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VECTOR BUNDLE VALUED HARMONIC FORMS AND IMMERSIONS OF RIEMANNIAN MANIFOLDS

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The purpose of this paper is to discuss an application of the theory of vector bundle valued harmonic forms on a Riemannian manifold to the study of immersions.

Let M be a Riemannian manifold and E a Riemannian vector bundle over M. Then we can define in a natural way the Laplacian $\square$ operating on E-valued differential forms and we can express the scalar product $\langle\square\theta, \theta\rangle$, where θ is an E-valued p-form, in terms of curvature and covariant differentials. Moreover, if M is compact, we obtain, by integrating over M, a formula analogous to Bochner's for ordinary (i.e. real valued) differential forms.

Let f be an immersion of M into a Riemannian manifold M'. We may regard the second fundamental form α of (M, f) as a Hom $(T(M), N(M))$-valued 1-form. Assuming that M' is of constant sectional curvature, we shall prove that the second fundamental form α is harmonic, i.e. $\square\alpha=0$, if the mean curvature normal of (M, f) is parallel. In particular, if the immersion f is a minimal immersion, then α is harmonic. Conversely, if M is compact and if α is harmonic, then the mean curvature normal is parallel. We obtain from this result together with the formula of Bochner type the results of Simons [5], Chern [1], Nomizu-Smyth [4] and Erbacher [2] proved by them in different ways. In a future paper we shall discuss the case where M is a Kähler manifold.

1. Let M be an n-dimensional Riemannian manifold and E a vector bundle over M with a metric along the fibers and a covariant differentiation D_X satisfying

$$X\langle\varphi, \psi\rangle = \langle D_X, \varphi\rangle + \langle\varphi, D_X\psi\rangle$$

for any vector field X and any sections φ and ψ of E. A vector bundle E with these properties will be called a *Riemannian vector bundle.*

We shall denote $C^p(E)$ the real vector space of all E-valued differential p-forms on M. We define an operator

$$\partial : C^p(E)\to C^{p+1}(E), (p = 0, 1, \cdots)$$

by the formula

$$(\partial\theta)(X_1,\cdots,X_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}D_{X_i}(\theta(X_1,\cdots,\hat{X}_i,\cdots,X_{p+1}))$$
$$+\sum_{i<j}(-1)^{i+j}\theta([X_i,X_j],X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{p+1}),$$

where X_i's denote vector fields on M. The covariant derivative $D_X\theta$ of $\theta\in C^p(E)$ is an E–valued p–form such that

$$(D_X\theta)(X_1,\cdots,X_p) = D_X(\theta(X_1,\cdots,X_p))-\sum_{i=1}^{p}\theta(X_1,\cdots,\nabla_X X_i,\cdots,X_p),$$

where $\nabla_X X_i$ denotes the covariant derivative of the vector field X_i in the Riemannian manifold M.

For an E–valued 1–form θ we have the formula

$$(\partial\theta)(X, Y) = (D_X\theta)(Y)-(D_Y\theta)(X)$$

The covariant differential $D\theta$ of θ is an E–valued $(p+1)$–tensor defined by

$$(D\theta)(X_1,\cdots,X_p, X) = (D_X\theta)(X_1,\cdots,X_p).$$

We define an operator

$$\partial^* : C^p(E)\to C^{p-1}(E) \qquad (p>0)$$

as follows. Let $x \in M$ and let $\{e_1,\cdots,e_n\}$ be an orthonormal basis of the tangent space $T_x(M)$ of M at x. For any $p-1$ tangent vectors $u_1,\cdots,u_{p-1}$ at x, put

$$(\partial^*\eta)_x(u_1,\cdots,u_{p-1}) = -\sum_{k=1}^{n}(D_{e_k}\eta)_x(e_k,u_1,\cdots,u_{p-1}),$$

where $(D_{e_k}\eta)_x$ denotes the value of $D_X\eta$ at x for any vector field X such that $X_x=e_k$. Then $(\partial^*\eta)_x$ is an alternating $(p-1)$–linear map of $T_x(M)$ into E_x, the fiber of E over x, and the assignment $x \to (\partial^*\eta)_x$ defines an E–valued $(p-1)$–form $\partial^*\theta$. For any E–valued 0–form θ, we define $\partial^*\theta = 0$.

The *Laplacian* $\square$ for E–valued differential forms is defined as

$$\square = \partial\partial^*+\partial^*\partial.$$

The *curvature* $\tilde{R}$ of the covariant differentiation D in E is a Hom (E, E)–valued 2–forms given by

$$\tilde{R}(X, Y)\varphi = D_X(D_Y\varphi)-D_Y(D_X\varphi)-D_{[X,Y]}\varphi$$

for any section φ of E and for any vector fields X and Y in M. We shall denote by $\langle\theta,\eta\rangle$ the scalar product of two E–valued p–forms, that is, $\langle\theta, \eta\rangle$ is the smooth function on M given by

$$\langle\theta, \eta\rangle(x) = \sum_{i_1,\dots,i_p=1}^{n} \langle\theta(e_{i_1},\cdots,e_{i_p}),\eta(e_{i_1},\cdots,e_{i_p})\rangle,$$

where $\{e_1,\cdots,e_n\}$ denotes an orthonormal basis of $T_x(M)$.

