda_3)\psi_3\\
&\quad+g(\varepsilon)h(\varepsilon)\,((\bar{A}_1-\lambda_1)\psi_2+(\bar{A}_2-\lambda_2)\psi_1)\\
&\quad+h(\varepsilon)i(\varepsilon)\,((\bar{A}_2-\lambda_2)\psi_3+(\bar{A}_3-\lambda_3)\psi_2)\\
&\quad+i(\varepsilon)g(\varepsilon)\,((\bar{A}_3-\lambda_3)\psi_1+(\bar{A}_1-\lambda_1)\psi_3)\,.
\end{aligned} \tag{10.6}$$

By (10.5), (10.6) and Lemma 10.1, we see that

$$\|(\bar{\boldsymbol{P}}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon)\|_2\leqq C(g(\varepsilon)^2+\varepsilon^4)\leqq C\varepsilon^4(1+\varepsilon^{-2\sigma})\,.$$

Therefore we get the following.

Lemma 10.2. *For a constant C independent of ε,*

$$\|(\bar{\boldsymbol{P}}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon)\|_2\leqq C\varepsilon^4(1+\varepsilon^{-2\sigma}) \tag{10.7}$$

holds.

On the other hand, by Lemmas 8.1, 10.1 and Theorem 4, we see that

$$\begin{aligned}&\|(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)(\chi_\varepsilon\psi(\varepsilon))\|_{2,\varepsilon}\\&\leqq C(\varepsilon^4(1+\varepsilon^{-\sigma})+|g(\varepsilon)|\varepsilon^2+|h(\varepsilon)|\varepsilon+|i(\varepsilon)|\varepsilon^{-3/p})\qquad (p>3)\\&\leqq C\varepsilon^4(1+\varepsilon^{-\sigma}).\end{aligned}$$

Therefore, we get the following.

Lemma 10.3. *For a constant C independent of ε,*

$$\|(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)(\chi_\varepsilon\psi(\varepsilon))\|_{2,\varepsilon}\leqq C\varepsilon^4(1+\varepsilon^{-\sigma}) \tag{10.8}$$

holds.

11. Proof of Theorem 5

We put

$$J_\varepsilon(x;v)=(\chi_\varepsilon\bar{\boldsymbol{P}}_\varepsilon v-\boldsymbol{P}_\varepsilon\chi_\varepsilon v)(x) \tag{11.1}$$

for $v\in L^p(\Omega)$.
Then, we see that

$$\begin{aligned}\Delta J_\varepsilon(x;v)&=0 \qquad x\in\Omega_\varepsilon\\ J_\varepsilon(x;v)&=0 \qquad x\in\partial\Omega.\end{aligned} \tag{11.2}$$

As we get (8.9), we have

$$\begin{aligned}&J_\varepsilon(x;v)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}J_\varepsilon(x;v)_{|x=\varepsilon e_1}\\&=\sum_{n=9}^{11}I_n(\varepsilon;v)-k\varepsilon^\sigma(I_{12}(\varepsilon;v)+I_{13}(\varepsilon;v)),\end{aligned} \tag{11.3}$$

where $I_9(\varepsilon;v)$ is given by $I_0(\varepsilon;\tilde{f})$ in (8.9) with $f=\xi_\varepsilon\hat{\chi}_\varepsilon v=(\xi_\varepsilon-\chi_\varepsilon)v$ and

$$\begin{aligned}I_{10}(\varepsilon;v)&=(\boldsymbol{G}\hat{\chi}_\varepsilon v)(x)-(\boldsymbol{G}\xi_\varepsilon\hat{\chi}_\varepsilon v)(\tilde{w})-\varepsilon\frac{\partial}{\partial w_1}(\boldsymbol{G}\xi_\varepsilon\hat{\chi}_\varepsilon v)(\tilde{w})\\ I_{11}(\varepsilon;v)&=-(4\pi/3)\mu_j\varepsilon^3G(x,\tilde{w})(\boldsymbol{G}\xi_\varepsilon v)(\tilde{w})\\ I_{12}(\varepsilon;v)&=\frac{\partial}{\partial x_1}(\boldsymbol{G}\hat{\chi}_\varepsilon v)(x)-\Big(\frac{\partial}{\partial w_1}+\varepsilon\frac{\partial^2}{\partial w_1^2}\Big)(\boldsymbol{G}\xi_\varepsilon\hat{\chi}_\varepsilon v)(\tilde{w})\\ I_{13}(\varepsilon;v)&=-(4\pi/3)\mu_j\varepsilon^3\frac{\partial}{\partial x_1}G(x,\tilde{w})(\boldsymbol{G}\xi_\varepsilon v)(\tilde{w})\end{aligned}$$

for $\tilde{w}=0$, $x=\varepsilon e_1$.

