



Title	Vector bundle valued harmonic forms and immersions of Riemannian manifolds
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Citation	Osaka Journal of Mathematics. 1971, 8(1), p. 1-13
Version Type	VoR
URL	https://doi.org/10.18910/7603
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VECTOR BUNDLE VALUED HARMONIC FORMS AND IMMERSIONS OF RIEMANNIAN MANIFOLDS

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(Received June 1, 1970)

The purpose of this paper is to discuss an application of the theory of vector bundle valued harmonic forms on a Riemannian manifold to the study of immersions.

Let M be a Riemannian manifold and E a Riemannian vector bundle over M . Then we can define in a natural way the Laplacian \square operating on E -valued differential forms and we can express the scalar product $\langle \square\theta, \theta \rangle$, where θ is an E -valued p -form, in terms of curvature and covariant differentials. Moreover, if M is compact, we obtain, by integrating over M , a formula analogous to Bochner's for ordinary (i.e. real valued) differential forms.

Let f be an immersion of M into a Riemannian manifold M' . We may regard the second fundamental form α of (M, f) as a $\text{Hom}(T(M), N(M))$ -valued 1-form. Assuming that M' is of constant sectional curvature, we shall prove that the second fundamental form α is harmonic, i.e. $\square\alpha = 0$, if the mean curvature normal of (M, f) is parallel. In particular, if the immersion f is a minimal immersion, then α is harmonic. Conversely, if M is compact and if α is harmonic, then the mean curvature normal is parallel. We obtain from this result together with the formula of Bochner type the results of Simons [5], Chern [1], Nomizu-Smyth [4] and Erbacher [2] proved by them in different ways. In a future paper we shall discuss the case where M is a Kähler manifold.

1. Let M be an n -dimensional Riemannian manifold and E a vector bundle over M with a metric along the fibers and a covariant differentiation D_X satisfying

$$X\langle\varphi, \psi\rangle = \langle D_X\varphi, \psi\rangle + \langle\varphi, D_X\psi\rangle$$

for any vector field X and any sections φ and ψ of E . A vector bundle E with these properties will be called a *Riemannian vector bundle*.

We shall denote $C^p(E)$ the real vector space of all E -valued differential p -forms on M . We define an operator

$$\partial : C^p(E) \rightarrow C^{p+1}(E), (p = 0, 1, \dots)$$

by the formula

$$(\partial\theta)(X_1, \dots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} D_{X_i}(\theta(X_1, \dots, \hat{X}_i, \dots, X_{p+1})) \\ + \sum_{i < j} (-1)^{i+j} \theta([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{p+1}),$$

where X_i 's denote vector fields on M . The covariant derivative $D_X\theta$ of $\theta \in C^p(E)$ is an E -valued p -form such that

$$(D_X\theta)(X_1, \dots, X_p) = D_X(\theta(X_1, \dots, X_p)) - \sum_{i=1}^p \theta(X_1, \dots, \nabla_X X_i, \dots, X_p),$$

where $\nabla_X X_i$ denotes the covariant derivative of the vector field X_i in the Riemannian manifold M .

For an E -valued 1-form θ we have the formula

$$(\partial\theta)(X, Y) = (D_X\theta)(Y) - (D_Y\theta)(X)$$

The covariant differential $D\theta$ of θ is an E -valued $(p+1)$ -tensor defined by

$$(D\theta)(X_1, \dots, X_p, X) = (D_X\theta)(X_1, \dots, X_p).$$

We define an operator

$$\partial^* : C^p(E) \rightarrow C^{p-1}(E) \quad (p > 0)$$

as follows. Let $x \in M$ and let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space $T_x(M)$ of M at x . For any $p-1$ tangent vectors u_1, \dots, u_{p-1} at x , put

$$(\partial^*\eta)_x(u_1, \dots, u_{p-1}) = - \sum_{k=1}^n (D_{e_k}\eta)_x(e_k, u_1, \dots, u_{p-1}),$$

where $(D_{e_k}\eta)_x$ denotes the value of $D_X\eta$ at x for any vector field X such that $X_x = e_k$. Then $(\partial^*\eta)_x$ is an alternating $(p-1)$ -linear map of $T_x(M)$ into E_x , the fiber of E over x , and the assignment $x \rightarrow (\partial^*\eta)_x$ defines an E -valued $(p-1)$ -form $\partial^*\theta$. For any E -valued 0-form θ , we define $\partial^*\theta = 0$.