Now we prove the following

Theorem 1. *Let θ be an E-valued 1-form. Then*

$$\langle\Box\theta, \theta\rangle = \frac{1}{2}\Delta\langle\theta, \theta\rangle + \langle D\theta, D\theta\rangle + A,$$

where Δ denotes the Laplacian of the Riemannian manifold M and A denotes a smooth function in M defined as follows:

$$A(x) = \sum_{i,j} \langle(\tilde{R}(e_j, e_i)\theta(e_j), \theta(e_i)\rangle + \sum_i \langle\theta(S(e_i)), \theta(e_i)\rangle,$$

where $\{e_1,\cdots,e_n\}$ is an orthonormal basis of $T_x(M)$ and S denotes the endomorphism of $T_x(M)$ defined by the Ricci tensor S of M, i.e. $S(e_i) = \sum_k S_{ki}\, e_k$.

Proof. Fix a point $x \in M$ and let $\{e_1,\cdots,e_n\}$ be an orthonormal basis of $T_x(M)$. We can choose n vector fields $E_1,\cdots,E_n$ in M such that $E_i(x) = e_i$ and $(\nabla_{E_k} E_i)_x = 0$ for $i, k = 1,\cdots,n$. Then, because $\nabla_{e_s} E_i$ are zero for $i, s = 1,\cdots, n$, we have

$$\begin{aligned}(\partial^*\partial\theta)(e_i) &= -\sum_s (D_{e_s}\partial\theta)(e_s, e_i) = -\sum_s D_{e_s}((\partial\theta)(E_s, E_i)) \\ &= -\sum_s D_{e_s}((D_{E_s}\theta)(E_i) - (D_{E_i}\theta)(E_s)) \\ &= \sum_s (D_{E_s}D_{E_i}\theta)(e_i) - \sum_s (D_{E_s}D_{E_s}\theta)(e_i).\end{aligned}$$

On the other hand, $\partial^*\theta = -\sum_{s,t} g^{st}(D_{E_t}\theta)(E_s)$ where (g^{st}) is the inverse matrix of the matrix $(g(E_s, E_t))$, we have

$$\begin{aligned}(\partial\partial^*\theta)(e_i) = D_{e_i}(\partial^*\theta) &= -\sum_{s,t} (e_i g^{st})(D_{e_t}\theta)(e_s) - \sum_{s,t} \delta^{st} e_i((D_{E_t}\theta)(E_s)) \\ &= -\sum_s e_i((D_{E_s}\theta)(E_s)) = -\sum_s (D_{E_i}D_{E_s}\theta)(e_s),\end{aligned}$$

because $\nabla_{e_i} E_k = 0$ at x.

Therefore we obtain

$$(\Box\theta)(e_i) = \sum_s ((D_{E_s}D_{E_i} - D_{E_i}D_{E_s})\theta)(e_s) - \sum_s (D_{E_s}D_{E_s}\theta)(e_i).$$

Since $[E_s, E_i] = 0$ at x, we have

$$\begin{aligned}((D_{E_s}D_{E_i} - D_{E_i}D_{E_s})\theta)(e_s) &= (([D_{E_s}, D_{E_i}] - D_{[E_s,E_i]})\theta)\,(e_s) \\ &= \tilde{R}(e_s, e_i)(\theta(e_s)) - \theta(R(e_s, e_i)e_s).\end{aligned}$$

Therefore

$$\begin{aligned}\langle \square\theta, \theta\rangle &= \sum_i \langle((\square\theta)\theta(e_i), (e_i)\rangle \\ &= \sum_{s,t} \langle \tilde{R}(e_s, e_i)\theta(e_s), \theta(e_i)\rangle + \sum_i \theta(S(e_i), \theta(e_i)) \\ &\quad - \sum_{s,t} \langle (D_{E_s}D_{E_s}\theta)(e_i), \theta(e_i)\rangle.\end{aligned}$$

Now by a local computation we see that

$$\begin{aligned}&- \sum_{s,i} \langle (D_{E_s}D_{E_s}\theta)(e_i), \theta(e_i)\rangle \\ &= \langle D\theta, D\theta\rangle(x) + \frac{1}{2}(\Delta\langle\theta, \theta\rangle)(x).\end{aligned}$$

Thus we have proved that

$$\langle \square\theta, \theta\rangle = \frac{1}{2}\Delta\langle\theta, \theta\rangle + \langle D\theta, D\theta\rangle + A.$$

Corollary 1. *Let θ be an E-valued 1-form. Assume that $\square\theta = 0$ and $\Delta\langle\theta, \theta\rangle = 0$. Then we have $A \leqq 0$ everywhere on M.*

Assume now that M is compact and oriented. Then we can define the inner product (θ, η) of two E-valued p-forms by

$$(\theta, \eta) = \int_M \langle\theta, \eta\rangle * 1.$$

Then we obtain from Theorem 1 the following corollary.