It is easy to see that

$$|I_9(\varepsilon;v)|\leqq C(\varepsilon^2+\varepsilon^{3+\sigma})\|v\|_p \qquad (p>3) \tag{11.4}$$

$$|I_{11}(\varepsilon;v)|\leqq C\varepsilon^2\|v\|_p \qquad (p>3/2) \tag{11.5}$$

(11.6) $$|I_{13}(\varepsilon; v)| \leqq C\varepsilon \|v\|_p \qquad (p>3/2).$$

We have

(11.7) $$\begin{aligned} |I_{10}(\varepsilon; v)| &\leqq C\Big(\int_{B_\varepsilon} |x-y|^{-p'} dy\Big)^{1/p'}_{|x=\varepsilon e_1} \|v\|_p \\ &\quad + C\Big(\int_{B_\varepsilon} |\tilde{w}-y|^{-p'} dy\Big)^{1/p'} \|v\|_p \\ &\quad + C\varepsilon\Big(\int_{B_\varepsilon} |\tilde{w}-y|^{-2p'} dy\Big)^{1/p'} \|v\|_p \\ &\leqq \begin{cases} C\varepsilon^{2-3/p} \|v\|_p & (3/2<p<\infty) \\ C\varepsilon^2 \|v\|_\infty & (p=\infty), \end{cases} \end{aligned}$$

and

(11.8) $$\begin{aligned} |I_{12}(\varepsilon; v)| &\leqq C\Big(\int_{B_\varepsilon} |x-y|^{-2p'} dy\Big)^{1/p'}_{|x=\varepsilon e_1} \|v\|_p \\ &\quad + C\Big(\int_{B_\varepsilon} |\tilde{w}-y|^{-2p'} dy\Big)^{1/p'} \|v\|_p \\ &\quad + C\varepsilon\Big(\int_{B_\varepsilon} |\tilde{w}-y|^{-3p'} dy\Big)^{1/p'} \|v\|_p \\ &\leqq C\varepsilon^{1-3/p} \|v\|_p \qquad (3<p<\infty). \end{aligned}$$

Summing up these facts, we have

$$|J_\varepsilon(x; v) - k\varepsilon^\sigma \frac{\partial}{\partial x_1} J_\varepsilon(x; v)|_{x=\varepsilon e_1} \leqq C\varepsilon^{1+\sigma-3/p} \|v\|_p$$

for $p>3$. By Proposition 3.2, we get the following.

Lemma 11.1. *There exists a constant C independent of ε such that*

(11.9) $$\|J_\varepsilon(\cdot\,; v)\|_{2,\varepsilon} \leqq C\varepsilon^{3-3/p} \|v\|_p$$

holds for any $v \in L^p(\Omega)$ ($p>3$).

Next we estimate $\|J_\varepsilon(\cdot\,; \varphi_j)\|_{2,\varepsilon}$. We see that

(11.10) $$I_{12}(\varepsilon; \varphi_j) = I_4(\varepsilon) + \sum_{n=14}^{16} I_n(\varepsilon),$$

where

$$\begin{aligned} I_{14}(\varepsilon) &= \Big(\frac{\partial}{\partial w_1} + \varepsilon \frac{\partial^2}{\partial w_1^2}\Big)(S(1-\xi_\varepsilon)\hat{\chi}_\varepsilon \varphi_j)(\tilde{w}) \\ I_{15}(\varepsilon) &= \frac{\partial}{\partial w_1} \int_{B_\varepsilon} L(w, y)(1-\xi_\varepsilon(y))\varphi_j(y) dy_{|w=0} \\ I_{16}(\varepsilon) &= \varepsilon \frac{\partial^2}{\partial w_1^2} \int_{B_\varepsilon} L(w, y)(1-\xi_\varepsilon(y))\varphi_j(y) dy_{|w=0}. \end{aligned}$$

Since $S(x, y)\in C^\infty(\Omega\times\Omega)$,

$$|I_{14}(\varepsilon)|\leqq C\varepsilon^2\,. \tag{11.11}$$

In section 8, we have already showed the following.

$$|I_4(\varepsilon)|\leqq C\varepsilon^2 \tag{11.12}$$

$$\frac{\partial}{\partial w_1}\int_{B_\varepsilon} L(w, y)\varphi_j(y)dy = O(\varepsilon^2) \tag{11.13}$$

$$\frac{\partial^2}{\partial w_1^2}\int_{B_\varepsilon} L(w, y)\varphi_j(y)dy = -(1/3)\varphi_j(\tilde{w})+O(\varepsilon^2) \tag{11.14}$$

for $w=\tilde{w}=0$.