The *Laplacian* \square for E -valued differential forms is defined as

$$\square = \partial\partial^* + \partial^*\partial.$$

The *curvature* \tilde{R} of the covariant differentiation D in E is a $\text{Hom}(E, E)$ -valued 2-forms given by

$$\tilde{R}(X, Y)\varphi = D_X(D_Y\varphi) - D_Y(D_X\varphi) - D_{[X, Y]}\varphi$$

for any section φ of E and for any vector fields X and Y in M . We shall denote by $\langle\theta, \eta\rangle$ the scalar product of two E -valued p -forms, that is, $\langle\theta, \eta\rangle$ is the smooth function on M given by

$$\langle \theta, \eta \rangle(x) = \sum_{i_1, \dots, i_p=1}^n \langle \theta(e_{i_1}, \dots, e_{i_p}), \eta(e_{i_1}, \dots, e_{i_p}) \rangle,$$

where $\{e_1, \dots, e_n\}$ denotes an orthonormal basis of $T_x(M)$.

Now we prove the following

Theorem 1. *Let θ be an E -valued 1-form. Then*

$$\langle \square \theta, \theta \rangle = \frac{1}{2} \Delta \langle \theta, \theta \rangle + \langle D\theta, D\theta \rangle + A,$$

where Δ denotes the Laplacian of the Riemannian manifold M and A denotes a smooth function in M defined as follows :

$$A(x) = \sum_{i,j} \langle (\tilde{R}(e_j, e_i)\theta(e_j), \theta(e_i)) \rangle + \sum_i \langle \theta(S(e_i)), \theta(e_i) \rangle,$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis of $T_x(M)$ and S denotes the endomorphism of $T_x(M)$ defined by the Ricci tensor S of M , i.e. $S(e_i) = \sum_k S_{ki} e_k$.

Proof. Fix a point $x \in M$ and let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_x(M)$. We can choose n vector fields E_1, \dots, E_n in M such that $E_i(x) = e_i$ and $(\nabla_{E_k} E_i)_x = 0$ for $i, k = 1, \dots, n$. Then, because $\nabla_{e_s} E_i$ are zero for $i, s = 1, \dots, n$, we have

$$\begin{aligned} (\partial^* \partial \theta)(e_i) &= - \sum_s (D_{e_s} \partial \theta)(e_s, e_i) = - \sum_s D_{e_s} ((\partial \theta)(E_s, E_i)) \\ &= - \sum_s D_{e_s} ((D_{E_s} \theta)(E_i) - (D_{E_i} \theta)(E_s)) \\ &= \sum_s (D_{E_s} D_{E_i} \theta)(e_i) - \sum_s (D_{E_s} D_{E_s} \theta)(e_i). \end{aligned}$$

On the other hand, $\partial^* \theta = - \sum_{s,t} g^{st} (D_{E_t} \theta)(E_s)$ where (g^{st}) is the inverse matrix of the matrix $(g(E_s, E_t))$, we have

$$\begin{aligned} (\partial \partial^* \theta)(e_i) &= D_{e_i} (\partial^* \theta) = - \sum_{s,t} (e_i g^{st}) (D_{E_t} \theta)(e_s) - \sum_{s,t} \delta^{st} e_i ((D_{E_t} \theta)(E_s)) \\ &= - \sum_s e_i ((D_{E_s} \theta)(E_s)) = - \sum_s (D_{E_i} D_{E_s} \theta)(e_s), \end{aligned}$$

because $\nabla_{e_i} E_k = 0$ at x .

Therefore we obtain

$$(\square \theta)(e_i) = \sum_s ((D_{E_s} D_{E_i} - D_{E_i} D_{E_s}) \theta)(e_s) - \sum_s (D_{E_s} D_{E_s} \theta)(e_i).$$

Since $[E_s, E_i] = 0$ at x , we have

$$\begin{aligned} ((D_{E_s} D_{E_i} - D_{E_i} D_{E_s}) \theta)(e_s) &= ([D_{E_s}, D_{E_i}] - D_{[E_s, E_i]}) \theta(e_s) \\ &= \tilde{R}(e_s, e_i)(\theta(e_s)) - \theta(R(e_s, e_i)e_s). \end{aligned}$$

Therefore

$$\begin{aligned}
\langle \square\theta, \theta \rangle &= \sum_i \langle (\square\theta)\theta(e_i), (e_i) \rangle \\
&= \sum_{s,i} \langle \tilde{R}(e_s, e_i)\theta(e_s), \theta(e_i) \rangle + \sum_i \theta(S(e_i), \theta(e_i)) \\
&\quad - \sum_{s,i} \langle (D_{E_s}D_{E_s}\theta)(e_i), \theta(e_i) \rangle.
\end{aligned}$$

Now by a local computation we see that

$$\begin{aligned}
& - \sum_{s,i} \langle (D_{E_s}D_{E_s}\theta)(e_i), \theta(e_i) \rangle \\
&= \langle D\theta, D\theta \rangle(x) + \frac{1}{2}(\Delta\langle\theta, \theta\rangle)(x).
\end{aligned}$$

Thus we have proved that

$$\langle \square\theta, \theta \rangle = \frac{1}{2}\Delta\langle\theta, \theta\rangle + \langle D\theta, D\theta \rangle + A.$$

Corollary 1. *Let θ be an E -valued 1-form. Assume that $\square\theta = 0$ and $\Delta\langle\theta, \theta\rangle = 0$. Then we have $A \leq 0$ everywhere on M .*

Assume now that M is compact and oriented. Then we can define the inner product (θ, η) of two E -valued p -forms by

$$(\theta, \eta) = \int_M \langle \theta, \eta \rangle * 1.$$

Then we obtain from Theorem 1 the following corollary.

