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TIGHT 4-DESIGNS

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The purpose of this paper is to fill a gap in [1]. The result is as follows:

Theorem. 1. *There exist only finitely many tight 4-designs.*

Actually our result is considerably more explicit. Furthermore we do not think that there exist tight 4-designs other than Witt designs.

[1] and [2] will be quoted as (T) and (C). We use the same notation as in (T).

1. $d_1 = e - 1$

Lemma 1. *Put $w = v - 2k$. Then we have that*

$$(1) \quad \begin{aligned} w^2 &= v^2 - 2(2ae + a + 1)v + 2(6ae + 3a - 1) \\ &= (v - 2)^2 - 2(2ae + a - 1)(v - 3). \end{aligned}$$

Proof. By (C), (5) we have that

$$(2) \quad \begin{aligned} (a^2 - 1)k^2 - \{4a^2e^2 + 4a(a - 2)e + (3a - 1)(a - 1)\}k \\ + 2a^2e(e + 1)(2ae + a - 3) = 0. \end{aligned}$$

Since k and $v - k$ are solutions of (2), we have that

$$(3) \quad (a^2 - 1)k(v - k) = 2a^2e(e + 1)(2ae + a - 3)$$

and

$$(4) \quad (a^2 - 1)v = 4a^2e^2 + 4a(a - 2)e + (3a - 1)(a - 1).$$

Since $w^2 - v^2 = -4k(v - k)$, (1) follows from (3) and (4).

The next lemma is crucial.

Lemma 2. *There exists a positive integer N such that*

$$(5) \quad (2ae + a - 1)N = (a^2 - 1)^2.$$

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Proof. By (30) and (47) of (T) we have that

$$(6) \quad 2(v-2)y = 2kx = (v-w)(v-w-2ae-a-1).$$

By Lemma 1 the right-hand side of (6) equals

$$(7) \quad \begin{aligned} & (v-w)^2 - (v-w)(2ae+a+1) \\ &= v^2 + w^2 - 2vw - (2ae+a+1)v + (2ae+a+1)w \\ &= 2v^2 - (2w+6ae+3a+3)v + 2(6ae+3a-1) + (2ae+a+1)w \\ &= (v-2)\{2(v-w) - 6ae - 3a + 1\} + (2ae+a-3)w. \end{aligned}$$

Now by (C), (3) we have that

$$(8) \quad (a^2-1)(v-2) = (2ae+a-1)(2ae+a-3).$$

Hence from (8) we have that

$$(9) \quad (2ae+a-3)w = \frac{(a^2-1)(v-2)w}{2ae+a-1}.$$

Furthermore, if a is odd, then the second factor of the first term of the right-hand side of (7) is even. Thus by (8), (9) and Lemma 1 we see that $2ae+a-1$ is a divisor of $(a^2-1)^2$.

From (5) we have that $N \equiv -1 \pmod{a}$. Hence we may put

$$(10) \quad N = Ma - 1,$$

where M is a positive integer. Then from (5) and (10) we have that

$$(11) \quad 2e(Ma-1) = (a-1)(a^2+a-1-M).$$

Lemma 3. *Assume that $d_1 < e-1$. Then we have that*

$$(12) \quad M < \frac{1}{4}a.$$

Proof. Assume that $M \geq \frac{1}{4}a$. Then from (11) of (T) and (11) we have that

$$(4a+2)(\frac{1}{4}a^2-1) \leq (a-1)(a^2+a-1-\frac{1}{4}a),$$

which implies that $a \leq 4$.

Lemma 4. $a \neq M^2 + M - 1$.

Proof. Assume that $a = M^2 + M - 1$. Then from (11) we have that $2e = M^3 + 2M^2 - M - 2$. By (C), (3) $a^2 - 1$ is a divisor of $4(e-a)(e-a+1)$. Now we have that $\frac{4(e-a)(e-a+1)}{a^2-1} = M^2 - 2M + 1 + \frac{4}{M+1}$. Thus $M=1$ or $M=4$. If $M=1$, then $a=1$. If $M=3$, then $a=11$ and $e=20$. Then from (C), (5) we have

that

$$k^2 - 1682k + 379456 = 0,$$

which has no rational integral solution.

From now on we assume that $d_1 < e - 1$, though some of the assertions may be independent from this assumption.