On the other hand, we see that

$$\Delta_w\int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)dy = -(\hat{\chi}_\varepsilon\xi_\varepsilon)(w)\,.$$

Since $\xi_\varepsilon(w)=0$ for $w\in B_{\varepsilon/2}$ and $\xi_\varepsilon(w)$ is rotationary invariant, we have

$$\int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)dy = \text{Constant} = O(\varepsilon^2) \qquad \text{for} \quad w\in B_{\varepsilon/2} \tag{11.15}$$

and

$$|\xi_\varepsilon(y)| = |\xi_\varepsilon(y)-\xi_\varepsilon(w)|\leqq C\varepsilon^{-1}|y-w| \tag{11.16}$$

for $w\in B_{\varepsilon/2}$, $y\in B_\varepsilon$.
Therefore, we have the following.

$$\begin{aligned}
&\frac{\partial}{\partial w_1}\int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)\varphi_j(y)dy_{|w=0}\\
&=\frac{\partial}{\partial w_1}\int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)(\varphi_j(y)-\varphi_j(w))dy_{|w=0}\\
&\qquad+\frac{\partial}{\partial w_1}(\varphi_j(w)\int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)dy)_{|w=0}\\
&=O(\varepsilon^2)
\end{aligned} \tag{11.16}$$

$$\begin{aligned}
&\frac{\partial^2}{\partial w_1^2}\int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)\varphi_j(y)dy_{|w=0}\\
&=\frac{\partial^2}{\partial w_1^2}\int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)(\varphi_j(y)-\varphi_j(w))dy_{|w=0}\\
&\qquad+\frac{\partial^2}{\partial w_1^2}(\varphi_j(w)\int_{B_\varepsilon} L(w, y)\xi_\varepsilon(y)dy)_{|w=0}\\
&=O(\varepsilon)
\end{aligned} \tag{11.17}$$

Summing up these facts, we have

$$I_{12}(\varepsilon;\varphi_j)=-(\varepsilon/3)\varphi_j(\tilde{w})+O(\varepsilon^2)\,. \tag{11.18}$$

It is easy to see that

$$I_{13}(\varepsilon;\varphi_j)=(\varepsilon/3)\varphi_j(\tilde{w})+O(\varepsilon^2)\,. \tag{11.19}$$

Thus, by (11.3), (11.4), (11.5), (11.7), (11.18) and (11.19), we have

$$|J_\varepsilon(x;\varphi_j)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}J_\varepsilon(x;\varphi_j)|_{x=\varepsilon e_1}\leqq C(\varepsilon^2+\varepsilon^{2+\sigma})\,.$$

By Proposition 3.2, we get the desired Theorem 5.

Furthermore, we have the following.

Lemma 11.2. *There exists a constant C independent of ε such that*

$$||J_\varepsilon(\cdot\,;\psi(\varepsilon))||_{2,\varepsilon}\leqq C\varepsilon^4(1+\varepsilon^{-\sigma}) \tag{11.20}$$

holds.

Proof. We recall that $\psi(\varepsilon)=\varphi_j+g(\varepsilon)\psi_1+h(\varepsilon)\psi_2+i(\varepsilon)\psi_3$. We put $p>3$ in Lemma 10.1. Then, (11.20) easily follows from Lemmas 10.1, 11.1 and Theorem 5. q.e.d.

REMARK. By neglecting $I_{11}(\varepsilon;v)$ and $I_{13}(\varepsilon;v)$ in (11.13), we have

$$||(\chi_\varepsilon\hat{P}_\varepsilon-P_\varepsilon\chi_\varepsilon)\varphi_j||_{2,\varepsilon}\leqq C\varepsilon^3\,, \tag{11.21}$$

where

$$\hat{P}_\varepsilon=\bar{P}_\varepsilon+(4\pi/3)\mu_j\varepsilon^3\bar{A}_1\,.$$

Since the remainder term of an asymptotic formula (1.4) is $O(\varepsilon^4)$ for $\sigma\leqq-2$, the estimate (11.21) is weak in the sense that the right hand side is $O(\varepsilon^3)$. Therefore, the existence of the term $(4\pi/3)\mu_j\varepsilon^3\,G(x,\tilde{w})G(\tilde{w},y)\xi_\varepsilon(x)\xi_\varepsilon(y)$ in (2.9) is essential to get Theorem 2.