Corollary 2. *Let θ be an E-valued 1-form such that $\square\theta = 0$. Then we have*

$$(D\theta, D\theta) + \int_M A * 1 = 0.$$

If $A \geqq 0$ everywhere on M, then we have $A \equiv 0$ and $D\theta = 0$.

We remark that the operator ∂^* is the adjoint operator of ∂, i.e.

$$(\partial\theta, \eta) = (\theta, \partial^*\eta)$$

for any $\theta \in C^p(E)$ and $\eta \in C^{p+1}(E)$ and hence we have

$$(\square\theta, \theta) = (\partial\theta, \partial\theta) + (\partial^*\theta, \partial^*\theta).$$

Therefore, if M is compact, $\square\theta = 0$ if and only if $\partial\theta = 0$ and $\partial^*\theta = 0$.

2. Let M be an n-dimensional Riemannian manifold isometrically immersed in a Riemannian manifold M' of dimension $n + p$. We shall denote by $N(M)$ and α the normal bundle and the second fundamental form of M [3]. The second fundamental form α is an $N(M)$-valued symmetric 2-form on M.

In the following we put

$$E = \mathrm{Hom}\,(T(M), N(M)) = T^*\,(M) \otimes N(M)$$

and we interprete α as an E–valued 1–form β as follows: For any vector field X in M, $\beta(X)$ is a section of E such that

$$\beta(X)\cdot Y = \alpha(X, Y)$$

for all vector field Y in M. Then we have

$$\beta(X)\cdot Y = \beta(Y)\cdot X.$$

We call also β the second fundamental form of M.

A metric along the fibres of E is defined naturally by the Riemann metrics of M and M' and a covariant derivation D_X in E is also naturally defined by the covariant differentiation ∇_X in M and $D_X^{\perp}$ in $N(M)$, where for any normal vector ξ of M, $D_X^{\perp}\xi$ is defined as the normal component of $\nabla_X{}'\xi$, where $\nabla_X{}'$ denote the covariant differentiation in the Riemannian manifold M' (See [3]).

Let φ be a section of E. We may regard φ as an $N(M)$–valued 1–form on M and we have

$$(D_X\varphi)(Y) = D_X^{\perp}(\varphi(Y)) - \varphi(\nabla_X Y)\,,$$
$$\langle D_X\varphi, \psi\rangle + \langle \varphi, D_X\psi\rangle = X\langle \varphi, \psi\rangle$$

for any sections φ and ψ of E.

The following Proposition 1 may be considered as an interpretation of the equation of Codazzi in our formalism.

Proposition 1. *Assume that M' is a Riemannian manifold of constant sectional curvature. Then the second fundamental form β of M satisfies the equation* $\partial\beta = 0$.

Proof. By a straightforward computation we see that

$$\begin{aligned}(\partial\beta(X, Y))(Z) = &\{D_X^{\perp}(\alpha(Y, Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)\}\\ &- \{D_Y^{\perp}(\alpha(X, Z)) - \alpha(\ {}_Y X, Z) - \alpha(X, \nabla_Y Z)\}\end{aligned}$$

and the right hand side is 0 by [3, Vol. II, P. 25, Cor. 4.4].

For each normal vector $\nu \in N_X(M)$ we define an endomorphism A_ν of $T_x(M)$ by the formula

$$\langle A_\nu(u), v\rangle = \langle \beta(u)v, \nu\rangle$$

for any tangent vectors $u, v \in T_x(M)$. The *mean curvature normal* η of M is a normal vector field in M such that

$$\frac{1}{n}\mathrm{Tr}\ A_\nu = \langle \nu, \eta(x)\rangle$$

for any $\nu \in N_x(M)$ and $x \in M$.

M is said to be *minimal* in M' if the mean curvature normal vanishes at each point, that is, if $\mathrm{Tr}\ A_\nu = 0$ for any $\nu \in N_x(M)$ and $x \in M$.

We say that M has a *constant mean curvature* if the mean curvature normal η is parallel, that is, $D_X^\perp \eta = 0$ for any vector field X in M.

Let ν be a normal vector field. Then we have $\mathrm{Tr}\ A_\nu = n\langle \nu, \eta\rangle$ and hence $X\cdot \mathrm{Tr}\ A_\nu = n\{\langle D_X^\perp \nu, \eta\rangle + \langle \nu, D_X^\perp \eta\rangle\}$. Therefore M has a constant mean curvature, if and only if

$$X\cdot \mathrm{Tr} A_\nu = \mathrm{Tr} A_{D_X^\perp \nu}$$

for any normal vector field ν and any vector field X in M.