Corollary 2. *Let θ be an E -valued 1-form such that $\square\theta = 0$. Then we have*

$$(D\theta, D\theta) + \int_M A * 1 = 0.$$

If $A \geq 0$ everywhere on M , then we have $A \equiv 0$ and $D\theta = 0$.

We remark that the operator ∂^* is the adjoint operator of ∂ , i.e.

$$(\partial\theta, \eta) = (\theta, \partial^*\eta)$$

for any $\theta \in C^p(E)$ and $\eta \in C^{p+1}(E)$ and hence we have

$$(\square\theta, \theta) = (\partial\theta, \partial\theta) + (\partial^*\theta, \partial^*\theta).$$

Therefore, if M is compact, $\square\theta = 0$ if and only if $\partial\theta = 0$ and $\partial^*\theta = 0$.

2. Let M be an n -dimensional Riemannian manifold isometrically immersed in a Riemannian manifold M' of dimension $n + p$. We shall denote by $N(M)$ and α the normal bundle and the second fundamental form of M [3]. The second fundamental form α is an $N(M)$ -valued symmetric 2-form on M .

In the following we put

$$E = \text{Hom}(T(M), N(M)) = T^*(M) \otimes N(M)$$

and we interpret α as an E -valued 1-form β as follows: For any vector field X in M , $\beta(X)$ is a section of E such that

$$\beta(X) \cdot Y = \alpha(X, Y)$$

for all vector field Y in M . Then we have

$$\beta(X) \cdot Y = \beta(Y) \cdot X.$$

We call also β the second fundamental form of M .

A metric along the fibres of E is defined naturally by the Riemann metrics of M and M' and a covariant derivation D_X in E is also naturally defined by the covariant differentiation ∇_X in M and D_X^\perp in $N(M)$, where for any normal vector ξ of M , $D_X^\perp \xi$ is defined as the normal component of $\nabla_X' \xi$, where ∇_X' denote the covariant differentiation in the Riemannian manifold M' (See [3]).

Let φ be a section of E . We may regard φ as an $N(M)$ -valued 1-form on M and we have

$$\begin{aligned} (D_X \varphi)(Y) &= D_X^\perp(\varphi(Y)) - \varphi(\nabla_X Y), \\ \langle D_X \varphi, \psi \rangle + \langle \varphi, D_X \psi \rangle &= X \langle \varphi, \psi \rangle \end{aligned}$$

for any sections φ and ψ of E .

The following Proposition 1 may be considered as an interpretation of the equation of Codazzi in our formalism.

Proposition 1. *Assume that M' is a Riemannian manifold of constant sectional curvature. Then the second fundamental form β of M satisfies the equation $\partial\beta = 0$.*

Proof. By a straightforward computation we see that

$$\begin{aligned} (\partial\beta(X, Y))(Z) &= \{D_X^\perp(\alpha(Y, Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)\} \\ &\quad - \{D_Y^\perp(\alpha(X, Z)) - \alpha(\nabla_Y X, Z) - \alpha(X, \nabla_Y Z)\} \end{aligned}$$

and the right hand side is 0 by [3, Vol. II, P. 25, Cor. 4.4].

For each normal vector $\nu \in N_x(M)$ we define an endomorphism A_ν of $T_x(M)$ by the formula

$$\langle A_\nu(u), v \rangle = \langle \beta(u)\nu, v \rangle$$

for any tangent vectors $u, v \in T_x(M)$. The mean curvature normal η of M is a

normal vector field in M such that

$$\frac{1}{n} \text{Tr } A_\nu = \langle \nu, \eta(x) \rangle$$

for any $\nu \in N_x(M)$ and $x \in M$.

M is said to be *minimal* in M' if the mean curvature normal vanishes at each point, that is, if $\text{Tr } A_\nu = 0$ for any $\nu \in N_x(M)$ and $x \in M$.

We say that M has a *constant mean curvature* if the mean curvature normal η is parallel, that is, $D_X^\perp \eta = 0$ for any vector field X in M .