12. Proof of Theorem 2

Now we are in a position to prove Theorem 2. As in section 7, by Lemmas 10.2, 10.3 and 11.2, we have

$$||(G_\varepsilon-\lambda(\varepsilon))\,(\chi_\varepsilon\psi(\varepsilon))||_{2,\varepsilon}\leqq C\varepsilon^4(1+\varepsilon^{-2\sigma})\,.$$

Since $||\psi(\varepsilon)||_{2,\varepsilon}\in(1/2,2)$ for small ε, there exists at least one eigenvalue $\lambda^*(\varepsilon)$ of G_ε satisfying

$$|\lambda^*(\varepsilon)-\lambda(\varepsilon)|\leqq C\varepsilon^4(1+\varepsilon^{-2\sigma})\,. \tag{12.1}$$

We here represent λ_1, λ_2, λ_3 explicitly as follows.

$$(12.2)\qquad \lambda_1 = \left(\int_\Omega G(w,y)\xi_\varepsilon(y)\varphi_j(y)dy\right)^2_{|w=\tilde{w}} = \mu_j^{-2}\varphi_j(\tilde{w})^2+O(\varepsilon^2)$$

$$(12.3)\qquad \lambda_2 = \sum_{n=1}^{3}\left(\frac{\partial}{\partial w_n}\int_\Omega G(w,y)\xi_\varepsilon(y)\varphi_j(y)dy\right)^2_{|w=\tilde{w}}$$

$$(12.4)\qquad \lambda_3 = \sum_{m,n=1}^{3}\left(\frac{\partial^2}{\partial w_m\partial w_n}\int_\Omega G(w,y)\xi_\varepsilon(y)\varphi_j(y)dy\right)^2_{|w=\tilde{w}}$$

Since $\xi_\varepsilon(y)=0$ for $y\in B_{\varepsilon/2}$,

$$(12.5)\qquad |\lambda_3|\leqq C\left(\int_{\Omega\setminus B_{\varepsilon/2}}|y-\tilde{w}|^{-3}dy\right)^2\leqq C|\log\varepsilon|^2\,.$$

On the other hand, by (8.25) and (11.15), we see that

$$\begin{aligned}
&\frac{\partial}{\partial w_n}\int_\Omega G(w,y)(1-\xi_\varepsilon(y))\varphi_j(y)dy\\
&=\frac{\partial}{\partial w_n}\int_\Omega S(w,y)(1-\xi_\varepsilon(y))\varphi_j(y)dy\\
&\quad+\frac{\partial}{\partial w_n}\int_\Omega L(w,y)(1-\xi_\varepsilon(y))(\varphi_j(y)-\varphi_j(w))dy\\
&\quad+\frac{\partial}{\partial w_n}\left(\varphi_j(w)\int_\Omega L(w,y)(1-\xi_\varepsilon(y))dy\right)\\
&=O(\varepsilon)\qquad (n=1,2,3)
\end{aligned}$$

for $w=\tilde{w}=0$.

Thus, we have

$$(12.6)\qquad \lambda_2=\mu_j^{-2}\,|\mathrm{grad}\,\varphi_j(\tilde{w})|^2+O(\varepsilon)\,.$$

From (12.2), (12.5) and (12.6),

$$(12.7)\qquad \lambda(\varepsilon)=\mu_j^{-1}-\mu_j^{-2}(Q_j\varepsilon^{2-\sigma}+R_j\varepsilon^3)+O(\varepsilon^{3-2\sigma}+\varepsilon^{4-\sigma}+\varepsilon^4)\,,$$

where Q_j, R_j are as mentioned before.

By (12.1), (12.7) and the fact (9.7), we see that $\lambda^*(\varepsilon)$ must be $\mu_j(\varepsilon)^{-1}$. Then,

$$(12.8)\qquad \begin{aligned}&|\mu_j(\varepsilon)^{-1}-\mu_j^{-1}(1-\mu_j^{-1}(Q_j\varepsilon^{2-\sigma}+R_j\varepsilon^3))|\\&\leqq C(\varepsilon^{3-2\sigma}+\varepsilon^4)\end{aligned}$$

holds.

Theorem 2 easily follows from (12.8).

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