Proposition 2. *Let M' be a Riemmanian manifold of constant sectional curvature. Then the second fundamental form β of M satisfies the equation $\partial^*\beta = 0$ if and only if M has a constant mean curvature.*

Proof. Let x be a point in M and let $\{e_1,\cdots,e_n\}$ be an orthonormal basis of $T_x(M)$. Let $E_1,\cdots,E_n$ be vector fields in a neighborhood of x such that $(E_i)_x = e_i$ and $\nabla_{E_i}E_k = 0$ at x for $i, k = 1,\cdots,n$. Let(g^{st}) the inverse matrix of the matrix $(\langle E_s, E_t\rangle)$. Then $\partial^*\beta = -g^{st}(D_{E_t}\beta)(E_s)$[1] and$(\partial^*\beta)\cdot E_k = -g^{st}(D_{E_t}\beta)(E_s)\cdot E_k$. Since$(D_{E_t}\beta)(E_s)E_k = D_{E_t}^\perp(\alpha(E_s, E_k)) - \alpha(\nabla_{E_t}E_s, E_k) - \alpha(E_s, \nabla_{E_t}E_k)$ and since α is symmetric, we get $(D_{E_t}\beta)(E_s)\ E_k = (D_{E_t}\beta)(E_k)E_s$. On the other hand, by Proposition 1, we have $\partial\beta = 0$ and hence $(D_{E_t}\beta)(E_k) = (D_{E_k}\beta)(E_t)$, hence $(D_{E_t}\beta)(E_s)E_k = (D_{E_k}\beta)(E_t)\cdot E_s$. Therefore, for any normal vector field ν, we have

$$\begin{aligned}\langle(\partial^*\beta)\cdot E_k, \nu\rangle &= -g^{st}\langle(D_{E_k}\beta)(E_t)E_s, \nu\rangle \\ &= -g^{st}\{\langle D_{E_k}^\perp(\alpha(E_t, E_s)), \nu\rangle - \langle\alpha(\nabla_{E_k}E_t, E_s), \nu\rangle \\ &\qquad - \langle\alpha(E_t, \nabla_{E_k}E_s), \nu\rangle\}.\end{aligned}$$

Now

$$\begin{aligned}&g^{st}\langle D_{E_k}^\perp(\alpha(E_t, E_s), \nu\rangle \\ &= g^{st}\{E_k\langle\alpha(E_t, E_s), \nu\rangle - \langle\alpha(E_t, E_s), D_{E_k}^\perp\nu\rangle \\ &= E_k(g^{st}\langle\alpha(E_t, E_s), \nu\rangle) - (E_k g^{st})\langle\alpha(E_t, E_s), \nu\rangle - g^{st}\langle\alpha(E_t, E_s), D_{E_k}^\perp\nu\rangle \\ &= E_k(T_r A_\nu) - T_r A_{D_{E_k}^\perp \nu} - E_k g^{st}\cdot\langle\alpha(E_t, E_s), \nu\rangle.\end{aligned}$$

1) We omit here the summation signs.

Since $\nabla_{E_k}E_i = 0$ at x, we have $E_k g^{st} = 0$ at x. Therefore we get from the above that

$$\langle(\partial^*\beta)E_k, \nu\rangle(x) = \operatorname{Tr} A_{D^{\perp}_{E}\nu} - E_k(\operatorname{Tr} A_\nu)$$

at x for $k = 1, 2, \cdots, n$ and hence for any vector field X we have $\langle(\partial^*\beta)X, \nu\rangle(x) = \operatorname{Tr} A_{D^{\perp}_{X}\nu} - X(\operatorname{Tr} A_\nu)$at x. Since x is an arbitrary point of M and ν is an arbitrary normal vector field, we see from the above equation that $\partial^*\beta = 0$ if and only if M has a constant mean curvature.

From Propositions 1 and 2 we get the following

Theorem 2. *Let M be a Riemannian manifold immersed isometrically into a Riemannian manifold M' of constant sectional curvature. Let β be the second fundamental form of M regarded as a Hom $(T(M), N(M))$-valued 1-form. Then β satisfies the equation $\Box\beta = 0$, if M has a constant mean curvature. Conversely, if M is compact and orientable and $\Box\beta = 0$, then M has a constant mean curvature.*

3. We shall discuss in this section some applications of Theorems 1 and 2. Let M be a Riemannian manifold immersed isometrically into a Riemannian manifold M' of constant sectional curvature c. Let $x \in M$ and let $\{e_1, \cdots, e_n\}$ and $\{\nu_1, \cdots, \nu_p\}$ be orthonormal bases of $T_x(M)$ and $N_x(M)$ respectively. We shall denote by $A_a (a = 1, 2, \cdots, p)$ the endomorphism of $T_x(M)$ defined by$\langle A_a u, v\rangle = \langle\beta(u)\cdot v, \nu_a\rangle$ and put $A_a \cdot e_i = \sum_j (A_a)^j_i e_j$. Then we have the following Gauss equation:

$$R_{klij} = c\{\delta_{ki}\delta_{lj} - \delta_{kj}\delta_{li}\} + \sum_a \{(A_a)^k_i (A_a)^l_i - (A_a)^k_j (A_a)^l_i\}, \tag{3.1}$$

where R_{klij} denote the components of the curvature tensor with respect to the basis $\{e_1, \cdots, e_n\}$ of $T_x(M)$. Then the endomorphism S of $T_x(M)$ defined by $S(e_j) = \sum_l S_{lj}(e_l)$ with $S_{ej} = \sum_k R_{klkj}$ is of the form

$$S = c(n-1)I + \sum_a (\operatorname{Tr} A_a)A_a - \sum_a A_a^2, \tag{3.2}$$

where I denotes the identity endomorphism of $T_x(M)$.