Let ν be a normal vector field. Then we have $\text{Tr } A_\nu = n \langle \nu, \eta \rangle$ and hence $X \cdot \text{Tr } A_\nu = n \{ \langle D_X^\perp \nu, \eta \rangle + \langle \nu, D_X^\perp \eta \rangle \}$. Therefore M has a constant mean curvature, if and only if

$$X \cdot \text{Tr } A_\nu = \text{Tr } A_{D_X^\perp \nu}$$

for any normal vector field ν and any vector field X in M .

Proposition 2. *Let M' be a Riemmanian manifold of constant sectional curvature. Then the second fundamental form β of M satisfies the equation $\partial^* \beta = 0$ if and only if M has a constant mean curvature.*

Proof. Let x be a point in M and let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_x(M)$. Let E_1, \dots, E_n be vector fields in a neighborhood of x such that $(E_i)_x = e_i$ and $\nabla_{E_i} E_k = 0$ at x for $i, k = 1, \dots, n$. Let (g^{st}) the inverse matrix of the matrix $(\langle E_s, E_t \rangle)$. Then $\partial^* \beta = -g^{st} (D_{E_t} \beta)(E_s)^{1)}$ and $(\partial^* \beta) \cdot E_k = -g^{st} (D_{E_t} \beta)(E_s) \cdot E_k$. Since $(D_{E_t} \beta)(E_s) E_k = D_{E_t}^\perp (\alpha(E_s, E_k)) - \alpha(\nabla_{E_t} E_s, E_k) - \alpha(E_s, \nabla_{E_t} E_k)$ and since α is symmetric, we get $(D_{E_t} \beta)(E_s) E_k = (D_{E_t} \beta)(E_k) E_s$. On the other hand, by Proposition 1, we have $\partial \beta = 0$ and hence $(D_{E_t} \beta)(E_k) = (D_{E_k} \beta)(E_t)$, hence $(D_{E_t} \beta)(E_s) E_k = (D_{E_k} \beta)(E_t) \cdot E_s$. Therefore, for any normal vector field ν , we have

$$\begin{aligned} \langle (\partial^* \beta) \cdot E_k, \nu \rangle &= -g^{st} \langle (D_{E_k} \beta)(E_t) E_s, \nu \rangle \\ &= -g^{st} \{ \langle D_{E_k}^\perp (\alpha(E_t, E_s)), \nu \rangle - \langle \alpha(\nabla_{E_k} E_t, E_s), \nu \rangle \\ &\quad - \langle \alpha(E_t, \nabla_{E_k} E_s), \nu \rangle \}. \end{aligned}$$

Now

$$\begin{aligned} &g^{st} \langle D_{E_k}^\perp (\alpha(E_t, E_s)), \nu \rangle \\ &= g^{st} \{ E_k \langle \alpha(E_t, E_s), \nu \rangle - \langle \alpha(E_t, E_s), D_{E_k}^\perp \nu \rangle \\ &= E_k (g^{st} \langle \alpha(E_t, E_s), \nu \rangle) - (E_k g^{st}) \langle \alpha(E_t, E_s), \nu \rangle - g^{st} \langle \alpha(E_t, E_s), D_{E_k}^\perp \nu \rangle \\ &= E_k (T_r A_\nu) - T_r A_{D_{E_k}^\perp \nu} - E_k g^{st} \cdot \langle \alpha(E_t, E_s), \nu \rangle. \end{aligned}$$

1) We omit here the summation signs.

Since $\nabla_{E_k} E_i = 0$ at x , we have $E_k g^{st} = 0$ at x . Therefore we get from the above that

$$\langle (\partial^* \beta) E_k, \nu \rangle(x) = \text{Tr } A_{D_{\frac{1}{E}} \nu} - E_k (\text{Tr } A_\nu)$$

at x for $k = 1, 2, \dots, n$ and hence for any vector field X we have $\langle (\partial^* \beta) X, \nu \rangle(x) = \text{Tr } A_{D_X \nu} - X(\text{Tr } A_\nu)$ at x . Since x is an arbitrary point of M and ν is an arbitrary normal vector field, we see from the above equation that $\partial^* \beta = 0$ if and only if M has a constant mean curvature.

From Propositions 1 and 2 we get the following

Theorem 2. *Let M be a Riemannian manifold immersed isometrically into a Riemannian manifold M' of constant sectional curvature. Let β be the second fundamental form of M regarded as a $\text{Hom}(T(M), N(M))$ -valued 1-form. Then β satisfies the equation $\square \beta = 0$, if M has a constant mean curvature. Conversely, if M is compact and orientable and $\square \beta = 0$, then M has a constant mean curvature.*