By Lemma 3 we may put

$$(13) \quad a = zM + L,$$

where z is a positive integer and $0 \leq L < M$. Now from (11) we have that $-2e \equiv M + 1 \pmod{a}$. So we may put

$$(14) \quad 2e = Ka - M - 1,$$

where K is a positive integer.

Lemma 5. $2e = za + a - M - 1$.

Proof. By (11) and (14) we have that $K(Ma - 1) = a^2 - 2 + M^2$. By (13) this implies that

$$(15) \quad K = z + \frac{La - 2 + M^2 + z}{Ma - 1}.$$

By Lemma 3 the last term of (15) must be equal to 1. So Lemma 5 follows from (11) and (14). Moreover, from (15) we have that

$$(16) \quad Ma + 1 = La + M^2 + z.$$

Lemma 6. *We have that*

$$(17) \quad 2e < a^{3/2} + a.$$

Proof. If $z \geq M$, then by (13) $M^2 \leq a$. So from (16) we have that $L = M - 1$. Then from (13) and (16) we have that $a = M^2 + z - 1 = zM + M - 1$. Thus $M = z$ and $a = M^2 + M - 1$ contradicting Lemma 4. Thus $M > z$ and so by (13) we have that $a < M^2$. Then by (13) we have that

$$2e = za + a - M - 1 \leq (a^2 - La) + M + a - M - 1 < a^{3/2} + a.$$

Lemma 7. $2ae + a - 1$ divides $(J + 1)(2e - a + 1)$. (See p. 519 of [1].)

Proof. Since $a(2e - a + 1) = (2ae + a - 1) - (a^2 - 1)$, the greatest common divisor of $2ae + a - 1$ and $a^2 - 1$ equals that of $2e - a + 1$ and $a^2 - 1$. Thus by Lemma 2 we see that $2ae + a - 1$ divides $(2e - a + 1)^2$. Since $(2e - a + 1)^2 + 2(2ae + a - 1) = (a^2 - 1) + (v - 2) - (J + 1)$, by (8) and Lemma 2 we see that $2ae + a - 1$ divides $(a^2 - 1)(J + 1)$ and hence $(2e - a + 1)(J + 1)$.

Lemma 8. *We have that*

$$(18) \quad 2e > a^{3/2} + a.$$

Proof. By Lemma 7 and (C), (2) we have that $J+1 > a$. So by (C), (1) we have that

$$4(e-a)(e-a+1) > (a+3)(a^2-1),$$

which implies that

$$2e > a^{3/2} + 2a - 2 > a^{3/2} + a.$$

(17) and (18) are obviously in contradiction. Thus we obtain that $d_1 = e - 1$.

2. Diophantine equation

Two cases are remaining, namely Cases 2 and 3 of (C), § 1.

Lemma 9. *Case 2 cannot occur.*

Proof. By (iii) of (C), § 1, Case 2 and (C), (1) we have that $J = -2$. Since Lemma 7 holds without the assumption that $d_1 < e - 1$, we see that $2ae + a - 1$ divides $2e - a + 1$. This implies that $a = 1$.

Thus by (C), § 1 we finally have that

$$(19) \quad v = 4e^2 + 4e + 2$$

and

$$(20) \quad a^2 - 8ae + 4e^2 - 4a + 4e + 3 = 0.$$

By (19) and (20) we have that

$$(21) \quad 2(v-2) = (2e-a+1)(6e-a+3)$$

and

$$(22) \quad 2(2ea+a-1) = (2e-a+1)^2.$$

Then by (1), (21) and (22) we obtain that

$$(23) \quad 4f^2 = (6e-a+3)^2 - 4(4e^2+4e-1),$$

where f is a positive integer.

Since $(4e+2-a)^2 = 3(2e+1)^2 - 2$ by (20), from (23) we have that $(2f^2-3)^2 = (2e+1)^2 \{3(2e+1)^2 - 2\}$. Thus $2e+1$ and f are integral rational solutions of

$$(24) \quad 3X^4 - 4Y^4 - 2X^2 + 12Y^2 - 9 = 0.$$

Since (24) has only finitely many integral rational solutions ([3], p. 276), it follows that there exist only finitely many possibilities for a and e being pa-

parameters of tight 4-designs.

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