Let K be the scalar curvature of M. Then $K(x) = \operatorname{Tr} S = c(n-1)n + \sum_a (\operatorname{Tr} A_a)^2 - \sum_a \operatorname{Tr} A_a^2$. The value $\eta(x)$ at x of the mean curvature normal η is given by $\eta(x) = \frac{1}{n}\sum_a \operatorname{Tr} A_a \cdot \nu_a$ and hence $n^2\langle\eta, \eta\rangle(x) = \sum_a (\operatorname{Tr} A_a)^2$. Analogously we have $\langle\beta, \beta\rangle(x) = \sum_a \operatorname{Tr} A_a^2$. Hence we get

$$K = c(n-1)n + n^2\langle\eta, \eta\rangle - \langle\beta, \beta\rangle, \tag{3.3}$$

where β and η denotes the second fundamental form and the mean curvature normal of M respectively. For any Riemannian vector bundle E over M we have defined the endomorphism $\tilde{R}(u, v)$ of the fiber E_x, where $u, v \in T_x(M)$. Let $E = \mathrm{Hom}(T(M), N(M)$ and let $\varphi \in E_x$. Then $\tilde{R}(u, v)\ \varphi$ is an element of $E_x = \mathrm{Hom}\ (T_x(M), N_x(M))$ such that

$$(\tilde{R}(u, v)\varphi)(w) = R^{\perp}(u, v)(w)\varphi) - \varphi(R(u, v)w), \tag{3.4}$$

where $u, v, w \in T_x(M)$ and $R^{\perp}$ denotes the curvature of the Riemannian vector bundle $N(M)$.

Let ν be a normal vector of M at x and let N be a normal vector field such that $N_x = \nu$. Let X and Y be vector fields in M such that $X_x = u$ and $Y_x = v$. Then we have

$$R^{\perp}(u, v)\nu = (D_X^{\perp} D_Y^{\perp} - D_Y^{\perp} D_X^{\perp} - D^{\perp}{}_{[X,Y]})\ N$$

at x.

Denote by ∇' the covariant derivation in the ambiant space M'. Then we have

$$\nabla_X' Y = \nabla_X Y + \alpha(X, Y),$$
$$\nabla_X' N = -A_N(X) + D_X^{\perp} N.$$

We see from these two equations that the normal component $(R'(X, Y)N)^{\perp}$of $R'(X, Y)N$, where R' denotes the curvature tensor of M_i, is equal to $R^{\perp}(X, Y)\ N - \alpha(A_N(Y), X) + \alpha(A_N(X), Y)$. Since M' is of constant curvarute $R'(X, Y)\ N = c\{\langle N, Y\rangle\ X - \langle N, X\rangle Y\} = 0$ and hence we get $R^{\perp}(X, Y)N = -\alpha(A_N(X), Y) + \alpha(A_N(Y), X)$. Thus we have

$$R^{\perp}(u, v)\nu = -\alpha(A_\nu u, v) + \alpha(A_\nu v, u).$$

In particular

$$R^{\perp}(u, v)\nu_a = -\alpha(A_a u, v) + \alpha(u, A_a v).$$

Since $\alpha(A_a u, v) = \sum_b \langle \alpha(A_a u, v), \nu_b \rangle \nu_b = \sum_b (A_b A_a u, v)\ \nu_b$

and $\alpha(u, A_a v) = \sum_b \langle A_b u, A_a v \rangle \nu_b = \sum_b \langle A_a A_b u, v \rangle \nu_b$

we get

$$R^{\perp}(u, v)\nu_a = \sum_b \langle [A_a, A_b]u, v \rangle \nu_b. \tag{3.5}$$

Now by Theorem 1, we have

$$\langle \Box\beta, \beta \rangle = \frac{1}{2} + \Delta\langle \beta, \beta \rangle + \langle D\beta, D\beta \rangle + A,$$

where

$$A(x)=\sum_{i,j}\langle\tilde{R}(e_j,e_i)\beta(e_j),\beta(e_i)\rangle+\sum_i\langle\beta(S(e_i),\beta(e_i)\rangle. \tag{3.6}$$

Now

$$\begin{aligned}\sum_i\langle\beta(S(e_i),\beta(e_i)\rangle&=\sum_{i,j}\langle\alpha(S(e_i),e_j),\alpha(e_i,e_j)\rangle\\&=\sum_{i,j,a}\langle A_a(S(e_i)),e_j\rangle\langle A_a(e_i),e_j\rangle=\sum_a\mathrm{Tr}(SA_a^2)\end{aligned}$$

and by (3.2) we get

$$\begin{aligned}&\sum_i\langle\beta(S(e_i),\beta(e_i)\rangle\\&=c(n-1)\sum_a\mathrm{Tr}\,A_a^2+\sum_{a,b}\mathrm{Tr}\,A_a\cdot\mathrm{Tr}(A_aA_b^2)-\sum_{a,b}\mathrm{Tr}(A_a^2A_b^2).\end{aligned}\tag{3.7}$$