3. We shall discuss in this section some applications of Theorems 1 and 2. Let M be a Riemannian manifold immersed isometrically into a Riemannian manifold M' of constant sectional curvature c . Let $x \in M$ and let $\{e_1, \dots, e_n\}$ and $\{v_1, \dots, v_p\}$ be orthonormal bases of $T_x(M)$ and $N_x(M)$ respectively. We shall denote by A_a ($a = 1, 2, \dots, p$) the endomorphism of $T_x(M)$ defined by $\langle A_a u, v \rangle = \langle \beta(u) \cdot v, v_a \rangle$ and put $A_a \cdot e_i = \sum_j (A_a)_i^j e_j$. Then we have the following Gauss equation:

$$(3.1) \quad R_{kl ij} = c\{\delta_{ki}\delta_{lj} - \delta_{kj}\delta_{li}\} + \sum_a \{(A_a)_i^k (A_a)_l^j - (A_a)_j^k (A_a)_i^l\},$$

where $R_{kl ij}$ denote the components of the curvature tensor with respect to the basis $\{e_1, \dots, e_n\}$ of $T_x(M)$. Then the endomorphism S of $T_x(M)$ defined by $S(e_j) = \sum_i S_{ij}(e_i)$ with $S_{ej} = \sum_k R_{klij}$ is of the form

$$(3.2) \quad S = c(n-1)I + \sum_a (\text{Tr } A_a) A_a - \sum_a A_a^2,$$

where I denotes the identity endomorphism of $T_x(M)$.

Let K be the scalar curvature of M . Then $K(x) = \text{Tr } S = c(n-1)n + \sum_a (\text{Tr } A_a)^2 - \sum_a \text{Tr } A_a^2$. The value $\eta(x)$ at x of the mean curvature normal η is given by $\eta(x) = \frac{1}{n} \sum_a \text{Tr } A_a \cdot v_a$ and hence $n^2 \langle \eta, \eta \rangle(x) = \sum_a (\text{Tr } A_a)^2$. Analogously we have $\langle \beta, \beta \rangle(x) = \sum_a \text{Tr } A_a^2$. Hence we get

$$(3.3) \quad K = c(n-1)n + n^2 \langle \eta, \eta \rangle - \langle \beta, \beta \rangle,$$

where β and η denotes the second fundamental form and the mean curvature normal of M respectively. For any Riemannian vector bundle E over M we have defined the endomorphism $\tilde{R}(u, v)$ of the fiber E_x , where $u, v \in T_x(M)$. Let $E = \text{Hom}(T(M), N(M))$ and let $\varphi \in E_x$. Then $\tilde{R}(u, v) \varphi$ is an element of $E_x = \text{Hom}(T_x(M), N_x(M))$ such that

$$(3.4) \quad (\tilde{R}(u, v)\varphi)(w) = R^\perp(u, v)(w)\varphi - \varphi(R(u, v)w),$$

where $u, v, w \in T_x(M)$ and R^\perp denotes the curvature of the Riemannian vector bundle $N(M)$.

Let ν be a normal vector of M at x and let N be a normal vector field such that $N_x = \nu$. Let X and Y be vector fields in M such that $X_x = u$ and $Y_x = v$. Then we have

$$R^\perp(u, v)\nu = (D_X^\perp D_Y^\perp - D_Y^\perp D_X^\perp - D^\perp_{[X, Y]}) N$$

at x .

Denote by ∇' the covariant derivation in the ambient space M' . Then we have

$$\begin{aligned} \nabla_{X'} Y &= \nabla_X Y + \alpha(X, Y), \\ \nabla_{X'} N &= -A_N(X) + D_X^\perp N. \end{aligned}$$

We see from these two equations that the normal component $(R'(X, Y)N)^\perp$ of $R'(X, Y)N$, where R' denotes the curvature tensor of M' , is equal to $R^\perp(X, Y)N - \alpha(A_N(Y), X) + \alpha(A_N(X), Y)$. Since M' is of constant curvarute $R'(X, Y)N = c\{\langle N, Y \rangle X - \langle N, X \rangle Y\} = 0$ and hence we get $R^\perp(X, Y)N = -\alpha(A_N(X), Y) + \alpha(A_N(Y), X)$. Thus we have

$$R^\perp(u, v)\nu = -\alpha(A_u, v) + \alpha(A_v, u).$$

In particular

$$R^\perp(u, v)\nu_a = -\alpha(A_a u, v) + \alpha(u, A_a v).$$

Since $\alpha(A_a u, v) = \sum_b \langle \alpha(A_a u, v), \nu_b \rangle \nu_b = \sum_b \langle A_b A_a u, v \rangle \nu_b$

and $\alpha(u, A_a v) = \sum_b \langle A_b u, A_a v \rangle \nu_b = \sum_b \langle A_a A_b u, v \rangle \nu_b$

we get

$$(3.5) \quad R^\perp(u, v)\nu_a = \sum_b \langle [A_a, A_b]u, v \rangle \nu_b.$$

Now by Theorem 1, we have

$$\langle \square \beta, \beta \rangle = \frac{1}{2} + \Delta \langle \beta, \beta \rangle + \langle D\beta, D\beta \rangle + A,$$

where

$$(3.6) \quad A(x) = \sum_{i,j} \langle \tilde{R}(e_j, e_i) \beta(e_j), \beta(e_i) \rangle + \sum_i \langle \beta(S(e_i)), \beta(e_i) \rangle.$$

Now

$$\begin{aligned} \sum_i \langle \beta(S(e_i)), \beta(e_i) \rangle &= \sum_{i,j} \langle \alpha(S(e_i), e_j), \alpha(e_i, e_j) \rangle \\ &= \sum_{i,j,a} \langle A_a(S(e_i)), e_j \rangle \langle A_a(e_i), e_j \rangle = \sum_a \text{Tr}(SA_a^2) \end{aligned}$$

and by (3.2) we get

$$\begin{aligned} (3.7) \quad & \sum_i \langle \beta(S(e_i)), \beta(e_i) \rangle \\ &= c(n-1) \sum_a \text{Tr } A_a^2 + \sum_{a,b} \text{Tr } A_a \cdot \text{Tr}(A_a A_b^2) - \sum_{a,b} \text{Tr}(A_a^2 A_b^2). \end{aligned}$$

On the other hand,

$$\begin{aligned} & \sum_{i,j} \langle \tilde{R}(e_j, e_i) \beta(e_j), \beta(e_i) \rangle \\ &= \sum_{i,j,k} \langle R^\perp(e_j, e_i) \alpha(e_j, e_k), \alpha(e_i, e_k) \rangle - \sum_{i,j,k} \langle \alpha(e_j, R(e_j, e_i) e_k), \alpha(e_i, e_k) \rangle \\ &= \sum_{i,j,k} \sum_{a,b} \langle A_a e_j, e_k \rangle \langle A_b e_i, e_k \rangle \langle R^\perp(e_j, e_i) v_a v_b \rangle \\ &\quad - \sum_{i,j,k} \sum_a \langle A_a e_j, R(e_j, e_i) e_k \rangle \langle A_a e_i, e_k \rangle \end{aligned}$$

and by (3.5), the first term equals $\sum_{a,b} \text{Tr}(A_a A_b [A_a, A_b]) = -\sum_{a,b} \text{Tr}(A_a^2 A_b^2) + \sum_{a,b} \text{Tr}(A_a A_b)^2$ and by the Gauss equation (3.1) the second term equals $-c \sum_a (\text{Tr } A_a)^2 + c \sum_a \text{Tr}(A_a^2) - \sum_{a,b} (\text{Tr}(A_a A_b))^2 + \sum_{a,b} \text{Tr}(A_a A_b)^2$.

Therefore we have

$$\begin{aligned} (3.8) \quad & \sum_{i,j} \langle \tilde{R}(e_j, e_i) \beta(e_j), \beta(e_i) \rangle \\ &= c \sum_a \text{Tr } A_a^2 - c \sum_v (\text{Tr } A_a)^2 - \sum_{a,b} \text{Tr}(A_a^2 A_b^2) - \sum_{a,b} (\text{Tr}(A_a A_b))^2 + \\ &\quad + 2 \sum_{a,b} \text{Tr}(A_a A_b)^2 \end{aligned}$$

Then we get from (3.6), (3.7) and (3.8) that

$$\begin{aligned} (3.9) \quad A(x) &= cn \sum_a \text{Tr } A_a^2 - c \sum_a (\text{Tr } A_a)^2 - \sum_{a,b} (\text{Tr}(A_a A_b))^2 \\ &\quad + \sum_{a,b} \text{Tr } A_a \cdot \text{Tr}(A_a A_b^2) + \sum_{a,b} \text{Tr}[A_a, A_b]^2. \end{aligned}$$

Now let $\lambda_1^{(a)}, \dots, \lambda_n^{(a)}$ be eigen-values of A_a and let $\{e_1^{(a)}, \dots, e_n^{(a)}\}$ be an orthonormal basis of $T_x(M)$ such that $A_a e_i^{(a)} = \lambda_i^{(a)} e_i^{(a)}$ ($i=1, \dots, n, a=1, \dots, p$).

We shall denote by $K_{ij}^{(a)}$ the sectional curvature for the 2-plane spanned by $e_i^{(a)}$ and $e_j^{(a)}$, $i \neq j$.