On the other hand,

$$\begin{aligned}&\sum_{i,j}\langle\tilde{R}(e_j,e_i)\beta(e_j),\beta(e_i)\rangle\\&=\sum_{i,j,k}\langle R^{\perp}(e_j,e_i)\alpha(e_j,e_k),\alpha(e_i,e_k)\rangle-\sum_{i,j,k}\langle\alpha(e_j,R(e_j,e_i)e_k),\alpha(e_i,e_k)\rangle\\&=\sum_{i,j,k}\sum_{a,b}\langle A_ae_j,e_k\rangle\langle A_be_i,e_k\rangle\langle R^{\perp}(e_j,e_i)\nu_a\nu_b\rangle\\&-\sum_{i,j,k}\sum_a\langle A_ae_j,R(e_j,e_i)e_k\rangle\langle A_ae_i,e_k\rangle\end{aligned}$$

and by (3.5), the first term equals $\sum_{a,b}\mathrm{Tr}(A_aA_b[A_a,A_b])=-\sum_{a,b}\mathrm{Tr}(A_a^2A_a^2)+\sum_{a,b}\mathrm{Tr}(A_aA_b)^2$ and by the Gauss equation (3.1) the second term equals $-c\sum_a(\mathrm{Tr}\,A_a)^2+c\sum_a\mathrm{Tr}(A_a^2)-\sum_{a,b}(\mathrm{Tr}(A_aA_b))^2+\sum_{a,b}\mathrm{Tr}(A_aA_b)^2$.

Therefore we have

$$\begin{aligned}&\sum_{i,j}\langle\tilde{R}(e_u,e_i)\beta(e_j),\beta(e_i)\rangle\\&=c\sum_a\mathrm{Tr}\,A_a^2-c\sum_v(\mathrm{Tr}\,A_a)^2-\sum_{a,b}\mathrm{Tr}(A_a^2A_a^2)-\sum_{a,b}(\mathrm{Tr}(A_aA_b))^2+\\&\quad+2\sum_{a,b}\mathrm{Tr}(A_aA_b)^2\end{aligned}\tag{3.8}$$

Then we get from (3.6), (3.7) and (3.8) that

$$\begin{aligned}A(x)=cn\sum_a\mathrm{Tr}\,A_b^2-c\sum_a(\mathrm{Tr}\,A_a)^2-\sum_{a,b}(\mathrm{Tr}(A_aA_b))^2\\+\sum_{a,b}\mathrm{Tr}\,A_a\cdot\mathrm{Tr}(A_aA_b^2)+\sum_{a,b}\mathrm{Tr}[A_a,A_b]^2.\end{aligned}\tag{3.9}$$

Now let $\lambda_1^{(a)},\cdots,\lambda_n^{(a)}$ be eigen-values of A_a and let $\{e_1^{(a)},\cdots,e_n^{(n)}\{$ be an orthonormal basis of $T_x(M)$ such that $A_ae_i^{(a)}=\lambda_i^{(a)}e_i^{(a)}(i=1,\cdots,n,\ a=1,\cdots,p)$.

We shall denote by $K_{ij}^{(a)}$ the sectional curvature for the 2–plane spanned by $e_i^{(a)}$ and $e_j^{(a)}$, $i\neq j$.

We show that

$$A(x)=\sum_v\sum_{i<j}(\lambda_i^{(a)}-\lambda_j^{(a)})^2K_{ij}^{(a)}+\frac{1}{2}\sum_{a,b}\mathrm{Tr}[A_a,A_b]^2. \tag{3.10}$$

We write $A(x)$ in the following form:

$$\begin{aligned}A(x)=B(x)&+\sum_{a\neq b}\mathrm{Tr}\,A_a\cdot\mathrm{Tr}(A_aA_b^2)-\sum_{a\neq b}(\mathrm{Tr}(A_aA_b))^2+\\&+\sum_{a,b}\mathrm{Tr}[A_a,A_b]^2,\end{aligned} \tag{3.11}$$

where

$$B(x)=\sum_a\{cn\ \mathrm{Tr}\,A_a^2-c(\mathrm{Tr}\,A_a)^2-(\mathrm{Tr}\,A_a^2)^2+\mathrm{Tr}\,A_a\cdot\mathrm{Tr}A_a^3\}. \tag{3.12}$$