We show that

$$(3.10) \quad A(x) = \sum_v \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 K_{ij}^{(a)} + \frac{1}{2} \sum_{a,b} \text{Tr}[A_a, A_b]^2.$$

We write $A(x)$ in the following form:

$$(3.11) \quad A(x) = B(x) + \sum_{a \neq b} \text{Tr } A_a \cdot \text{Tr}(A_a A_b^2) - \sum_{a \neq b} (\text{Tr}(A_a A_b))^2 + \sum_{a,b} \text{Tr}[A_a, A_b]^2,$$

where

$$(3.12) \quad B(x) = \sum_a \{cn \text{Tr } A_a^2 - c(\text{Tr } A_a)^2 - (\text{Tr } A_a^2)^2 + \text{Tr } A_a \cdot \text{Tr } A_a^3\}.$$

Now by a lemma of Nomizu-Smyth [4] we have

$$(3.13) \quad \begin{aligned} & cn \text{Tr } A_a^2 - c(\text{Tr } A_a)^2 - (\text{Tr } A_a^2)^2 + \text{Tr } A_a \cdot \text{Tr}(A_a)^3 \\ &= \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 (c + \lambda_j^{(a)} \lambda_j^{(a)}) \end{aligned}$$

for each a . Now fix an index a and let

$$A_b e_i^{(a)} = \sum_j (A_b)_i^j e_j^{(a)} \quad (b=1,2,\dots,p)$$

Then we have $(A_a)_j^i = \delta_j^i \lambda_j^{(a)}$ and hence

$$(3.14) \quad (A_a A_b)_j^i = \lambda_i^{(a)} (A_b)_j^i, (A_b A_a)_j^i = (A_b)_j^i \lambda_j^{(a)}.$$

By the equation of Gauss we have

$$\begin{aligned} K_{ij}^{(a)} &= R(e_i^{(a)}, e_j^{(a)}, e_i^{(a)}, e_j^{(a)}) \\ &= c + \sum_b (A_b)_i^i (A_b)_j^j - \sum_b (A_b)_j^i (A_b)_i^j. \\ &= c + \lambda_i^{(a)} \lambda_j^{(a)} + \sum_{b \neq a} (A_b)_i^i (A_b)_j^j - \sum_b (A_b)_j^i (A_b)_i^j. \end{aligned}$$

Hence we have

$$\begin{aligned} & (\lambda_i^{(a)} - \lambda_j^{(a)})^2 (c + \lambda_i^{(a)} \lambda_j^{(a)}) \\ &= (\lambda_i^{(a)} - \lambda_j^{(a)})^2 K_{ij}^{(a)} + \sum_b (\lambda_i^{(a)} - \lambda_j^{(a)})^2 (A_b)_j^i (A_b)_i^j \\ &\quad - \sum_{b \neq a} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 (A_b)_i^i (A_b)_j^j. \end{aligned}$$

This equality holds also for $i = j$ trivially if we define $K_{ii}^{(a)} = 0$.

Then by (3.14)

$$\begin{aligned} & \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 (c + \lambda_i^{(a)} \lambda_j^{(a)}) = \frac{1}{2} \sum_{i,j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 (c + \lambda_i^{(a)} \lambda_j^{(a)}) \\ &= \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 K_{ij}^{(a)} - \frac{1}{2} \sum_b \sum_{i,j} (\lambda_i^{(a)} - \lambda_j^{(a)}) (A_b)_j^i (\lambda_j^{(a)} - \lambda_i^{(a)}) (A_b)_i^j \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \sum_{b \neq a} \left\{ \sum_{i,j} (\lambda_i^{(a)})^2 (A_a)_i^i \sum_j (A_b)_j^j - 2 \sum_i \lambda_i^{(a)} (A_b)_i^i \sum_j \lambda_j^{(a)} (A_b)_j^j \right. \\
& \quad \left. + \sum_i (A_b)_i^i \sum_j \lambda_j^{(a)} (A_b)_j^j \right\} \\
& = \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 K_{ij}^{(a)} - \frac{1}{2} \sum_b \text{Tr}[A_a, A_b]^2 \\
& - \sum_{b \neq a} \{ \text{Tr } A_b \cdot \text{Tr}(A_a^2 A_b) - (\text{Tr}(A_a A_b))^2 \}.
\end{aligned}$$

Then we obtain from (3.11), (3.12) and (3.13) the equality (3.10).

Now we cite the following two lemmas from [1].

Lemma 1. *Let A and B be symmetric $n \times n$ matrices. Then*

$$\text{Tr}[A, B]^2 \geq -2 \text{Tr } A^2 \cdot \text{Tr } B^2,$$

and the equality holds for non-zero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiple of \tilde{A} and \tilde{B} respectively, where

$$(3.15) \quad \tilde{A} = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right), \quad \tilde{B} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array} \right).$$

Lemma 2. *Let A_1, A_2 and A_3 be $n \times n$ symmetric matrices and if*

$$\text{Tr}[A_a, A_b]^2 = -2 \text{Tr } A_a^2 \cdot \text{Tr } A_b^2$$

for $1 \leq a < b \leq 3$, then at least one of the matrices A_a must be zero.