Now by a lemma of Nomizu-Smyth [4] we have

$$\begin{aligned}&\mathrm{cn}\ \mathrm{Tr}\,A_a^2-c(\mathrm{Tr}\,A_a)^2-(\mathrm{Tr}\,A_a^2)^2+\mathrm{Tr}\,A_a\cdot\mathrm{Tr}(A_a)^3\\&=\sum_{i<j}(\lambda_i^{(a)}-\lambda_j^{(a)})^2(c+\lambda_j^{(a)}\lambda_j^{(a)})\end{aligned} \tag{3.13}$$

for each a. Now fix an index a and let

$$A_be_i^{(a)}=\sum_j(A_b)_i^je_j^{(a)}\quad(b=1,2,\cdots,p)$$

Then we have $(A_a)_j^i=\delta_j^i\lambda_j^{(a)}$ and hence

$$(A_aA_b)_j^i=\lambda_i^{(a)}{}_b(A_b)_j^i,\ (A_bA_a)_j^i=(A_b)_j^i\lambda_j^{(a)}. \tag{3.14}$$

By the equation of Gauss we have

$$\begin{aligned}K_{ij}^{(a)}&=R(e_i^{(a)},\mathrm{e}_j^{(a)},e_i^{(a)},e_j^{(a)})\\&=c+\sum_b(A_b)_i^i(A_b)_j^j-\sum_b(A_b)_j^i(A_b)_i^j.\\&=c+\lambda_i^{(a)}\lambda_j^{(a)}+\sum_{b\neq a}(A_b)_i^i(A_b)_j^j-\sum_b(A_b)_j^i(A_b)_i^j.\end{aligned}$$

Hence we have

$$\begin{aligned}&(\lambda_i^{(a)}\lambda-{}_j^{(a)})^2(c+\lambda_i^{(a)}\lambda_j^{(a)})\\&=(\lambda_i^{(a)}-\lambda_j^{(a)})^2K_{ij}^{(a)}+\sum_b(\lambda_i^{(a)}-\lambda_j^{(a)})^2(A_b)_j^i(A_b)_i^j\\&-\sum_{b\neq a}(\lambda_i^{(a)}-\lambda_j^{(a)})^2(A_b)_i^i(A_b)_j^j.\end{aligned}$$

This equality holds also for $i=j$ trivially if we define $K_{ii}^{(a)}=0$.

Then by (3.14)

$$\sum_{i<j}(\lambda_i^{(a)}-\lambda_j^{(a)})^2(c+\lambda_i^{(a)}\lambda_j^{(a)})=\frac{1}{2}\sum_{i,j}(\lambda_i^{(a)}-\lambda_j^{(a)})^2(c+\lambda_i^{(a)}\lambda_j^{(a)})$$

$$=\sum_{i<j}(\lambda_i^{(a)}-\lambda_j^{(a)})^2K_{ij}^{(a)}-\frac{1}{2}\sum_b\sum_{i,j}(\lambda_i^{(a)}-\lambda_j^{(a)})(A_b)_j^i(\lambda_j^{(a)}-\lambda_i^{(a)})(A_b)_i^j$$

$$-\frac{1}{2}\sum_{b\neq a}\{\sum_{i,j}(\lambda_i^{(a)})^2(A_a)_i^i\sum_j(A_b)_j^j-2\sum_i\lambda_i^{(a)}(A_b)_i^i\sum_j\lambda_j^{(a)}(A_b)_j^j$$
$$+\sum_i(A_b)_i^i\sum_j\lambda_j^{(a)}(A_b)_j^j\}$$
$$=\sum_{i<j}(\lambda_i^{(a)}-\lambda_j^{(a)})^2K_{ij}^{(a)}-\frac{1}{2}\sum_b\mathrm{Tr}[A_a,A_b]^2$$
$$-\sum_{b\neq a}\{\mathrm{Tr}\,A_b\cdot\mathrm{Tr}(A_a^2A_b)-(\mathrm{Tr}(A_aA_b))^2\}.$$

Then we obtain from (3.11), (3.12) and (3.13) the equality (3.10).

Now we cite the following two lemmas from [1].

Lemma 1. *Let A and B be symmetric $n\times n$ matrices. Then*

$$\mathrm{Tr}[A,B]^2\geqq-2\mathrm{Tr}\,A^2\cdot\mathrm{Tr}\,B^2,$$

and the equality holds for non-zero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiple of $\tilde{A}$ and $\tilde{B}$ respectively, where

$$\tilde{A}=\left(\begin{array}{cc|c}0&1&\\1&0&0\\\hline\multicolumn{2}{c|}{0}&0\end{array}\right),\quad\tilde{B}=\left(\begin{array}{cc|c}1&0&\\0&-1&0\\\hline\multicolumn{2}{c|}{0}&0\end{array}\right).\tag{3.15}$$

Lemma 2. *Let A_1, A_2 and A_3 be $n\times n$ symmetric matrices and if*

$$\mathrm{Tr}[A_a,A_b]^2=-2\mathrm{Tr}\,A_a^2\cdot\mathrm{Tr}\,A_b^2$$

for $1\leqq a<b\leqq 3$, then at least one of the matrices A_a must be zero.

By Lemma 1, we have

$$\frac{1}{2}\sum_{a,b}\mathrm{Tr}[A_a,A_b]^2\geqq-\sum_{a\neq b}\mathrm{Tr}\,A_a^2\cdot\mathrm{Tr}\,A_b^2=-2\sum_{a<b}\mathrm{Tr}\,A_a^2\cdot\mathrm{Tr}\,A_b^2.$$

Put $S_a=\mathrm{Tr}\,A_a^2$. Then $\sum_a S_a=\langle\beta,\beta\rangle(x)$.

Since

$$0\leqq\sum_{a<b}(S_a-S_b)^2=\sum_{a<b}(S_a^2+S_b^2)-2\sum_{a<b}S_aS_b$$
$$=(p-1)\sum_a S_a^2-2\sum_{a<b}S_aS_b$$
$$=(p-1)\{(\sum_a S_a)^2-2\sum_{a<b}S_aS_b\}-2\sum_{a<b}S_aS_b$$
$$=(p-1)\langle\beta,\beta\rangle^2(x)-2p\sum_{a<b}S_aS_b$$

we have

$$-2\sum_{a<b} S_a S_b \geqq -\frac{(p-1)}{p}\langle\beta, \beta\rangle^2(x)$$

ane here the equality holds if and only if $S_a = S_b$ for $a,b=1,\cdots,p$. Therefore we get

$$\frac{1}{2}\sum_{a,b}\mathrm{Tr}[A_a, A_b]^2 \geqq -\frac{(p-1)}{p}\langle\beta, \beta\rangle^2(x) \tag{3.16}$$

and the equality holds if and only if $\mathrm{Tr}\, A_a^2 = \mathrm{Tr}\, A_b^2 = \mathrm{Tr}\, A_b^2$ for $a, b=1,\cdots,p$ and either A_a are all zero except possibly one of them or A_a are all zero except two of them, say A_1 and A_2, and they can be transformed simultaneously by an orthogonal matrix into scalar multiple of the matrices of the form (3.15). Thus we obtain from (3.10) the inequality

$$A(x) \geqq \sum_a \sum_{i<j} (\lambda_i^{(a)}-\lambda_j^{(a)})^2 K_{ij}^{(a)} - \frac{p-1}{p}\langle\beta, \beta\rangle^2(x). \tag{3.17}$$

Assume now that the scalar curvatures of M are bounded below by a positive constant d. Then

$$\sum_a \sum_{i<j} (\lambda_i^{(a)}-\lambda_j^{(a)})^2 K_{ij}^{(a)} \geqq d\sum_a \sum_{i<j} (\lambda_i^{(a)}-\lambda_j^{(a)})^2$$

and

$$\sum_{i<j} (\lambda_i^{(a)}-\lambda_j^{(a)})^2 = (n-1)\mathrm{Tr}\, A_a^2 - 2\sum_{i<j}\lambda_i^{(a)}\lambda_j^{(a)}$$
$$-2\sum_{i<j}\lambda_i^{(a)}\lambda_j^{(a)} = \mathrm{Tr}\, A_a^2 - (\mathrm{Tr}\, A_a)^2$$

and hence

$$\sum_a \sum_{i<j} (\lambda_i^{(a)}-\lambda_j^{(a)})^2 = n\langle\beta, \beta\rangle(x) - n^2\langle\eta, \eta\rangle(x),$$

where η denotes the mean curvature normal of M. Thus we get the following inequality

$$A \geqq \left(dn - \frac{p-1}{p}\langle\beta, \beta\rangle\right)\langle\beta, \beta\rangle - dn^2\langle\eta, \eta\rangle \tag{3.17}$$

at each point of M.

We obtain from Corollaries 1 and 2 of Theorem 1 and Theorem 2 the following

Theorem 3. *Let M be an n-dimensional, Riemannian manifold with sectional curvatures bounded below by a positive constant d. Assume that M is immersed in a Riemannian manifold M' of constant sectional curvature of dimension $n+p$ and that M has a constant mean curvature. Then, if M is compact and orientable or if the length of the second fundamental form β of M is constant, then we have*

$$0 \geqq A \geqq \left\{dn - \frac{p-1}{p}\langle\beta, \beta\rangle\right\}\langle\beta, \beta\rangle - dn^2\langle\eta, \eta\rangle \tag{3.18}$$

at each point of M, where η denotes the mean curvature normal of M which is parallel and $\langle\eta, \eta\rangle$ is a constant.

Now assume M is compact and oriented and let $k = \langle\eta, \eta\rangle$. Then integrating both sides of the inequality (3.18) we obtain

$$dn^2k\int_M *1 \geqq \int_M \left\{dn - \frac{p-1}{p}\langle\beta, \beta\rangle\right\}\langle\beta, \beta\rangle *1$$

and we have the equality here if and only if

$$dn^2k = \left\{dn - \frac{p-1}{p}\langle\beta, \beta\rangle\right\}\langle\beta, \beta\rangle$$

and this implies also that $A = 0$ and that β is parallel by Theorem 1. Then $\langle\beta, \beta\rangle$ must satisfy the quadratic equation $(p-1)x^2 - p\,dn\,x + pn^2k = 0$ and since the discriminant of this equation should be positive we should have the inequality

$$d \geqq \frac{4k(p-1)}{p}.$$

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