By Lemma 1, we have

$$\frac{1}{2} \sum_{a,b} \text{Tr}[A_a, A_b]^2 \geq - \sum_{a \neq b} \text{Tr } A_a^2 \cdot \text{Tr } A_b^2 = -2 \sum_{a < b} \text{Tr } A_a^2 \cdot \text{Tr } A_b^2.$$

Put $S_a = \text{Tr } A_a^2$. Then $\sum_a S_a = \langle \beta, \beta \rangle(x)$.

Since

$$\begin{aligned}
0 & \leq \sum_{a < b} (S_a - S_b)^2 = \sum_{a < b} (S_a^2 + S_b^2) - 2 \sum_{a < b} S_a S_b \\
& = (p-1) \sum_a S_a^2 - 2 \sum_{a < b} S_a S_b \\
& = (p-1) \{ (\sum_a S_a)^2 - 2 \sum_{a < b} S_a S_b \} - 2 \sum_{a < b} S_a S_b \\
& = (p-1) \langle \beta, \beta \rangle^2(x) - 2p \sum_{a < b} S_a S_b
\end{aligned}$$

we have

$$-2 \sum_{a < b} S_a S_b \geq -\frac{(p-1)}{p} \langle \beta, \beta \rangle^2(x)$$

ane here the equality holds if and only if $S_a = S_b$ for $a, b=1, \dots, p$. Therefore we get

$$(3.16) \quad \frac{1}{2} \sum_{a, b} \text{Tr}[A_a, A_b]^2 \geq -\frac{(p-1)}{p} \langle \beta, \beta \rangle^2(x)$$

and the equality holds if and only if $\text{Tr } A_a^2 = \text{Tr } A_b^2 = \text{Tr } A_b^2$ for $a, b=1, \dots, p$ and either A_a are all zero except possibly one of them or A_a are all zero except two of them, say A_1 and A_2 , and they can be transformed simultaneously by an orthogonal matrix into scalar multiple of the matrices of the form (3.15). Thus we obtain from (3.10) the inequality

$$(3.17) \quad A(x) \geq \sum_a \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 K_{ij}^{(a)} - \frac{p-1}{p} \langle \beta, \beta \rangle^2(x).$$

Assume now that the scalar curvatures of M are bounded below by a positive constant d . Then

$$\sum_a \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 K_{ij}^{(a)} \geq d \sum_a \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2$$

and

$$\begin{aligned} \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 &= (n-1) \text{Tr } A_a^2 - 2 \sum_{i < j} \lambda_i^{(a)} \lambda_j^{(a)} \\ -2 \sum_{i < j} \lambda_i^{(a)} \lambda_j^{(a)} &= \text{Tr } A_a^2 - (\text{Tr } A_a)^2 \end{aligned}$$

and hence

$$\sum_a \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 = n \langle \beta, \beta \rangle(x) - n^2 \langle \eta, \eta \rangle(x),$$

where η denotes the mean curvature normal of M . Thus we get the following inequality

$$(3.17) \quad A \geq \left(dn - \frac{p-1}{p} \langle \beta, \beta \rangle \right) \langle \beta, \beta \rangle - dn^2 \langle \eta, \eta \rangle$$

at each point of M .

We obtain from Corollaries 1 and 2 of Theorem 1 and Theorem 2 the following

Theorem 3. *Let M be an n -dimensional, Riemannian manifold with sectional curvatures bounded below by a positive constant d . Assume that M is immersed in a Riemannian manifold M' of constant sectional curvature of dimension $n+p$ and that M has a constant mean curvature. Then, if M is compact and orientable or if the length of the second fundamental form β of M is constant, then we have*

$$(3.18) \quad 0 \geq A \geq \left\{ dn - \frac{p-1}{p} \langle \beta, \beta \rangle \right\} \langle \beta, \beta \rangle - dn^2 \langle \eta, \eta \rangle$$

at each point of M , where η denotes the mean curvature normal of M which is parallel and $\langle \eta, \eta \rangle$ is a constant.

Now assume M is compact and oriented and let $k = \langle \eta, \eta \rangle$. Then integrating both sides of the inequality (3.18) we obtain

$$dn^2 k \int_M *1 \geq \int_M \left\{ dn - \frac{p-1}{p} \langle \beta, \beta \rangle \right\} \langle \beta, \beta \rangle *1$$

and we have the equality here if and only if

$$dn^2 k = \left\{ dn - \frac{p-1}{p} \langle \beta, \beta \rangle \right\} \langle \beta, \beta \rangle$$

and this implies also that $A = 0$ and that β is parallel by Theorem 1. Then $\langle \beta, \beta \rangle$ must satisfy the quadratic equation $(p-1)x^2 - p \, dn \, x + pn^2 k = 0$ and since the discriminant of this equation should be positive we should have the inequality

$$d \geq \frac{4k(p-1)}{p}.$$

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